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THE SHAPE OF THE NUCLEAR PHOTO-RESONANCE IN DEFORMED NUCLEI

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Abstract: The characteristic features of photonuclear resonance spectra of deformed nuclei are discussed on the basis of some simple estimates in which we employ nucleonic wave functions in the approximate, "asymptotic" form. The likelihood, in different regions of deformed nuclei, of a resolution of the resonance spectrum into two separate peaks is discussed. Finally, in the appendix, we exhibit a convenient operator method for deriving various nuclear matrix elements in the representation based on the "asymptotic" wave functions.

1. Introduction

In an ellipsoidal potential the frequency of particle motion is higher for motion along the symmetry axis than for motion perpendicular to that axis. Thus, one expects that the nuclear dipole photo-resonance will be split or at least broadened in deformed nuclei, and indeed such an effect has recently been observed ††. This effect has been discussed earlier by D. H. Wilkinson (ref. [5])), who calculated the actual resonance shape for a few representative nuclei, employing the detailed wave functions of ref. [6]). The present calculations are based on the more approximate "asymptotic" wave functions (see the appendix). In the latter case one can obtain simple estimates in a closed form, which may facilitate the understanding of the different physical effects that determine the resonance shape.

If we start from an axially symmetric, but non-isotropic, harmonic oscillator potential for the single-particle motion, we would expect the photo-resonance to consist of two lines, one at $\hbar\omega_z$, corresponding to excitation of particle motion along the symmetry axis, and one at $\hbar\omega_\perp$ for excitation of motion in a plane perpendicular to the symmetry axis. Since all the well-known non-spherical nuclei have a prolate shape, ω_z is smaller than $\omega_\perp$. Also, the total strength of the line at $\hbar\omega_\perp$ is twice that of the line at $\hbar\omega_z$.

These lines will be broadened by a number of effects. First, the nucleus

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†† For a summary of the theoretical and experimental data on this effect see ref.[1]); cf. also refs. [2–4]).

may be set into rotation as a consequence of its interaction with the photon. If the ground-state spin is I_0, the rotational states with total angular momenta I_0+1, I_0 and I_0-1 may be excited. In the rare-earth nuclei and for the heavy elements the energy difference between these states will be only a few hundred keV; however, in the region of deformed nuclei around $A = 25$ these rotational energies should amount to about 2 MeV (for $I_0 = \frac{5}{2}$). In any case for $I_0 = 0$ (as in even nuclei) the E1 process can excite only states with $I = 1$.

A second source of broadening of the lines is associated with the deviations from a harmonic oscillator potential. It is easy to see that the types of deviation which have been especially considered [6]) will tend systematically to broaden the line at $\hbar\omega_\perp$ more than the line at $\hbar\omega_z$. Thus, the spin-orbit force depends, in first approximation, only on the component of the orbital angular momentum of the particle along the symmetry axis, Λ; for the transitions corresponding to $\hbar\omega_z$ this angular momentum is not changed, and the corresponding line is not split by this term. The transition with $\hbar\omega_\perp$ is associated with a change $\Delta\Lambda = \pm 1$, and there is thus a splitting of the high-frequency line due to the spin-orbit force. Also deviations from the harmonic oscillator potential that are proportional to the square of the angular momentum of the particle, $\mathbf{l}^2$, are found to broaden the high-frequency line more than the lower one. It is the main purpose of this note to consider these effects somewhat more quantitatively in order to obtain a better estimate of the systematics to be expected for the dipole resonance in deformed nuclei.

Finally, another important source of broadening should be mentioned. The final state of motion described by a single, highly excited nucleon moving with respect to the rest of the nucleus is of course very far from a stationary state of the system; the excited nucleon may either fly out of the nucleus or, more likely, make a collision with one of the other nucleons, leading eventually to the complicated many-particle states of motion characteristic of the compound nucleus. This mechanism, which tends to damp the single-particle motion, will thus also lead to a broadening of the photo-resonance. One may obtain an estimate of these effects from the magnitude of the absorptive part of the optical potential used to describe the interaction of nucleons with the nucleus. From this source we might expect a broadening of a few MeV. The absorptive potential is an increasing function of excitation energy in this region; in the energy interval between $\hbar\omega_z$ and $\hbar\omega_\perp$ one might expect a change of the order of about 30 %.

2. Photo Excitation Spectrum in the Asymptotic Limit

The sum of the transition intensities of the E1 transitions leading from a state (I, K) associated with a wave function $\chi_K D^I_{MK}$ to all members of a

rotational band based on an intrinsic state $\chi_{K'}$ is proportional to the squared matrix element of the multipole operator involving only the *intrinsic* wave functions of initial and final states:

$$T(\mathrm{E1}; IK \to K') = \tfrac{16}{9}\pi \frac{1}{\hbar}\left(\frac{\omega}{c}\right)^3 \left| \int \chi^*_{K'} M(\mathrm{E1}, K'-K)\chi_K \, \mathrm{d}\tau \right|^2. \quad (1)$$

As a basis for a description of the intrinsic states we first assume an average potential of a pure harmonic oscillator type. An anisotropy of the oscillator potential corresponding to $\omega_z \neq \omega_\perp = \omega_x = \omega_y$ removes some of the degeneracy, and the states can now be characterized by the quantum number n_z, the number of nodal planes perpendicular to the z-axis, in addition to the quantum number N, the total number of nodes of the nucleonic wave function.

Further, Λ, the component of orbital angular momentum along the nuclear symmetry axis, is a constant of the motion by virtue of the assumed rotational symmetry about the z-axis. We may characterize the state of the nucleonic spin by the quantum number $\Sigma = \pm\frac{1}{2}$, representing the component of spin along the symmetry axis. And finally Λ and Σ add up to K, the component of total angular momentum characterizing the intrinsic state. There is of course in this description a complete degeneracy among the different values of Σ and Λ (consistent with a certain value of $n_\perp = N - n_z$). The wave functions may be written in product form as

$$|N n_z \Lambda\Sigma\rangle = |z; n_z\rangle |x\, y; n_\perp \Lambda\rangle |\Sigma\rangle, \quad (2)$$

where $|z; n_z\rangle$ denotes that the state vector associated with the quantum number n_z depends solely on the z-coordinate, etc.

We first note from eq. (1) that there occur two groups of E1 transitions in this limit: one group represents transitions with $\Delta K = 0$ and corresponds to the multipole operator z, the other group corresponds to $\Delta K = \pm 1$ and the operator $2^{-\frac{1}{2}}(x \pm iy)$. From the form of the wave function (2) it is immediately apparent that the $\Delta K = 0$ transitions obey the selection rules $\Delta n_\perp = \Delta\Lambda = \Delta\Sigma = 0$ and $\Delta n_z = \pm 1$ and involve a change of energy equal to $\hbar\omega_z$. The other types of transitions, $\Delta K = \pm 1$, are associated with the selection rules $\Delta n_z = \Delta\Sigma = 0$, $\Delta n_\perp = \pm 1$, $\Delta\Lambda = \pm 1$. The fundamental frequency of these transitions is $\hbar\omega_\perp$. This simplified situation is illustrated in fig. 1, where we have taken as an example a filled $N = 2$ shell from which nucleons are excited into the empty $N = 3$ shell. As a consequence of the fact that the transitions are associated with ΔK equal to both $+1$ and -1, the $\hbar\omega_\perp$ transition line is twice as strong as the $\hbar\omega_z$ line in this description.

Introduction of other terms in the nucleon potential of the type $-(2\mathbf{l}\cdot\mathbf{s} + \mu\mathbf{l}^2)\kappa\hbar\omega_0$ will now remove the Λ and Σ degeneracy. However, the "asymptotic quantum" numbers $(N, n_z, \Lambda, \Sigma)$ still remain useful as long as

these added terms are small compared with the splitting, $\hbar\omega_\perp - \hbar\omega_z$, caused by the deformation of the nuclear field. This condition seems to be fairly well satisfied (i.e. with an accuracy of about 10 % for the greatly deformed nuclei, and thus we may still use the wave function (2) as a first approximation.

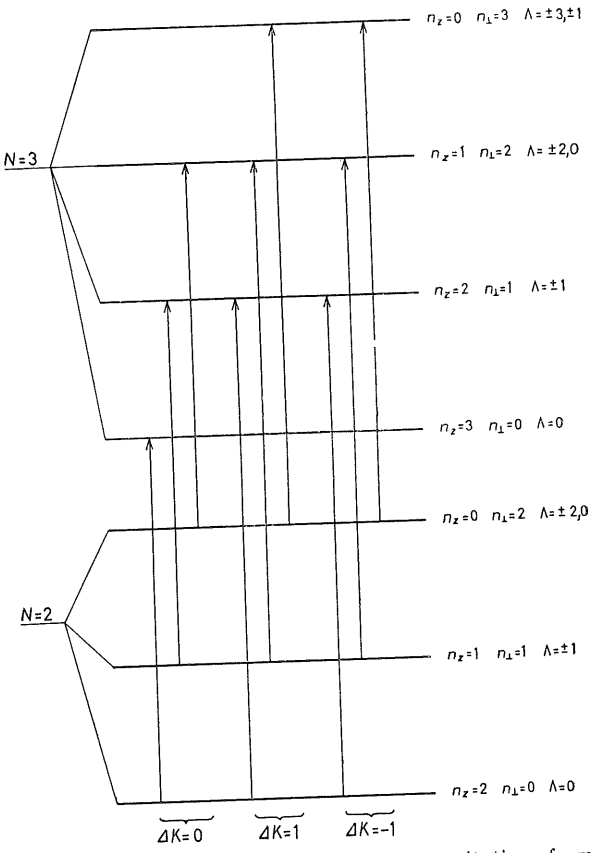

Fig. 1. The diagram illustrates possible single-particle E1 excitations from a filled harmonic oscillator shell $N = 2$ into an empty $N = 3$ shell. Part of the spherical degeneracy of the N shell is removed by a quadrupole deformation of the oscillator field having $\omega_z < \omega_\perp$. In this example, however, in contrast to the case of fig. 2, the levels are assumed degenerate in Λ and Σ.

The energy eigenvalues of each intrinsic orbital may now be written (to lowest approximation)

$$E(n_z n_\perp \Lambda\Sigma) = (n_\perp+1)\hbar\omega_\perp + (n_z+\tfrac{1}{2})\hbar\omega_z - \kappa\hbar\omega_0[2\Lambda\Sigma+\mu(2n_z n_\perp+2n_z+n_\perp+\Lambda^2)]. \quad (3)$$

2.1 TRANSITION LINES CORRESPONDING TO $\Delta K = 0$

The multipole operator corresponding to $\Delta K = 0$, being associated with the selection rules $\Delta n_\perp = \Delta\Lambda = \Delta\Sigma = 0$ and $\Delta n_z = \pm 1$, connects states

with energy differences

$$E(n_z+1,\, n_\perp \Lambda\Sigma)-E(n_z n_\perp \Lambda\Sigma) = \hbar\omega_z+\delta E^{\mathrm{I}} 2\mu\kappa\hbar\omega_0, \tag{4}$$

where

$$\delta E^{\mathrm{I}} = -(n_\perp+1). \tag{4a}$$

The splitting of the group is caused by the $\mathbf{l}^2$-term and corresponds to the occurrence of the quadratic terms proportional to $n_\perp n_z$ in eq. (3).

The different transitions within the group $\Delta K = 0$ and the corresponding intensities are listed below in table 1.

TABLE 1

n_z	$n_\perp$	$-\delta E^{\mathrm{I}}$	degeneracy	$2\lvert$matr. elem.$\rvert^2$	deg. $\times\, 2\lvert$matr. elem.$\rvert^2$
0	N	$N+1$	N+1	1	$(N+1)\cdot 1$
1	$N-1$	N	N	2	$N\cdot 2$
2	$N-2$	$N-1$	$N-1$	3	$(N-1)\cdot 3$
·	·	·	·	·	·
·	·	·	·	·	·
n_z	$N-n_z$	$N-n_z+1$	$N-n_z+1$	n_z+1	$(N-n_z+1)\cdot(n_z+1)$
·	·	·	·	·	·
·	·	·	·	·	·
$N-1$	1	2	2	N	$2\cdot N$
N	0	1	1	$N+1$	$1\cdot(N+1)$

The intensity of the transitions from a state of given n_z and characterized by $\Delta K = 0$ is thus proportional to the following expression:

$$f_z(n_z) = (N-n_z+1)(n_z+1). \tag{5}$$

We can now combine (5) with the expression (4) for the energies of the various transitions to obtain the line shape for the transitions with $\Delta K = 0$, i.e. those transitions which for $\kappa = 0$ would have the energy $\hbar\omega_z$. Several examples illustrating the effect of the added terms in broadening the line are shown in the figures. The line width may be characterized by the mean square deviation

$$\sigma^2 = \frac{\sum I(E)(E-\bar{E})^2}{\sum I(E)} = (\mu\kappa\hbar\omega_0)^2 \tfrac{1}{5} N(N+4). \tag{6}$$

2.2. TRANSITION LINES CORRESPONDING TO $\Delta K = \pm 1$

The $\Delta K = \pm 1$ transitions will be broadened by both the $\mathbf{l}^2$ and the $\mathbf{l}\cdot\mathbf{s}$ term in the nuclear potential. The energy differences connected with these transitions are then given in the asymptotic limit as

$$E(n_z,\, n_\perp+1,\, \Lambda\pm 1,\, \Sigma)-E(n_z n_\perp \Lambda\Sigma) = \hbar\omega_\perp+2\mu\kappa\hbar\omega_0\cdot\delta E^{\mathrm{II}}, \tag{7}$$

where

$$\delta E^{\mathrm{II}} = -(N+1) \mp \frac{1}{2\mu} \Sigma + 2 \begin{Bmatrix} s \\ r \end{Bmatrix}, \quad \text{for } \Delta\Lambda = \pm 1, \tag{7a}$$

with

$$2r = n_\perp + \Lambda, \quad 2s = n_\perp - \Lambda, \tag{7b}$$

s and r thus corresponding, in eq. (7a), to $\Delta\Lambda = +1$ and $\Delta\Lambda = -1$ respectively.

The quantum numbers r and s are especially convenient for the discussion of the spectrum and matrix elements in the asymptotic limit (cf. the appendix). Considering first, for instance, the case of $\Delta K = +1$, the energy

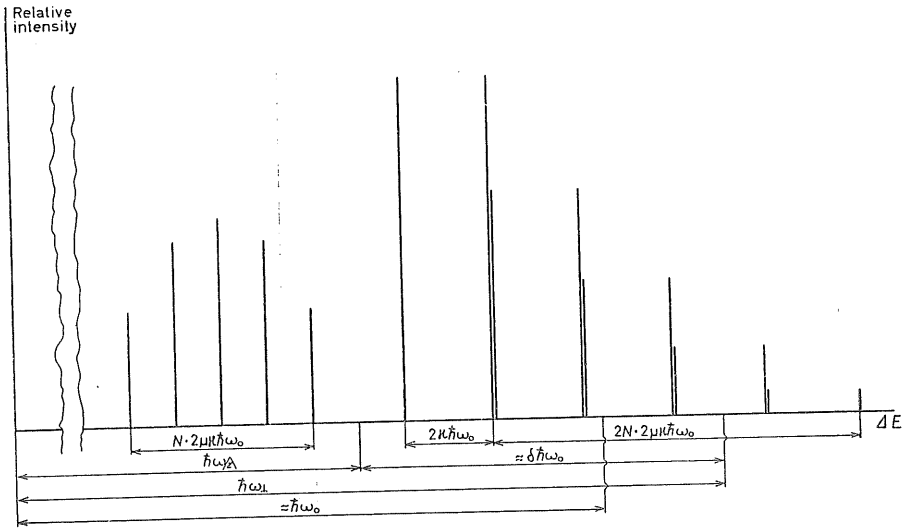

Fig. 2. The diagram exhibits the theoretical line spectrum in the giant-resonance region, calculated on the basis of the single-particle energies of a harmonic oscillator potential, where diagonal terms of $\mathbf{l}\cdot\mathbf{s}$ and $\mathbf{l}^2$ have been included. The effect of these added terms is to remove the degeneracy of the single-particle spectrum in Λ and Σ (the components of angular momentum and spin) and thereby to broaden the two fundamental peaks of energy $\hbar\omega_z$ and $\hbar\omega_\perp$ (cf. fig. 1).

differences are expressible solely in terms of Σ and s (see eq. (7)). Each such energy difference is associated with a degeneracy in r, since r may be $0, 1, \ldots, N-s$, and each of these transitions is in its turn connected with an intensity $\propto(r+1)$. The total oscillator strength attributed to the radiation energy $E(s, \Sigma)$ is thus proportional to

$$\sum_{r=0}^{N-s} (r+1) = \tfrac{1}{2}(N-s+1)(N-s+2) = f_\perp(s). \tag{8}$$

We obtain the line shape by combining $f_\perp(s)$ with the expression (7) and letting s take on each of the allowed values $0 \leqq s \leqq N$. There are then two

line spectra of the type $f_\perp(s)$, transposed an energy $\kappa\hbar\omega_0$ with respect to one another, corresponding to the two values of $\Sigma = \pm\frac{1}{2}$ (see eq. (7)).

The intensity distribution corresponding to the case of $\Delta K = -1$ is identical with the $\Delta K = +1$ spectrum. One may note that for this latter spectrum Σ corresponds to $-\Sigma$ and r to s in the former case.

We may now write the total intensity distribution corresponding to $\Delta K = \pm 1$ as

$$\sum_x f_\perp(x) = \sum_{x=0}^{N} \{(N+1-x)(N+2-x)\} + \sum_{x=\frac{1}{4\mu}}^{N+\frac{1}{4\mu}} \left(N+1+\frac{1}{4\mu}-x\right)\left(N+2+\frac{1}{4\mu}-x\right), \tag{9}$$

where the integral values of x correspond to equidistant spectral lines with a spacing equal to $4\mu\kappa\hbar\omega_0$. For this line shape we have a mean square width

$$\sigma^2 = \left[\tfrac{3}{5}N(N+4)+\left(\frac{1}{2\mu}\right)^2\right](\mu\kappa\hbar\omega_0)^2. \tag{10}$$

2.3. CHARACTERISTICS OF THE TOTAL RESONANCE SPECTRUM

Comparing the two peaks associated with $\hbar\omega_\perp$ and $\hbar\omega_z$, for instance for the case of $\mu \approx 0.5$ and $N \approx 4-5$ (corresponding to the "rare earth" region and the "heavy element" region of nuclei, displaying large equilibrium deformations), one finds a ratio between the mean square widths of the order 1.7.

In figs. 3 and 4 we have smeared out the sharp lines given by this model over energy intervals of the order 0.5 MeV in the heavy-element region †. Fig. 3 illustrates the expected type of resonance spectrum at the beginning of the "rare earth" region of deformed nuclei ††, say Tb^{159}, corresponding to a very large deformation $\delta \approx 0.35$. Fig. 4 is representative of the expected resonance spectra of the heavy elements beyond $A \approx 230$ [and also of nuclei

† We have so far, in deriving the spectral distributions (5) and (9), implicitly assumed that the single-particle states are filled through the whole shell N_0. Only nucleons of the shell N_0 may then be excited and lifted up to the shell N_0+1.

We have thus neglected the fact that, owing to the spin-orbit forces, these shells may largely overlap. The result will be that some orbitals of the N_0+1 shell are filled before the N_0 shell is completed. This effect will tend to slightly broaden both resonance peaks in a similar way. It can be approximately provided for by the use of an intermediate value of N between N_0 and N_0+1 in calculating the widths.

Furthermore, in two of the regions of large nuclear deformation the neutron and proton numbers differ considerably, and actually the N_0 shell is filled for protons simultaneously with the N_0+1 shell for neutrons. Of course one may take this effect into account simply by calculating the neutron and proton spectra (with the respective effective charges) separately and finally adding them up. This correction is small, of the order $1/N$, and parallel for the two resonance peaks.

†† Cf. the new experimental results by E. G. Fuller and M. S. Weiss (ref. [4])), which may indicate a resolved double peak for Tb and Ta.

at the end of the "rare earth" region, say Ta^{181}], where the deformation is smaller, $\delta \approx 0.25$, and the width of each of the two peaks is larger,

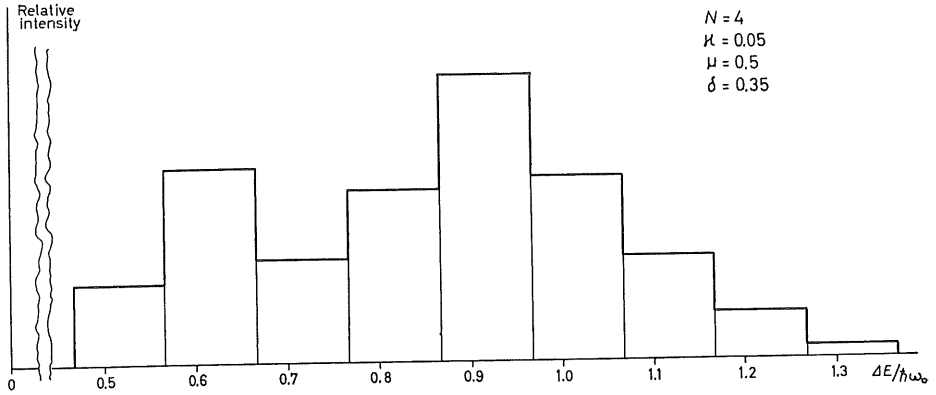

Fig. 3. Histogram plotted on the basis of the theoretical line spectrum of fig. 2 for the case of $N = 4$ and $\delta \approx 0.35$. The shell parameters of this spectrum are chosen so as to be applicable to nuclei at the beginning of the rare-earth region, say $153 \lesssim A \lesssim 170$. It appears that it might be possible to resolve experimentally the two peaks of the spectrum in this region of nuclei.

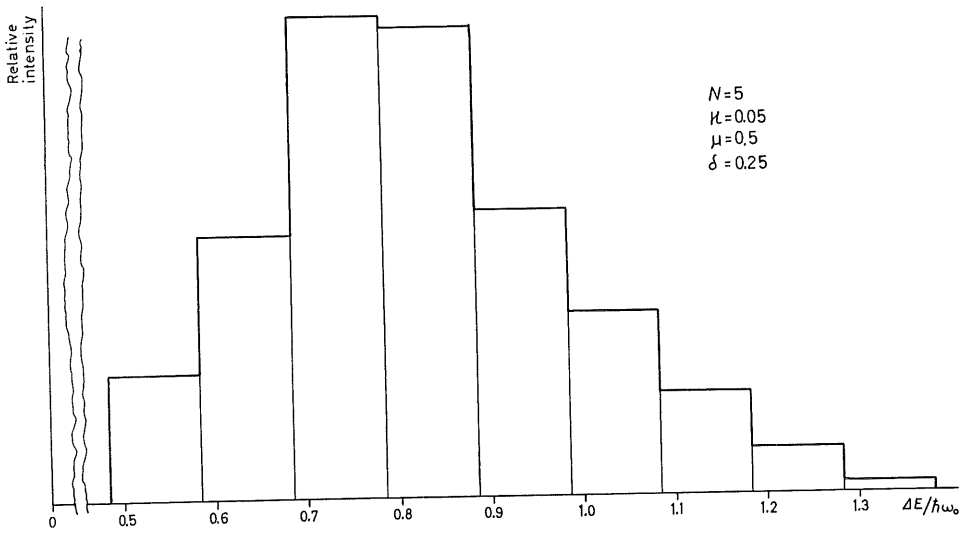

Fig. 4. The same as fig. 3 with the parameters $N = 5$ and $\delta \approx 0.25$. The theoretical spectrum is meant to show the general features of giant-resonance spectra in the regions of deformed nuclei $A \gtrsim 230$ and $170 \lesssim A \lesssim 190$.

owing to a larger effective N-value. Here the two peaks appear no longer to be resoluble, even when the broadening due to additional effects discussed in the introduction is neglected.

Appendix

CONSTRUCTION OF ASYMPTOTIC STATES †

It is well known that one can construct one-dimensional oscillator wave functions by the aid of the following creation and annihilation operators:

$$\Gamma_z^* = \frac{1}{\sqrt{2}}\left(\zeta - \frac{\partial}{\partial\zeta}\right), \tag{A.1a}$$

$$\Gamma_z = \frac{1}{\sqrt{2}}\left(\zeta + \frac{\partial}{\partial\zeta}\right), \tag{A.1b}$$

which obey the commutation relation

$$[\Gamma_z, \Gamma_z^*] = 1. \tag{A.2}$$

The dimensionless coordinate ζ is related to the coordinate z as follows:

$$\zeta = \sqrt{\frac{M\omega_z}{\hbar}}\, z.$$

A normalized state corresponding to n_z oscillator quanta can then be expressed by means of the operator Γ_z^*:

$$|n_z\rangle = \frac{1}{\sqrt{n_z!}}(\Gamma_z^*)^{n_z}|0\rangle, \tag{A.3}$$

where $|0\rangle$ is the "vacuum" state, corresponding to $n_z = 0$.

We denote by Γ_x and Γ_y etc. the corresponding operators in the x and y dimensions. They obviously fulfil relations analogous to (A.2). It is useful to consider Γ_x^*, Γ_y^*, Γ_z^*, and Γ_x, Γ_y, Γ_z, as components of the vector operators $\boldsymbol{\Gamma}^*$ and $\boldsymbol{\Gamma}$.

We now want to construct eigenstates corresponding to a three-dimensional oscillator with rotational symmetry only about the z-axis, i.e. with frequencies ω_x and ω_y equal to one another (denoted $\omega_\perp$), but different from ω_z. The energy only depends on $n_\perp (= n_x + n_y)$ and n_z. Of the $(n_\perp+1)$-fold degenerate eigenstates with given $n_\perp$ one may then form linear combinations which are eigenfunctions of the z-component of the angular momentum operator L_z.

Such states are denoted $|n_\perp \Lambda\rangle$, where Λ is the eigenvalue of L_z, and may now be constructed in analogy with (A.3) by the aid of the following operators:

$$\Gamma_+^* = \frac{1}{\sqrt{2}}(\Gamma_x^* + i\Gamma_y^*) = R^*, \tag{A.4a}$$

$$\Gamma_+ = \frac{1}{\sqrt{2}}(\Gamma_x + i\Gamma_y) = S, \tag{A.4b}$$

† *Note added in proof:* Similar operator methods have been employed by Dr. B. Bayman in constructing wave functions for the case considered here as well as for the three-dimensional isotropic case with L^2 and L_z being constants of the motion (private communication from B. Bayman). Cf. also ref. [7]).

$$\Gamma_-{}^* = \frac{1}{\sqrt{2}}(\Gamma_x{}^* - i\Gamma_y{}^*) = S^*, \tag{A.4c}$$

$$\Gamma_- = \frac{1}{\sqrt{2}}(\Gamma_x - i\Gamma_y) = R, \tag{A.4d}$$

which, apart from a sign, are the conventional spherical components of the vector operators Γ^* and Γ.

These operators obey the commutation relation

$$[S, S^*] = [R, R^*] = 1 \tag{A.5}$$

in analogy with (A.2). All other commutators vanish identically.

The selection rules of the operators (A.4a), . . . , (A.4d) may be studied in table A.1. Thus, for instance, $\Gamma_+{}^*$ increases the number of quanta in $n_\perp$

TABLE A1

	$\Delta n_\perp$	$\Delta\Lambda$	$\Delta r = \Delta\frac{1}{2}(n_\perp+\Lambda)$	$\Delta s = \Delta\frac{1}{2}(n_\perp-\Lambda)$
$\Gamma_+{}^* = R^*$	+1	+1	+1	0
$\Gamma_+ = S$	−1	+1	0	−1
$\Gamma_-{}^* = S^*$	+1	−1	0	+1
$\Gamma_- = R$	−1	−1	−1	0

by one unit, which is obvious, since it is linear and homogeneous in $\Gamma_x{}^*$ and $\Gamma_y{}^*$. Furthermore it increases Λ by one unit, which is apparent from the fact that, apart from a sign, it is the +1 component of the *vector* operator Γ^*.

One may also directly verify the relation

$$L_z\{\Gamma_+{}^*|n_z\Lambda\rangle\} = (\Lambda+1)\{\Gamma_+{}^*|n_z\Lambda\rangle\}. \tag{A.6}$$

As is clearly indicated by the table, the quantum number associated with R and the hermitian conjugate R^* is $r = \frac{1}{2}(n_\perp+\Lambda)$, while $s = \frac{1}{2}(n_\perp-\Lambda)$ may be associated with S and S^*.

In summary, we now have a system of operators, R, R^*, S, S^*, and Γ_z, $\Gamma_z{}^*$, which fulfil the commutation relations (A.2) and (A.5), while all other commutators between any two of these operators vanish. The eigenvalues associated with these operator pairs are r, s and n_z respectively.

The total three-dimensional eigenfunction may now be constructed as

$$|r\rangle|s\rangle|n_z\rangle = \frac{1}{\sqrt{r!}}(R^*)^r|0\rangle\frac{1}{\sqrt{s!}}(S^*)^s|0\rangle\frac{1}{\sqrt{n_z!}}(\Gamma_z{}^*)^{n_z}|0\rangle. \tag{A.7}$$

In a representation in terms of the previously used quantum numbers $n_\perp$ and Λ one may rewrite the $x-y$ part of the wave function

$$|n_\perp\Lambda\rangle = \frac{1}{\sqrt{(\frac{1}{2}(n_\perp+\Lambda))!\,(\frac{1}{2}(n_\perp-\Lambda))!}}(\Gamma_-{}^*)^{\frac{1}{2}(n_\perp-\Lambda)}(\Gamma_+{}^*)^{(\frac{1}{2}n_\perp+\Lambda)}|00\rangle. \tag{A.8}$$

One may note at this point that the quantum numbers r and s provide a useful basis for the classification of the asymptotic states. We have so far

labelled the latter by $n_\perp$ and Λ, allowing $n_\perp$ to take on the values 0, 1, 2, . . ., N; for a given $n_\perp$-value, permissible values of Λ are

$$\Lambda = \pm n_\perp, \qquad \pm(n_\perp-2), \qquad \pm(n_\perp-4), \ldots, 0 \text{ or } 1,$$

depending on whether $n_\perp$ is even or odd. If one uses the alternative classification, one must let r take on all values 0, 1, 2, . . ., N, while for a given r the quantum number s can assume all values 0, 1, 2, . . ., $(N-r)$.

MATRIX ELEMENTS

Matrix elements of various transition operators between asymptotic states are most easily derived by expressing the operators in terms of R and S. Thus

$$\left(\frac{\hbar}{M\omega_\perp}\right)^{-\frac{1}{2}}(x+iy) = \xi+i\eta = S+R^*, \tag{A.9a}$$

$$\left(\frac{\hbar}{M\omega_\perp}\right)^{-\frac{1}{2}}(x-iy) = \xi-i\eta = S^*+R. \tag{A.9b}$$

Using the relations

$$R^*|r\rangle = \sqrt{r+1}\,|r+1\rangle \tag{A.10}$$

and

$$R|r\rangle = \sqrt{r}\,|r-1\rangle, \tag{A.11}$$

which hold, in accordance with the commutation relations (A.2) and (A.5), for all the operator pairs R, R^* and S, S^* as well as Γ_z, Γ_z^*, one immediately obtains the matrix elements

$$\langle r+1,\, s|\xi+i\eta|rs\rangle = \sqrt{r+1} \tag{A.12a}$$

$$\langle r,\, s-1|\xi+i\eta|rs\rangle = \sqrt{s}. \tag{A.12b}$$

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