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MASTER THESIS II

CONTINUOUS TIME PROCESSES IN TIMES OF CRISIS: THE CASE OF GBM AND CEV MODELS

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ABSTRACT

This research aims at studying continuous time models within different stock market environments. We assume that the modeling of continuous time processes may be altered whether an equity market is experiencing a crisis or a pre-crisis period. As a benchmark index, the S&P500 has been chosen for this study and the sampling periods in question include the Black Monday of 1987, the Dot-Com of 2001, and the more recent 2007 Financial Crisis. Among the continuous time processes family, this study covers the Geometric Brownian Motion (GBM) and the Constant Elasticity of Variance (CEV). After estimating and analyzing their respective parameters using the Maximum Likelihood Estimation method, both the Jarque-Bera normality test and the Likelihood Ratio are performed on the two models. Unlike most research that support the use of CEV over GBM, this research test outcomes show that there is no strong argument that could favor the addition of a discount factor i.e. CEV over the Black-Scholes based process, the GBM model.

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PURPOSE	Two of the most popular models in option pricing and market predictions are the so called Geometric Brownian Motion (GBM) and the Constant Elasticity of Variance (CEV). This study is aimed at identifying and comparing their degree of signaling of equity market collapse. The S&P500 is taken as the case study for equity market because of its recognized influence on world stock markets. In assessing the GBM and CEV level of accuracy, the two mentioned models have their equivalent estimators and stochastic processes being put under scrutiny using maximized likelihood estimation method, likelihood ratio test and Jarque-Bera normality test.

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I. Introduction

What started as a US mortgage and housing market crisis spread out to the rest of the financial world to become known as the global financial crisis, the most severe crisis within the post world war II era according to many top economists (Business Wire, 2009). Many theoretical foundations within the fields of financial economics started to be more questioned regarding their predictability and accuracy. Nouriel Rubini, one of the few economists who predicted the upcoming of a worldwide crisis to the International Monetary Fund's conference in front of his peers, has claimed that the large majority of economists failed to identify recessionary signals. According to Rubini, in 97% of the cases, forecasts from top economists fail to predict the coming of a recession a year before of its occurring. Many economists have failed to signal recessionary behaviors two months before the start of the downturn. Within the rare cases that economists predict an economic downturn, its impact is highly underestimated (New York Times, August 15, 2008). This lack of predictability among top economists raises questions on the validity of conventional models under usage.

Two of the most popular models in option pricing and market predictions are the so-called Geometric Brownian Motion (GBM) and the Constant Elasticity of Variance (CEV). This study is aimed at identifying and comparing their degree of signaling equity market collapse. The S&P500 is taken as the case study for equity market because of its recognized influence on world stock markets. In assessing the GBM and CEV level of accuracy, the two mentioned models have their equivalent estimators and stochastic processes being put under scrutiny using maximized likelihood estimation method, likelihood ratio test and Jarque-Bera normality test.

The first part of this paper lists the different recent studies on the models in question under the literature review section. The theoretical background comes next and is related to the GBM and CEV models. Section 3 presents the methodology used in this study and the different assumptions being taken to build the financial models. The fourth section presents charts and graphs summarizing the different calculation and distribution outcomes as well as additional comments about the results. The next part of this study analyses and compares the performed tests, parameters estimation results and hypothesis testing on both the GBM and CEV across the various sampling periods. The final section, Conclusion and Recommendations for Further Studies, sums up the results of this study and suggests proposals and methods to improve the models, the estimation techniques, and the stochastic process distributions.

II. Literature Review

i. Geometric Brownian Motion Model

Since Osborne (1959) introduced the Brownian motion to be used on common stock prices, the geometric Brownian motion has been widely accepted as “the model” for growth in the price of a stock over time. In fact, Hull (2000) refers to it as 'the model for stock prices'. The GBM uses are well known in the Black-Scholes Model, one of the most important concepts in modern financial theory. Black and Scholes (1973) mentioned their option pricing formula for the first time in their paper, "The Pricing of Options and Corporate Liabilities". One of the assumptions for their conceptual framework which is of interest for our research is that stock prices follow a geometric Brownian motion with constant drift and volatility.

Hassett and Metcalf (1995) argue that mean-reverting processes are appropriate for most "real option" investment models. They further argue that using a GBM process is justified since mean-reversion has two opposing effects. Firstly, it brings the investment trigger closer, and secondly it reduces the conditional volatility, which thereby lowers the likelihood of reaching that trigger. They concluded in their work that cumulative investment in general is unaffected by the use of a mean reversion process rather than geometric Brownian motion.

Sarkar (2002) extends on the work by Hassett and Metcalf (1995) and adds a third factor, the effect of mean-reversion on systematic risk. One of the major findings by Sarkar is that mean-reversion does have a significant impact on investment. It is also found that it is inappropriate to use the GBM process to approximate a mean-reverting process.

A considerable number of studies have suggested the need to generalize the GBM model by introducing the possibility of jumps and allowing the volatility to be a stochastic process. The first jump-diffusion model was introduced by Merton (1976). The work by Merton included an addition of a compound Poisson process as a model for the jumps to the Brownian motion part of the GBM model. In Merton’s model the jumps occur randomly, with a certain average frequency, at the time when the jump occurs, the logarithmic price jump is drawn independently from a normal distribution with two parameters describing the mean jump size and jump size volatility.

Lewis (2002) argues that there are some problems with the use of the geometric Brownian motion. On the plus side, the author mentions that the GBM: 1) is consistent with securities having limited liabilities 2) has uncorrelated returns and 3) is very tractable computationally. The problem according to Lewis (2002) is that it relies on normal distribution, and that there are too many outliers for this assumption to hold.

In Leon *et al.* (2002), they center their research on a very simple Lévy process, where they take the sum of a Brownian motion and *k* independent Poisson processes, as to obtain a weak derivative interpretation and useful formulas for their work. Additionally, they also approximate the Lévy process used as a jump-diffusion model by the means of a simple Lévy process.

In Espinosa and Vives' (2006), the authors present a generalization of the traditional Lévy-Merton jump diffusion model, allowing discrete stochastic volatility. They use a method which is based on quadratic variation. This is done in order to estimate jump instants and jump amplitudes.

ii. Constant Elasticity of Variance Model

Concerning the literature covering the CEV, a large number of studies have covered its use within the finance and financial economics fields. Research on CEV has mainly focused on the application of CEV models to price hybrid options, as well as an alternative for lognormal processes.

In their pricing method of convertible bonds and American credit spread options, Ruxing and Shenghong (2009) have used the CEV process within an extended model. The latter accounts for movements in equity prices and interest rates, which affect the default probability, as well as for the negative correlation between the volatility of the stock and its price. For this purpose, they suggested to use a trinomial tree model with embedded call and put options. As mentioned by authors, the model can also be applied to other hybrid derivatives such credit default swaps.

Das and Sundaram (2007) have studied the effect of adding a leverage component to different cases of CEV models on credit default swaps spreads. They concluded that increases in the leverage component are accompanied by increases in the spreads. The authors also found out that the total volatility is a more important factor than the leverage effect on the credit default swaps pricing.

Chan, Choy and Lee (2007) contested the use of normal distribution in the discretization of the CEV diffusion process for interest rate data, with the case of two popular CEV processes, Ornstein–Uhlenbeck and the square-root processes. Several heavy-tailed and symmetrical distributions; namely the Student-t distribution, the logistic distribution and the exponential family of distributions; had their robustness tested relative to the normal distribution to identify their validity as alternative distributions. The outcome of the research was that normality assumption in the CEV model diffusion process may be less appropriate than the above-mentioned heavy tails distributions.

Svoboda-Greenwood (2009) studied alternatives to the lognormal process in modeling the dynamics of interest rates and equity prices. Using CEV process as an alternative to lognormal process, she came up with the conclusion that the CEV option pricing method holds similar results and is appropriate for valuing vanilla derivatives. In fact, significant pricing errors seem to appear after the 10 year maturity level. However, the CEV process pricing may be inaccurate in determining the values of more complex forms of derivatives.

Ren-Raw et al. (2009) test for the accuracy of CEV model relative to the stochastic volatility model in pricing European and American options. The CEV model appears to have better results for in-sample, out-of-the-sample and implied volatility tests than the stochastic volatility model within the category of short term and out-of-the-money European options. For pricing American options, the simple CEV model is considered being a better alternative than more complex stochastic volatility models in terms of applicability, implementation cost and computational speed. The authors recommend the use of CEV models while dealing with complex path-dependent options and credit risk models.

Chan and Ng (2007) have proposed a CEV model that account for the long-memory pattern exhibited in financial markets. For this purpose, they constructed a fractional CEV model and replaced the Wiener process with a fractional Brownian motion. The added component does not depend on the strike price, but hold the same implied volatility pattern as the classical CEV model. With this fractional transformation of the classical CEV model, the authors identified a European option pricing formula with volatility skew pattern also being revealed.

Davydov and Linetsky (2001) have studied the significant errors in hedging and pricing originated from the use of the standard geometric Brownian motion for pricing path-dependent options such as barrier and lookback options. They argue that the implied volatilities of the option prices with different strike prices are “not constant but vary with strike price”, implying the presence of an implied volatility smile that cannot be captured by a lognormal process. With the use of the CEV model, the authors have identified significantly deviating results of the barrier and lookback option prices and hedge ratios from the lognormal process models. The closed form CEV pricing formula for the path-dependent options allow for faster and more accurate prices and hedge ratios outcomes.

Under the same basis, Boyle and Tian (1999) have used trinomial method for CEV process approximation and test its accuracy for the determination of lookback and barrier options prices. Testing different parameter values of the CEV model to measure the accuracy of their model, the authors concluded that the difference between CEV and Black and Scholes is minimal for standard options, but widens for path-dependent options, making the CEV model a much better alternative than the Black and Scholes for lookback and barrier options.

In their pricing of warrants, Lauterbach and Schultz (1990) have studied the Black and Scholes model, in parallel with its alternatives, which included the CEV model. They argue that the constant variance assumption of the Black and Scholes model allow for biases in the pricing of almost all the sampling warrants and periods. Using different changes to the Black and Scholes model to find the alternative for warrant pricing, Lauterbach and Schultz could only find more accurate predictions when adjusting the model to implied equity standard deviations. This adjustment is equivalent to the CEV model, in this study a square-root CEV, since the Black and Scholes is considered to be a special case of the CEV model. In addition, the authors identified an important drawback of the CEV model is that it may not be accurate when the parameters are not known. In their own words, “the relevant empirical question is whether the CEV model provides sufficient improvement in price forecasts to overcome the noise associated with estimating an additional parameter”. They recommend the use of CEV model rather than Black and Scholes for pricing warrants.

Beckers (1980) extended the Black and Scholes' log-normality assumption testing to two special cases of the CEV model, the square root and the absolute models. Although Beckers' results show in general that the two CEV class cases are better models to describe the stock price movements than the Black and Scholes' lognormal model, none of the model can be applied uniformly among all the stock prices series.

III. Theoretical Framework

This section states the theories, as well as previous research, associated with the process models, index price transformations, stochastic integral calculations and estimation methods used in this index price study.

i. Brownian Motion

In 1828, the botanist Robert Brown was the first to identify the random movements of pollen in water. Since then, the Brownian movement has been widely used in the fields of biology, physics, economics and management systems. It also holds a primary role in theoretical finance. In fact, the movements described by the Brownian motion can be replicated to forecast the stock market movements. The Brownian motion has the following properties (Karatzas & Shreve, 1991, pp.169-175):

- $W = \{W_t, F_t; 0 \leq t < \infty\}$
- $W_0 = 0$
- $W_t - W_S$: is independent of $F_s; 0 \leq s \leq t$
- $E[W_t - W_s] = 0$ and $V[W_t - W_s] = t - s; 0 \leq s \leq t$

W_t denotes the Wiener process, also referred to as Brownian motion, which is a type of Markov stochastic process. The Markov process is particularly fundamental in theoretical finance since historical data are irrelevant but rather the present value of a variable. In other words, the likelihood of any future state does not depend on the past states but only on the present states, which is of particular importance while using memoryless series. From the properties of the Wiener process, it can be seen that mean change is equal to zero and the variance of change is equal to the time interval (Hull, 2000).

ii. Logarithmic Return

As early as the first financial empirical studies, Working (1934) and Kendall (1953) identified that stock and commodity prices are nearly impossible to forecast starting from historical data of the price series. To overcome the difficulty of predicting movement of prices that resides in their non-stationarity characteristic, with very high correlation and increasing variance with time, Osborne (1959) suggested the use of the logarithmic return. Thus, instead of using the prices process, the returns process (X_t) is more suitable:

$$\log(S_t) = \log(S_{t-1}) + X_t$$
$S_t \equiv$ index price at time t
$X_t \equiv$ independent series of errors; $E(X_t) = 0$

X_t is then independent of the past index prices changes. In other terms, X_t can be expressed as the logarithmic change of the index price as such:

$$\Delta \log(S_t) = \log(S_t) - \log(S_{t-1}) = X_t$$

Therefore and in parallel with Osbourne's study about the subject (1959), the logarithmic index prices changes are generated from a normally distributed random error with zero mean and constant variance.

iii. Geometric Brownian Motion

In financial time series, the literature and research have focused on the determination of a model on three major families of price processes: the autoregressive conditional heteroscedasticity (ARCH) model group, the stochastic volatility (SV) model group, and the random walk model group.

Samuelson (1965) was the first to suggest the use of the GBM model as a method to illustrate price behavior. The GBM is of particular use for this study because this stochastic process only allows for non-negative values, as it is the case for index price of an equity market. Here is the form of the stochastic differential equation (SDE) of the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$
$\mu \equiv$ instantaneous expected return (constant)
$\sigma \equiv$ instantaneous volatility (constant)

By using logarithmic return on Samuelson's model, the following is obtained:

$$d \log S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

Recently, geometric Brownian motion has been extensively used as a model for the stock prices, commodity prices and growth in demand for products and services. The geometric Brownian motion is also the model behind the Black-Scholes European options pricing formula as well as far more complex derivatives. Various recent studies invoked the geometric Brownian motion in real options analysis (Nembhard et al., 2002; Thorsen, 1998; Benninga & Tolkowsky, 2002…), in representing future demand in capacity studies (Whitt, 1981; Lieberman, 1989; Ryan, 2004…). In general, its popularity was motivated from the assumption that random changes over time follow a geometric Brownian motion process (Marathe & Ryan, 2005). The geometric Brownian motion is particularly useful for this index price study because the process in question assumes that percentage changes are independent and identically distributed over equal and non-overlapping time length (Luenberger, 1995; Ross, 2000). Unlike the standard Brownian motion's constant drift term, the geometric Brownian motion assumes that the instantaneously expected rate of return is constant. Therefore, the constant instantaneous expected drift assumption of the standard Brownian process is substituted with the constant expected rate of return in the geometric Brownian process (Hull, 2000). This is of particular importance since the drift function is substituted with observable quantities, the rate of return and the index price.

On the other hand, the accuracy of the geometric Brownian motion has been put into question and has constituted the subject of debate between scholars. Watteel-Sprague (2000) questioned the normality assumption of the geometric Brownian motion stochastic process in past studies by showing that the high kurtosis of the returns process implies non-normal distribution (p.11). She adds that the assumption of independence of the geometric Brownian motion is invalid because of the autocorrelations identified in squared returns process (p.15), making the GBM real option pricing and predictability of price movements null. In fact, daily returns are characterized by volatility clustering, refuting the argument of constant volatility of returns (p.93). A study by Ross (1999) revealed that the data on the price of crude oil are not compatible with the geometric Brownian motion assumptions. Thorsen (1998) stated that the geometric Brownian motion process assumptions should be tested first. Marathe & Ryan (2005), checked for the geometric Brownian motion process fit. Likewise, they tested for the normality and independence of the logarithmic returns. They found out that the geometric Brownian motion assumptions were consistent with some data sets and inconsistent with others, concluding for the necessity of assumptions' testing.

One of the main objectives of this study is to confirm or refute the different mentioned studies about the validity of GBM processes in predicting stock market movements. On one side of the literature review on the GBM, it reveals that the process is a good tool for real option pricing as the random changes over time follows a GBM process. Other authors, cited in the second paragraph of this section, doubt the accuracy of the assumptions surrounding GBM models such as the normality of the distribution and the constant volatility of returns. This study is aimed at testing the validity of GBM processes, which are widely popular within the financial economics community in option pricing.

iv. Constant Elasticity of Variance

In finance, a large part of the academic literature assumes that asset prices follow a geometric Brownian motion. This assumption implies that the future price of the asset follow a log-normal distribution. Work done by Davydov & Linetsky (2001) on path dependent options, and thereby the underlying assets, show that implied volatilities computed from market prices are not constant, but vary with the strike price. The same variation is observed over the underlying assets. The variation is known as the implied volatility smile or frown, depending on the shape. This variation is not captured by the log-normal assumption with constant elasticity, and hence the geometric Brownian motion. In fact, several authors (Black, 1976; Christie, 1982; Nelson, 1991) have empirically identified a short-term phenomenon to stock prices, consisting of negative correlation between index price movements and the volatility. Davydov & Linetsky (2001) went further in extending the basic model by the use of a CEV model, to include the variation. The results from Davydov & Linetsky (2001) shows that financial institutions that use the standard geometric Brownian motion assumption are exposed to significant pricing and hedging errors. The use of CEV diffusion to model asset prices was introduced to finance for the first time by Cox back in 1975. The advantage of the CEV model relative to the geometric Brownian motion (GBM) model is that it allows for the leverage effect (Engle & Lee, 1992; Gallant et al. 1993). The SDE form of the CEV model with the leverage effect, or discount factor, is as follows:

$$dS_t = \mu S_t dt \ + \ \sigma S_t^{\delta} dW_t \qquad 0 < \delta < 1$$
$$\delta \equiv \ discount\ factor$$

v. GBM Model vs. CEV Model

The proponents of the CEV model over the GBM argue that the GBM is invalid only because the latter lacks to capture the implied volatility smile, implying a reverse relationship between the price movements and the volatility. In other words, adding a discount factor strengthens the model. As matter of fact, this study tries to answer the question whether one of the models is mostly suitable to predict the price movements of an equity market in the caliber of the S&P500 index or none of them.

Both the GBM and the CEV models belong to the following general continuous-time specification (Chan et al., 1992):

$$dS_t = (\lambda + \mu S_t)dt + \sigma S_t^{\delta} dW_t$$

In the case of the GBM, $\lambda = 0$ and $\delta=1$ while the CEV only displays $\lambda = 0$. From the stochastic differential equation forms of the GBM and CEV models, it could be observed that the GBM is a special case of the CEV. In fact, the GBM model is equivalent to the CEV form when $\delta=1$. A testing of the two models is thus necessary to assess the strength of the specific model (GBM) relative to the more general one (CEV). Another objective aimed from the testing of the GBM over the CEV is to assess whether or not the presence of an additional parameter, namely the discount factor, reinforces or simply doesn't add any strength to the model. The testing of the GBM over the CEV will identify whether today's stock price affects the stochastic process component of the CEV model or not. To evaluate the strength of a model relative to another two tests are suggested: the Likelihood Ratio Test and the Jarque-Bera Test. Those two tests are described in more details in the next section of the paper.

IV. Methodology

The Methodology section presents the data sampling methods to identify the different stages of the S&P500 as well as the logarithmic transformation on the input data. Then, both the models under scrutiny, the GBM and the CEV, are assumed to be in discrete time using a discretization scheme.

i. Data Collection

The data used in this study consist of time series of the daily S&P500 index at the close of the market (4:00 pm Eastern Time). The data has been obtained from DataStream's database, covering the period of March 31st, 1979 to September 8th, 2009. It consists of the "S&P500 COMPOSITE - PRICE INDEX", which is a value-weighted index of a large cross-section of listed stocks. It is comprised of the 500 largest publicly held companies. Almost all of the stocks included in the index are among the 500 American stocks with the largest market capitalizations. The index does also include a minority of non-U.S. companies (9 as of March 26th, 2009). The companies included in the S&P500 are selected by a committee and they have to live up to certain requirements of sufficient liquidity and being publicly traded, leaving out some of the largest U.S. companies. The committee selects the companies in the S&P500 so they are representative of various industries in the United States economy. This justifies using the S&P500 as an overall health indicator of the overall economy, and it is actually one of ten key variables included in the Index of Leading Indicators.

After a careful analysis of the stock index graph, three post-World War II periods when the S&P500 index declined sharply were identified: 1987, 2001 and 2007. In order to better measure the effect of the model within different stock environments (bull and bear markets); the data samples were split into two time periods: the "pre-crisis period" which is referred to as the bull market period of each of the crises and the "crisis period" for the bear market. For the three identified stock market crises, the pre-crisis and crisis periods include 500 trading days each. One full business year is assumed as being equivalent to 250 trading days.

After observing the evolutionary stock movements of the S&P500, the following process for the determination of the bull (pre-crisis) and bear (crisis) markets were identified. The first step is to look for the peak stock index day in 1987, 2001 and 2007. This day is called Day 0. After that, the sampling periods prior and subsequent to Day 0 were chosen.

The period prior to Day 0 is the pre-crisis period and the period subsequent to Day 0 is the crisis period.

Again, each of the pre-crisis and crisis periods is split in two sub-periods: period I and II. Period I and II are equally divided: 250 trading days each.

Derived from the daily index price of the S&P500 Composite, the logarithmic return is preferably used as a basis for future calculations, illustrated in the following:

$$r_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$$

$r_t \equiv$ logarithmic return

$S_t \equiv$ price index at time t

In the calculations of the daily stock returns, the period log returns $\left(\ln\frac{S_t}{S_{t-1}}\right)$ are used rather than simple discrete returns $\left(\frac{S_t - S_{t-1}}{S_{t-1}}\right)$. The major reason behind the use of log returns within quantitative finance is that unlike the simple discrete returns, log returns are time additive. So, if the log-return for period n is needed, the sum of each of the consecutive daily returns summed will equal the return of period n. More or less all financial assets have limited liability, meaning that the maximum loss one can incur is a negative 100%. This constitutes a problem since the arithmetic Brownian motion *r(t)*, the price of some financial asset over any time interval follows a normal distribution, and over real time the *r(t)* could become negative, which would contradict the assumption that any financial asset's maximum loss would be -100%. Hence, the price *r(t)* would be negative. This problem is easily eliminated by defining the price of the financial asset *r(t)* as the natural logarithm of *S(t)*. By using this definition, the Brownian motion no longer violates the assumption of non-negativity, since the price $S(t)= e^{r(t)}$ is always non-negative (Campbell et al., p.347-348, 1997).

ii. Itô's Lemma

a. GBM Model

Even in continuous time, the standard Brownian motion cannot be differentiated and cannot be monotonic with probability 1 (Karatzas & Shreve, 1991). The need of a stochastic integral is necessary. The Itô's integral has been one of the most popular stochastic integrals in theoretical finance since it "reflects the notion that agents cannot anticipate future price movements" (Watteel-Sprague, 2000, p.21). Applying the Itô's change of variable on the continuous geometric Brownian motion:

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dW$$

Using Itô's lemma, the geometric Brownian motion model can be solved to get:

$$S_t = S_0 e^{\left[\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right]}$$

Assuming that the dynamics of the index price process is $r(t) = \ln S_t$ and by using Itô's lemma stochastic differential equation, a GBM with a mean and standard deviation proportional to the index price S is obtained as such:

$$dS = \frac{\partial S}{\partial r} dr + \frac{1}{2} \frac{\partial^2 S}{\partial r^2} (dr)^2$$

$$dS_t = e^{\ln S_t} + d \ln S_t + \frac{1}{2} e^{\ln S_t} (d \ln S_t)^2$$

$$dS = \left(\mu - \frac{1}{2} \sigma^2 \right) S dt + \sigma S dz$$

with $dz = \lim_{\Delta t \to 0} (z_{t+\Delta t} - z_t) \Rightarrow \Delta z_t = z_{t+\Delta t} - z_t$

$z_{t+\Delta t} - z_t \sim N(0, \Delta t)$ and $\Delta t = 1$

Therefore, the index price change $\frac{dS}{S}$ behaves as a random walk process, with μ being the constant instantaneous mean return (expected return) and σ^2 being the constant instantaneous return variance. The following illustrates the index price change dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t$$

$$d \ln S_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz_t$$

$$\theta = (\mu; \sigma)$$

b. CEV Model

The same steps of Itô’s integral on the GBM model can be applied on the CEV model with one significant difference. Unlike the GBM model and as mentioned in the theoretical framework section of this study, the stochastic differential equation of the CEV model includes a leverage effect, or discount factor (δ), so the corresponding index price dynamics is as follows:

$$\frac{dS_t}{S_t} = \mu dt + \sigma S_t^{\delta-1} dW_t$$

$$d\ln S_t = \left(\mu - \frac{1}{2}\left(\sigma S_t^{\delta-1}\right)^2\right)dt + \sigma S_t^{\delta-1} dW_t$$

$$0 < \delta < 1$$

iii. Discretization Scheme

a. GBM model

To be able to transfer the continuous process into a discrete form, a discretization scheme is needed. The finite element discretization is considered to be the most used method for dynamic analysis of complex structures. Discretization is used in this study as a simplification of continuous time processes and thus allowing for the numerical evaluation needed. Applied first on the GBM model, the discretization scheme performed is represented as follows:

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

$$\Rightarrow \Delta \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta W_t$$

$$\Rightarrow \Delta W_t = \frac{\Delta \ln S_t - \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma} \sim N\left(0, \Delta t\right)$$

$$\Delta W_t = \frac{\ln\left(\frac{S_t}{S_{t-\Delta t}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma} \sim N\left(0, \Delta t\right)$$

where ΔW_t iid $N\left(0;\Delta t\right)$ with $\Delta t = 1$

b. CEV model

Basically, the same discretization steps applied with the GBM model can be applied for the CEV model:

$$d\ln S_t = \left(\mu - \frac{1}{2}\left(\sigma S_t^{\delta-1}\right)^2\right)dt + \sigma S_t^{\delta-1} dW_t$$

$$\Rightarrow \Delta \ln S_t = \left(\mu - \frac{1}{2}\left(\sigma S_t^{\delta-1}\right)^2\right)\Delta t + \sigma S_t^{\delta-1} \Delta W_t$$

$$\Rightarrow \Delta W_t = \frac{\Delta \ln S_t - \left(\mu - \frac{\left(\sigma S_t^{\delta-1}\right)^2}{2}\right)\Delta t}{\sigma S_t^{\delta-1}} \sim N\left(0, \Delta t\right)$$

$$\Delta W_t = \frac{\ln\left(\frac{S_t}{S_{t-\Delta t}}\right) - \left(\mu - \frac{\left(\sigma S_t^{\delta-1}\right)^2}{2}\right)\Delta t}{\sigma S_t^{\delta-1}} \sim N\left(0, \Delta t\right)$$

and ΔW_t iid $N\left(0;\Delta t\right)$ with $\Delta t = 1$

iv. Maximum Likelihood Estimation

The next step is to perform an estimation of the parameter vector θ, with $\hat{\theta}$ being the estimator of the parameter vector $\theta = \left(\mu, \sigma\right)$ for the GBM model and $\theta = \left(\mu, \sigma, \delta\right)$. In order to estimate $\overline{\theta}$, the maximum likelihood estimation method is performed. The maximum likelihood estimation is useful for calculations based on historical data, as well as for data which are not equally sp

aced, like *S(t)* sampled at non-stochastic dates $t_0 < t_1 < \ldots < t_n$. This is the case for financial time series such as stock prices since markets are closed during weekends and holidays, which yields to irregular sampling intervals. *S(t)* being by assumption a continuous time Markov process, irregular sampling does not create any conceptual problem. The joint density function f of the sample is given by the following product:

$$f\ (P_0, \ldots, P_n; \theta) = f_0(P_0; \theta)\ \prod_{k=1}^{n} f\ (P_k, t_k \left| P_{k-1}, t_{k-1}; \theta\right.)$$

In order to define the maximum likelihood estimator $\overline{\theta}$, let $\ln L(\theta)$ denote the log-likelihood function. The logarithm of the joint density function of P_0, …, P_n viewed as a function of θ is:

$$\ln L(\theta) \equiv \sum_{k=0}^{n} \ln f_k$$

$$\bar{\theta} \equiv \arg\max \ln L(\theta)$$
$$\theta \in \Theta$$
$$\Theta \equiv \text{parameter space of parameters } \theta$$

As previously mentioned, the Wiener process (ΔW_t) is assumed to follow a normal distribution with mean 0 and variance Δt. Starting from the corresponding continuous density function of the Wiener process, a general formulation of the log-likelihood function is obtained:

$$g(\Delta W) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{1}{2\Delta t}(\Delta W)^2}$$
$$g(\Delta W_1, \Delta W_2, ..., \Delta W_T) = g(\Delta W_1)\, g(\Delta W_2) ... g(\Delta W_T) \qquad \text{iid over time}$$
$$= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\Delta t}} \exp\left[-\frac{1}{2\Delta t}(\Delta W_t)^2\right]$$
$$\Rightarrow \ln L(\theta) = \sum_{t=1}^{T}\left[-\frac{1}{2}\ln(2\pi\Delta t) - \frac{(\Delta W_t)^2}{2\Delta t}\right]$$
$$\ln L(\theta) = -\frac{T}{2}\ln(2\pi\Delta t) - \frac{1}{2\Delta t}\sum_{t=1}^{T}(\Delta W_t)^2$$
$$\Delta W_t \sim N(0;\Delta t) \Leftrightarrow Var[\Delta W] = \Delta t$$
$$\text{let } \Delta t = 1$$

In this study, the estimation of the parameters is derived through the Excel Solver application. This is due to the fact that the expression $\left[-\frac{T}{2}\ln(2\pi\Delta t)\right]$ is unrelated to the parameters under study within the $\ln L(\theta)$. Thus, the $\left[-\frac{T}{2}\ln(2\pi\Delta t)\right]$ expression does not hold any importance for the parameters estimation, which explains that it has been removed from the log-likelihood function so that its transformed version becomes:

$$\ln L_{tr}(\theta) = -\frac{1}{2\Delta t}\sum_{t=1}^{T}(\Delta W_t)^2$$

After identifying the general formulation of the log-likelihood function and letting $\Delta t = 1$, the specific log-likelihood function of the GBM and CEV models can be found by replacing the equivalent Wiener process such that:

$$\text{GBM: } \ln L_{tr}(\mu,\sigma) = -\frac{1}{2}\sum_{t=1}^{T}\left[\frac{\ln\left(\frac{S_t}{S_{t-\Delta t}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma}\right]^2$$

$$\text{CEV: } \ln L_{tr}(\mu,\sigma,\delta) = -\frac{1}{2}\sum_{t=1}^{T}\left[\frac{\ln\left(\frac{S_t}{S_{t-\Delta t}}\right) - \left(\mu - \frac{\left(\sigma S_t^{\delta-1}\right)^2}{2}\right)\Delta t}{\sigma S_t^{\delta-1}}\right]^2$$

To be able to find the optimal estimators for each of the models, the log-likelihood has to be maximized with respect to each of the estimators $\frac{\partial \ln L_{tr}(\theta)}{\partial \theta} = 0$. The latter allows identifying the optimal parameters for the models in use.

v. Likelihood Ratio Test

The likelihood ratio test measures the validity of a model over another in terms of the number of parameters and the level of restriction. It judges the relative strength of models with different numbers of parameters. In this study, the likelihood ratio test assesses whether adding a discount factor strengthens the CEV model relative to the GBM model or not. The likelihood ratio test also aims at testing the unrestricted model, namely the CEV model with $0 < \delta < 1$, against the restricted model, namely the GBM model with $\delta = 1$. The GBM is the restricted one since it constitutes a special case of the CEV with $\delta = 1$.The distribution of the likelihood ratio is asymptotically $\chi^2(1)$ and is measured as follows:

$$D(\theta) = -2\left[\ln L_{tr,GBM}(\theta_{GBM}) - \ln L_{tr,CEV}(\theta_{CEV})\right]$$

The result of the likelihood ratio is tested on each of the periods for statistical significance within a chi-square distribution, with degree of freedom equaling 1 and three different significance levels (10%, 5% and 1%). In this study, a simple-vs.-simple hypothesis test is used (Mood et al., 1974) with the following null and alternative hypotheses:

$H_0 : \delta = 1$ for the GBM
$H_1 : \delta \neq 1$ for the CEV

vi. Normality Tests

In order to determine whether the errors are normally distributed, a normality test is needed. Based on Rakotomalala's (2008) study on different normality tests, it was decided to choose the Jarque-Bera normality test for our data series. The test is based on the coefficient measures of the skewness and kurtosis. It evaluates the spread degree between the observed skewness and kurtosis with the expected ones for a normally distributed series. Since the Jarque-Bera test lacks precision for small number of observations, which is compensated by the 250 observations on each of the series in this study. Under the null hypothesis (normality), the test statistic is distributed according to a chi-square distribution with 2 degrees of freedom. The null hypothesis is a joint hypothesis that the skewness is 0 and the kurtosis is 3.

V. Results Summary

i. Parameters Estimation

The results displayed on the chart summarize the estimated parameters for both the GBM and CEV models after the maximum likelihood estimation:

Table 1 – Estimated Parameters of GBM and CEV for Pre-Crisis and Crisis

	Pre-Crisis 87		Day 0	Crisis 87	
	GBM	CEV		GBM	CEV
σ	0,5970	1,5448		0,8036	2,8516
μ	0,1773	0,4324		0,3161	1,3732
δ	_	0,91		_	0,90
lnL(θ)	-0,02961	-0,01724		-0,09981	-0,02837
Date	04-11-1985 ... 02-10-1987		05-10-1987	06-10-1987 ... 04-09-1989	
	Pre-Crisis 01		**Day 0**	**Crisis 01**	
	GBM	CEV		GBM	CEV
σ	0,4648	3,2110		0,7471	2,1603
μ	0,1044	0,9585		0,2732	0,7322
δ	_	0,88		_	0,92
lnL(θ)	-0,10446	-0,01415		-0,05447	-0,02430
Date	04-10-1999 ... 31-08-2001		03-09-2001	04-09-2001 … 04-08-2003	
	Pre-Crisis 07		**Day 0**	**Crisis 07**	
	GBM	CEV		GBM	CEV
σ	0,3802	2,2860		0,7116	2,5575
μ	0,0707	0,7113		0,2469	0,8805
δ	_	0,91		_	0,91
lnL(θ)	-0,03738	-0,00370		-0,05808	-0,01975
Date	08-11-2005 … 08-10-2007		09-10-2007	10-10-2007 … 08-09-2009	

Various observations could be raised regarding the parameters results, but let us first say that for the CEV model, we consider its estimated σ as a better comparison tool against the GBM's standard deviation since the CEV standard deviation has both price and discount factor components integrated in it, while the GBM's standard deviation does not.

Regardless of the crash years, the estimated μ and σ of the GBM model, within the same period, are smaller than their CEV counterparts. The GBM estimated μ and σ also show lower values during pre-crisis, than during the crisis period. In that sense, the CEV figures display less consistency than the GBM numbers, possibly because of the additional factor (δ). The latter is both close to 1 and relatively stable from the pre-crisis to the crisis. The case of 2001, where δ moved from 0,88 to 0,92, had the pre-crisis μ and σ higher than the crisis period. On the other hand, the 1987 and 2007 cases where the discount factor was more or less constant, the estimated mean and σ of the pre-crisis were smaller than the crisis figures. This may suggest that a small change in the discount factor from a period to another has a significant effect on the rest of the CEV parameters. Also, it seems that the introduction of the discount factor leads to magnified figures of μ and σ relative to the GBM parameters results.

Related to the figures obtained after calculating the max likelihood estimation that all the cases display negative numbers close to zero. The corresponding results for the GBM are always smaller than their CEV equivalents.

ii. Likelihood Ratio Test

The likelihood ratio test results on the strength of the CEV model (unrestricted model) relative to the GBM model (restricted model) are shown in the following chart:

Table 2 – LR Test Statistics and Critical Values

	LR Test
Pre-Crisis 87	0,0247
Crisis 87	0,1429
Pre-Crisis 01	0,1806
Crisis 01	0,0603
Pre-Crisis 07	0,0674
Crisis 07	0,0767

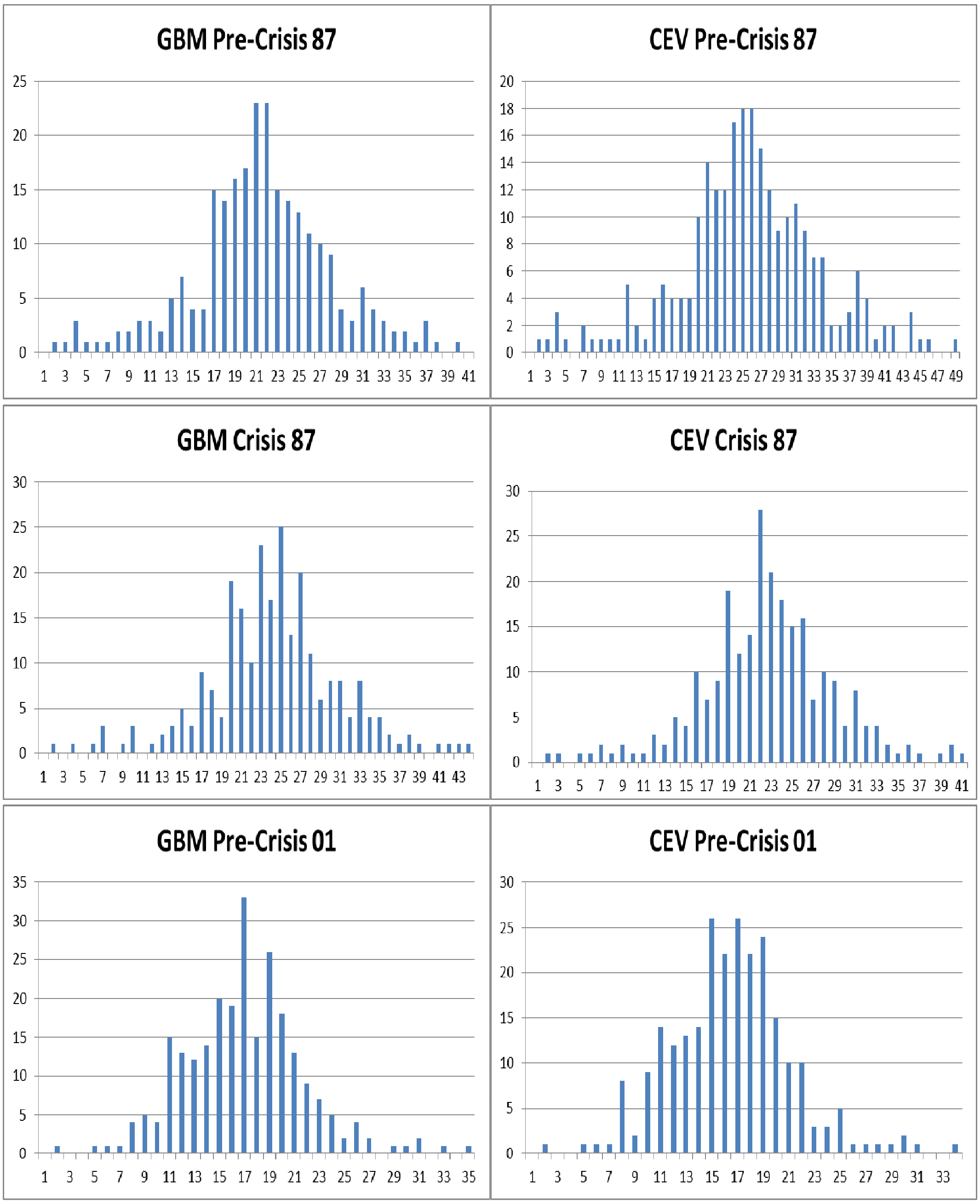
Critical Values	
*X^2(10%;1)**	2,70554
*X^2(5%;1)***	3,84146
*X^2(1%;1)****	6,63490

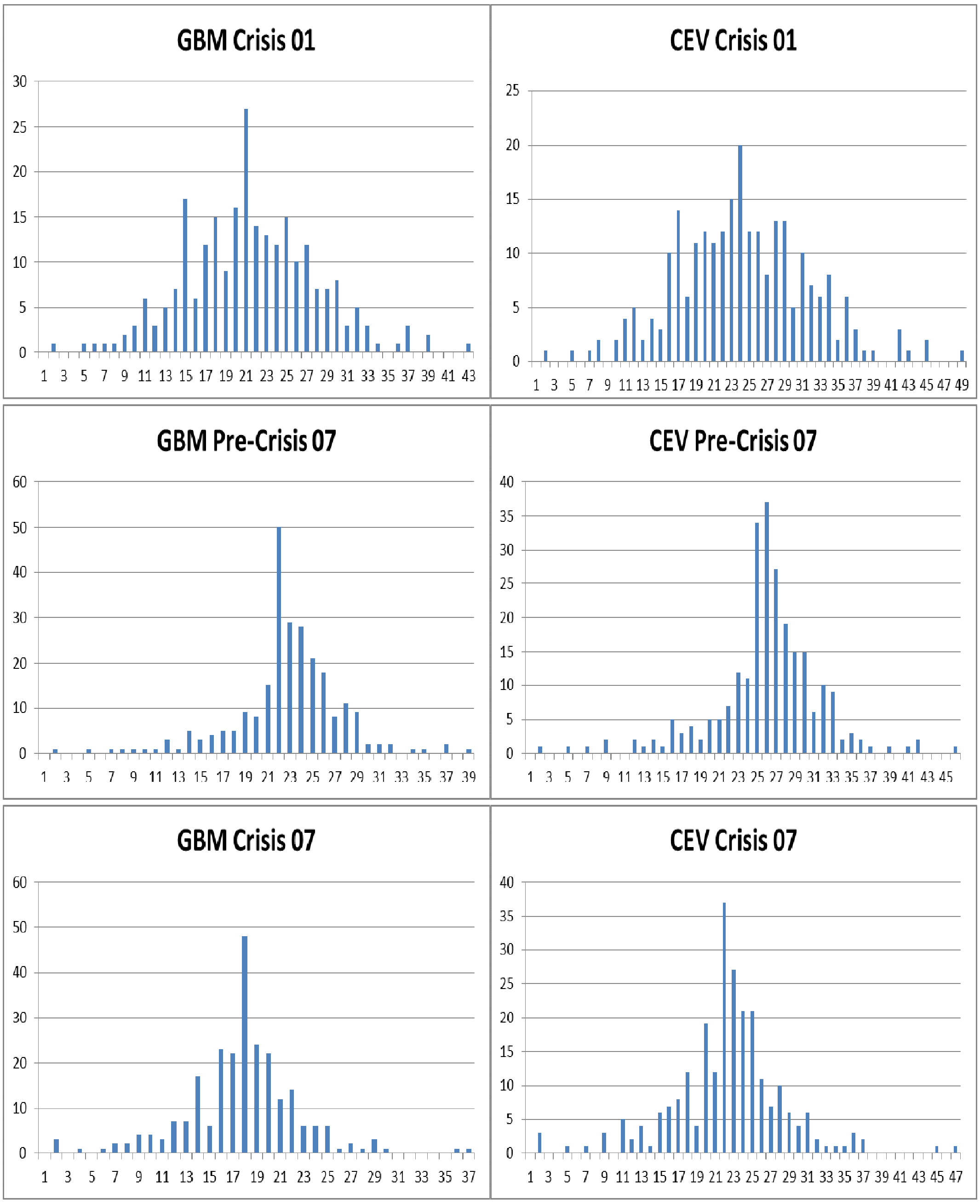
First, the numbers are all positive which correspond to the chi-distribution that goes from 0 to infinity. The results of the likelihood ratio test show that the LR test statistics are smaller than the critical values in the three significance levels: 1%, 5% and 10%. This tendency includes the results covering the pre-crises and crises periods under scrutiny. Not only is there a prevalence in the acceptance of the null hypothesis (δ=1), but also the LR test statistics seem to be very close to the minimum figure in the chi-square distribution curve, namely zero.

iii. Normality Tests

The graphs below illustrate the distribution of the Wiener process according the sampling period and the model in question. The X-axis of the graphs represents the bin, while the Y-axis represents the frequency of the Wiener process calculated from the maximum likelihood estimation of the parameters:

Figure 1- Distribution of the Wiener Process





From the graphs, it could be seen that some of the cases look closer to a bell shaped curve than others. Yet, a simple graph observation could not lead to the conclusion that the Wiener process of each of the distributions holds normality properties or not. The Jarque-Bera test is necessary to test the assumed normality of the Wiener process under study. The chart below indicates the outcomes obtained from the Jarque-Bera test on the dW_t as well as the critical values $\chi^2_{1\%}(2), \chi^2_{5\%}(2)$ and $\chi^2_{10\%}(2)$:

Table 3 – Jarque-Bera Test Results

		Jarque-Bera
Pre-Crisis 87	**GBM*****	6,2858
	CEV***	6,1398
Crisis 87	**GBM**	13,2981
	CEV	12,9845
Pre-Crisis 01	**GBM**	16,2863
	CEV	16,5214
Crisis 01	**GBM***	2,2599
	CEV*	2,0690
Pre-Crisis 07	**GBM**	114,7913
	CEV	113,2306
Crisis 07	**GBM**	64,2357
	CEV	67,0959

A number of remarks can be drawn from the Jarque-Bera test chart. Few of the cases display normality, namely the pre-crisis period of 1987 and the crisis period of 2001. The former failed to reject normality at 1% significance level (P-value of 4,32% for GBM and 4,64% for CEV), while the latter accepted the null hypothesis of normality as high as 10% significance level (P-value of 32,31% for GBM and 35,54% for CEV). Within the normality period cases, the CEV model shows slightly more strength than the GBM, but not strong enough to imply a conclusion from it. In general, whether showing normality or not, it can be seen that both the GBM and CEV's P-values are nearly similar within the same study period.

VI. Analysis

This section analyses the different outcomes obtained from the log-likelihood estimation, the likelihood ratio test and the Jarque-Bera normality test.

An analysis of the parameters' results from the maximum likelihood estimation of the GBM and CEV models among the different time-series provides with the followings. First and from the ML estimated parameters, the mean returns of the CEV relative to the GBM show that the CEV model is effectively having a tendency of estimated mean returns higher than the GBM's. The addition of the discount factor in the model has a tradeoff effect on the mean. Not only does the discount factor have a supposed leverage effect on the standard deviation, but also on the mean return within the different series.

The standard deviation results show that, the CEV's estimated σ is higher than the GBM's. This allows saying that the leverage effect added to the CEV model has an impact on the estimated standard deviation. In other words, adding the discount factor leads to a higher estimated standard deviation, confirming the leverage effect property of δ. The GBM's estimated standard deviation, also display a rise from the pre-crisis to the crisis ranges. The expansionary period of the S&P500 index held a more volatile logarithmic return than during the recessionary period. The pre-crisis times had experienced sharper movements of the equity market, which can be interpreted as a sign for an upcoming general decline in stock markets, and subsequently, to a crisis. Both the GBM and CEV's estimated σ increases from the pre-crisis to the crisis ranges. Remarks related to the discount factor results could be that it tend to display values close to the GBM case of $\delta=1$. The lowest discount factor among the pre-crisis series is $\delta=0.88$ and among the crisis series is $\delta=0.90$. In a recessionary environment for the equity market, the CEV model is more inclined to the special case of the GBM, although the difference seems unsubstantial.

The last analysis of the table of parameters' results concerns the mean return-variance movements. Both for the GBM and CEV, an increase in the logarithmic mean returns is accompanied with a rise in the estimated standard deviation. The intuition behind this result is that the higher return, the higher the risk.

Concerning the likelihood ratio test, the results are straightforward in the sense that they convey to the same conclusion. The null hypothesis model is considered to be better than the model related to the alternative hypothesis. As such, the CEV model is a better predictor for the movements of the S&P500 index according to the LR tests. The addition of a parameter relative to the GBM model reinforces the strength of the CEV model. However, the LR test statistics are also relatively low and close to the lower-tail, and thus to the rejection area of the null hypothesis. This can be due to the Type II error, where there is a tendency for failing to reject rather than rejecting the null hypothesis. A general consensus about the CEV being a better model than the GBM cannot be affirmed.

Both the Jarque-Bera normality test results and the distribution graphs show that the Wiener process of the CEV model and the GBM model does not show normality. Both the CEV & GBM Jarque-Bera test statistics hold values that are beyond the critical value for most of the series, and thus rejects the null hypothesis of normality, except for the 2001 crisis and pre-crisis of 2007. The Wiener process of the CEV is not normally distributed in 4 out of 6 series. For the GBM model, the CEV model test statistic fall within the rejection area for 4 out of 6 series, so both the CEV and GBM show normality in 2 out of 6 series, and for both of them it applies that it is the 2001 crisis and pre-crisis of 2007. From simple graph observation and test statistic, it can observed that the GBM and CEV model, show similar trends, and the test statistics are too close to make any conclusions on, they only thing that can me mentioned is the fact that the CEV test statistic is lower than the corresponding GBM statistic in 4 out of 6 series, but in terms of statistical significance, the conclusion was as the above mentioned, 2 out of 6 series show normality for both the CEV and GBM.

To sum up, the addition of the discount factor seems to strengthen the CEV model relative to the GBM model. The likelihood ratio test confirms that the CEV model is a good alternative to the GBM model for S&P500 market predictions. Moreover, the Wiener process of the CEV model is as normally distributed as the GBM model and nothing can be said as to one model being closer to normality than the other.

VII. Conclusion and Recommendation for Further Studies

Although this study partly illustrated the limitations of two popularized models within the fields of option pricing and financial economics, various outcomes cam be implied from the log-likelihood estimation, the likelihood ratio test and the Jarque-Bera normality test. The Geometric Brownian motion is of particular interest within the financial research community as it is the basis for the widely-used Black and Scholes method for option pricing, while the Constant Elasticity of Variance has been presented as a better alternative to Black and Scholes mainly because it breaks the unlikely assumption of constant equity volatility, but rather supports that the equity volatility should move in an opposite direction than the equity prices.

Indeed, this study tried to compare and contrast the two models in three different stock market periods in terms of their predictability of such exceptional movements in the S&P500 equity market, namely the 1987, the 2001 and the 2007 crises. The CEV model seems to exhibit a better model, from the LR test, than the GBM with the addition of the leverage effect factor, the strength of the results also imply that this can be considered as a sufficient and strong argument for the adoption of the CEV relative to the GBM. Another conclusion concerns the distribution of the corresponding Wiener process. In overall, the GBM models have shown normality features in 2/6 cases, with a test statistic being relatively close for a statistical adoption of normality, and the CEV model has exhibited the exact same normality properties for its stochastic process.

Few recommendations for further studies on lognormal processes for financial market modeling can be stated. 250 trading days as a sampling period can be behind inaccuracies estimating parameters, especially in high volatility environments, before and during stock market crashes. Considering the use of a discretization scheme in this study as a simplification of continuous time processes, two limitations can be raised. Shorter samples should then be considered for a better accuracy of the parameters' estimation. The necessity of applying a comparison of the CEV and GBM on continuous time should be recognized. Another recommendation could be the use of non-parametric estimation methods. These methods could solve the problems related to sensitive leverage effect estimation, being ranged from 0 to 1 while maximized likelihood functions can hold high numerical values. In addition, the distribution testing of the Wiener process could be extended to non-normal distributions.

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