Estimating and Testing Risk Approaches:
A Technical Analysis using Affine Term Structure Models, Monte Carlo Simulation and
GARCH Method

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Abstract

This paper investigates if the Log-Normal Mean-Reverting Ornstein-Uhlenbeck spot price (LNMROU) and the Vasicek (1977) process can forecast Value-at-Risk (VaR) using the Monte Carlo method. The results from LNMROU are validated against Delta-Normal-GARCH (DNG) and Historical Simulation (HS) which are well known approaches for VaR estimations. The backtested results indicated that HS and DNG are good measures for VaR estimation and that LNMROU failed in capturing price changes in the stock index market. The Vasicek (1977) proved to be a good model for forecasting VaR.
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List of Acronyms

ACF  
    Autocorrelation function
AR  
    Autoregressive
ARCH  
    Autoregressive Conditional Heteroskedasticity
ARMA  
    Autoregressive Moving Average
DNG  
    Delta-Normal-GARCH Approach
DW  
    Durbin-Watson
GARCH  
    Generalized Autoregressive Heteroskedasticity
HS  
    Historical Simulation Approach
LNMROU  
    Log-Normal Mean-Reverting Ornstein-Uhlenbeck Spot Price Process
MCM  
    Monte Carlo Method
MLE  
    Maximum Likelihood
OLSR  
    Ordinary Least Squares Regression
SDE  
    Stochastic Differential Equation
VaR  
    Value-at-Risk

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1 Introduction

A well known and popular measure for estimating risk exposure is Value-at-Risk (VaR), which is a proxy over how much we expect to lose in nominal amounts, given a certain and previously determined confidence level. The Basel commission in Switzerland requires that banks and financial institutions calculate and accumulate capital corresponding to 99% VaR during a 10 day period (Hull, 2005).

My intentions with this paper are to investigate different approaches for analyzing and forecasting risky outcomes and the general VaR models I have chosen to include in my study are: Monte Carlo method (MCM), Delta-Normal-GARCH (DNG) and Historical Simulation (HS). Those VaR models are well known within finance and applied on a regular basis by risk managers all around the world. Thus, large financial institutions have a great need for superior risk models that are able to forecast unforeseen fluctuations in the markets.

This paper research if the Log-Normal Mean-Reverting Ornstein-Uhlenbeck spot price (LNMROU) and the Vasicek (1977) process can forecast Value-at-Risk (VaR) using the Monte Carlo method.

MCM is an advanced and complex method which rests upon a stochastic differential equation (SDE). It yields a complete probability distribution over future outcomes and is often used to mimic the real world scenarios (Hull, 2005). Thus, it is reasonable to assume that those models used in MCM should be accurate. However, the LNMROU and the Vasicek (1977) process are well-known SDE and mean reverting models. They are often used when pricing commodities and short-term bonds but they are not normally associated with predicting VaR.

However, one substantial issue when estimating risk might be – is it worth the extra cost of applying more advanced models like MCM instead of more common easy-to-use models such as HS and DNG?

In chapter 2 my methodology is presented. In chapter 3 data analysis is carried out and my results are discussed. Finally, in chapter 4 my conclusions are presented.
2 Methodological Framework

To achieve my research goal, to evaluate if the Log-Normal Mean-Reverting Ornstein-Uhlenbeck spot price process (LNMROU) and the Vasicek (1977) process can forecast Value-at-Risk (VaR), I use the Monte Carlo Method (MCM). I also use Delta-Normal-GARCH (1.1) (DNG) and Historical Simulation (HS) methods to see if easy-to-use models can compare with more advanced methods like MCM.

For use with the MCM a LNMROU and the Vasicek (1977) short interest model is applied. I have chosen to use, in respective order, OMX30 Stockholm stock market index\(^1\) and SSVX3M\(^2\) short interest rate. The study is thus based on two assets, one without mean reversion and one with mean reversion. For parameter estimation I use Ordinary Least Squares Regression (OLSR) and Maximum Likelihood Estimation (MLE). For validation I use Christoffersens (1998) and Kupiec’s (1995) binomial backtest. All programming is made in Excel, and for some computations Add-In macros like Data-Analysis and the Solver tool pack are applied. In the following chapters I will explain these processes in detail.

2.1 Mean Reversion Models

Within the family of affine term structure mean reversion models, we can find The Vasicek (1977) model which is similar to an Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process has been derived from what is called Itô’s Lemma (Copeland et al, 2005), a stochastic differential equation (SDE) often applied in Monte Carlo methods. The Ornstein-Uhlenbeck process:

\[ \frac{ds}{s} = \mu_B dt + \sigma_B dW, \quad W \sim \text{IID}(0,1) \]  \hspace{1cm} 2.1

\( \mu_B \) is the constant drift and \( \sigma_B \) is the volatility (Craine, 2000). The error term \( dW \) is a standard Wiener process with independent and identically distributed (IID) properties. A Weiner process rests upon a Markov stochastic process with standard normal properties assumed to

\(^1\) OMX30 Stockholm is a Swedish stock index containing the thirty most traded stocks at the Stockholm Stock Exchange.
\(^2\) SSVX3M is a 90 days interest rate used for pricing short term zero-coupon bonds.
follow a Geometric Brownian Motion (Hull, 2005). The special aspect of mean reversion models is that they have a built-in speed of adjustment. This works as a force for reversing the rate back to its long term mean.

2.1.1 Log – Normal Mean-Reverting Ornstein-Uhlenbeck Spot Price Process

Boogert and Jong (2006) as well as Bjerksund et al. (2008) employ the Log-Normal Mean-Reverting Ornstein-Uhlenbeck spot price process when valuating gas storage and succeeded with good results. They attribute their results to strong mean reversion in the gas spot market which limits the variance of the price distribution. Their model also performs well up to a few years and they suggest using a multi-factor model for describing the long term price levels. The authors claim that their model captures more of the real-world flexibility value compared to single one-factor model produce.

They describe a one-factor model for calculating the log spot prices and they assume that the spot price dynamics follow:

$$\frac{dS(t)}{S(t)} = \kappa[\mu(t) - \ln S(t)] dt + \sigma dW(t), \quad \kappa, \sigma > 0. \tag{2.2}$$

where the mean reversion rate $\kappa = \alpha$ and the spot price volatility $\sigma$ are greater than zero. The term $\mu(t)$ is the long term mean. Bjerksund et al. (2008) refer to this model as the log-normal mean-reverting Ornstein-Uhlenbeck price process. Let $X(t): = \ln S(t)$ and the spot price is transformed into:

$$dX(t) = \kappa [\mu(t) - x(t) - \frac{\sigma^2}{2}] dt + \sigma dW(t) \tag{2.3}$$

The general form for Vasicek (1977) according to Spykens (2010) is:

$$x(t) = e^{-at} \left( x_0 + \frac{\sigma^2}{2a} \right) - \frac{\sigma^2}{2a} + \alpha e^{-at} \int_{u=0}^{t} e^{-at} \mu du + \sigma e^{-at} \int_{u=0}^{t} e^{-at} dW(u)$$
According to Bjerksund et al. (2008), the conditional expectation for the $log S(t) = x(t)$ is:

$$E_0[x(t)] = e^{-at} \left( x(t_0) + \frac{\sigma^2}{2a} - \frac{\sigma^2}{2a} + a e^{-at} \int_{u=0}^{t} e^{-a(t-u)} \mu du \right)$$ \hspace{1cm} 2.4

and the conditional variance is written as:

$$Var_0[x(t)] = Var_0(\sigma e^{-at} \int_{u=0}^{t} e^{-a(t-u)} dW(u))$$ \hspace{1cm} 2.5

2.1.2 Ornstein-Uhlenbeck (The Vasicek) Model

The Vasicek (1977) model is one of the earliest stochastic models for estimating short-term interest rates. It works with a general form of an affine term structure model known as the Ornstein Uhlenbeck process (the Gaussian case). Vasicek (1977) emphasizes three assumptions:

i. The spot rate follows a diffusion process;
ii. The price of a (discount) bond depends on the spot rate over its term;
iii. The market is efficient.

He also considers “a market in which investors buy and issue default free claims on a specified sum of money to be delivered at a given future date” (Vasicek, 1977, p. 177).

Those claims correspond to discount bonds. The price of a bond is denoted as $P(t, s)$ at time $t$ maturating at time $s$, where $t \leq s$.

$$P(s, s) = 1$$ \hspace{1cm} 2.6

$$R(t, T) = -\frac{1}{T} \log P(t, t+T), \quad T > 0.$$  

$$R(t, T) = \frac{1}{T} \int_{t}^{t+T} F(t, \tau) d\tau.$$
In the eq. 2.6, \( R(t,T) \) corresponds to the yield to maturity on a bond with a maturity date \( s = (t + T) \), which also refers the term structure at time \( t \). \( F(t, \tau) \) refers to the forward rate and the instantaneous spot rate for borrowing and lending is defined as:

\[
r(t) = R(t,0) = \lim_{T \to 0} R(t,T)
\]

The spot rate is denoted \( r(t) \) and the loan will go on increasing with \( W \) during the time to maturity.

\[
dW = Wr(t)dt.
\]

Since the spot rate is a stochastic process it will vary over time and so will also the bond price (but the bond will always be worth a certain amount of cash at the end of maturity). The rate of return \( R(t) \) is assumed to be a continuous function of time \( t \) and to follow a Markov process, which is a diffusion process implying that the spot price characterizes its current value. This can be described by a stochastic differential equation (affine term structure model):

\[
dr = f(r,T)dt + \rho(r,t)dz
\]

known as Itô’s Lemma (Copeland et al., 2005). In the eq. 2.7, \( z(t) \) is a standard Wiener process with variance \( dt \). The functions \( f(r,T) \) and \( \rho(r,t) \) refers to drift and variance of the rate process \( r(t) \).

The price of a bond \( P(t,s) \) is determined by the partial equilibrium model representing the demand and supply theorem. It also rests upon the several assumptions of efficient markets, thus there exists no transaction costs, all information is reflected in the bond’s price and finally every investor acts rationally (homogenous expectation and no profitable riskless arbitrage possibilities).
In order to illustrate the general model, the Vasicek (1977) interest rate model considers some assumptions e.g.:

i. The market price of risk $g(t, r)$ is constant

$$g(t, r) = q$$  \hspace{1cm} \text{(2.8)}

ii. The spot rate follows an Ornstein-Uhlenbeck process proposed by Merton (1971) and it possesses stationary increments.

iii. In eq. 2.9 below, the first term is the instantaneous drift which keeps pulling the rate towards its long term mean $\mu$ with speed of adjustment $\kappa$. Since the model has a stochastic element with constant variance $\rho^2$ the process keeps fluctuating around its long-term mean.

$$dr = \kappa(\mu - r)dt + \rho dz$$

$$r_{t+1} = r_t + \kappa(\mu - r_t)dt + \sigma \sqrt{t-t}$$  \hspace{1cm} \text{(2.9)}

One of the major drawbacks of Vasicek's model is that it allows for negative interest rates to occur (which is not common). This disadvantage is taken into consideration and adjusted by Cox et al. (1985) in the Cox-Ingersoll-Ross (CIR) short interest rate model. The conditional expectations and the variance of the model are given by:

$$E_0[x_t] = \mu + (x_0 - \mu)e^{-at}$$

$$Var_0[x_t] = \frac{\sigma^2}{2\alpha}(1 - e^{-2at}), \alpha > 0.$$  

---

1 Cox et al. (1985) is a non-Gaussian model and it includes a square root of the interest rate term, which yields an insurance against negative rates ($r(t) > 0$):

$$dr = \kappa(\mu - r(t)) + \sigma \sqrt{r(t)}dW(t).$$
2.2 Value-at-Risk

VaR is often criticized for being inconsistent and this is a drawback which is connected to VaR as a coherent risk measure. Most concepts within the financial industry today assume that past returns follow a normal distribution, known as the parametric Gaussian approach. Obviously, according to Craine et al. (2000), displays high frequency data often display excess kurtosis (fat tailed distributions), skewness and volatility clustering. Hence, to capture these characteristics with a simple parametric model is very difficult, because these models tend to underestimate actual volatility. To capture volatility clustering, models like GARCH are often employed (Craine et al., 2000). To avoid ad hoc assumptions, Pattarathammas et al. (2008) suggests Historical Simulation, since it has many advantages. Firstly, it is very easy to implement. Secondly, it relaxes the assumption of normality in the returns distribution and it also allows for heavy tails.

In this paper time series data sampled from Nasdaq OMX Nordics and Thomson Reuters database are used. The data contains daily spot price observations of the OMX30 Stockholm (1995-2009) and also SSVX3M (1995-2003).

In the MCM for LNMROU I use a test window of 366 observations is applied when estimating the OLSR and MLE parameters. When estimating the Vasicek (1977) the whole sample of 1251 observations (1995-2003) is used.

Furthermore, the VaR I computed with DNG estimated with the MLE method and also HS. In both DNG and HS I use an ad hoc test window containing 1759 observations.

2.2.1 Monte Carlo Method

MCM is a well known and accepted approach within risk management. It is often employed when predicting future unknown events, assuming that historical outcomes reflect the future. MCM starts with a stock price process, assuming risk neutrality (Hull, 2005):

\[ dS = \mu Sdt + \sigma Sdz \]  \hspace{1cm} 2.10
where $dz$ is a Wiener process, $\hat{\mu}$ is the expected return and $\sigma$ is the volatility. From Itô’s lemma the process follows by the log of spot price $\ln S (t)$.

$$d \ln S = (\hat{\mu} - \frac{\sigma^2}{2}) dt + \sigma dz$$

$$\ln S_{t+1} = S_t + (\hat{\mu} - \frac{\sigma^2}{2}) dt + \sigma dz$$

When calculating Value-at-Risk (VaR) for a portfolio it is normal to proceed from a fixed factor model and then generate several possible outcomes for the future, using normally distributed random numbers $\varepsilon \sim N(0,1)$. To get a complete probability distribution over future outcomes, thousands of iterations for every possible outcome are used. This gives a sequence of thousand possible directions which the spot price path can take from $t_i$ to $t_{i+n}$. To obtain VaR, the price change in the stock $dS$ is chosen and the confidence level which corresponds to the risk level is selected (Asgharian and Nordén, 2007).

### 2.2.2 Historical Simulation

Historical Simulation (HS) assumes that future outcomes reflect the past. HS is estimated for every asset “i” during the past days before $t$.

$$R_{s,i} = \left[ \frac{p_{s,i} - p_{s,i-1}}{p_{s,i-1}} \right]$$  \hspace{1cm} 2.11

According to eq. 2.11 “$S$” historical observations are obtained and VaR is estimated taking the percentile which corresponds to the risk level (Asgharian and Nordén, 2007).

$$VaR = R_{s}^c V_{t-1}$$  \hspace{1cm} 2.12

In eq. 2.12, $R_{s}^c$ corresponds to an $\alpha$ – percent (one-sided) confidence level in $R_{s,p}$. The $V_{t-1}$ represents the assets price at time $t - 1$.

Historical Simulation is a straight forward approach to work with and it does not take any statistical assumptions into consideration. However, a drawback with the model is that it places the same weight on all observations in the time series and it also assumes constant volatility (Asgharian and Nordén, 2007).
2.2.3 Delta-Normal-GARCH Approach

Delta-Normal-GARCH (DNG) is often referred to as the main alternative to Historical Simulation (HS), DNG is parametric approach and HS is non-parametric. The methodology for Delta-Normal model is straight forward, still there are some issues to take into consideration. One of the most important aspects concerns the choice of volatility model. There are a lot of volatility models to select between, some assumes constant volatility and other let volatility be conditional (Hull, 2005). This paper uses a conditional approach, known as Generalized Autoregressive Conditional Heteroskedasticity (GARCH). There are several GARCH models. One of the most common used is GARCH (1.1) which also is applied in this thesis.

The equation for Delta-Normal method is (Asgharian and Nordén, 2007):

\[ VaR = \sigma_x C_\alpha V \]  \hspace{1cm} 2.13

where \( \sigma_x \) represents the daily volatility, \( C_\alpha \) the confidence level and \( V \) is the market value of the asset.

2.3 Models of Changing Volatility

Most financial models assumes linearity, however economic behavior can be non-linear, e.g. investors’ attitude towards risk as well as expected returns on financial instruments (Campbell et al, 1997).

2.3.1 GARCH (1.1)

GARCH belongs to univariate ARCH processes introduced by Engel (1982), these models are non-linear both in mean and variance. In finance, \( \eta_{t+1} \) is assumes to be the innovation in asset returns, \( \sigma_t^2 \) is the conditional variance of \( \eta_{t+1} \) at time \( t \) which is equivalent to \( \eta_{t+1}^2 \) with a normal distribution.

\[ \eta_{t+1}^2 \sim N(0, \sigma_t^2). \]
When studying asset returns it is common that large returns are followed by even larger returns and vice versa. It seems that asset returns are serially correlated and to capture this serial correlation ARCH processes are often used. The most general GARCH model belongs to GARCH (1.1) which looks as follows:

\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \eta_t^2 \]  

It could also be written as an ARCH process:

\[ \eta_{t+1}^2 = \omega + (\alpha + \beta) \eta_t^2 + (\eta_{t+1}^2 - \sigma_t^2) - \beta(\eta_t^2 - \sigma_{t-1}^2) \]

GARCH (1.1) is exactly an ARMA (1.1)\(^4\) for squared innovations (Campbell et al., 1997).

Since asset returns are stochastic, Campbell et al. (1997) emphasizes that with response to non-normality of the returns we assume that returns follow a diffusion process. GARCH belongs to continuous-time models and should be estimated at every moment of time during the test period.

2.4 Mathematical and Statistical Approaches

Estimating parameters, this paper apply Ordinary Least Squares Regression (OLSR) and Maximum Likelihood Estimation (MLE) referring to Spykens (2010), Sheikh (2007), Bjerksund et al. (2008) and Herrala (2009).

2.4.1 Time Series Modeling

When investigating time series some concepts are very important to consider, for instance whether a time series is stationary or not, since non-stationary increments could result in very misleading conclusions regarding future events (Brooks, 2008).

\(^4\) ARMA (1.1) stands for Autoregressive Moving Average of first order.
2.4.2 Ordinary Least Squares Regression

Ordinary Least Squares Regression (OLSR) is an estimator requiring that the data sample fulfill the properties presented below (Brooks, 2008):

\[ E[\hat{\theta}] = \theta \]

**Interpretation:**

1. \( y = X\beta + \varepsilon \)  
   Functional form for the regression function
2. \( E[\varepsilon] = 0 \)  
   The errors including a zero mean
3. \( E[\varepsilon\varepsilon'] = \sigma^2 I \)  
   The variance is constant over time
4. \( E[\varepsilon|X] = 0 \)  
   Unbiasedness, \( \varepsilon \) and \( X \) should be independent
5. \( \text{Pr} [\text{Rank } (X) = \mathcal{R}] = 1 \)  
   Full rank, the inverse of \([XX'] \) exist, \(([XX']^{-1}) \)
6. \( \varepsilon \sim N(0, \sigma^2 I) \)  
   The error term is normally distributed with zero mean and covariance matrix \( \sigma^2 I \).

Are those assumptions fulfilled, the OLSR estimator is BLUE\(^5\), and this implies that the estimator is:

**Consistent** (estimates converge to its true value as the sample size goes large):

\[ \lim_{n \to \infty} \text{Pr} [|\hat{\theta} - \theta| > \delta] = 0 \quad \forall \delta > 0 \quad 2.15 \]

**Unbiasedness:**

\[ E[\hat{\alpha}] = \alpha \quad E[\hat{\beta}] = \beta \quad 2.16 \]

**Efficiency:**

The estimator has the smallest variance among all estimators, thus it has flatter tails.

\[ \hat{\theta} = \text{arg min}_\theta \sum E[\varepsilon \varepsilon'] \quad 2.17 \]

---

\(^5\) BLUE involves that our estimator is Best Linear Unbiased Estimator and contains least variance of all available estimators.
2.4.3 Maximum Likelihood Estimator

The most immediate method for obtaining the maximum likelihood estimator $\hat{\theta}$, is to use historical data (time-series data or cross-sectional data). Historical observations of $x(t)$ are sampled at non-stochastic dates, $t_i, t_{i+1}, ..., t_{i+n}$. Since, financial institutions are closed at weekends, time-series yields irregular sampling intervals. However, $x(t)$ assumes to be a Markov process, which implies that irregular sampling do not cause any trouble. The joint density function is given by (Campbell et al., 1997):

$$f(x_0, x_1, ..., x_n; \theta) = f_0(x_0; \theta) \prod_{k=1}^{n} f((x_k, t_k \mid x_{k-1}, t_{k-1}; \theta))$$

Where $x_k \equiv x(t_k)$, $f_0(x_0)$ refers to the marginal density function of $x_0$ and $f((x_k, t_k \mid x_{k-1}, t_{k-1}; \theta))$ contributes with the conditional density function of $x_k$ given $x_{k-1}$ also called the transition density function.

To estimate the Maximum Likelihood Estimator (MLE) $\hat{\theta}$, Campbell et al. (1997) defines the log-likelihood function $\mathcal{L}(\theta)$ as:

$$\mathcal{L}(\theta) = \sum_{k=0}^{n} \log f_k$$

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta)$$

2.18

Under the condition that $\hat{\theta}$ is consistent, the following normal limiting distribution must be fulfilled:

$$\sqrt{n}(\hat{\theta} - \theta) \sim N(0, N^{-1}J^{-1})$$

where $J$ is the information matrix and the variance is given by:

$$J = \frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta \partial \theta'}$$

$$\text{Var}[\hat{\theta}] \approx \frac{1}{n} J^{-1}(\theta)$$
2.4.4 Ordinary Least Squares Regression for Log-Normal Mean-Reverting
Ornstein-Uhlenbeck Spot Price Process

Bjerksund et al. (2008) describes the price process model as:

\[
\frac{dS(t)}{S(t)} = \kappa [\mu(t) - \ln S(t)] dt + \sigma dW(t)
\]

\[
\ln S(t) = x(t_i)
\]

\[
dx(t_i) = x(t_{i+1}) - x(t_i)
\]

\[
x(t_{i+1}) = x(t_i) e^{-\alpha \Delta t} + \left( \mu - \frac{\sigma^2}{2\alpha} \right) (1 - e^{-\alpha \Delta t}) + \sigma e^{-\alpha t_{i+1}} \int_t^{t_{i+1}} e^{\alpha s} dW_s
\]

where \( \kappa \) is the mean reversion rate, the term \( \mu(t) \) is the long term mean and \( \sigma \) representing the volatility. These terms are constants.

Referring to Spykens (2010), it is necessary to convert the eq. 2.19 above into:

\[
Y = mX + c + \epsilon
\]

From eq. 2.20 OLSR is applied to minimize the variance of the residual \( \epsilon \), assuming that:

\[
\epsilon = \epsilon(t_i)
\]

by minimizing the variance, the estimators \( \hat{\sigma}, \hat{\mu} \) and \( \hat{\alpha} \) are obtained.

\[
x(t_{i+1}) = x(t_i) e^{-\alpha \Delta t} + \left( \mu - \frac{\sigma^2}{2\alpha} \right) (1 - e^{-\alpha \Delta t}) + \sigma e^{-\alpha t_{i+1}} \int_t^{t_{i+1}} e^{\alpha s} dW_s
\]

through subtracting both sides with \( x(t) \) the following is obtained:

\[
x(t_{i+1}) - x(t_i) = x(t_i) e^{-\alpha \Delta t} + \left( \mu - \frac{\sigma^2}{2\alpha} \right) (1 - e^{-\alpha \Delta t}) + \sigma e^{-\alpha t_{i+1}} \int_t^{t_{i+1}} e^{\alpha s} dW_s - x(t)
\]

and by comparing eq. 2.21 with eq. 2.22, \( m, c, \) and \( \epsilon \) are obtained:
Let the regression function be \( Y = x_{t_{i+1}} - x_{t_i} \) where \( X \) is a \([n \times 1]\) vector matrix, with intercept \( c \) and slope \( m \). Note that the estimator’s \( \hat{\sigma}, \hat{\mu} \) and \( \hat{\alpha} \) must be estimated using OLSR.

\[
Y = \begin{bmatrix}
x_{t_1} - x_{t_0} \\
x_{t_2} - x_{t_1} \\
x_{t_3} - x_{t_2} \\
\vdots \\
x_{t_n} - x_{t_{n-1}}
\end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_{n-2} \\ 1 & x_{n-1}\end{bmatrix}
\]

By first solving the estimators \( \hat{\sigma}, \hat{\mu} \) and \( \hat{\alpha} \) (see eq. 2.23-2.24), one can obtain \( \hat{\sigma}, \hat{\mu} \) and \( \hat{\alpha} \).

\[
Y = mX + c
\]

\[
\theta_{\text{OLSR}}(\hat{\alpha}, \hat{\mu}, \hat{\sigma}) = \begin{pmatrix} \hat{c} \\ \hat{m} \end{pmatrix} = (X'X)^{-1}X'Y
\]

The estimated parameters \( c \) and \( m \) from eq. 2.23 and 2.24 are then applied in eq. 2.27 and 2.28 below.

\[
\hat{\alpha} = \frac{\ln(m + 1)}{\Delta t} \tag{2.27}
\]

\[
\hat{\mu} = \frac{\hat{c}}{1 - e^{\hat{\alpha}\Delta t}} + \frac{\hat{\sigma}^2}{2\hat{\alpha}} \hat{\sigma}^2 = \left( \frac{2\hat{\alpha}}{1 - e^{2\hat{\alpha}\Delta t}} \right) \left( \text{VaR}(Y) \cdot \frac{\text{Cov}(X,Y)}{\text{Var}(X)} \right) \tag{2.28}
\]

2.4.5 Maximum Likelihood Estimation for Log-Normal Mean-Reverting Ornstein-Uhlenbeck Spot Price Process

The conditional expectation for the log spot price is (Spykens, 2010):

\[
v(t_{t+1}) = E(x(t_{t+1})|x(t_0)) \tag{2.29}
\]
the conditional variance is defined as:

\[ w^2(t_{i+1}) = \sigma^2 e^{-2\alpha t_i} \left\{ \int_{t(i)}^{t_{i+1}} e^{as} dW(s) \right\}^2 \]

\[ w^2(t_i) = \frac{\sigma^2}{2\alpha} [e^{-2\alpha t_i}] \]

and the natural logarithm of the log-likelihood is given by:

\[ \ln L(\alpha, \mu, \sigma^2) = -\frac{n}{2} \ln 2\pi - \sum_{i=1}^{n} \left( \frac{\ln w^2_i}{2} + \frac{1}{2\omega^2_i} (x_{t_i} - v_{t_i})^2 \right) \]

\[ \theta = \arg \max_{\theta} \ln L(\alpha, \mu, \sigma^2) \]

To obtain the MLE estimators given above, it is necessary to maximize the log-likelihood function using the OLSR estimators as start values.

2.4.6 Maximum Likelihood Estimator for GARCH (1,1.)

To estimate GARCH (1,1), parameter estimation first has to be carried out. The most common way of doing this is to apply MLE. This method obtains parameters by maximizing the log-likelihood ratio. The following equations are used referring to Asgharian (2010).

\[ r_t = \sum_{i=1}^{k} b_i x_{i,t} + \eta_t \]

\[ \eta_t | \Omega_{t-1} \sim N(0, \sigma_t^2) \]

\[ \sigma_t^2 = \omega + \alpha \eta_t^2 \]

\[ \ln L(b_i, \omega, \alpha) = \sum_{t=1}^{T} \left[ -\frac{1}{2} \ln(2\pi \sigma_t^2) - \frac{r_t^2}{2\sigma_t^2} \right] \]

\[ \theta = \arg \max_{\theta} \ln L(b_i, \omega, \alpha) \]

where the error term \( \eta_t \) is conditional to covariance matrix \( \Omega_{t-1} \). The MLE estimator should be maximized for the parameters \( b_i \), \( \omega \) and \( \alpha \).
2.4.7 Ordinary Least Squares Regression for the One-Factor Vasicek (1977) Model

OLSRR consider a linear relationship between the time series observations where $\varepsilon \sim N(0, \sigma^2 I)$ represents a random shock and $I$ is the identity matrix included in the covariance matrix (Spykens, 2010).

$$ x_{t+1} = ax_{t+1} + b + \varepsilon $$  \hspace{1cm} 2.31

where:

$$ a = e^{-\alpha \Delta t} \quad b = \mu(1 - e^{-\alpha \Delta t}) \quad \varepsilon_{sd} = \sigma \sqrt{-\frac{2\ln(\alpha)}{\Delta t(1-\alpha^2)}} $$  \hspace{1cm} 2.32

To simplify the calculations the following methodology is applied (Spykens, 2010):

$$ s(x) = \sum_{i=1}^{n} x_{t_i - 1} \quad s(y) = \sum_{i=1}^{n} x_{t_i} \quad s(xx) = \sum_{i=1}^{n} x_{t_i - 1}^2 \quad s(xy) = \sum_{i=1}^{n} x_{t_i} x_{t_i - 1} $$  \hspace{1cm} 2.33

$$ \hat{\alpha} = \frac{n(s(y)x - s(x)s(y))}{ns(xx) - s(x)^2} \quad \hat{\beta} = -\frac{s(y) - as(x)}{n} \quad \hat{\varepsilon}_{st} = \sqrt{\frac{ns(yy) - s(y)^2 - \hat{\alpha}(ns(xy) - sl(x)s(y))}{n(n-2)}} $$

$$ \hat{\alpha} = \frac{b_{st} - (s(sty) - s(st)x s(y))}{s(xx) - s(x)^2} \quad \hat{\beta} = \frac{ny}{n[1 - \frac{(n(s(sty) - s(st)x s(y))}{n(s(xx) - s(x)^2)^2}]} \quad \hat{\sigma} = \frac{\sqrt{-2\ln(\hat{\beta})(s(sty) - s(st)x s(y))}}{\sqrt{\Delta t[1 - \frac{(n(s(sty) - s(st)x s(y))}{n(s(xx) - s(x)^2)^2}]}} $$  \hspace{1cm} 2.34

2.4.8 Maximum Likelihood for the One-Factor Vasicek (1977) Model

Below, two ways of estimating the one-factor Vasicek (1977) are presented:

1.

Method one starts with presenting the conditional density function (Spykens, 2010):
\[ f_i = (x_{t_i}; \alpha, \mu, \sigma) = (2\pi \sigma^2)^{-1/2} \exp \left[ -\frac{(x_{t_i} - x_{t_{i-1}} e^{-\alpha \Delta t} - \mu(1 - e^{-\alpha \Delta t}))^2}{2\sigma^2} \right] \]

where

\[ \sigma^2 = \frac{\sigma^2}{1 - e^{-2\alpha \Delta t}} \quad 2.35 \]

Further the MLE log-likelihood function is given by equation:

\[ \ln L(x_i; \alpha, \mu, \sigma) = -\frac{n}{2} \ln(2\pi) - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_{t_i} - x_{t_{i-1}} e^{-\alpha \Delta t} - \mu(1 - e^{-\alpha \Delta t})^2 \quad 2.36 \]

To achieve MLE estimators \( \hat{\mu} \) and \( \hat{\alpha} \), eq. 2.37-2.38 need to be solved:

\[ \hat{\mu} = \frac{n s(xy) - s(x) s(y) + n\hat{\mu}^2}{n(s(x^2) - s(x)s(y))} \quad 2.37 \]

\[ \hat{\alpha} = -\frac{1}{\Delta t} \ln \left( \frac{s(xy) - s(x) s(y) + n\hat{\mu}^2}{s(x^2) - 2\hat{\mu}(x) - n\hat{\mu}^2} \right) \quad 2.38 \]

\[ \hat{\sigma}^2 = \frac{1}{n} \left[ s(yy) - 2e^{-\hat{\alpha} \Delta t} s(xy) + e^{-2\hat{\alpha} \Delta t} s(ss) - 2\hat{\mu}(1 - e^{-\hat{\alpha} \Delta t}) \left( s(y) - e^{-\hat{\alpha} \Delta t} s(x) \right) + n\hat{\mu}^2 (1 - e^{-\hat{\alpha} \Delta t})^2 \right] \]

and to obtain \( \hat{\sigma} \) eq. 2.39 is solved:

\[ \hat{\sigma} = \hat{\sigma} \sqrt{\frac{2\hat{\alpha}}{1 - e^{-2\hat{\alpha} \Delta t}}} \quad 2.39 \]

2.

In method two the log-likelihood function is given by (Herrala, 2009):

\[ L = \prod_{t_i=1}^{n} \left[ 2\pi \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha \Delta t}) \right]^{-1/2} \exp(-0.5\nu^2(r_{t_i}, r_{t_{i+1}}, t_i)) \quad 2.40 \]

where:

\[ E[r_{t_i}] = (\mu + (r_{t_i} - \mu) e^{-\alpha \Delta t}) \]
\[ \nu a r(r_{t_i}) = \frac{\sigma^2}{2\sigma} (1 - e^{-2\lambda t_i}) \]

\[ v(r_{t_i}, r_{t_i+1}, t_i) = \frac{r_{t_i+1} - E[r_{t_i}]}{\sqrt{\text{var}(r_{t_i})}} = \frac{r_{t_i+1} - (r_{t_i} - \mu)e^{-\lambda t_i}}{\sqrt{\text{var}(r_{t_i})}} \]

\[ \ln L = -\frac{N-1}{2} \ln 2\pi - \frac{N-1}{2} \ln \left( \frac{\sigma^2}{2\sigma} (1 - e^{-2\lambda t_i}) \right) - \frac{1}{2} \sum_{i=1}^{N} \left( \frac{v^2(r_{t_i}, r_{t_i+1}, t_i)}{\text{var}(r_{t_i})} \right) \]

\[ \theta = \arg \max_{\theta} \ln L(\theta) \]

Eq. 2.42 represents the variance of the returns and when \( T \to \infty \) the term \( (1 - e^{-2\lambda t_i}) \to 1 \) (Herrala, 2009).

2.4.9 Pricing Bonds

Bonds fall within the area of fixed-income securities and a bond can promise a certain amount of money in the future, given that it is held until its maturity. There are two kinds of bonds, Zero-coupon bonds and Coupon bonds. In this thesis only the first one is considered. Zero-coupon bonds (or discount bonds) gives one single payment at the maturity date (known as the bonds face value). Campbell et al. (1997) use the equation below to describe the market price of a bond as:

\[ P_{nt} = \frac{H_{ext}}{(1 + R_{nt})} \]

where \( R_{nt} \) is the \( n \)-period (annual) spot rate at time \( t \). Zero-coupon bonds are often given at a maturity less than an year and Coupon bonds for longer than an year.

2.5 Christoffersens (1998) Conditional Backtest

Christoffersens (1998) backtest is one of the more commonly used backtests within finance. The test takes both independency and unconditional likelihood ratios into consideration (see eq. 2.44, 2.45 and 2.46). These likelihood ratios assume that there should be no temporal patterns in the exceedance series known as clusters, thus earlier occasions should not affect
future occasions. Ignoring independency could result in a Type II error, which means that a bad model is accepted.

The hypothesis for a correct model is given by:

\[ L(I, p) = p^{r_1}(1 - p)^{T - r_1} \]

\[ L(I, \hat{p}_1) = \hat{p}_1^{r_1}(1 - \hat{p}_1)^{T - r_1} \]

\[ \hat{p}_1 = \frac{T}{T} \]

\[ LR_{uc} = 2(\ln L(I, \hat{p}_{01}) - \ln L(I, p)) \]

where \( T \) is the sample size, \( r_1 \) the exceedance, \( r_0 \) the non-exceedance and \( p \) represents the confidence level \((1 - \alpha)\). The likelihood function is chi-square distributed with one degree of freedom. Let \( n_{ij} \) be the days of which “i” (0) was followed by “j” (1), and where 0 indicated no exceedance. Let \( \pi_{ij} \) be the probability for state “ij” to occur. The independent likelihood ratio is given by:

\[ \hat{p}_{01} = \frac{\hat{r}_{01}}{\hat{r}_0} \]

\[ \hat{p}_{11} = \frac{\hat{r}_{11}}{\hat{r}_1} \]

\[ LR_{ind} = 2(\ln L(I, \hat{p}_{01}, \hat{p}_{11}) - \ln L(I, \hat{p}_1)) \sim \chi^2(1) \]

\[ LR_{cc} = 2(\ln L(I, \hat{p}_{01}, \hat{p}_{11}) - \ln L(I, p)) \sim \chi^2(2) \]

\[ L(I, \hat{p}_{01}, \hat{p}_{11}) = (1 - \hat{p}_1)^{r_0 - r_{01}} \hat{p}_{01}^{r_{01}} \]

The \( LR_{cc} \) is chi-squared distributed with two degrees of freedom and it is able to test both for coverage and independency (Christoffersen et al., 2004).
2.6 Kupiec’s (1995) Binomial Approach

When backtesting VaR, another commonly used approach is Kupiec’s binomial backtest (Dowd, 2007).

\[
Prob(x, n|p) = \binom{n}{x} p^x (1-p)^{n-x} \tag{2.47}
\]

where eq. 2.47 concerns the binomial probability with a sample size of \( n \) observations, \( x \) exceedance and a confidence level \( p \) (where \( p = (1 - \alpha) \) and \( \alpha \) is the significance level). The model is rejected if the estimated p-value is less than the confidence level.
3 Data analysis and Preliminary Results

In this section all results are presented as well as analyzed. Two assets are used for parameter and VaR estimation and they are all presented in section 3.1 respectively in section 3.2. The VaR-models are analyzed and backtested applying Christoffersen (1998) and Kupiec’s (1995) backtest (see Table 5-6 and 11-12).

3.1 OMX 30 Stockholm

In this paper, time series data sampled from NASDAQ OMX Nordic is used. The data contains daily observations of OMX30 Stockholm\(^6\) from 1995-2009. In figure 3.1 the \(x\) – axis = time and \(y\) – axis = price in SEK.

![Figure 3.1 Spot prices for OMX30 Sthlm](image)

<table>
<thead>
<tr>
<th>Table 1 Data properties</th>
<th>(OMX30 Sthlm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E[R(t)]:)</td>
<td>-0.000610</td>
</tr>
<tr>
<td>(\text{Var}[R(t)]:)</td>
<td>0.0250 %</td>
</tr>
<tr>
<td>Volatility (\sigma) (daily):</td>
<td>1.59 %</td>
</tr>
<tr>
<td>Volatility (\sigma) (yearly):</td>
<td>23.8 %</td>
</tr>
<tr>
<td>Skewness:</td>
<td>-0.0352</td>
</tr>
<tr>
<td>Kurtosis:</td>
<td>2.95</td>
</tr>
<tr>
<td>Durbin-Watson:</td>
<td>2.01</td>
</tr>
</tbody>
</table>

\(^6\) OMX 30 Stockholm is a stock market index containing the thirty mostly traded stocks on the Stockholm Stock Exchange (NASDAQ OMX Nordic).
Table 1 shows that the OMX30 returns are almost normally distributed. However, some negative skewness occurs and the excess kurtosis is 2.95 (leptokurtic). This contributes with small increase in risk for long-term investors, since kurtosis tend to raise the risk exposure in the tails (heavy tails).

According to the DW-test, the errors seem to be “white noise” distributed, thus $DW \approx 2$ which indicates no autocorrelation. Loss of autocorrelation is important for the Delta-Normal model, which applies a GARCH (1,1) process, which assumes that the data follows an AR (1) and a MA (1).

The $VR(2) = 0.996$, shows that there only exists some very small mean-reversion in the sample and according to earlier research (see Bjerksund et al., 2008), loss of mean-reversion can result in that the LNMROU model fails to forecast future outcomes.

Research has previously been carried out with good results on commodities having strong mean reversion. Thus there might be some indications that mean reversion helps the LNMROU model to yield better results. However, the purpose of this paper is not to test term structure models on assets that they already are proven to work on, rather it is to investigating if other assets works.

### Table 2 Results mean reversion regression

<table>
<thead>
<tr>
<th>Regression Statistics</th>
<th>(OMX30 Sthlm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple R</td>
<td>0.0319</td>
</tr>
<tr>
<td>R Square</td>
<td>0.00101</td>
</tr>
<tr>
<td>Adjusted R Square</td>
<td>0.000730</td>
</tr>
<tr>
<td>Standard Error for the residual (STYDX)</td>
<td>13.1</td>
</tr>
<tr>
<td>Volatility for residuals in % (=STYDX/Long run mean)</td>
<td>-0.0157 %</td>
</tr>
<tr>
<td>Price volatility (daily)</td>
<td>0.0158</td>
</tr>
<tr>
<td>Price volatility (yearly)</td>
<td>0.250</td>
</tr>
<tr>
<td>Mean reversion speed (negative slope)</td>
<td>-0.00149</td>
</tr>
<tr>
<td>Long run mean</td>
<td>-840</td>
</tr>
</tbody>
</table>
3.1.1 Log-Normal Mean-Reverting Ornstein-Uhlenbeck Spot Price Process.

When estimating Log-normal mean reverting Ornstein-Uhlenbeck spot price process (LNMROU), two estimators are employed, OLSR and MLE. The parameter estimations from applying OLSR are presented in Table 3. Those estimates are used in LNMROU for simulation.

Table 3 OLSR parameter estimation results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>1.50</td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>0.122</td>
</tr>
<tr>
<td>$\hat{\sigma}^2$</td>
<td>0.0514</td>
</tr>
</tbody>
</table>

Fig. 3.2 displays simulated spot price path using the parameters from the OLSR in Table 3. To estimating VaR, a thousand different price paths for each outcome were taken in consideration, and the 90th, 95th and 99th percentile were chosen for VaR estimation. In figure 3.2 the $x$ – axis = time and $y$ – axis = price in SEK.

Figure 3.2 Monte Carlo Simulation using OLSR estimator

In Fig. 3.3 the blue line corresponds to the OMX30 log spot price changes. For long positions: the (dark red) one corresponds to the 90 % VaR, the (dark green) to 95 % VaR and finally the (plum) to 99 % VaR. For short positions: the (orange) one corresponds to the 90 % VaR, the
(grey) to the 95 % VaR and the (light red) to 99 % VaR. The diagram shows that LNMROU tend to underestimate the risk exposure for all VaR levels. In Figure 3.3: $y - axis = d \ln P$ and the $x - axis = time$.

**Figure 3.3 Monte Carlo Method applying the Log-Normal Mean-Reverting Ornstein-Uhlenbeck spot price process using OLSR**

When computing VaR, every log spot price change between $t_i$ and $t_{i+1}$ is taken into consideration, and the value that corresponds to the selected confidence level constitutes VaR. Regards to the $x - axis$, it can be seen that the time series data is almost symmetric around 0, with some negative skewness.

In the Fig. 3.4 the green horizontal line represents the 90 % VaR, which is exactly the x-value corresponding to the 10 “cumulative %”. For 90 % VaR, the $1 - (10 \text{ cumulative } \%) = 0.90$. 
When estimating MLE, OLSR estimates are used as start values before minimizing the log-likelihood function. This can be carried out using the Excel Solver function, however the Solver tends to pull the variance towards zero and that is not reasonable. Another option for solving the optimization is MATLAB (for instance fminsearch. However, previous studies show that it is not for certain that MATLAB-methods yield any better results compared to Excel (Spykens, 2010).

### Table 4 Parameter estimation results MLE

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>1.50</td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>0.122</td>
</tr>
<tr>
<td>$\hat{\sigma}^2$</td>
<td>0.000</td>
</tr>
</tbody>
</table>

3.1.2 Historical Simulation Approach

In HS a sliding test window containing 1759 observations is used and the percentile that corresponds to each VaR-level is selected. In Fig. 3.5 the blue line corresponds to OMX30 spot price returns. For long positions: the (dark red) one corresponds to the 90 % VaR, the (dark green) to 95 % VaR and finally the (plum) to 99 % VaR. For short positions: the (orange) one corresponds to the 90 % VaR, the (grey) to the 95 % VaR and the (light red) to 99 % VaR. HS contributes with good results at all VaR-levels. In Fig. 3.5: $y = axis = R(t)$ and the $x = axis = time$. 

![Figure 3.4 Cumulative distribution of the change in log spot price](image)
3.1.3 Delta-Normal-GARCH (1.1) Approach

In Fig. 3.6 the blue line corresponds to the changes in the OMX30 spot price \((dP)\). For long positions: the (dark red) one corresponds to the 90 % VaR, the (dark green) to 95 % VaR and finally the (plum) to 99 % VaR. For short positions: the (orange) one corresponds to the 90 % VaR, the (grey) to the 95 % VaR and the (light red) to 99 % VaR.

The figure 3.6 demonstrates that the DNG approach performs well at all confidence levels and it also shows that GARCH (1.1) responds well to jumps in the stock price. This effect arises from its advantage allowing the volatility to change over time. \(y - axis = dP\) and the \(x - axis = time\).

The red color indicates “rejection” of the “null” and the green color means “acceptance” of the model. All likelihood ratios are tested for 5 % significance during a p-value test in Excel:

\[ p - value \equiv CHIDIST (LR; M) \]

where \( M \) is the number of degrees of freedom and if \( p - value < 0.05 \) the model is rejected. For the Kupiec´s binomial approach the model is rejected if \( p - value \) is less than the confidence level (Dowd, 2004).

In table 5 and 6, it is shown that Kupiec´s binomial test show that it accepts more VaR approaches and confidence levels than Christoffersen (1998) conditional backtest (Christoffersen et al., 2004). This arises from the fact that Kupiec´s test do not take independence into consideration, which increase risk of type II errors (Westerlund, 2005). Type II error implies that a false model is not rejected. In Table 5, HS take only unconditional likelihood ratios into consideration.

Table 5 Results from Christoffersens (1998) backtests

<table>
<thead>
<tr>
<th>Approaches</th>
<th>LR</th>
<th>Left tail (long position)</th>
<th>Right tail (short position)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DNA</td>
<td>Uncond.</td>
<td>0.154 0.138 0.213</td>
<td>0.0267 0.684 0.386</td>
</tr>
<tr>
<td></td>
<td>Ind.</td>
<td>0.00451 0.648 4.89E-05</td>
<td>4.74E-06 0.00432 0.0588</td>
</tr>
<tr>
<td></td>
<td>Cond.</td>
<td>0.00640 0.301 0.000121</td>
<td>2.44E-06 0.0157 0.115</td>
</tr>
<tr>
<td>HS</td>
<td>Uncond.</td>
<td>0.0372 0.843 0.0934</td>
<td>0.000217 0.228 0.567</td>
</tr>
<tr>
<td>LNMROU</td>
<td>Uncond.</td>
<td>1.566E-10 1.686E-13 3.219E-24</td>
<td>0.00247 0.000153 9.88E-13</td>
</tr>
<tr>
<td></td>
<td>Ind.</td>
<td>0.307 0.870 5.21E-07</td>
<td>0.332 0.740 0.000194</td>
</tr>
<tr>
<td></td>
<td>Cond.</td>
<td>7.63E-10 1.56E-12 1.41E-28</td>
<td>0.00640 0.000729 8.68E-15</td>
</tr>
</tbody>
</table>

Table 6 Kupiec´s (1995) binomial backtest

<table>
<thead>
<tr>
<th>Approaches</th>
<th>Left tail (long position)</th>
<th>Right tail (short position)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90% 95% 99%</td>
<td>90% 95% 99%</td>
</tr>
<tr>
<td>DNG</td>
<td>p-value 0.997 0.299 1.409E-05</td>
<td>1.000 0.997 0.0211</td>
</tr>
<tr>
<td></td>
<td>x % 8.03 % 5.24 % 2.11 %</td>
<td>6.89 % 3.59 % 1.48 %</td>
</tr>
</tbody>
</table>
In the Vasicek (1997) model, time series data sampled from Thomson Reuters via Riksbanken (Swedish Central Bank) are used. The data contains daily SSVX3M observations from 1998-2004.

The simulation procedure takes place in the period (2003-2004). The simulated rates are then used to estimate future bond prices, assuming a bond with a term structure of 90 days to maturity and a face value of 1 SEK. These simulations assumes an investor with an investment horizon of one day where he buys the bond with 90 days left to maturity and sells it the next day (with 89 days left to maturity). An investor buying and selling bonds is exposed to a price risk and a reinvestment risk, where the price of bond goes up when interest rates goes down, and vice versa.

### Table 7 Data properties

<table>
<thead>
<tr>
<th>Properties</th>
<th>(SSVX3M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E[R(t)]:</td>
<td>0.0387</td>
</tr>
<tr>
<td>Var[R(t)]:</td>
<td>0.00002000</td>
</tr>
<tr>
<td>Volatility σ (daily):</td>
<td>0.00445</td>
</tr>
<tr>
<td>Skewness:</td>
<td>-0.804</td>
</tr>
<tr>
<td>Kurtosis:</td>
<td>-0.366</td>
</tr>
<tr>
<td>VR(2)</td>
<td>1.88</td>
</tr>
<tr>
<td>Numbers of observations (n):</td>
<td>1253</td>
</tr>
</tbody>
</table>

As can be seen in Table 7, the SSVX3M data has a mean of 0.0387 which is close to the long term mean estimated with OLSR and MLE (also see Table 8 and 9). The time-series contain some negative skewness and platykurtic (excess kurtosis < 0) and this tend to result in an overestimation of the risk. Consequently, models tend to be rejected due to lower exceedances. The Variance ratio VR (2), indicates strong mean reversion, commonly occurring among interest rates. Strong mean reversion in the sample indicates the relevance of
using affine term structure models (also see Bjerksund et al., 2008 for related research). In the mean reversion regression below in Table 8, Blanco and Soronow (2001) has been used as a starting point for the estimation procedure.

Table 8 Results mean reversion regression

<table>
<thead>
<tr>
<th>Regression Statistics</th>
<th>(SSVX3M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple R</td>
<td>0.0444</td>
</tr>
<tr>
<td>R Square</td>
<td>0.00197</td>
</tr>
<tr>
<td>Adjusted R Square</td>
<td>0.00118</td>
</tr>
<tr>
<td>Standard Error residuals (STYDX)</td>
<td>0.0295</td>
</tr>
<tr>
<td>Volatility (=STYDX/Long run mean)</td>
<td>0.00818</td>
</tr>
<tr>
<td>Price volatility (daily)</td>
<td>0.00786</td>
</tr>
<tr>
<td>Price volatility (yearly)</td>
<td>0.124</td>
</tr>
<tr>
<td>Mean reversion speed (negative slope)</td>
<td>-0.00294</td>
</tr>
<tr>
<td>Long run mean (in percent)</td>
<td>3.60</td>
</tr>
<tr>
<td>Numbers of observations</td>
<td>1252</td>
</tr>
</tbody>
</table>

Table 8, shows that our data has a negative speed of adjustment of -0.00294 % and a long run mean of the interest rate is 0.036, which is the same results as obtained by MLE (method 2, see Table 10). However, the data has a daily volatility of 0.0079 and an annual of 0.12. This diverges from estimates obtained from the OLSR and MLE estimations below, which optimized the volatility term to 0.0056. According to the table 8, forecasted volatility should be approximately 0.0082 of the forecast rate level.

Estimation procedure for this section is similar to the estimations in chapter 3.1, except that the simulated outcomes (interest rates) are used for pricing bonds with 89 and 90 days left to maturity. However, VaR is estimated in the same manner.

3.2.1 The One-Factor Vasicek (1977) Interest Rate Model

When estimating the one-factor Vasicek (1977) interest model two estimators are used, OLSR and MLE. The MLE approach applies both an analytical and a numerical approach. The estimation results for the OLSR are presented in Table 9.
These parameters are applied when simulating and estimating VaR with a Monte Carlo method. In Fig. 3.7 the blue line corresponds to the changes in the bond prices estimated with SSVX3M. For long positions: the (dark red) one corresponds to the 90 % VaR, the (dark green) to 95 % VaR and finally the (plum) to 99 % VaR. For short positions: the (orange) one corresponds to the 90 % VaR, the (grey) to the 95 % VaR and the (light red) to 99 % VaR. In Figure 3.7: y-axis = dP and the x-axis = time.

Fig. 3.7 graphically illustrated VaR-results and it is clear that the model overestimates the left tail distribution at all confidence levels. The three plots with the largest price change, indicates that there are some VaR exceedances in the right tail, but they are not so many.

In the MLE method 2, the OLSR parameters are used as start values (to avoid division by zero) when maximizing the log-likelihood function in Excel solver (though variation of $\alpha, \mu$ and $\sigma$).
Table 10 Parameter estimation results (MLE)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Method 1</th>
<th>Method 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.02</td>
<td>1.079</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.0359</td>
<td>0.0360</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.00564</td>
<td>0.00564</td>
</tr>
</tbody>
</table>

The analytical MLE (method 1) yielded exactly the same results as OLSR (compare Table 9 and 10). The results from the three methods almost yielded the same results, which should indicate a correct estimation.

In Fig. 3.8 the blue line corresponds to bond estimated with SSVX3M. For long positions: the (dark red) one corresponds to the 90 % VaR, the (dark green) to 95 % VaR and finally the (plum) to 99 % VaR. For short positions: the (orange) one corresponds to the 90 % VaR, the (grey) to the 95 % VaR and the (light red) to 99 % VaR. In Figure 3.8: $y - axis = dP$ and the $x - axis = time$.

![Monte Carlo Method applying the Vasicek (1977) using MLE](image)

Since the parameter estimation yielded almost the same results, Fig. 3.7 and 3.8 can be considered to be near identical.
3.2.2 Mean Reversion

Figure 6, 7 Simulation of SSVX3M interest rate estimated with OLSR/MLE (method 1) and MLE (method 2)

Fig. 6 and 7 demonstrates the mean reversion properties obtained in the interest rates, thus the interest rates clearly revert to the level of the long term mean ($\mu = 0.0359$). The OLSR turned out to yield the same parameters as MLE (method 1’s). In Fig. 6 the blue line is the simulated interest rates and the red line the long term mean.

3.2.3 Kupiec´s (1995) Binomial Approach

Table 11 Backtest of Vasicek (1977) using OLSR/MLE (method 1)

<table>
<thead>
<tr>
<th>Approaches</th>
<th>Left tail (long position)</th>
<th>Right tail (short position)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90%</td>
<td>95%</td>
</tr>
<tr>
<td>p-value</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>Z</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$x%$</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The Vasicek (1977)

Table 12 Backtest of Vasicek (1977) using MLE (method 2)

<table>
<thead>
<tr>
<th>Approaches</th>
<th>Left tail (long position)</th>
<th>Right tail (short position)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90%</td>
<td>95%</td>
</tr>
<tr>
<td>p-value</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>Z</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$x%$</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The Vasicek (1977)

According to the Kupiec´s (1995) binomial backtest for a short term bond, it is obvious that the Vasicek (1977) VaR model performs well at all levels.
Vasicek (1977) model performs much better compared to LNMROU. This is due to the different properties related to the assets, where SSVX3M interest rates prove that higher amounts of mean reversion tend to yield better VaR-results when applying term-structure models.
4 Conclusion

The conclusions drawn from this paper is that it can be seen clearly that Log-Normal Mean-Reverting Ornstein-Uhlenbeck spot price process (LNMROU) performed poorly at forecasting VaR. The Christoffersen (1998) backtest for LNMROU proved that it failed in capturing price changes in the stock index market. The Vasicek (1977) model backtested with Kupiec’s (1995) binomial approach, proved to be a good model for forecasting VaR.

These results also affirm that it is very important to investigate the data statistically for mean reversion before applying affine term-structure models including a mean-reversion term. Another conclusion made, is that it is not always necessary to use advanced models such as LNMROU and Vasicek (1977) for forecasting VaR. The drawbacks of using advanced term structure models are that they are very time consuming require a deep technological knowledge to utilize and the parameterization of the models tend to be sensitive to changes in the estimation window.

The conclusions are based on the outcome of this project as well as previous research within the field of asset valuation.
References

Literature


Internet Sources


Energy Information Administration http://www.eia.gov/dnav/ng/ng_pri_fut_s1_d.htm (Accessed 30 April)


