Grand Unified Theories: \( SU(5) \), \( SO(10) \)
and supersymmetric \( SU(5) \)

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Abstract

A brief summary of the Standard Model of particle physics is presented with some of its problems listed. After this some necessary concepts and mathematics are introduced. Then the simplest Grand Unified Theory, the minimal $SU(5)$ will be presented in some depth, with its predictions stated, the reasons it fails to describe reality given and an extension of the Higgs sector making it viable is presented. Then a more complex unification group, the $SO(10)$ group, and an extension of $SU(5)$ by adding Supersymmetry are discussed. Lastly the current experiment probing nucleon decay, the Super-Kamiokande, is described as well as some of the proposed experiments that are to detect proton decays. These experimental limits on the proton lifetime are compared to the predictions in the summary.
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Motivation

Before starting with this thesis I had only decided that I wanted to do something on particle physics, as that is the branch of physics I find most fascinating. So I contacted Else Lytken, who had been my lecturer in particle physics. She told me that the possible projects involving data analysis were unrealistic in ten weeks full-time studies if not already knowing the relevant programming language. But it was possible to do a literature study in some interesting area of particle physics. After considering it I decided to do a literature study. So I went to a meeting with Else, having given some thought to what parts of particle physics I would like to devote my thesis to. Without having really decided, my first thought was a literature study on grand unified theories. So speaking with Else again, we decided on such a project.

At the start of the project my knowledge of grand unified theories was very limited; the courses I had taken had mentioned them, giving some of the motivation for them and a few of their predictions, but not much more. So the choice was superficial; I might as well have picked some other topic, such as supersymmetry. The more I have learnt about grand unified theories the more I have come to like it though, so in retrospect I am satisfied with the choice. This project has also made me wait with anticipation for the coming decades, as then the next-generation of proton decay experiments will have improved the limits on the possible grand unified theories, and perhaps some grand unified theory is actually proved to be a description of reality.
1 Introduction

In this section a very short review of the Standard Model (SM) will be presented, a fuller treatment at an undergraduate level can be found in [1] and, at a slightly higher level, [2]. As is customary in particle physics natural units are used in this text, i.e. $\hbar = c = 1$.

1.1 The Standard Model

SM is the exceptionally successful theory of the elementary particles and their interactions, excluding gravity. It contains the three other forces: electromagnetism, the weak force and the strong force. The theory is formed from a free particle Lagrangian, $\mathcal{L}$. Actually $\mathcal{L}$ is what is called the Lagrangian density in classical mechanics, which is connected to the Lagrangian, $L$, by $L = \iiint \mathcal{L} \, dx \, dy \, dz$. To find the interactions $\mathcal{L}$ is gauged, which means that it is made gauge invariant under certain internal symmetries.

The Lagrangian describing a system will have extra degrees of freedom, not affecting the equations of motion; these are what are called gauges. The transformations that takes the Lagrangian from one gauge to another are called gauge transformations. So gauge invariance is the invariance of the Lagrangian under the gauge transformations. The advantage of this is that the interactions of a theory can be deduced from the assumption that it is to be invariant under a certain type of gauge transformation.

One internal symmetry is under the group transformations called $U(1)$. This invariance is possible to make plausible from elementary knowledge of quantum mechanics. As all observables in quantum mechanics will be invariant to a global change of phase, i.e. all kets transform like $|\phi \rangle \rightarrow e^{i\alpha} |\phi \rangle$ and the bras then transform as $\langle \phi | \rightarrow (e^{i\alpha} |\phi \rangle)^\dagger = \langle \phi | e^{-i\alpha}$. The observable corresponding to an operator $A$ takes the form:

$$\langle \phi | A |\phi \rangle \rightarrow \langle \phi | e^{-i\alpha} A e^{i\alpha} |\phi \rangle.$$

For a global transformation $\alpha$ will just be a constant, so that it commutes with any operator $A$, which then leaves the observable unchanged. When this is turned into a local symmetry $\alpha$ is allowed to be a function of space-time, i.e. $\alpha \rightarrow \alpha (x^\mu)$.

The internal symmetries of the SM are:

- $U(1)$, phase invariance, it is the group with multiplication as group product of all complex numbers with 1 as absolute value.

- $SU(2)$ and $SU(3)$, where $SU(n)$ is the group with matrix multiplication as group product, consisting of $n \times n$ unitary matrices with determinant 1.

The complete gauge group of the SM is then $SU(3)_c \times SU(2)_L \times U(1)_Y$, the subscripts here are denoting which indices the group acts on, $c$ is for color indices, $L$ is as it only acts on left-handed states and $Y$ is because it is the hypercharge of the particles that determine their coupling to the gauge boson of the $U(1)$ group.
Invariance under the internal symmetries is achieved through the use of a covariant derivative, which for the SM is:

\[ D^\mu = \partial^\mu - ig_1 \frac{Y}{2} B^\mu - ig_2 \frac{\tau_j}{2} W_j^\mu - ig_3 \frac{\lambda_a}{2} G_a^\mu. \] (1)

Here \( B^\mu \), \( W_j^\mu \) and \( G_a^\mu \) are the fields of the groups \( U(1)_Y \), \( SU(2)_L \) and \( SU(3)_c \) respectively, where \( j \) runs over 1, 2, 3 and \( a \) over 1 to 8. These are linear combinations of the fields of the gauge bosons, which are the last four of table 1. The photon, \( \gamma \), and the \( Z^0 \)-boson are combinations of \( B^\mu \) and \( W^3_\mu \), while the charged \( W \)-bosons are combinations of \( W^1_\mu, W^2_\mu \), and the eight gluons, \( g \), are the \( G_a^\mu \) fields. The \( Y \) is just a scalar, called the hypercharge, \( \tau_j \) and \( \lambda_a \) are the Pauli spin matrices and the Gell-mann Zweig matrices, respectively, these matrices are all given in section 3.1. The couplings \( (g_1, g_2, g_3) \) are:

\[ g_1 = \frac{e}{\cos(\theta_W)} \] (2)

and

\[ g_2 = \frac{e}{\sin(\theta_W)}, \] (3)

where \( e \) is the electron charge and \( \theta_W \) is the weak mixing angle. The weak mixing angle is just a measure of how much the photon and the \( Z \)-boson are composed of the \( B^\mu \) and of the \( W^3_\mu \) fields, according to:

\[ B^\mu = A^\mu \cos(\theta_W) + Z^\mu \sin(\theta_W), \] (4)

\[ W^3_\mu = -A^\mu \sin(\theta_W) + Z^\mu \cos(\theta_W). \] (5)

The third coupling, \( g_3 \), is that of the color group, so it determines the strength of the strong interaction. A useful definition is \( \alpha_i = \frac{g^2_i}{4\pi} \), as it simplifies a lot of expressions since the square of the couplings occurs often.

The particles in the SM are found in table 1. It contains only the first family of the fermions (fermions are particles with half-integer spin, there are only spin 1/2 fundamental particles in the SM), even though there are three families, but the particles of the other two families just mirror those of the first, having the same quantum numbers, but larger masses. There are also anti-particles of the fermions, but it is the right-handed antiparticle (the L/R subscript in the table refers to left- and right-handed particles, respectively) that have non-zero \( T_3 \), which is the third component of weak isospin. Spin transforms under a different \( SU(2) \) group, so \( T_3 \) can be understood from this. For spin this quantum number would correspond to the \( z \)-projection of the spin, \( J_z \). Like \( J_z \), \( T_3 \) can be any integer or half-integer, in an \( SU(2)_L \) doublet one state would have \( T_3 = +1/2 \) and one \( T_3 = -1/2 \), and in a triplet \( T_3 = +1, 0, -1 \), and so on. \( T_3 \) along with the hypercharge, \( Y \), of a particle, determine their coupling to the photon and the \( Z \)-boson fields. The handedness of particles is determined by whether the projection of the spin of the particle is in the same direction.
Table 1: Particles in the Standard Model, only the first generation of fermions. All masses are from the Particle Data Group (PDG) [3].

<table>
<thead>
<tr>
<th>Particle</th>
<th>Spin</th>
<th>Mass</th>
<th>Electric Charge, ( Q )</th>
<th>Weak Isospin 3(^{rd}) comp., ( T_3 )</th>
<th>Color mult.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_L^- )</td>
<td>1/2</td>
<td>0.511 MeV</td>
<td>(-1)</td>
<td>(-1/2)</td>
<td>Singlet</td>
</tr>
<tr>
<td>( e_R^- )</td>
<td>1/2</td>
<td>0.511 MeV</td>
<td>(-1)</td>
<td>0</td>
<td>Singlet</td>
</tr>
<tr>
<td>( \nu_e )</td>
<td>1/2</td>
<td>&lt; 2 eV</td>
<td>0</td>
<td>+1/2</td>
<td>Singlet</td>
</tr>
<tr>
<td>( u_L )</td>
<td>1/2</td>
<td>2.3 MeV</td>
<td>+2/3</td>
<td>+1/2</td>
<td>Triplet</td>
</tr>
<tr>
<td>( d_L )</td>
<td>1/2</td>
<td>4.8 MeV</td>
<td>(-1/3)</td>
<td>(-1/2)</td>
<td>Triplet</td>
</tr>
<tr>
<td>( u_R )</td>
<td>1/2</td>
<td>2.3 MeV</td>
<td>+2/3</td>
<td>0</td>
<td>Triplet</td>
</tr>
<tr>
<td>( d_R )</td>
<td>1/2</td>
<td>4.8 MeV</td>
<td>(-1/3)</td>
<td>0</td>
<td>Triplet</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>Singlet</td>
</tr>
<tr>
<td>( W^\pm )</td>
<td>1</td>
<td>80.4 GeV</td>
<td>±1</td>
<td>±1</td>
<td>Singlet</td>
</tr>
<tr>
<td>( Z^0 )</td>
<td>1</td>
<td>91.2 GeV</td>
<td>0</td>
<td>*</td>
<td>Singlet</td>
</tr>
<tr>
<td>( g )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Octet</td>
</tr>
</tbody>
</table>

as the momentum of the particle or in the opposite. The left-handed electron and the neutrino, and the left-handed quarks (up and down) form \( SU(2)_L \) doublets:

\[
L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad Q = \begin{pmatrix} u \\ d \end{pmatrix}_L,
\]

while the right-handed electron and quarks form \( SU(2)_L \) singlets. So the SM does not treat left- and right-handed fermions equally. One thing to note about table 1 is that there is no right-handed neutrino, \( \nu_R \); this is because only left-handed neutrinos have been seen, so \( \nu_R \) is proposed to be a singlet under all interactions, and as such not interact at all. But depending on how the neutrino mass is realized it might exist, for example Dirac neutrino mass requires a right-handed neutrino, and, as will be seen, \( \nu_R \) is present in some Grand Unified Theories (GUTs). The stars in the column for \( T_3 \) are there as the photon, \( \gamma \), and the \( Z \)-boson are not weak eigenstates, they are the mass eigenstates, so they do not have a definite weak quantum number.

Gauging the Lagrangian does not give any mass terms, so all particles should be massless if that was the whole story. As that is not observed there must be something more. Just introducing mass terms in the Lagrangian is not possible because it does not preserve the \( SU(2)_L \) symmetry, as a standard mass term for fermions is:

\[
m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R),
\]

but the left-handed fermions are \( SU(2)_L \) doublets and the right-handed are singlets, so the product of their wavefunctions is not an \( SU(2)_L \) singlet, so this term is not \( SU(2)_L \)
invariant. This is solved by what is called spontaneous symmetry breaking, which is the introduction of a complex scalar field, called the Higgs field, in an $SU(2)_L$ doublet:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.$$ 

The Lagrangian for the Higgs field will be:

$$\mathcal{L}_\phi = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2,$$

where letting $\partial_\mu \to D_\mu$ (defined in equation (1)) makes it invariant under the SM gauge group. The neutral component of the Higgs field will then be given a non-zero vacuum expectation value (v.e.v.). If the charged component was given a v.e.v. charges could disappear into the vacuum, i.e. charge would not be a conserved quantity, which goes against observations. This will break $SU(2)_L \times U(1)_Y$ down to $U(1)_{EM}$, the QED gauge group. This field will break the $SU(2)_L$ symmetry as only one of the components has a v.e.v. and the $U(1)_Y$ will be broken due to the hypercharge, $Y$, of the Higgs being non-zero ($Q = T_3 + Y/2 = -1/2 + Y/2$). Operating with the charge operator, $Q$, on $\phi^0$ will yield zero of course, so $\phi^0$ will be invariant under the electromagnetic linear combination of $SU(2)_L$ and $U(1)_Y$. The bosons associated with this will then remain massless, while the other bosons in $SU(2)_L$ and $U(1)_Y$ become massive. Therefore we end up with massive $W^\pm$ and $Z^0$, but a massless $\gamma$, which is what is observed. The Higgs boson will also give mass terms to the quarks and the leptons, except for the neutrino, as that would require a right-handed neutrino.

As of the writing of this, the latest results from CERN have just been revealed. A new particle with a mass of about 126 GeV has been found that is consistent with being the Higgs boson, with a statistical significance at the level of 5$\sigma$ from both the CMS and the ATLAS experiments \[4, 5\].

### 1.1.1 Running Couplings

For a given interaction, for example photon exchange between two charged particles, there will be the tree level diagram with just two vertices, one for each of the charged particles and then the photon as a propagator, as can be seen in figure 1. But in addition to this there will be higher order corrections, as the photon will be able to create a particle-antiparticle pair, which then annihilates into a photon again, seen in figure 2.

There can of course be any number of fermion loops, if all orders are included. The physically measured cross-section for photon exchange will contain all of the loop corrections. These corrections will not have finite contributions, which is a problem, as the theory should not predict infinite cross sections. This is solved by what is called renormalization, this allows an approximate relation for the coupling constants as functions of the energy
Figure 1: Scattering of two charged particles by photon exchange.

Figure 2: Loop correction for particle scattering.
scale, $q^2$, given that they have been measured at a scale $\mu^2$:

$$\frac{1}{\alpha_i(q^2)} = \frac{1}{\alpha_i(\mu^2)} + \frac{b_i}{4\pi} \ln \left( \frac{q^2}{\mu^2} \right), \quad (7)$$

the $b_i$ are constants that depend on how many fermions and bosons enter in the loops. The constants for the three couplings are:

$$b_1' = -\frac{4n_F}{3}, \quad (8)$$

$$b_2 = \frac{22}{3} - \frac{4n_F}{3}, \quad (9)$$

$$b_3 = 11 - \frac{4n_F}{3}, \quad (10)$$

where $n_F$ is the number of families. The prime in $b_1'$ is due to normalization, this will be explained in the subsection on predicting the weak mixing angle in section 3 about $SU(5)$ unification.

In [3] the couplings have been extrapolated from the low energy values using (7), the resulting plot is the left one in figure 3. It can be seen that the couplings, while not unifying perfectly, do get quite close at a very high energy scale, $Q \sim 10^{15}$ GeV. While this could just be a coincidence, it is rather striking that the three couplings meet at roughly the same energy scale. The caption mentions SUSY, which is a shorthand for supersymmetry, which will be discussed in section 5.

1.2 Problems

Despite the successes of the SM it leaves several questions unanswered and it contains a lot of arbitrariness, so it is likely that there is a more fundamental theory, giving answers to the questions and reasons for what is arbitrary in the SM. One of the extensions of the SM are Grand Unified Theories (GUTs). This is a class of models where the electroweak and strong interactions are unified into a single interaction, with a larger gauge symmetry.

The arbitrariness found in the SM are:

- Why is the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$?
- Why are there three families of leptons and quarks?
- Why are left-handed and right-handed fermions treated differently?
- How come the magnitude of the electrons and the protons charge is equal?
- There are rather many free parameters in the SM, is there any connection between them? Parameters such as masses of quarks, leptons, neutrinos, the Higgs boson and the $Z$-boson, mixing angles and the coupling constants.
Unexplained things in the SM are:

- It does not explain gravity.

- Why is the matter content of the universe highly asymmetric with regards to the ratio of matter to antimatter? It is of order \(10^{-9}\) [3].

- The cosmological constant problem, which is that the vacuum energy contribution from the Higgs boson is far larger than what is found by astronomical observations. Experimentally the vacuum energy density is \(\sim 10^{-48}\) GeV\(^4\) [7]. The v.e.v. of the Higgs boson is required to be \(v = 2M_W/g_2 \approx 246\) GeV. Its contribution to the vacuum energy density is \(V_0 = -\lambda v^4/2 \approx 2 \cdot 10^9\lambda\) GeV\(^4\), for \(\lambda \sim 1\), this is of the order of \(10^9\) GeV\(^4\). This can be remedied by having a bare vacuum energy density almost canceling the contribution from the Higgs v.e.v., though. But that would require fine-tuning of the bare energy density of one part in \(10^9/10^{-48} = 10^{57}\). This makes the theory unattractive, since it can be argued that naturalness is a desirable quality of a theory.

- What is dark matter? From rotation of galaxies the existence of a non-luminous type of matter can be inferred. It makes up \(\sim 22\%\) of the energy in the universe [3].

The Higgs sector is not consistent:
• The gauge hierarchy problem: the gauge hierarchy is that there is such a large difference in energy scale from the $W^\pm$ and $Z^0$ masses up to the Planck scale, $m_{Pl} = 1.22 \cdot 10^{19}$ GeV, which is where gravity has a strength of order unity. In the SM radiative corrections will destroy this hierarchy, so that excessive fine-tuning is required. This is explained in slightly more detail in the part on SUSY.

• What is the origin of the masses of the quarks, the charged leptons and the neutrinos?

1.3 Motivation for Grand Unification

As quarks and leptons are not connected in the SM, their hypercharge is determined by the experimentally observed charge of the particles and their weak interactions, but these are not connected, so there is no reason for the quark charges to be such that the proton and the electron charges of equal magnitude. This is an experimental fact though, to incredible precision, [8] gives the limit $0.8(8) \cdot 10^{-21} e$ on the difference of their charges. As Grand Unified Theories (GUTs) embeds the gauge group of the SM into a larger gauge group the hypercharge of the leptons and the quarks will no longer be independent.

There is also the fact, mentioned previously in connection with running couplings, that if the coupling constants are extrapolated to extremely high energies, on the order of $10^{14} - 10^{16}$ GeV they seem to unify, which hints of grand unification, unless it is but a coincidence.

2 Group theory, Lie algebras and representations

This section is only meant to give a very short introduction to the parts of group theory necessary for this report, for a more rigorous treatment see either [9] or [10]. From now on the indices $c$ and $L$ will be dropped, as only the color $SU(3)$ group and the weak $SU(2)$ group will be discussed.

2.1 Group Theory

A set of objects, $g$, called elements, constitute a group, $G$, if, with a product operation (denoted by $*$), the following four criteria are met:

1. **Closure:** If $\forall g_1, g_2 \in G : g_1 * g_2 = g_3 \in G$.

2. **Associativity:** $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

3. **Identity element:** there $\exists e$, satisfying: $e * g = g * e = g$, $\forall g \in G$.

4. **Inverse element:** $\forall g$ there $\exists g^{-1}$ with the property $g * g^{-1} = g^{-1} * g = e$. 
Groups are called Abelian if the product operation commutes, i.e. if $\forall g_1, g_2 \in G : g_1 * g_2 = g_2 * g_1$, and non-Abelian if there $\exists g_1, g_2 \in G$ such that: $g_1 * g_2 \neq g_2 * g_1$.

The groups that will be relevant for this review of Grand Unification are:

- **Special Unitary group**: the group formed by special unitary matrices under matrix multiplication, such matrices are $n \times n$-matrices fulfilling: $U^\dagger U = U U^\dagger = 1_n$ and $\det(U) = +1$, where $U$ is a matrix belonging to $\text{SU}(n)$ ($1_n$ is the $n \times n$ unit matrix).

- **Special Orthogonal group**: $O(n)$ is the group of rotations in $n$ dimensions in Euclidean space, its elements can be represented by orthogonal $n \times n$-matrices, i.e. matrices $R$ satisfying $R^T R = R R^T = 1$, with the group product being matrix multiplication. All orthogonal matrices will have a determinant of either $+1$ or $-1$, if it is $+1$ it is called a proper rotation, as it can be described by a continuous rotation from the initial configuration to the final one, while $\det(R) = -1$ are improper rotations, which require an inversion of the coordinate system. In $SO(n)$ the $S$ stands for special and it is the group of only proper rotations.

2.2 Lie algebra

In a continuously generated group any element of the group can be reached from the identity element by applying an infinite number of infinitesimal transformations. An element differing infinitesimally from the identity element is then given by:

$$g(\alpha) = 1 + i\alpha^a T^a + O(\alpha^2),$$

where $\alpha^a$ are infinitesimal parameters defining in which direction from the identity element we are moving in and $T^a$ are the generators of the group. A group with this structure is called a Lie group. The generators $T^a$ will then span the vector space of infinitesimal transformations, hence their commutator will be a linear combination of generators, i.e.:

$$[T^a, T^b] = if^{abc} T^c,$$

where $f^{abc}$ are called structure constants. This vector space along with the commutator of the generators is called a Lie algebra. Close enough to the identity element the commutators will specify the product operation.

If there is a generator that commutes with all the other generators for the group it will itself generate an independent continuous Abelian group with the structure of $U(1)$. If there is no such generator the algebra is called semi-simple. It is called simple if, on top of being semi-simple, the generators cannot be divided into to sets of mutually commuting generators. Both of the groups of interest in this report, $SU(n)$ and $SO(n)$, are simple groups. A more thorough, but still brief, account of Lie algebras can be found in [11].

In a quantized gauge theory the generators are identified with the spin one bosons of the theory [12].
2.3 Representations

A representation of a group is a mapping, \( D(g) \), of the elements, \( g \in G \), that satisfies:

- \( D(e) = 1_n \), where \( 1_n \) is the unit matrix of size \( n \times n \),
- \( D(g_1)D(g_2) = D(g_1 \ast g_2) \), for all \( g_1, g_2 \in G \).

The word multiplet is used in some of the literature instead of representation.

Particles can be identified with representations of the group, which specifies their transformation properties under the group in question.

2.3.1 Representations of \( SU(n) \)

This part follows "Section 38. \( SU(n) \) multiplets and Young diagrams" in [3] quite closely, but reducing the information to the minimum required for this report, see that source for a slightly more in depth explanation of this section.

For \( SU(n) \) groups the representations can be assigned \( n - 1 \) integers \((\alpha_1, \ldots, \alpha_{n-1})\) that determines the representation. Taking the conjugate of a representation will give us \((\alpha_{n-1}, \ldots, \alpha_1)\). From these integers the number of particles in the representation can be calculated. As this will only be used for \( SU(5) \) that is the only formula to be shown in this report, see [3] for the formulas for \( n = 2, \ldots, 5 \) (the general formula is easily identified from the pattern of those). For an \( SU(5) \) representation determined by \((\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) the number of particles will be:

\[
N = \left( \prod_{i=1}^{4} \frac{\alpha_i + 1}{1} \right) \cdot \left( \frac{\alpha_1 + \alpha_2 + 2}{2} \right) \cdot \left( \frac{\alpha_2 + \alpha_3 + 2}{2} \right) \cdot \left( \frac{\alpha_3 + \alpha_4 + 2}{2} \right) \times \\
\times \left( \frac{\alpha_1 + \alpha_2 + \alpha_3 + 3}{3} \right) \cdot \left( \frac{\alpha_2 + \alpha_3 + \alpha_4 + 3}{3} \right) \cdot \left( \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 4}{4} \right) \quad (11)
\]

The number of particles will be used to name representations later, they will then be written in bold text, so \(5\) is the \((1, 0, 0, 0)\) of \( SU(5) \) and for the conjugate of that a bar will be used, i.e. \(\bar{5}\) is \((0, 0, 0, 1)\).

The representations also have a corresponding Young diagram, these are between 1 and \( n \) rows of boxes that are determined by use of the \( \alpha_i \) integers of the representation. The first number, \( \alpha_1 \), determines how many more columns of boxes there are in the first row as compared to the second, the second number by the same, but for the second row compared to the third, and so on. So for \((1, 0, \ldots, 0)\) there will always just be one box, as the first row has one more column of boxes than the second row, but the second row and all rows following will have the same number of boxes, zero. For \((0, 0, \ldots, 1)\) the \((n - 1):th\) row has one more box than the \( n:th\) row, so this will correspond to the diagram of a single column of \( n - 1 \) boxes. One more case is worth mentioning, that is the \((0, 0, \ldots, 0)\) representation,
which is the one with a single column with \( n \) boxes, from (11) it is seen that this is the singlet representation for \( SU(5) \) (this is true for all \( SU(n) \)).

\[
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Above are \((1,0,0,0), (0,0,1), (0,0,0,0)\) and \((1,1,0,0)\) shown for \( SU(5) \).

To be able to assign all particles to representations, when unifying in \( SU(5) \), it will be required to use larger representations than just the 5 and the \( \bar{5} \). The larger representations can be found by connecting these two with themselves or each other. This is why Young diagrams are introduced here, connecting representations is very simple to do with them. First it is useful to define admissible sequences. An admissible sequence is a sequence of letters where at every point the number of \( a \):s that have been in the sequence is larger than the number of \( b \):s, and the same holds for the number of \( b \):s compared to the number of \( c \):s and so forth. This means that \( abc \) is not admissible, but \( aabc \) is, and so is, for example \( aabcb \). To connect two representations using Young diagrams three steps are taken:

1. Both Young diagrams should be drawn, but the boxes for one of them is filled with letters, for the first row the boxes are filled with \( a \):s and for the second row with \( b \):s and so on. For \((0,1) \otimes (1,1)\) in \( SU(3) \) we have:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}

2. The \( a \) labelled boxes are added to the right of the unlabelled boxes, so that no column has more than one \( a \), no lower row has more columns than the one above and there are still at most \( n \) rows:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}, \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Now the \( b \):s are added in the same way, but when read from right to left, starting at the first row and going downwards, the letter sequence should be admissible. So the allowed diagrams will be:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}, \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}, \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

This step is performed for all letters, so now the \( c \):s would be added in the same way, and so forth.
3. The final result is then the Young diagrams that are obtained after the last letter has been added. So when the letters are removed, the result for this example is then:

\[
\begin{array}{ccc}
\fbox{1} & \fbox{1} & \\
\end{array}
\oplus
\begin{array}{ccc}
\fbox{2} & \\
\end{array}
\oplus
\begin{array}{ccc}
\fbox{3} & \\
\end{array},
\]

in the notation with 2 integers this would say:

\[
(0, 1) \otimes (1, 1) = (0, 1) \oplus (1, 2) \oplus (2, 0).
\]

So an extra filled column (one with \( n \) rows) does not correspond to a new representation.

A trick to keep in mind is that one can conjugate the two representations that are to be connected if that would make the three steps easier to do, this would simply have the effect of conjugating the final result. An example of this for \( SU(5) \) is if \((1, 0, 0, 0) \otimes (0, 0, 0, 1)\) is to be determined, that is:

\[
\begin{array}{ccc}
\fbox{1} & \fbox{1} & \\
\end{array}
\]

This would be quite tedious to do, so instead one could evaluate the much easier connection \((0, 0, 0, 1) \otimes (1, 0, 0, 0)\):

\[
\begin{array}{ccc}
\fbox{1} & \\
\end{array}
\oplus
\begin{array}{ccc}
\fbox{2} & \\
\end{array}
\oplus
\begin{array}{ccc}
\fbox{3} & \\
\end{array},
\]

and conjugate the result (for this case they are their own conjugates though):

\[
(0, 0, 0, 0) \oplus (1, 0, 0, 1) \rightarrow (0, 0, 0, 0) \oplus (1, 0, 0, 1).
\]

2.3.2 Fundamental representation of \( SU(n) \)

For \( SU(n) \) the fundamental representation will be a vector of \( n \) complex elements:

\[
\psi = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_n
\end{pmatrix},
\]

that transforms as:

\[
\psi \rightarrow U\psi,
\]
where the $U$ are the special unitary matrices of the group. This is the representation denoted by $(1, 0, ..., 0)$ in the previous section. There will also be a complex conjugate fundamental representation, $\psi^*$, formed by the complex conjugated elements of the fundamental representation, which transforms as:

$$\psi^* \rightarrow U^* \psi^*.$$ 

This will then be the $(0, ..., 0, 1)$ of the previous section. It is of course immaterial which representation is the conjugate and which is the fundamental one, that is just definition.

For the color group of the SM the quarks transforms as the fundamental representation of $SU(3)$, the color triplet, $\mathbf{3}$, and the anti-quarks transform as the conjugate representation, the color anti-triplet, $\overline{\mathbf{3}}$.

### 2.3.3 Adjoint representation of $SU(n)$

This representation can be formed by connecting the fundamental representation with the complex conjugated representation, this will give a singlet and an $(n^2 - 1)$-dimensional representation, where the latter is the adjoint representation. Connecting $(0, 0, ..., 1) \otimes (1, 0, ..., 0)$ is just forming the two possible Young diagrams from a column of $n - 1$ boxes and a single box, this will obviously just be a column of $n$ boxes, i.e. the singlet, and a column of $n - 1$ boxes, but with two boxes in the first row. Expressing this in terms of the $n - 1$ integers is just $(0, 0, ..., 0)$ for the singlet and $(1, 0, ..., 0, 1)$ for the adjoint representation, so we see that both of these representations are their own conjugates. This is obvious as which representation is the conjugate and which is not was immaterial, so then $\mathbf{n} \otimes \overline{\mathbf{n}} = \overline{\mathbf{n}} \otimes \mathbf{n}$ must be true. The gauge bosons of $SU(n)$ transform according to the adjoint representation, for $SU(3)$ the gluons transform as the color octet formed by $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$.

### 3 Grand Unification with $SU(5)$

Electromagnetic, weak, and strong interactions are each associated with a Lie algebra, together forming $SU(3)_c \times SU(2)_L \times U(1)_Y$. This can be embedded into a single Lie algebra, i.e. unifying the forces [10]. One possibility is the simple Lie algebra $SU(5)$, this was first explored in [13].

#### 3.1 Vector bosons

There will be 24 generators for $SU(5)$, associated with the vector bosons $V^a_\mu$, $a = 1, ..., 24$, they occur in the covariant derivative:

$$D^\mu = \partial^\mu - ig_5 T_a V^\mu_a.$$  

(12)
To get the subgroup $SU(3) \times SU(2)$ in order to get back SM at low energies, the basis where the color group acts on the first three columns and rows of the $5 \times 5$ generating matrices $T^a$ and the $SU(2)$ group acts on the last two columns and rows can be chosen. Using the normalization of the generators:

$$Tr(T^a T^b) = 2\delta^{ab},$$

(13)

the $SU(2)$ generators in $SU(5)$ are then given by just inserting the Pauli $2 \times 2$-matrices in the last two columns and rows of $T^{a=1,2,3}$, i.e.:

$$T^{a=1,2,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^{1,2,3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Where the Pauli matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These three generators will then be associated with the $W^1$, $W^2$ and $W^3$ bosons of $SU(2)$, the charged $W$-bosons being the linear combinations:

$$W^\pm = \frac{V^9 \pm iV^{10}}{\sqrt{2}}.$$

The $SU(3)$ generators in $SU(5)$ are then given in the same way, with the $3 \times 3$ Gell-Mann Zweig matrices, $\lambda^{a=1,\ldots,8}$, inserted in the first three columns and rows, i.e.:

$$T^{a=5,\ldots,12} = \begin{pmatrix} \lambda^a & 0 & 0 \\ 0 & \lambda^a & 0 \\ 0 & 0 & \lambda^a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Where the $\lambda^{a=1,\ldots,8}$ are:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
\[
\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\]
so \( V^a_{\mu = 1, \ldots, 8} \) will be the vector bosons of \( SU(3) \), the gluons. For the \( SU(n) \) groups the generators are traceless, proven in for example [2]. Using this, equation (13) and the fact that the generator associated with \( B_{\mu} \) should commute with \( T^a_{\mu = 1, 2, 3} \) and \( T^a_{\mu = 5, \ldots, 12} \), it will be the diagonal matrix:

\[
T^4 = \frac{1}{\sqrt{15}} \text{Diag}(-2, -2, -2, 3, 3).
\]

Now there are 12 vector bosons left, but all of the SM bosons have already been accounted for! These 12 will therefore be new bosons which will be giving new interactions. Their corresponding generators can be formed by

\[
l^{13} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad l^{14} = \begin{pmatrix} i & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad l^{15} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

and so forth up to \( l^{18} \), for 19 to 24 the \( l \):s are instead defined as:

\[
l^{19} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad l^{20} = \begin{pmatrix} 0 & -i \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad l^{21} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

and then \( L^{13+k} \) for \( k = 0, 1, \ldots, 11 \) is formed by:

\[
L^{1k} = \begin{pmatrix} 0_{3 \times 3} & l^{13+k} \\ (l^{13+k})^\dagger & 0_{2 \times 2} \end{pmatrix}, \quad k = 0, 1, \ldots, 11
\]

where \( 0_{n \times n} \) is just the \( n \times n \) zero matrix. The change in sign for the complex \( l \):s 19 to 24 will just change places of the particle and antiparticle. It is of course arbitrary which is defined as particle and antiparticle, but as they have different charge this makes the new bosons with positive electric charge the particles, and the negatively charged will then be the antiparticles. The new bosons will be:

\[
X^1_\mu = \frac{V^{13}_\mu - iV^{14}_\mu}{\sqrt{2}},
\]

\[
Y^1_\mu = \frac{V^{18}_\mu + iV^{19}_\mu}{\sqrt{2}},
\]

and similarly for the four additional \( X \) and \( Y \) bosons. They will carry a color charge and a fractional electric charge of \( Q_X = +\frac{4}{3} \) and \( Q_Y = +\frac{1}{3} \), and transform as color anti-triplets, the index 1 in (14) and (15) is an anti-color index.
Now a $5 \times 5$-matrix can be defined, $V_\mu$, by:

$$V_\mu = \frac{1}{\sqrt{2}} \sum_{a=1}^{24} V_\mu^a T^a,$$

(16)

this is then:

$$V_\mu = \begin{pmatrix}
    g_{11} - \frac{2B}{\sqrt{30}} & g_{12} & g_{13} & X^1 & \bar{Y}^1 \\
    g_{21} & g_{22} - \frac{2B}{\sqrt{30}} & g_{23} & X^2 & \bar{Y}^2 \\
    g_{31} & g_{32} & g_{33} - \frac{2B}{\sqrt{30}} & X^3 & \bar{Y}^3 \\
    X^1 & X^2 & X^3 & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\
    Y^1 & Y^2 & Y^3 & -\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & \end{pmatrix},$$

(17)

where the 4-vector index $\mu$ has been suppressed inside the matrix.

### 3.2 Charge quantization

Counting the fermions in table 1, with each quark counted as three, due to the three colors, gives a total of 15 fermions, and with their anti-particles there will be 30 fermions per family in the SM. The fermions of the first family have been listed in table 2, this is to make it easier to identify in which representations of $SU(5)$ they will be put in.

<table>
<thead>
<tr>
<th>Particle</th>
<th>$SU(2)$ multiplicity</th>
<th>$SU(3)$ multiplicity</th>
<th>Electric Charge, $Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = \begin{pmatrix} \nu_e \ e^- \end{pmatrix}_L$</td>
<td>2</td>
<td>1</td>
<td>$\begin{pmatrix} 0 \ -1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\bar{L} = \begin{pmatrix} e^+ \ \bar{\nu}_e \end{pmatrix}_R$</td>
<td>2</td>
<td>1</td>
<td>$\begin{pmatrix} +1 \ 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$e^+_R$</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$e^-_L$</td>
<td>1</td>
<td>1</td>
<td>$+1$</td>
</tr>
<tr>
<td>$Q = \begin{pmatrix} u \ d \end{pmatrix}_L$</td>
<td>2</td>
<td>3</td>
<td>$\begin{pmatrix} +2/3 \ -1/3 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\bar{Q} = \begin{pmatrix} d \ \bar{u} \end{pmatrix}_R$</td>
<td>2</td>
<td>3</td>
<td>$\begin{pmatrix} +1/3 \ -2/3 \end{pmatrix}$</td>
</tr>
<tr>
<td>$u_R$</td>
<td>1</td>
<td>3</td>
<td>$+2/3$</td>
</tr>
<tr>
<td>$d_R$</td>
<td>1</td>
<td>3</td>
<td>$-1/3$</td>
</tr>
<tr>
<td>$u_L$</td>
<td>1</td>
<td>3</td>
<td>$-2/3$</td>
</tr>
<tr>
<td>$d_L$</td>
<td>1</td>
<td>3</td>
<td>$+1/3$</td>
</tr>
</tbody>
</table>
The simplest representations in \( SU(5) \) are the fundamental representations \((1,0,0,0)\) and \((0,0,0,1)\), with, using \((11)\), \( 5 \) and \( \bar{5} \) particles, the \( \bar{5} \) will be:

\[
\bar{\psi} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ a^4 \\ a^5 \end{pmatrix}.
\]

As the color indices acts on the first three rows and \( SU(2) \) acts on the last two, \( a^{i=1,2,3} \) will be an \( SU(2) \) singlet and an \( SU(3) \) triplet, while \( a^{i=4,5} \) will be an \( SU(2) \) doublet and \( SU(3) \) singlet. A raising or lowering operator on this object cannot change the spin direction, so all the particles have to have the same helicity. For the first family, looking at table 2, the color triplet can only be identified with either of the right-handed quarks, as the antiquarks transform as color antitriplets. The \( SU(2) \) doublet must be identified with either \( L \) or with \( \bar{L} \), this is because the \( SU(2) \) representation is real, i.e. \( 2 = \overline{2} \) [10]. Assigning \( \bar{L} \) to \( a^{4,5} \) yields:

\[
\bar{\psi} = \begin{pmatrix} q_r \\ q_g \\ q_b \\ e^+ \\ \bar{\nu}_e \end{pmatrix}_R.
\]

The charge operator \( Q = T_3 + Y/2 \) is just a linear combination of the generators of \( SU(2) \) and \( U(1) \), and they were identified with generators of \( SU(5) \). Using this and the fact that generators for the \( SU(n) \) groups have to be traceless, the quarks in \( \bar{\psi} \) can be identified. So this requires that the eigenvalues of \( Q \) add up to zero, using this on \( \bar{\psi} \) gives:

\[
Q(\bar{\nu}_e) + Q(e^+) + 3Q(q) = 0 \Rightarrow Q(q) = -\frac{1}{3} e,
\]

hence it must be the right-handed down quark that is in this representation, this is the quantization of the down quark charge in \( SU(5) \). Finally we have that the \( \bar{5} \) (\( 5 \)) representation is:

\[
\bar{\psi} = \begin{pmatrix} d_r \\ d_g \\ e^+ \\ \bar{\nu}_e \end{pmatrix}_R, \quad \psi = \begin{pmatrix} d_r \\ d_g \\ d_b \\ \nu_e \\ e^- \end{pmatrix}_L.
\]

Using Young diagrams explained in section 2.3.1 larger representations can be built from the \( \bar{5} \) by connecting it with itself, i.e. taking \( \bar{5} \otimes 5 \), this gives:

\[
\begin{array}{c}
5 \\
\oplus \\
\end{array} \otimes 5 = \begin{array}{c}
5 \\
\oplus \\
\end{array}
\]
So:
\[(1, 0, 0, 0) \otimes (1, 0, 0, 0) = (2, 0, 0, 0) \oplus (0, 1, 0, 0),\]
in particle numbers this corresponds to, using (11):
\[5 \otimes 5 = 10 \oplus 15.\]

The 15 is the symmetric part and the 10 is the antisymmetric. The symmetric part can be excluded as it contains a sixtuplet of color, which does not fit with any of the SM states [14]. To obtain the elements of the antisymmetric matrix one can form:
\[\chi = \left(\begin{array}{ccccc}
d_r & d_g & d_b & e^+ & \bar{\nu}_e \\
d_r & 0 & a_{12} & a_{13} & a_{14} & a_{15} \\
d_g & 0 & 0 & a_{23} & a_{24} & a_{25} \\
e^+ & 0 & a_{34} & a_{35} & 0 \\
\bar{\nu}_e & 0 & 0 & 0 & 0
\end{array}\right),\]
where the zeros are due to the antisymmetry property of the representation, lower left part has been left out as its elements are not independent \((a_{ij} = -a_{ji})\). To determine the \(a_{ij}\) element the quantum numbers of charge, color and \(T_3\) for the \(i\):th object to the left of the matrix and the \(j\):th object on top of the matrix are added together. The resulting charge, color and \(T_3\) is then compared to the remaining fermions in the first generation and the fermion that has these quantum numbers is then \(a_{ij}\). So for \(a_{12}\) the charge is \(Q_{12} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}\), the color is \(c_{12} = r + g = \bar{b}\) and \(T^3_{12} = 0 + 0\), this corresponds to one of the anti-colors of a color anti-triplet and \(SU(2)\) singlet. Looking at table 2 one sees that \(a_{12}\) must then be the anti-blue anti-up quark. For \(a_{14}\) the charge is \(Q_{14} = -\frac{1}{3} + 1 = +\frac{2}{3}\), the color is just \(c_{14} = r\) and \(T^3_{14} = 0 + \frac{1}{2} = \frac{1}{2}\), so it is in a color triplet and an \(SU(2)\) doublet, which is just the red up quark. Apart from showing examples of how the elements are found this also gives the quantization of the up quarks charge, it must be \(Q_u = Q_d + Q_e^+ = +\frac{2}{3}\).

Identifying all of the elements finally gives:
\[\chi = \frac{1}{\sqrt{2}} \left(\begin{array}{ccccccc}
0 & \bar{u}_b & -\bar{u}_g & -u_r & -d_r \\
-\bar{u}_b & 0 & \bar{u}_g & -u_g & -d_g \\
\bar{u}_g & -\bar{u}_r & 0 & -u_b & -d_b \\
0 & u_r & u_g & u_b & 0 & -e^+ \\
d_r & d_g & d_b & e^+ & 0
\end{array}\right)_L,
\]
\[\bar{\chi} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccccccc}
n_0 & u_b & -u_g & -\bar{u}_r & -\bar{d}_r \\
-u_b & 0 & u_r & -\bar{u}_g & -\bar{d}_g \\
0 & u_r & -u_g & -\bar{u}_b & -\bar{d}_b \\
\bar{u}_r & \bar{u}_g & \bar{u}_b & 0 & -e^-\\
\bar{d}_r & \bar{d}_g & \bar{d}_b & e^- & 0
\end{array}\right)_R,\] (19)
where the minuses are from a more thorough treatment (for example in [12]) and the factor in front is just a conventionally used normalization factor. This procedure is repeated for the remaining two families in the same way, so that every family of fermions is given by the representations $\bar{5} \oplus 10$, and its conjugate.

So we see now that in $SU(5)$ the charges of the quarks are fixed in relation to that of the electron and that $Q_p + Q_e^- = 0$ is no longer a mystery, since a proton has the quark content of $(uud)$.

### 3.3 Prediction of $\theta_W$, $\alpha_3$ and the B-test

The energy scale at which the couplings unify is defined as $M_{GUT}$, see the left plot of figure 3. The couplings do not unify perfectly here, but they are close to each other at $M \sim 10^{15}$ GeV. When the couplings become equal $Cg_1 = g_2 = g_3 \equiv g$, the reason that there is a constant $C$ in front of $g_1$ is that the generators for $SU(2)$ and for $SU(3)$ are normalized in the same way, but the generator for $U(1)$ is not. In the introduction, section 1.1.1, the normalization was mentioned, here it will be derived. Finding this constant will directly give $\tan(\theta_W)$ at the unification scale, as that is by definition just $g_2/g_1$. To derive it at the scale that experiments are probing a bit more work is required, as will be seen. Instead of calculating $\sin^2(\theta_W)$ at low energy the strong coupling constant, $\alpha_3$, is calculated. This is due to the inputs required changes and they have different experimental uncertainties, $\sin^2(\theta_W)$ and $\alpha_{EM}$ are known very precisely, but $\alpha_3$ is not know to the same precision. Hence $\alpha_3$ will be solved for when finding the low energy prediction.

Expanding the sum over $a$ in (12) will give:

$$D^\mu = \partial^\mu - ig_5 T_a V_a^{\mu} = \partial^\mu - ig_5 (T_3 W_3^{\mu} + T_4 B^{\mu} + ...),$$  \hspace{1cm} (20)

just like (1) for the SM, the $T_a$ here are the 24 generators of $SU(5)$. We know that two of these correspond to the $W_3^{\mu}$ and the $B^{\mu}$ generators from section 3.1. Now $W_3^{\mu}$ and $B^{\mu}$ should be expressed in terms of the mass eigenstates $A^\mu$ and $Z^\mu$ instead, according to the equations (4) and (5). The $Z^\mu$ terms can be dismissed. This is due to the goal is to compare the expressions for the photon field, $A^\mu$, in $SU(5)$ with that of the SM. Doing this allows for the normalization constant to be solved for. Putting (4) and (5) in (20) and keeping only the $A^\mu$ terms gives:

$$-g_5(-\sin(\theta_W)T_3 + \cos(\theta_W)T_4)A^\mu = g_5 \sin(\theta_W)(T_3 - \cot(\theta_W)T_4)A^\mu.$$  \hspace{1cm} (21)

We know that the charge should be a linear combination of the $SU(5)$ generators corresponding to $SU(2)$ and $U(1)$ symmetries, $Q = T_3 + cT_4$ where $c$ is some coefficient that is unknown, as the hypercharge $Y$ is not normalized in the same fashion as $SU(2)$ and $SU(3)$ are. The linear combination before $A^\mu$ should be $eQ$, because we know the coupling of electromagnetism, so now we can determine $c$ by comparison with the expression for $Q$:

$$g_5 \sin(\theta_W)(T_3 - \cot(\theta_W)T_4)A^\mu = e(T_3 + cT_4)A^\mu,$$
so:

\[ c = -\cot(\theta_W), \quad (22) \]

\[ e = g_5 \sin(\theta_W). \quad (23) \]

If the trace of \( Q^2 \) is taken:

\[ \text{Tr} \ (Q^2) = \text{Tr} \ ((T_3 + cT_4)^2) = (1 + c^2) \text{Tr} \ (T_3^2). \quad (24) \]

Here (13) has been used twice, once to eliminate the cross-term and once to get rid of \( T_4 \).

Using the representation \( \bar{\psi} \) the trace of both \( Q^2 \) and of \( T_2^3 \) are easily calculated, as they are known for the particles:

\[ \frac{1}{1 + c^2} = \frac{\text{Tr} \ (T_3^2)}{\text{Tr} \ (Q^2)} = \frac{\frac{1}{4} + \frac{1}{4} + 3 \cdot 0}{0 + 1 + 3 \cdot \frac{1}{5}} = \frac{3}{8}. \quad (25) \]

From (22):

\[ \frac{1}{\tan^2(\theta_W)} = c^2 \Rightarrow c^2 + 1 = \frac{\cos^2(\theta_W)}{\sin^2(\theta_W)} + 1 = \frac{\cos^2(\theta_W)}{\sin^2(\theta_W)} + \sin^2(\theta_W) \Rightarrow c^2 + 1 = \frac{\cos^2(\theta_W) + \cos^2(\theta_W) \sin^2(\theta_W) + \sin^4(\theta_W)}{\sin^2(\theta_W)} \quad (26) \]

As:

\[ \cos^4(\theta_W) = \cos^2(\theta_W)(1 - \sin^2(\theta_W)) \Rightarrow \cos^2(\theta_W) = \cos^4(\theta_W) + \cos^2(\theta_W) \sin^2(\theta_W) \]

Using this in (26) gives:

\[ c^2 + 1 = \frac{\cos^4(\theta_W) + 2 \cos^2(\theta_W) \sin^2(\theta_W) + \sin^4(\theta_W)}{\sin^2(\theta_W)} = \frac{\cos^2(\theta_W) + \sin^2(\theta_W))^2}{\sin^2(\theta_W)} = \frac{1}{\sin^2(\theta_W)} \]

So finally:

\[ \sin^2(\theta_W) = \frac{1}{c^2 + 1} = \frac{3}{8} \quad (27) \]

This derivation is based on the derivation found in [15], but the starting point, the covariant derivative, is from [2].

As couplings run, i.e. they change with the momentum transferred, \( \mu \), the result of (27) is only valid at the unification scale and not at a lower scale where the mixing angle is actually measured, so in order to compare with experiments the dependence on the scale \( \mu \) must be found. The following is based on [16]. To find the \( \mu \) dependence one can use the decoupling theorem, which states that if there are particles at two mass scales \( m << M \) in a renormalizable Lagrangian field theory, an effective field theory can be used for calculation instead involving only the lighter particles of mass \( m \). The heavier particles will then only
affect normalization parameters and rescaling of the couplings in the theory. With this the three couplings \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) will be determined by (7) for running couplings from the Standard Model. Adding the Higgs contributions to the \( b_i \)'s changes them to:

\[
\begin{align*}
  b'_1 &= -\frac{4n_F}{3} - \frac{n_h}{10}, \\
  b_2 &= \frac{22}{3} - \frac{4n_F}{3} - \frac{n_h}{6}, \\
  b_3 &= 11 - \frac{4n_F}{3}
\end{align*}
\] (28)

where \( n_h \) is the number of Higgs \( SU(2) \) doublets (for the minimal case there is just one). By the definition of \( g_1 \) and \( g_2 \), in (2) and (3), along with

\[
\alpha_{EM} = \frac{e^2}{4\pi},
\]

we have that:

\[
\alpha_{EM} = \sin^2 (\theta_W) \alpha_2 = \frac{3}{5} \cos^2 (\theta_W) \alpha_1',
\] (31)

the factor coming from the previous part of this section. Taking the difference between (7) for \( i \) and \( j \) with \( \mu^2 = M_{GUT}^2 \), and exploiting the fact that at \( M_{GUT} \) the couplings are unified, yields:

\[
\frac{1}{\alpha_i (q^2)} - \frac{1}{\alpha_j (q^2)} = \frac{B_{ij}}{4\pi} \ln \left( \frac{q^2}{M_{GUT}^2} \right),
\] (32)

where \( B_{ij} = b_i - b_j \). For \( i = 1 \) and \( j = 2 \) this becomes:

\[
\frac{1}{\alpha_1' (q^2)} - \frac{1}{\alpha_2 (q^2)} = \frac{B_{12}}{4\pi} \ln \left( \frac{q^2}{M_{GUT}^2} \right),
\]

using equation (31) to express the couplings in terms of \( \alpha_{EM} \) and the trigonometric identity yields:

\[
\frac{8}{5\alpha_{EM}} \left( \frac{3}{8} - \sin^2 (\theta_W) (q) \right) = \frac{B_{12}}{4\pi} \ln \left( \frac{q^2}{M_{GUT}^2} \right) \Rightarrow
\] (33)

\[
M_{GUT} = q \exp \left[ \frac{16\pi \sin^2 (\theta_W) (q) - \frac{3}{8}}{5 \frac{\alpha_{EM} B_{12}}{\alpha_{EM} B_{12}}} \right],
\] (34)

Using (32) again, but with \( i = 2, j = 3 \) and (33) to remove the \( \ln \left( \frac{q^2}{M_{GUT}^2} \right) \) factor, yields:

\[
\frac{1}{\alpha_2 (q^2)} - \frac{1}{\alpha_3 (q^2)} = \frac{B_{23}}{B_{12} \frac{8}{5\alpha_{EM}}} \left( \frac{3}{8} - \sin^2 (\theta_W) (q) \right),
\]
using this, and equation (31) again gives:

\[ \frac{\sin^2(\theta_W)}{\alpha_{EM}} - \frac{1}{\alpha_3} = \frac{B_{23}}{B_{12}} \frac{8}{5\alpha_{EM}} \left( \frac{3}{8} - \sin^2(\theta_W)(q) \right), \]

from this one can get either:

\[ \frac{1}{\alpha_3} = \frac{1}{\alpha_{EM}} \left[ \left( 1 + \frac{8}{5} \frac{B_{23}}{B_{12}} \right) \sin^2(\theta_W)(q) - \frac{3}{5} \frac{B_{23}}{B_{12}} \right] \]  

(35)

or what is called the B-test, which is a test of whether the couplings actually unify:

\[ \frac{B_{23}}{B_{12}} = \frac{5 \sin^2(\theta_W)(q) - \frac{\alpha_{EM}}{\alpha_3}}{\frac{3}{8} - \sin^2(\theta_W)(q)}. \]  

(36)

One could also have solved (32) for \( \sin^2(\theta_W) \), but would then end up with an expression with \( \alpha_3(q) \) which has a large uncertainty compared to \( \alpha_{EM} \) and \( \sin^2(\theta_W)(q) \).

Using \( \sin^2(\theta_W)(m_Z) = 0.23116(13) \), \( \alpha_{EM}^{-1}(m_Z) = 127.916(15) \) and \( \alpha_3(m_Z) = 0.1184(7) \) [3] in the right-hand side of (36) yields:

\[ \frac{B_{23}}{B_{12}} = 0.718(5). \]  

(37)

Calculating the left-hand side of (36) from (28), (29) and (30) with \( n_F = 3 \) and \( n_h = 1 \) gives:

\[ \frac{B_{23}}{B_{12}} = 0.5275. \]

So the \( SU(5) \) model with only the SM particles and one Higgs doublet, i.e. no additional particles between the weak scale and the unification scale, fails the B-test badly. If there were additional particles the \( b/s \) would change, so unification is still possible if there are enough new particles between the weak scale and the scale of unification. This is achieved by adding more Higgses in a larger representation in [16], this will be mentioned in more detail in section 3.7. Using \( m_Z = 91.1876(21) \) GeV [3] in (34) gives \( M_{GUT} = 1 \cdot 10^{13} \) GeV.

More Higgs doublets would increase the \( B \)-ratio, which could give exact unification. In [17] this was done, in total 6 were required, but this would reduce the unification scale to \( M_{GUT} < 10^{14} \) GeV which is inconsistent with proton decay (see next subsection and section 6).

### 3.4 Proton decay

The new bosons of \( SU(5) \), mentioned in section 3.1, \( X_a^{+\frac{1}{3}} \) and \( Y_a^{+\frac{1}{3}} \) will allow additional vertices, the ones for the first generation are seen in figure 4. So for a baryon this will allow the constituent quarks to annihilate and create the new bosons, which could then create a
lepton and a quark, for example, so that the baryon could decay. The proton will be able to decay to, for example, an electron and a pion, $p \rightarrow e^+ \pi^0$ [13], two of the possible tree level Feynman diagrams can be seen in figure 5. There are more possible decays, to other mesons and with a muon instead of an electron in the final state, but the electron/pion final state is the one with largest branching ratio for $SU(5)$. Other modes are for example $p \rightarrow e^+ \omega$ and $p \rightarrow e^+ \rho^0$. The neutron will also be able to decay through baryon number violating channels, with similar final states as the proton. A rough estimate for the decay rate of the proton can be achieved by approximating it with that for the muon:

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} = \frac{1}{384\pi} \frac{\alpha_2^2 m_\mu^5}{m_W^4},$$

skipping the factor $1/384\pi$, as this is only to be a very rough estimate, using the unified coupling instead of $\alpha_2$, the proton mass instead of the muon mass and the $X$ or $Y$ mass instead of the $W$ mass. With the new bosons having masses on the order of the scale of grand unification $\sim 10^{15}$ GeV and a unification coupling of $\sim 1/42$ it becomes:

$$\tau_p \approx 3.7 \cdot 10^{31} \text{ yr}.$$

A more rigorous treatment is found in [18] where the mode with largest branching ratio is $p \rightarrow e^+ \pi^0$, which has a lifetime of order of:

$$\tau_{p \rightarrow e^+ \pi^0} \sim 10^{28.5} - 10^{31.5} \text{ yr.}$$

(38)

This has been excluded by the experiments performed at the Super-Kamiokande water Cherenkov detector. Increasing the scale of unification would help with the proton lifetime.
Figure 5: Two of the possible lowest-order diagrams for proton decay into a positron and a neutral pion in $SU(5)$.

If the scale was around $\sim 10^{16}$ GeV, proton decay by $X$ or $Y$ boson exchange would be suppressed enough for the model to survive the experimental limits. As proton lifetime is one of the fundamental tests of grand unification it will be mentioned in connection to all the models mentioned in this report. Due to this the experimental limits and all predictions of the proton lifetime are summarized and compared in the end of the report, section 7, instead of this being scattered over all the sections. The simplest $SU(5)$ theory, i.e. a unification with the $SU(5)$ gauge group with minimal Higgs content, is excluded by the absence of proton decay at current limits. It is also ruled out by the B-test, as was seen in the previous section.
3.5 Breaking $SU(5)$ to $SU(3)_c \times SU(2)_L \times U(1)_Y$

In order to break the electroweak symmetry a Higgs doublet is required, which can be in the $5_H$ or the $\bar{5}_H$ representations. Along with a scalar Higgs color triplet, the Higgs doublet will also give mass to the quarks and the leptons. The Higgs triplet can mediate proton decay, which would give the proton a very short lifetime, in contradiction with experiments, so this triplet needs a mass of $\sim 10^{11}\text{ GeV}$ in order to be consistent with proton lifetime limits. [3] The Higgs representation will be:

$$5_H = \begin{pmatrix} h_r \\ h_g \\ h_b \\ h^+ \\ -h^0 \end{pmatrix},$$

which is given a v.e.v. in the $h^0$ direction.

Breaking the $SU(5)$ symmetry down to the $SU(3)_c \times SU(2)_L \times U(1)_Y$ can be achieved by giving a v.e.v. in the $B_\mu$ direction to an adjoint 24-dimensional Higgs representation [12, 19] (which can be constructed by $5 \otimes \bar{5} = 24 \oplus 1$, using Young diagrams and (11)). This will result in the $X$ and $Y$ bosons getting masses, but all the SM bosons will remain massless, their mass will still be given by the electroweak symmetry breaking.

3.6 Yukawa coupling unification

The Yukawa couplings are the strength of the coupling of the fermions to the Higgs field. The mass of a fermion is given by:

$$m_f = \lambda_f v/\sqrt{2},$$

where $v$ is the vacuum expectation value for the Higgs field and $\lambda_f$ the Yukawa coupling. The connection between the quarks and the leptons in GUTs lead to relations between these, such as:

$$\lambda_d = \lambda_e, \quad \lambda_s = \lambda_\mu, \quad \lambda_b = \lambda_\tau,$$

at the unification scale $M_{GUT}$. From the first two we will have:

$$\frac{\lambda_s}{\lambda_d} = \frac{\lambda_\mu}{\lambda_e}.\quad \frac{m_s}{m_d} = \frac{m_\mu}{m_e}.$$
Using the masses $m_e = 0.510998928(11)$ MeV, $m_\mu = 105.6583715 \pm 4.8^{+0.7}_{-0.3}$ MeV and $m_s = 100^{+30}_{-20}$ MeV [3] gives:

\[
\frac{m_s}{m_d} = 21(7), \quad \frac{m_\mu}{m_e} = 206.768284(8),
\]

so this is in disagreement with experiments.

It is possible to reconcile this with experiments though, this can be done by introducing an additional Higgs representation, a 45-dimensional one (can be constructed by $10 \otimes 10 = \bar{5} \oplus 45 \oplus 50$), this was first done in [20]. The mass relations are then changed to:

\[
m_\mu = 3m_s, \quad 3m_e = m_d, \quad m_\tau = m_b,
\]

so that:

\[
\frac{m_s}{m_d} = \frac{1}{9} \frac{m_\mu}{m_e},
\]

which is in agreement with experiments. This problem is also soluble by other means [21].

### 3.7 A possible save for non-SUSY SU(5)

In [16] it is shown that both unification and a long enough proton lifetime can be obtained in SU(5) by introducing a 15_H Higgs representation. The masses of the new Higgses would be lower than the unification scale, $M_{GUT}$, in order to change the $b$'s used when calculating the running coupling, (28), (29) and (30). Doing this postpones proton decay so that current limits cannot rule it out. In [16] they find the upper limit to be $\tau_p \leq 1.4 \cdot 10^{36}$ years, this will be compared to experimental limits in section 7.

### 3.8 Conclusions on SU(5)

The problems that were mentioned as motivation for grand unification are obviously solved, the equal magnitude of the electrons and the protons charge and a single gauge group.

While minimal SU(5) (only $5_H$ or $\bar{5}_H$ and $24_H$ in the Higgs sector) is not the model describing reality, as it fails to unify the couplings, and predict the proton lifetime and Yukawa coupling relations, it still shows the promising features of grand unification. The problems faced by SU(5) are:

- **Unification of the couplings**: As could be seen in section 3.3 the minimal SU(5) fails the B-test, i.e. the couplings do not unify at any scale. This can be remedied by either introducing a 15_H [16] or by the introduction of an additional symmetry, called Supersymmetry (SUSY), which will be discussed in section 5.

- **Neutrino masses**: As there is no left-handed anti-neutrino predicted by SU(5) the neutrinos cannot attain mass like the leptons and quarks, unless they are introduced
The smallness of the neutrino mass can be explained by the so-called Seesaw mechanisms. In type I Seesaw models, two right-handed neutrinos need to be introduced and in type II the $15_H$ can give rise to the masses. [16]

- Yukawa unification gave mass relations that are incompatible with experiments (section 3.6), but this could be remedied by adding a $45_H$.

- Doublet-triplet splitting (2/3-splitting): This is the problem that the Higgs color triplet and the Higgs $SU(2)$ doublet in $5_H$ (or $\bar{5}_H$) must have very different masses. The triplet must have a large mass, $\sim 10^{11}$ GeV, in order to keep the proton lifetime longer than current experimental limits. But the Higgs doublet must be light, mass of order of the weak scale, for it to break the electroweak symmetry at the right energy scale.

As there is baryon number violation in GUTs they supply one of the requirements to produce baryon asymmetry in the universe. More of this can be found in [22, 23, 24] in which GUT lepto- and baryogenesis are dealt with to some extent. The cosmological implications of adding a $15_H$ is discussed in [16], with a right-handed neutrino included in the model. It is shown that this gives a mass constraint of $\lesssim 10^6$ GeV on one of the leptoquarks, allowing it to be reachable, possibly, by the LHC, or the next generation collider experiments.

One thing to note is that despite the high energy scale of grand unification, a few orders of magnitude below the Planck scale, it does not address gravity.

$SU(5)$ is the simplest possible GUT model. Grand unification could be realized by another gauge group in nature, such as $SO(10)$; this model will be dealt with briefly in the following section. There is also the possibility of having grand unification in higher dimensions, which has some promising features, such a model can be found in for example [25, 26].

4 Grand Unification with $SO(10)$

As could be seen in section 3.2 the left-handed SM fermions fit nicely in the $5$ and $10$ representations, no additional fermions had to be introduced to fill them. This is only true for $SU(5)$, for all other groups additional matter states have to be added [10]. This might seem like a reason that only $SU(5)$ is possible, but it is conceivable that the Higgs sector responsible for breaking the unified group also gives large masses to some of the particles, which would then explain why they have not yet been found. In the group that will be discussed in this section, $SO(10)$, such a new particle is required, it is identified as the right-handed neutrino.
4.1 Fermions

The groups $SO(2n)$ will have embedded in them $SU(n)$, so for the group $SO(10)$ this is $SU(5)$. $SO(10)$ will have a 16-dimensional spinor representation $\mathbf{16}$ which transforms like $\mathbf{10} \oplus \mathbf{5} \oplus \mathbf{1}$ under $SU(5)$, hence the $\mathbf{16}$ can contain the SM matter states according to (18) and (19), and the singlet being identified with $\nu_R$. So this group allows for complete unification, with all fermions from the same family fitting in one representation. All the states can be represented by a ket $|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle$, where $\epsilon_i = \pm$, with the condition that there are an even number of $+:s$, the states are shown in table 3. By the known interactions of the SM vector bosons one can see from table 3 that the charged $SU(2)$ bosons will flip one weak index from $+$ to $-$ and the other from $-$ to $+$ and the gluons will do the same but for two color indices. The vector bosons of $SU(5)$ can be seen to flip one color index and one weak index, allowing states to change into each other within their $SU(5)$ representation, but not across different representations. Finally the remaining $SO(10)$ vector bosons will flip any two indices, in the combinations not yet mentioned. [3]

Table 3: The fermions in the $\mathbf{16}$ spinor representation of $SO(10)$.

<table>
<thead>
<tr>
<th>$SU(5)$ repr.</th>
<th>States</th>
<th>Color indices</th>
<th>Weak indices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\nu^c$</td>
<td>$-$ $+$ $-$</td>
<td>$-$ $-$</td>
</tr>
<tr>
<td>10</td>
<td>$\bar{u}_r$</td>
<td>$-$ $+$ $-$</td>
<td>$-$ $-$</td>
</tr>
<tr>
<td></td>
<td>$\bar{u}_g$</td>
<td>$+$ $-$ $-$</td>
<td>$-$ $-$</td>
</tr>
<tr>
<td></td>
<td>$u_b$</td>
<td>$+$ $+$ $-$</td>
<td>$-$ $-$</td>
</tr>
<tr>
<td></td>
<td>$u_r$</td>
<td>$+$ $-$ $-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$u_g$</td>
<td>$-$ $+$ $-$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$u_b$</td>
<td>$-$ $-$ $+$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$d_r$</td>
<td>$+$ $-$ $-$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$d_g$</td>
<td>$-$ $+$ $-$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$d_b$</td>
<td>$-$ $-$ $+$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$e^+$</td>
<td>$-$ $-$ $+$</td>
<td>$+$</td>
</tr>
<tr>
<td>5</td>
<td>$d_r$</td>
<td>$-$ $+$ $+$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$d_g$</td>
<td>$+$ $-$ $+$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$d_b$</td>
<td>$+$ $-$ $+$</td>
<td>$+$</td>
</tr>
<tr>
<td></td>
<td>$\nu_e$</td>
<td>$+$ $+$ $+$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$e^-$</td>
<td>$+$ $+$ $+$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

4.2 Predictions of $SO(10)$ unification

As mentioned one of its predictions is the right-handed neutrino, which has to be introduced ad hoc in $SU(5)$ or the SM. This is good as then neutrino masses can be explained by Seesaw models.
Unlike the SU(5) model the SO(10) group will not have a unique single-step breaking of its symmetry. It will be broken down to an intermediate group first, and then be broken down to the SM [27]. How the symmetry is broken will affect the proton lifetime and that the energy scales of the intermediate steps of symmetry breaking are unknown will introduce errors in the predictions of the proton lifetime. In [28] the proton lifetime has been calculated for four different breaking schemes, resulting in lifetimes of:

$$\tau_{p \to e^+\pi^0} \sim 10^{30-40} \text{ years}.$$ 

This span has been partially probed by experiments, see section 7.

In SO(n) the adjoint representation will have the dimension $n(n - 1)/2$ (for SU(n) it is $n^2 - 1$). So there are 45 vector bosons in SO(10), of these 12 are the SM vector bosons, 32 are new non-neutral vector bosons and the final one is a neutral vector boson, called $Z'$ [29]. The mass of the $Z'$ is $719 \text{ GeV} < m_{Z'} < 10^9 \text{ GeV}$ [30], so it is possible for this or the next-generation of accelerators to find it, if its mass is at the lower end of this interval.

## 5 Supersymmetric Grand Unification

### 5.1 Supersymmetry

Supersymmetry (SUSY) is a hypothesized additional symmetry between fermions and bosons. It proposes an operator $Q$ that changes a fermionic state into a bosonic one, keeping all other quantum numbers the same:

$$Q |f\rangle = |b\rangle,$$

$$\bar{Q} |b\rangle = |f\rangle.$$ 

This can obviously not be a perfect symmetry, as the existing particles do not have two states differing only in spin. It could still be possible, though, if the symmetry is broken, that the supersymmetric partners are so much heavier than the normal states that they have avoided detection. The simplest supersymmetric extension of the standard model is called the Minimal Supersymmetric Standard Model (MSSM), and its new states are shown in table 4. As some of the superpartners now have identical quantum numbers, for example the superpartners for the right- and left-handed quarks will both have zero spin, these states can mix.

One might now become slightly sceptical of SUSY, as so many new particles that have not been detected have to be introduced, so what is the motivation for SUSY? It provides:

- A solution to the gauge hierarchy problem, if SUSY is softly broken [31].

- Unification of the couplings is made possible again, as can be seen in the right plot of figure 3.

- It contains a dark matter candidate, this will not be discussed more in this report, but see for example the SUSY section of [3].
<table>
<thead>
<tr>
<th>Name</th>
<th>Particle</th>
<th>Superpartner</th>
<th>Spin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squarks</td>
<td>$q_L$</td>
<td>$\tilde{q}_L$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$q_R$</td>
<td>$\tilde{q}_R$</td>
<td>0</td>
</tr>
<tr>
<td>Sleptons</td>
<td>$l_L$</td>
<td>$\tilde{l}_L$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$l_R$</td>
<td>$\tilde{l}_R$</td>
<td>0</td>
</tr>
<tr>
<td>Sneutrino</td>
<td>$\nu_L$</td>
<td>$\tilde{\nu}_L$</td>
<td>0</td>
</tr>
<tr>
<td>Gauginos</td>
<td>$\gamma$</td>
<td>$\tilde{\gamma}$</td>
<td>1/2</td>
</tr>
<tr>
<td></td>
<td>$Z^0$</td>
<td>$\tilde{Z}^0$</td>
<td>1/2</td>
</tr>
<tr>
<td></td>
<td>$h^0$</td>
<td>$\tilde{h}^0$</td>
<td>1/2</td>
</tr>
<tr>
<td></td>
<td>$H^0$</td>
<td>$\tilde{H}^0$</td>
<td>1/2</td>
</tr>
<tr>
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<td>$\tilde{W}^\pm$</td>
<td>1/2</td>
</tr>
<tr>
<td></td>
<td>$H^\pm$</td>
<td>$\tilde{H}^\pm$</td>
<td>1/2</td>
</tr>
<tr>
<td>Gluino</td>
<td>$g$</td>
<td>$\tilde{g}$</td>
<td>1/2</td>
</tr>
</tbody>
</table>

### 5.1.1 Gauge hierarchy problem

The Higgs field will be part of the grand unification scale or possibly the Planck scale, where gravity must be included. Assuming there is no new physics between our low energy world at the scale of the weak interactions and either of these extremely high energies, the smallness of the Higgs mass is a problem. The cutoff scale for the SM, $\Lambda$ will be $\sim 10^{16}$ GeV or larger. For the scalar field to have a small mass term $-\mu^2 \sim -(100 \text{ GeV})^2$, in (6), which is required for the Higgs boson to give the correct masses for $W^\pm$ and the $Z^0$ bosons, the bare mass of the Higgs boson has to be of the order of $-\Lambda^2$ and radiative corrections then brings it down to $-\mu^2$. But for that to be possible the radiative corrections have to be fine-tuned to one part in $\frac{\Lambda^2}{\mu^2} \sim 10^{28}$! This is clearly a problem.

With SUSY this is solved by an exact SUSY forbidding a Higgs mass term. If this symmetry is weakly broken the Higgs can have a small mass. The radiative corrections from the fermions will be:

$$\delta m_H^2 \propto - \int_{m_f^2}^{\infty} dp^2,$$

and from the superpartners:

$$\delta m_H^2 \propto + \int_{m_f^2}^{\infty} dp^2.$$

The total contribution from $f$ and $\tilde{f}$ is then:

$$\delta m_H^2 \propto m_f^2 - m_{\tilde{f}}^2,$$
Figure 6: Loop cancellations in SUSY, the top loop is due to a fermion, $f$, and the lower is for its superpartner, $\tilde{f}$.

which is then a finite value, of order 1000 GeV as the mass of the superpartners is expected to be. The loops responsible for these corrections can be seen in figure 6.

This section is based on both [11] and [31].

5.1.2 Unification with SUSY

As additional particles are present with SUSY, the running of the couplings will be altered, since the $b$s will change value, when there are more fermions and bosons contributing to corrections. A change of the $b$s results in the slope changing, and as can be seen in figure 3 this actually improves upon the unification, resulting in exact unification fitting within $3\sigma$ [3] to experimental data. It can be made into a precise unification with a small threshold correction. Without SUSY the unification is about $12\sigma$ [3] away from exact unification, non-SUSY GUTs can still be saved by introducing incomplete GUT multiplets, by having a more complex breaking pattern or as mentioned in subsection 3.7 introducing a $15_H$ Higgs representation.

Since the slope is changed the unification scale is pushed upwards, so that in addition of making the couplings meet better, the proton lifetime is also improved. In SUSY GUTs the unification scale is on the order of $3 \cdot 10^{16}$ GeV, while for non-SUSY models it is on the
order of $10^{15}$ GeV. [3]

5.2 SUSY $SU(5)$

The Higgs content of the minimal SUSY $SU(5)$ is the same as that of the minimal non-
SUSY $SU(5)$, i.e. a 24$H$, but with both 5$H$ and 5$H$, which is required in order to give
both the down and the up quark masses. [3] In SUSY the particles and their superpartners
are assigned to supermultiplets (as representations are also called multiplets). The Higgs
representations here also means the supermultiplets, so they contain both the Higgses and
their superpartners.

5.2.1 Predictions of SUSY $SU(5)$

The strong coupling constant at the weak scale, $m_Z$, is predicted to be $\alpha_s(m_Z) = 0.129(10)$
[32], which is close to the experimental value $\alpha_s(m_Z) = 0.1184(7)$.

In SUSY $SU(5)$ the dominant decay mode will be $p \rightarrow K^{+} \nu_{\tau/\mu}$ [33], mediated by the
Higgs color triplets of the 5$H$ and 5$H$. By demanding unification of the couplings it is
possible to get upper and lower limits on the Higgs color triplet mass, which will then
give a limit on proton lifetime. In [34] this is done, the limit ends up being $\tau_{p \rightarrow K^{+} \nu} \leq
2.5 \cdot 10^{29}$ years, which is quite a bit below the current limit on this decay mode (see section
6). It has been noted, in [35], that this is not a realistic model as it keeps the bad prediction
of $m_d/m_s = m_e/m_{\mu}$, so the exclusion by proton decay limits is not that interesting. This
can be fixed by allowing non-renormalizable interactions that are due to the Planck scale,
i.e. it is so close to the grand unification scale that it might have an influence, this was
argued in [21]. With such interactions it was found in [35] that using the strongest bound
on the triplet mass, $m_T > 10^{17}$ GeV, that the proton lifetime is:

$$\tau_{p \rightarrow K^{+} \nu} \gtrsim 7.6 \cdot 10^{41} \text{ years.}$$

6 Experiments Probing GUT and SUSY

The scale of unification, $M_{GUT}$, is of order $10^{14-17}$ GeV [3, 16, 25] so it is not reachable
by todays accelerators. There is another very distinct signature of Grand Unified theories
though: as baryon number, $B$, is violated by necessity in essentially all Grand Unified
theories [3], there will be nucleon decay present if any Grand Unified theory is correct. As
of today there is no evidence of proton or neutron decay through baryon number violating
modes, i.e. there are only lower limits for the lifetime of the proton [36]. These lower limits
can then be used to exclude different Grand Unification models, if their prediction for the
proton lifetime is incompatible with the limits.

There are other predictions of Grand Unification though, such as magnetic monopoles
[37], and for example in [16] there will be light leptoquarks. Leptoquarks are particles with
both a baryon, \( B \), and a lepton number, \( L \). For \( SU(5) \) the \( X \) and \( Y \) bosons are leptoquarks. In \cite{16} the light leptoquarks will be some of the Higgses in the \( 15_H \) representation. For SUSY GUTs there will also be a SUSY spectrum. The superpartners cannot have too high mass though, as that would reintroduce the gauge hierarchy problem that SUSY was introduced to mend \cite{3}. As of the writing of this none of these three predictions have been observed, so there are only bounds on the masses and production cross-sections of these, see the appropriate sections of \cite{3} for the values of these bounds.

6.1 Experiments running today

To detect nucleon decays relevant to GUTs two kinds of detectors have been developed, these are water Cherenkov detectors and fine-grained tracking detectors \cite{38}. Currently running is the Super-Kamiokande detector, which is of the first kind. It has provided many of the most stringent lower limits for the different decay modes of protons and neutrons today \cite{36}. An example of the second kind is the Soudan 2 \cite{38}, which is out-of-use now as it stopped taking data in 2005.

The parts of the experiments described here will only be those related to testing of GUT, for example the Super- and Hyper-Kamiokande are and will, respectively, test neutrino physics, but that is not of interest for this report.

6.1.1 Super-Kamiokande

This water Cherenkov detector is made up of 50 kton of water, with a fiducial volume of 22.5 kton, which is the interior volume that is shielded by the rest of the volume from the background events. It is situated in the Kamioka mine, in Japan, 1000 m underground, to shield it from cosmic muon background events \cite{39}. As the proton, if it is unstable, has an extremely long lifetime, the age of the universe being vanishingly small in comparison, a lot of protons are required in order to see any events. The fiducial volume corresponds to \( 7 \cdot 10^{33} \) protons \cite{36}, so there will be enough events to detect proton decay with lifetimes a couple of orders higher than the upper bound of \( (38) \).

A ring imaging Cherenkov detector uses the Cherenkov photons that relativistic charged particles radiate when traversing a medium if their velocity is larger than the speed of light in the medium. The photons will be emitted in a cone around the direction of the trajectory of the charged particle, the angle is given by \( \cos(\theta_c) = 1/n\beta \). For a refractive index of \( n = 1.33 \) and \( \beta \equiv v/c \approx 1 \) (these are the values for the Super-Kamiokande), the angle will be about \( 42^\circ \). \cite{40}

The water is surrounded by photomultiplier tubes (PMTs) which detects the Cherenkov photons, and by measuring pulse height and timing with the PMTs the Cherenkov ring is reconstructed. This allows for the vertex, the direction, and the energy to be measured and for the particle to be identified. \cite{38} Particle identification is based on the angle of the ring and its pattern, it identifies particles as one of two types, electron-like or muon-like. They
are discernible as electrons and photons will create electromagnetic showers, distorting the ring slightly [36]. The typical resolution is 3.0% for the momentum of a 1 GeV electron [36] and 30 cm for the vertex of a muon with the same energy [38], the momentum resolution was at 4.1% though during the acquisition of the SK-II data (see below) [36].

The major decay modes that are investigated are \( p \rightarrow e^+\pi^0 \) for non-SUSY GUT models and \( p \rightarrow \bar{\nu}K^+ \) for SUSY GUT models [36]. For the first type of decay the \( \pi^0 \) and the \( e^+ \) are emitted back-to-back in the rest frame of the nucleus, with \( E = \frac{m_p}{2} \). The pion will quickly decay into two photons and two or three electron-like Cherenkov rings will be seen, one of the photon rings might not be visible depending on its direction. As momentum is measured the pion mass should be reconstructable from the two photon rings if they are both visible and the same for the proton mass using all three rings, so this gives additional constraints on whether an event actually is this decay. Finally the total momentum should also be small, as if it is a nucleon decay and not, for example an atmospheric neutrino, the momentum of the nucleon is small before the decay. The reconstructed mass constraints are 85 MeV < \( m_{\pi^0} \) < 185 MeV and 800 MeV < \( m_p \) < 1050 MeV and for the momentum \( P_{\text{tot}} < 250 \text{ MeV} \) [36]. For the second decay the neutrino will be invisible and the kaon will be below the Cherenkov threshold, the kaon will then stop and decay, it has highest branching ratio for decay into a muon and a muon antineutrino. As there are only two decay products in the decay of the proton the muon will be monochromatic. The selections for this are one muon ring with a decay electron, and the muon momentum, \( P_{\mu} \), is then fitted to simulations. [36] There is identification of this event using prompt \( \gamma \), if the proton is in an \(^{16}\text{O}\) atom the daughter nucleus will be an excited \(^{15}\text{N}\)\(^*\), which can deexcite through emission of a 6 MeV photon. Using this photon as a tagging then eliminates background. Selection for this is a muon-like ring with a decay electron and a muon momentum within 215 MeV < \( P_{\mu} < 260 \text{ MeV} \), and tagging from the photon. [36]

The Super-Kamiokande data is divided into three parts, SK-I, SK-II and SK-III, as it has been run for three periods of time at different capacities. First it was run between 1996 and 2001, for a total of 1489.2 days of livetime, corresponding to an exposure of 91.7 kton · yrs, with an inner PMT count of 11146. [36] The exposure allows for comparison between the different periods of data acquisition, as the PMT count varied so a comparison of only time is not possible. Then during preparations for the second period of data gathering a PMT imploded, and this caused a cascade of implosions as the implosion of a PMT caused its neighbors to implode as well, this destroyed over half of the PMTs [39]. So the second period of data collection, SK-II, only had an inner PMT count of 5182, and it was from 2002 to 2005, with a livetime of 798.6 days corresponding to 49.2 kton · yrs. For the last period, SK-III, between 2006 and 2008 the PMT count was up to the original of 11146, and it had a livetime of 518.1 days, which was 31.9 kton · yrs. [36]

The most stringent limits today for the major decay modes are from the Super-Kamiokande, and they are [36]:

\[
\tau_{p \rightarrow e^+\pi^0} > 1.0 \cdot 10^{34} \text{ yrs},
\]
\[ \tau_{p \rightarrow K^+ \bar{\nu}} > 3.3 \times 10^{33} \text{ yrs}, \]

at 90% CL.

6.2 Planned experiments

The next generation of proton decay experiments are being proposed now. There are three different types: water Cherenkov detectors, for example Hyper-Kamiokande; liquid argon detectors, for example GLACIER; and finally liquid scintillator detectors, such as LENA. There are many more proposed experiments, but this section will only contain the three examples mentioned.

6.2.1 Hyper-Kamiokande

This is the proposal to build another detector like the Super-Kamiokande (S-K), but of far larger scale, allowing for more precise limits of proton decay to be established. It is planned to have a total volume of 0.99 Mton, which is 20 times larger than S-K and its fiducial volume will be 0.56 Mton (25 times that of S-K). A proposed location for it is 8 km south of S-K, where it will be under 648 m of rock, protecting it from cosmic muons. It will have 99,000 PMTs, which is about 9 times the number used in S-K. The Hyper-Kamiokande could be constructed and become fully operational in 10 years. [40]

If no proton decay candidate events are detected, the limits that are expected after 10 years of data collection are:

\[ \tau_{p \rightarrow e^+ \pi^0} > 1.3 \times 10^{35} \text{ years} \]
\[ \tau_{p \rightarrow K^+ \bar{\nu}} > 2.5 \times 10^{34} \text{ years} \]

at 90% CL and:

\[ \tau_{p \rightarrow e^+ \pi^0} > 5.7 \times 10^{34} \text{ years} \]
\[ \tau_{p \rightarrow K^+ \bar{\nu}} > 1.0 \times 10^{34} \text{ years} \]

at 3σ CL. [40]

6.2.2 GLACIER

GLACIER is the proposed Giant Liquid Argon Charge Imaging ExpeRiment, where a liquid argon time-projection chamber (LAr TPC) of the size of 100 kton is to be used for searching for proton decays, neutrino astrophysics and long baseline neutrino oscillation experiments [41]. A TPC uses a gas or a liquid where ionizing radiation creates charge carriers, which then drift to electrodes, where a signal is detected. The signal has a pulse height proportional to the energy deposited by the radiation. By measuring the drift time spatial information is obtained along the drift path, and by segmenting the electrodes...
spatial information in the two directions perpendicular to the drift path is given. The pulse height gives specific energy loss $dE/dx$ which allows for particle identification if the momentum is known, for example by curvature of the trajectory if a magnetic field is applied. And finally by adding up all the ionization charge the total energy of the ionizing radiation can be determined. The TPC allows for the entire trajectory of the ionizing particle in the detector to be reconstructed. [42] To measure the drift time GLACIER will use prompt scintillation light as $T_0$ and then the time of the signal is noted. Particle identification will be achieved by both $dE/dx$ measurements and topology, i.e. identifying a particle by its decay products.

The advantages with a LAr TPC over a water Cherenkov detector are the good energy resolution, the possibility to reconstruct the event, and also the good background suppression and a good signal efficiency. It is expected that the efficiency will be better than for a Cherenkov detector, so that it can be smaller but still have a similar performance, a size of 100 kton is expected to be competitive with next generation Cherenkov detectors (of sizes of Mton). [43]

Like H-K it is expected to have limits at the order of $\tau \sim 10^{35}$ years [43]. It will be especially sensitive to decay channels with a kaon in the final state [41], making it a good complement to water Cherenkov detectors.

6.2.3 LENA

LENA is the Low-Energy Neutrino Astronomy experiment, it is a proposed large volume liquid scintillation detector, of size of 50 kton, to be used for particle astrophysics, geophysics and nucleon decay searches. Scintillation detectors work by incoming ionizing radiation exciting the atoms and molecules of the scintillation material, which then emits light when they deexcite. By coupling the material to a PMT, information about the incoming radiation can be gathered. This detector type can measure energy and it is also possible with some scintillation materials to identify particles by analyzing the pulse shape. [42]

An advantage of the liquid scintillation detector as compared to a Cherenkov detector is that it can directly detect the energy deposited by the kaon in the decay channel $p \to K^+ \bar{\nu}$, which in a Cherenkov detector needs to be reconstructed from the decay products of the kaon. The subsequent decay of the kaon also provides a good coincidence signal, that helps in suppressing the background due to atmospheric neutrinos. After 10 years of data collection a limit at $\tau > 4 \cdot 10^{34}$ years, is expected. [44]

7 Summary

The basic requirements for the models to be considered realistic are: that the gauge couplings unify, a long enough proton lifetime, in order to survive experimental bounds, and the fermion mass phenomenology has to be correct. As was seen in section 3 the minimal
SU(5) model fails at all three of these requirements. Adding a $15_H$ Higgs representation and allowing non-renormalizable interactions allowed non-SUSY SU(5) to become a realistic model. In minimal SO(10) all three requirements can be fulfilled without a larger Higgs sector or by introducing SUSY. By introducing SUSY the minimal SU(5) model is close to satisfying gauge coupling unification, and threshold effects can make the unification exact. It still fails the other two requirements, but by allowing non-renormalizable interactions both of them can be met.

Of the problems mentioned in section 1.2, the problems without or with an unsatisfactory answer are:

- Why is the gauge group of the world the one it is? While only a single gauge group is required, there is still no explanation why it is just SU(5) or SO(10), if they happen to be correct.

- Why are there three families? In none of the models mentioned this has been addressed. The first family has just been put in a representation of the gauge group, and then the same procedure has been done to the remaining two families.

- Why are left- and right-handed fermions treated differently? This is still unexplained, the left- and right-handed fermions are still put in different representations in order to comply with low energy experiments.

- How does gravity enter? This has not been addressed.

- Are the free parameters connected? The number of free parameters has not been reduced, there are still all the masses, the mixing angles and so on.

- The cosmological constant problem. This is still unanswered.

- Why are the Higgs representations the ones they are? Still no underlying reason for this, the Higgs representations are introduced ad hoc in order for the low energy gauge groups to be realized.

The questions that have been answered are:

- Why is $Q_e + Q_p = 0$? This has been answered by putting the leptons and the quarks in the same representations, which fixes the relation between the quark charges and the electron charge.

- Why is the universe asymmetric with respect to matter and anti-matter? This can conceivably be answered in grand unified theories, as they violate baryon number conservation.

- Gauge hierarchy problem. For non-SUSY GUTs this is a problem, but this is the main reason SUSY is introduced, as it provides a solution to it.
What is dark matter? With SUSY there are dark matter candidates, i.e. particles with the characteristics of dark matter.

Table 5: Proton lifetime predictions of the models mentioned and the limits from the experiments, with the last column giving the section in which the model/experiment was described.

<table>
<thead>
<tr>
<th>Model/Experiment</th>
<th>$\tau_p$, in years</th>
<th>Channel</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimal $SU(5)$</td>
<td>$10^{28.5-31.5}$</td>
<td>$e^+\pi^0$</td>
<td>3</td>
</tr>
<tr>
<td>$SU(5)$ with $15_H$</td>
<td>$\leq 1.4 \cdot 10^{36}$</td>
<td>Total</td>
<td>3.7</td>
</tr>
<tr>
<td>$SO(10)$</td>
<td>$10^{30-40}$</td>
<td>$e^+\pi^0$</td>
<td>4</td>
</tr>
<tr>
<td>Minimal SUSY $SU(5)$</td>
<td>$\leq 2.5 \cdot 10^{29}$</td>
<td>$K^+\bar{\nu}$</td>
<td>5.2</td>
</tr>
<tr>
<td>SUSY $SU(5)$ with non-renorm. int.</td>
<td>$\geq 7.6 \cdot 10^{41}$</td>
<td>$K^+\bar{\nu}$</td>
<td>5.2</td>
</tr>
<tr>
<td>S-K exp. (current)</td>
<td>$\geq 1.0 \cdot 10^{34}$</td>
<td>$e^+\pi^0$</td>
<td>6.1.1</td>
</tr>
<tr>
<td></td>
<td>$\geq 3.3 \cdot 10^{33}$</td>
<td>$K^+\bar{\nu}$</td>
<td></td>
</tr>
<tr>
<td>H-K exp. (future)</td>
<td>$\geq 1.3 \cdot 10^{35}$</td>
<td>$e^+\pi^0$</td>
<td>6.2.1</td>
</tr>
<tr>
<td></td>
<td>$\geq 2.5 \cdot 10^{34}$</td>
<td>$K^+\bar{\nu}$</td>
<td></td>
</tr>
<tr>
<td>GLACIER exp. (future)</td>
<td>$\sim 10^{35}$</td>
<td>$K^+\bar{\nu}$</td>
<td>6.2.2</td>
</tr>
<tr>
<td>LENA exp. (future)</td>
<td>$\geq 4 \cdot 10^{34}$</td>
<td>$K^+\bar{\nu}$</td>
<td>6.2.3</td>
</tr>
</tbody>
</table>

In table 5 all the lifetimes from the previous sections have been repeated, in order to make comparison easier. Now it is easy to see, as has been mentioned, that the minimal $SU(5)$ model has been excluded by proton decay, by a few orders of magnitude. But if the $15_H$ representation is included $SU(5)$ survives the bounds. As the limit is not too far off the next-generation limits, it will either be verified in the coming decades or be very close to being excluded.

For $SO(10)$ the span is quite large, due to the dependence of the proton lifetime on the breaking pattern of the group. As can be seen this has been partially probed by experiments, but the upper limit is very high, so next-generation experiments will not have a chance of excluding it completely.

With SUSY included the situation is very bad for the minimal $SU(5)$ model. This is not the only problem though, this model does not give the correct fermion masses, just as the minimal non-SUSY $SU(5)$ model did not. If non-renormalizable interactions are added in order to get the right fermion mass phenomenology the situation changes drastically. From being excluded it goes to being far from reachable by next-generation experiments. Hence there is no hope of it being tested and excluded in the coming decades.

Comparing the limits from the future experiments it is seen that GLACIER and LENA will be good complements to H-K, pushing the limits on the mode $p \rightarrow K^+\bar{\nu}$ close to that of $p \rightarrow e^+\pi^0$. The former channel is the dominant one for SUSY models and the latter for non-SUSY. As SUSY models are of more interest, due to gauge coupling still being possible in them, it is good to get better limits for the latter channel.

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Final thoughts

Is there anything I could have done differently during the writing process of this thesis? Or perhaps even before I started? The thing I have had the most trouble with was deciding how much to write on the different sections, especially the $SO(10)$ and the SUSY sections. I had many doubts about if I had written enough about those sections during the writing process, but now that I am done I am quite satisfied with the final result. There is probably not much I could have done to avoid this though, as I have had to learn the things I were to write I could not have started by deciding the extent of every section. Indeed, learning is an ongoing process, and it is connected to active documentation.

A course or two in both group theory and quantum field theory would also have made it easier to write. But I am not sure I would have liked to trade any of the courses I have read, as they have all to some extent, been useful. It could also be the other way round, that writing this thesis will give me better motivation and understanding when taking courses in group theory, particle physics or quantum field theory.

Have I learnt anything apart from the physics I have written about? One thing that I had not been exposed to much before is the use of databases to find relevant articles, and also to be able to sort through all the information available. At first I was not very good at finding or sorting out the irrelevant information, but I have improved a lot in that respect, now that I think back on how I fared in the beginning.

References


