THEME AND VARIATIONS ON THE 3N+1 PROBLEM

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Abstract. We investigate the dynamics of the Collatz map over various domains. In particular, cycles will be studied, and conditions for their existence will be provided.
1. Introduction

"Problems worthy of attack prove their worth by hitting back." - Piet Hein

Certainly, the $3n + 1$ problem or Collatz conjecture, have by now proven its worth. It has resisted sophisticated attacks for a long time and been deemed intractable by many [11]. Paul Erdős famously said of it: “Mathematics is not yet ready for such problems” [8], and he also offered 500$ for a solution. Its origins are somewhat hazy, dating back to the 1930’s. L. Collatz [11] reports in 1986 that he as a student (1928-1933), represented integer functions by graphs. He wanted to find simple graphs containing cycles, which could also be represented by integer functions. Eventually this led him to the function

\[
f(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\
3n + 1 & \text{if } n \equiv 1 \pmod{2},
\end{cases}
\]

and he conjectured (1937) that for all $n \in \mathbb{N}$, $f^C(n) = 1$ for some $C \in \mathbb{N}$. However, as he was unable to solve it, he did not publish the problem. A colleague of his circulated it by mouth during a visit to Syracuse university in the 1950’s, proposing the name “the Syracuse problem”. Apparently since the 1970’s – judging by the increase in the number of articles and publications on this and related problems – interest has picked up, and today it’s in parity with conjectures such as the twin prime or Goldbach. Part of what makes this problem so fascinating is that, in spite of how easy it is to state and understand, there have been little progress towards a proof. It has been checked by computer to be true for all $n < 5 \cdot 2^{60}$. Beyond that, we don’t have much but heuristic arguments. The problem can be split into two smaller conjectures (which in turn branch into weaker conjectures and problems); both of which are still unproven, though partial results exist.

Only one cycle conjecture.
The only cycle for the function $f$ over $\mathbb{N}$ is the trivial $(1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$.

No divergent trajectories conjecture.
There exists no $x \in \mathbb{N}$ such that $\lim_{n \to \infty} f^n(x) = \infty$.

J. C. Lagarias [8] showed that if such a trajectory exists, then it cannot grow too slowly.

To paraphrase Wirsching [11]: If one want’s to say something interesting about this problem, one may need to venture beyond the realm of the natural numbers. Indeed, the problem has been translated into a wide variety of settings. For example, I. Korec [7] relate the $3n + 1$ problem to certain structural properites of certain finite algebras. In [3], M. Chamberland notes that the function $g(x) = x + \frac{1}{4} - \frac{2x + 1}{4} \cos(\pi x)$, interpolates the $3n + 1$ function, opening up a new set of tools to analyze the problem with. Several authors [11] have considered the function $f$, over the $2$-adic integers, and observed ergodic properties among other things. In this spirit; we will in this paper investigate the $3n + 1$, and related functions, over various domains. But first, we recall some basic facts and terminology about discrete dynamical systems.

2. Preliminaries

A discrete dynamical system is essentially an iterated function over some set to it self. There need not be anything discrete about the domain, the “discrete” part simply refers to how the system evolves.
More generally: it’s a tuple \((G, M, T)\), \(G\) a group or semigroup, \(M\) a set and \(T\) a function, usually called the evolution function.

\[
T : G \times M \to M \\
(g, m) \mapsto T_g(m)
\]

Starting at a point \(x\) in our domain and iterating some given function \(f\), we get the orbit or trajectory of \(x\)

\[
T_f(x) := (x, f(x), f^2(x), \ldots, f^k(x), \ldots),
\]

where the exponent refers to function composition, i.e.

\[
f^k(x) = f(f^{k-1}(x)).
\]

**Definition 2.1.** A point \(x\) is called a periodic point of \(f\) with period \(k\) if \(f^k(x) = x\). A point \(x\) is called eventually periodic with period \(k\) if there exits \(N\) s.t \(f^{n+k}(x) = f^n(x)\) whenever \(n \geq N\).

**Definition 2.2.** Let \(C\) be a cycle with period \(k\), we denote the set of eventually periodic points with respect to this cycle \(W(C) := \{ x \in C : f^n(x) \in C \text{ for some } n \}\). In other words: it’s the set of all points that eventually converge to this cycle.

**Proposition 2.3.** Let \(C\) and \(B\) be two different cycles. Then \(W(C) \cap W(B) = \emptyset\).

*Proof.* This is an immediate consequence of the fact that two cycles necessarily are disjoint. \(\square\)

3. The \(3n+1\)-problem in \(\mathbb{Z}/k\mathbb{Z}\)

The ring \(\mathbb{Z}/k\mathbb{Z}\) is perhaps an unlikely setting for the \(3n+1\) conjecture if one wishes to gain insight into the original problem. The integers \(\mathbb{Z}\) and \(\mathbb{Z}/k\mathbb{Z}\) are quite different as domains and – apart from both being rings – do not share many structural properties (for non prime \(k\) anyway). Nevertheless, a case can be made for why this is a worthwhile pursuit. Consider the two questions below:

(1) Does there exist arbitrarly large \(k\) for which the \(3n+1\) conjecture is true?

(2) What can be said about the asymptotic density of the rings \(\mathbb{Z}/k\mathbb{Z}\) for which \(3n+1\) conjecture is true? If

\[
h(k) = \lvert \{ d \leq k : 3n+1 \text{ conjecture is true on } \mathbb{Z}/d\mathbb{Z} \} \rvert
\]

denotes the number of rings with modulus \(d \leq k\) for which the \(3n+1\) problem is true, then

\[
\lim_{k \to \infty} \frac{h(k)}{k} > 0?
\]

Clearly, a positive asymptotic density would imply (1), though we suspect this not to be the case. A “yes”, to the first question would have some interesting implications concerning cycles in the case over \(\mathbb{N}\). It would be a strong argument – if not a proof, of the non-existence of cycles other than the “trivial” one. To see this, note that if there exists a cycle \(C\) over \(\mathbb{N}\), then \(\max(C)\) also exists. As we can find \(k\) of arbitrary size, we choose \(k\) such that
Given this, we suspect that these questions may be just as hard to answer as the original. Rather than to focus on those, we will in this section investigate the various obstructions to the “success” of the $3n+1$ function over finite integer rings. There is an immediate gain to working over $\mathbb{Z}/k\mathbb{Z}$, namely, no divergent trajectories. After $k-1$ steps one must either have entered the trivial cycle $(1 \to 4 \to 2 \to 1)$, some other cycle or be stuck at some fixed point.

From here on and for the rest of this section, $k$ will denote the modulus. The results in this section concerns the $3n+1$ function $f$ over finite integer rings, i.e.

$$f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}.$$ 

The various calculations we are referring to throughout this section are all done by simple scripts we wrote in Maple.

**Proposition 3.1.** If $k \equiv 4 \pmod{6}$, or $k \equiv 2 \pmod{3}$ then the $3n+1$ conjecture is false.

**Proof.** Let $k = 6t + 4$, $t \in \mathbb{N}$ and take $n = 2t + 1$, then after one iteration $2t + 1 \rightarrow (2t + 3) + 1 = 6t + 4 \equiv 0 \pmod{k}$ and hence converges to the fix point 0. For the second congruence condition just note that $6t + 4 = 2(3t + 2) \equiv 0 \pmod{k}$, with $k$ now on the form $k = 3t + 2$. 

**Proposition 3.2.** If $k = 4t + 3$ then the Collatz map fails and there exists a fixed point $x \neq 0$.

**Proof.** Take $x = 2t + 1$, one iteration gives $6t + 4 \equiv 2t + 1 \pmod{k}$. 

While it is still undecided whether there exist any cycles other than the “trivial” $1 \to 4 \to 2 \to 1$, for the original problem, cycles abound for it over finite integer rings. One can, using experimental data, reverse engineer a lot of congruence conditions on the modulus for the existence of cycles of various period. We show the general procedure and provide some examples.

**Proposition 3.3.** Let $k = 50t - 1$, $t \in \mathbb{N}$. Then there exists a cycle of period 4 starting at $x = 16t - 1$.

**Proof.** For $k = 49$ we see that

$$15 \rightarrow 46 \rightarrow 23 \rightarrow 70 \equiv 21 \rightarrow 64 \equiv 15$$

constitutes a cycle of period 4. Now let $x$ be an arbitrary odd integer and follow the above.

$$x \rightarrow 3x + 1 \rightarrow (3x + 1)/2 \rightarrow (9x + 5)/2 - k \rightarrow (27x + 15)/2 - 3k + 1 - k = x$$
Solving the last equation:

\[
27x + 15 - 6k + 2 - 2k = 2x \iff 25x + 17 = 8k \iff x + 1 \equiv 0 \pmod{8} \iff x = 8t - 1
\]

To get \(k\) we just plug in \(x\) into the equation, which yields \(k = 25t - 1\). The expression \(\frac{2x+5}{2} - k\) is odd if and only if \(t\) is even and the result follows. \(\Box\)

**Proposition 3.4.** Let \(k = 454t + 37, t \in \mathbb{N}\). Then there exists a cycle of period 9 starting at \(x = 96t + 7\).

*Proof.* Starting at \(x = 96t + 7\) we get

\[
96t + 7 \to 288t + 22 \to 144t + 11 \to 432t + 34 \to 216t + 17
\]

\[
\to 648t + 52 - (454t + 37) = 194t + 15 \to 582t + 46 - (454t + 37)
\]

\[
= 128t + 9 \to 384t + 28 \to 192t + 14 \to 96t + 7.
\]

\(\Box\)

**Proposition 3.5.** Let \(k = 358t - 160\). Then there exists a cycle of period 11 starting at \(x = 72t - 41\).

*Proof.* A similar “diagram chase” as in Proposition 3.4 gives the proof. \(\Box\)

**Proposition 3.6.** Let \(k = 14t - 1, 46t - 1\) or \(38t, t \in \mathbb{N}\). Then \(k\) contains cycles of periods 3, 5, 6 respectively, starting at \(x = 4t - 1, x = 8t - 1\) and \(x = 8t - 1\).

Using these and similar results we can now construct new rings containing several cycles.

**Example 3.7.** Using the first two congruence conditions from Proposition 3.6 and by the Chinese Remainder Theorem we have:

\[
\begin{cases}
k = 46s - 1 \\
k = 14t - 1
\end{cases}
\iff
\begin{cases}
k \equiv -1 \pmod{46} \\
k \equiv -1 \pmod{14}
\end{cases}
\iff
\begin{cases}
k \equiv -1 \pmod{2} \\
k \equiv -1 \pmod{23} \\
k \equiv -1 \pmod{7}
\end{cases}
\]

which in this case has the obvious solution \(k = 2 \cdot 7 \cdot 23 - 1 = 321\) with our cycles starting at \(x = 4 \cdot 23 - 1\) and \(x = 8 \cdot 7 - 1\) respectively. Note that this will not always work. Picking the first and last congruence from Proposition 3.6 we end up with

\[
\begin{cases}
k \equiv -1 \pmod{2} \\
k \equiv 0 \pmod{2} \\
k \equiv 0 \pmod{19} \\
k \equiv -1 \pmod{7}
\end{cases}
\]

which clearly has no solutions.

**Definition 3.8.** An element \(n \in \mathbb{Z}/k\mathbb{Z}\) is called *repellent* if there does not exist any \(x\) such that \(f(x) = n\).

Definition 3.9. A cycle $C$ is called a repellor if there do not exist any trajectories that eventually converge to it. Put differently, $W(C) = \emptyset$. We define an upper-cycle to mean a cycle that is entirely contained in the set $A := \{ n : n \geq \lfloor k/2 \rfloor \}$.

Theorem 3.10. There exist no repellent upper-cycles of period $\geq 2$ for the $3n + 1$ function over the rings $\mathbb{Z}/k\mathbb{Z}$.

Proof. First note that if such a cycle exists, then it must contain only odd integers. Also note that if $k$ is even, then it will immediately fail since $3n + 1$ will always produce an even integer or an integer $\leq (k/2)$. So assume $k$ is odd.

The elements in such an cycle must be contained in the set

$$S := \{ n : n \equiv 1 \pmod{2}, \lfloor k/2 \rfloor \leq n \leq \lfloor 2k/3 \rfloor \},$$

where $[ ]$ denotes the integer part. The function $f_k(n) = 3n + 1 - k$ is increasing on this set and hence can’t generate a cycle.

Proposition 3.11. There exist no cycles with period $\geq 2$ with only odd integers.

Proof. The elements of such a cycle must be contained in the set

$$S := \{ x : x \equiv 1 \pmod{2}, \lfloor k/3 \rfloor \leq x \leq k \}.$$

The fate of these elements is enterily determined by the function

$$f_x(n) = 3^n x + \sum_{i=1}^{n-1} 3^i(1 - k) + 1 - k,$$

i.e. the n:th iterate of a point $x \in S$, which again is increasing on the set $S$.

Conjecture 3.12. There exist no repellent cycles for the $3n + 1$ function when the modulus is greater than 5.

Remark 3.13. This is trivially true for the $3n + 1$ function defined over $\mathbb{N}$. (Clearly true for the $qn + 1$ function as well, which we will discuss later.) To see this, suppose there exists a cycle $C$, then Period$(C) < \infty$ and so max$(C)$ exists and hence $2 \cdot \text{max}(C) \in W(C)$.

In general we found that rings congruent to zero (mod 12), seem to fair best with regard to the $3n + 1$ conjecture being true. Picking large multiples of 12 (basically at random), we could find $k$ as large as 3219264 for which the $3n + 1$ conjecture is true (we didn’t check further).

4. A $qn+1$ Generalization

Diophantine equations. It’s a fair question to ask what makes the “3”, in the $3n + 1$ problem so special. In fact, we can consider this problem to just be an instance of a larger class of dynamical systems of the form

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{qn + 1}{2} & \text{if } n \equiv 1 \pmod{2} \end{cases}$$
with \( q \) odd.

We will in this section consider the \( qn + 1 \) problem defined over \( \mathbb{N} \) and relate the existence of certain kinds of cycles — called circuits — to the solutions of exponential diophantine equations.

**Definition 4.1.** An iteration is called a circuit of the \( qn + 1 \) function if it looks like this.

\[
(8) \quad (n \xrightarrow{k} m \xrightarrow{\ell} n^*) := (n, T(n), \ldots, T^k(n) = m, T(m), \ldots, T^\ell(m) = n^*)
\]

with \( n < T(n) < \cdots < T^k(n) \) and \( m > T(m) > \cdots > T^\ell(m) \) and circuits being cycles when \( n = n^* \).

Basically, it’s a cycle with a subsequence of odd integers and a subsequence of just even integers, ending at the odd starting value \( n \).

Steiner[10] proved that for \( q = 3 \), the only cycle that is also a circuit is the only known cycle \((1,2)\). In [9], similar results where proved for the cases \( q = 5 \) and \( q = 7 \).

Building on these results we will derive a general formula for the Diophantine equations associated with the cyclic circuits for an arbitrary odd \( q \).

**Theorem 4.2.** The dynamical system defined by (7) has a cyclic-circuit \( T^Z(n) = n \) only if there exist \( \ell, k, h \in \mathbb{N} \) satisfying the following equation

\[
(2^k - q^k)h = 2^\ell - 1.
\]

**Proof.** Let \( \{n_i\}_{i=0}^k \) be the sequence of odd integers in the increasing component of a circuit, then we have

\[
qn_0 + 1 = 2n_1 \\
qn_1 + 1 = 2n_2 \\
\vdots \\
qn_{k-1} + 1 = 2n_k.
\]

Solving this recursion we end up with

\[
(9) \quad 2^k n_k = q^k n_0 + \sum_{i=0}^{k-1} q^{k-1-i}2^i.
\]

Using the identity

\[
x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})
\]

we can rewrite the sum in (9) as

\[
\sum_{i=0}^{k-1} q^{k-1-i}2^i = \frac{q^k - 2^k}{q - 2}.
\]

Setting \( n_0 = n, n_k = m \) and simplifying we get

\[
2^k(m(q - 2) + 1) = q^k(n(q - 2) + 1),
\]

which implies that

\[
(10) \quad m(q - 2) + 1 = q^kh, \\
n(q - 2) + 1 = 2^kh.
\]
must hold for some $h$.

Setting $m = 2^h n$ and solving we end up with the equation

$$(2^{k+\ell} - q^k)h = 2^\ell - 1.$$ \hfill \Box

Remark 4.3. As Ray Steiner noted in [9], the converse of this Theorem is not necessarily true. For example, consider the case $q = 9$. We have

$$(2^{k+\ell} - q^k)h = 2^\ell - 1$$

with a solution $k = 1, \ell = 3, h = 1$, but which does not yield integer values for $m$ and $n$.

Continuing with this example we can in fact show that this is the only solution.

**Theorem 4.4.** The only solution to the equation

$$(2^{k+\ell} - q^k)h = 2^\ell - 1$$

with $k, \ell, h \in \mathbb{N}$, is $k = 1, \ell = 3, h = 1$.

Before we prove this we need some results from number theory. The following deep Theorem is due to A. Baker [1]

**Theorem 4.5.** If $a_1, \ldots, a_n \geq 2$ are non-zero algebraic numbers with degrees and heights at most $d(\geq 4)$ and $A(\geq 4)$ respectively, and if there exist rational integers $b_1, \ldots, b_n$ with absolute values at most $B$, such that

$$0 < |b_1 \log(a_1) + \cdots + b_n \log(a_n)| < e^{-\delta B}$$

where $0 < \delta \leq 1$, and the logarithms have their principal values, then

$$B < (4^{n^2} \delta^{-1} d^{2n} \log(A))^{(2n+1)^2}.$$  

The second result we will need is the following of Legendre – which we will prove.

**Theorem 4.6.** Let $\theta$ be a real number, $p_n/q_n$ a convergent in its continued fraction expansion, and $a_{n+1}$ the corresponding partial quotient. Then

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left| \frac{\theta - p_n}{q_n} \right|.$$  

**Proof.** We have that

$$\left| \frac{\theta - p_n}{q_n} \right| > \left| \frac{p_{n+2} - p_n}{q_{n+2} - q_n} \right| = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}}.$$  

and we want to show that

$$\frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} > \frac{1}{(a_{n+1} + 2)q_n^2}.$$  

For this purpose,

$$(q_{n+2} - q_n)(a_{n+1} + 2)q_n > q_{n+2}q_{n+1} \iff (q_{n+2} - q_n)(q_{n+1} - q_{n-1} + 2q_n) > q_{n+2}q_{n+1},$$
where we used that \( q_{n+1} = a_{n+1}q_n + q_{n-1} \). Expanding and moving terms with negative coefficients to the right hand side and substituting \( q_{n+2} = a_{n+2}q_{n+1} + q_n \) and \( q_{n+1} = a_{n+1}q_n + q_{n-1} \) we get,

\[
2q_n(a_{n+2}q_{n+1} + q_n) + q_nq_{n-1} > q_{n+2}q_{n+1} + q_nq_{n-1} + 2q_n^2
\]

\[
\Leftrightarrow 2q_n(a_{n+2}(a_{n+1}q_n + q_{n-1}) + q_n) + q_nq_{n-1} > (a_{n+2}(a_{n+1}q_n + q_{n-1}) + q_n)q_{n-1} + (a_{n+1}q_n + q_{n-1})q_n + 2q_n^2.
\]

Comparing coefficients for \( q_n^2, q_{n-1}^2, q_nq_{n-1} \) and rearranging, the Theorem amounts to proving that the quadratic form below is positive for \( a_{n+1}, a_{n+2} \geq 1 \) and \( q_n > q_{n-1} \geq 1 \).

\[
Q(q_n, q_{n-1}) = (2a_{n+2} - 1)a_{n+1}q_n^2 - (a_{n+2}(a_{n+1} - 2) + 1)q_nq_{n-1} - a_{n+2}q_{n-1}^2.
\]

To this end; dividing both sides by \( q_{n-1}^2 \) and setting \( z = q_n/q_{n-1} \), we want to show that

\[
f(z) = (2a_{n+2} - 1)a_{n+1}z^2 - (a_{n+2}(a_{n+1} - 2) + 1)z - a_{n+2}
\]

is positive for \( z > 1 \). Setting \( z = 1 \), factoring yields,

\[
(a_{n+2} - 1)(a_{n+1} + 1) \geq 0,
\]

which is clearly true since \( a_{n+1}, a_{n+2} \geq 1 \). To get strict inequality, note that \( z = \frac{q_n}{q_{n-1}} > 1 \) for all \( n > 1 \). As \( f'(1) > 0 \) and the coefficient for \( z^2 \) is positive, \( f(z) \) is increasing and strict inequality follows.

We now prove Theorem 4.4

**Proof.** The proof will follow a similar procedure as in [9]. Essentially it boils down to reducing the inequality (16), to a linear form in logarithms, and then applying Theorem 4.5.

For \( k < 4 \) the only solution is the proposed \( k = 1, \ell = 3, h = 1 \), so we look at \( k \geq 4 \). We have that

\[
0 < 2^{\ell+k} - 9^k \leq 2^\ell - 1.
\]

\[
0 < 2^{\ell+k} - 9^k \leq 2^\ell - 1 < 2^\ell
\]

\[
\Rightarrow 0 < 2^{\ell+k} - 2^\ell < 9^k.
\]

Dividing the last two members by \( 2^{k+\ell} \) and taking logarithms, we get

\[
\log(\frac{2^k - 1}{2^k}) < \log(\frac{9^k}{2^{\ell+k}})
\]

\[
\Leftrightarrow (\ell + k)\log(2) - k\log(9) < \log(\frac{2^k}{2^{\ell+k}} - 1)
\]

Using the fact that

\[
\log(\frac{2^k}{2^{\ell+k} - 1}) < \frac{1}{2^k - 1} \text{ for } k \geq 1,
\]

we get,

\[
(k + \ell)\log(2) - k\log(9) < \frac{1}{2^k - 1}.
\]
Back to (16) we have,
\begin{equation}
9^k < 2^{\ell+k} \Rightarrow k \log(9) < (\ell + k) \log(2).
\end{equation}

Combined these yield,
\begin{equation}
0 < (k + \ell) \log(2) - k \log(9) < \frac{1}{2^k - 1}
\end{equation}

\[(21)\]

As \(\log(2)(2^k - 1) > 2^k\) for \(k \geq 4\) we see that \(\ell/k\) must be a convergent in the continued fraction expansion of \(\log_2(9/2)\) (See e.g. Theorem 15.9 in [2]). With the aid of Maple we find the first 8 convergents
\begin{equation}
(22)\quad 2, 11, 5, 13, 102, 6, 115, 332, 1443, 16205.
\end{equation}

which for \(k > 4\), are easily seen to not satisfy (21) or solve (11), and so we assume \(k > 7468\).

Next step is to derive a lower bound for the partial quotients of the continued fraction expansion of \(\log_2(9/2)\). Using \(\theta = \log_2(9/2)\), \(p_n = \ell\) and \(q_n = k\) in Theorem 4.6, we have
\begin{equation}
\frac{1}{(a_{n+1} + 2)k^2} < \left| \frac{\log_2(9/2) - \ell}{k} \right| < \frac{1}{k \log(2)(2^k - 1)}.
\end{equation}

Thus,
\begin{equation}
a_{n+1} > \frac{2^k - 1}{k} \log(2) - 2 > \frac{2^{7468} - 1}{7468} \log(2) - 2 > 10^{2244},
\end{equation}

which means that any further solutions must correspond to extremely large partial quotients. Lastly, we tie everything together by finding an upper bound for the partial quotients of the continued fraction expansion of \(\log_2(9/2)\). Using \(\theta = \log_2(9/2)\), \(p_n = \ell\) and \(q_n = k\) in Theorem 4.6, we have
\begin{equation}
0 < |(\ell + k) \log(2) - k \log(9)| < \frac{1}{2^k - 1} < e^{-0.001B}.
\end{equation}

And so we are now in a position to apply Theorem 4.5, with \(\delta = 0.001\), \(n = 2\), \(d = 4\) and \(A = 9\). With the degree of an algebraic number \(a\) being defined as usual, and the height of \(a\) is given by
\begin{equation}
(27)\quad H(a) = \max(\{|b_1|, |b_2|, ..., |b_d|\}),
\end{equation}

where \(\{b_i\}_{i=1}^d\) are the coefficients of the minimal polynomial of \(a\). Note that the degree of any rational integer will always be 1, but we put it to 4 as per instruction in Theorem 4.5. This gives us
\begin{equation}
(28)\quad B = \ell + k < (4^4 \cdot 10^3 \cdot 4^4 \cdot \log(9))^{25} < 10^{175}
\end{equation}

and hence \(k < 10^{175}\). Now it’s just a matter of computing the continued fraction expansion of \(\log_2(9/2)\) until \(q_n\) exceeds 10^{175}. If none of the partial quotients is larger than 10^{2244} for \(q_n < 10^{175}\), we are done – the proof is finished. We find that after computing the first 400 partial quotients, \(q_n\) exceeds the bound. The maximum of these partial quotients turn out to be 1928 – not even close. \(\square\)
**Proposition 4.7.** Let \( q \) be of the form \( q = 2^d - 1 \). Then the equation

\[
(2^{k+\ell} - q^k)h = 2^\ell - 1,
\]

has a solution \( \ell = d - k, h = 2^\ell - 1, k = 1 \), which is a cyclic circuit in its corresponding dynamical system.

**Proof.** It's straightforward to verify that the proposed values, \( \ell = d - k, h = 2^\ell - 1, k = 1 \), indeed constitutes a solution. For the second part, solving for \( n \) in (10), with \( m = 2^\ell n \) and the rest as before, the first equation yields

\[
n = \frac{2^d - 3}{2d - 3} = 1,
\]

which means that the solution corresponds to the cyclic circuit starting at \( n = 1 \). \( \square \)

5. **Polynomial analogue**

In this section we will briefly discuss the “Polynomial 3n+1”, as defined in [5], for polynomial rings over finite fields. In particular, a detailed proof of Theorem 5.2 in [5], will be provided. As alluded to in the introduction; a common approach when dealing with problems of apparently intractable nature, is to translate it to as many different settings as possible and try to solve analogous and related problems there. For example, both the Goldbach and the twin prime conjecture have analogues for polynomial rings over finite fields [4], though both are still unsolved there. The polynomial ring \( F_q[x] \) is a natural setting to create analogues for number theoretic problems. Indeed, the integer ring \( \mathbb{Z} \) and \( F_q[x] \) have a lot in common, for example, the density of irreducibles are about the same (for a deeper discussion on their similarities see [4]).

**Polynomial 3n+1:**

Let \( F \) be a field and let \( f(x) \in F[x], f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \). Then the polynomial 3n+1 is defined by:

\[
C_F(f(x)) := \begin{cases} 
(x - \frac{a_0}{a_j})f(x) + \frac{a_0^2}{a_j} & \text{if } f(0) \neq 0 \\
\frac{f(x)}{x} & \text{if } f(0) = 0,
\end{cases}
\]

where \( j \) is the smallest value so that \( 1 \leq j \leq n \) and \( a_j \neq 0 \).

The case \( F_2 \), which is the most similar to the integer version, is proved in [5]. We now prove the general case.

**Theorem 5.1.** For any monic polynomial \( f(x) \) of degree \( n \geq 1 \) over \( F \), \( C_F^{(I)}(f(x)) = 1 \) for some \( I \leq n^2 + 2n \).

**Proof.** We proceed by induction on the degree \( n \). For \( n = 1 \), \( f(x) = x + a_0 \). If \( a_0 = 0 \) we divide by \( x \) and are done. For \( a_0 \neq 0 \) one iteration yields,

\[
C_F((x + a_0) = (x - a_0)(x + a_0) + a_0^2 = x^2,
\]

which after dividing by \( x \) twice ends at 1. Now, assume the result holds for any monic polynomial \( f(x) \in F[x] \) of degree \( n - 1 \). Let \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \), if \( a_0 = 0 \) we iterate once and obtain a monic polynomial of degree \( n - 1 \), which by our induction
hypothesis converge to 1 in at most \((n - 1)^2 + 2(n - 1)\) iterations. If \(a_0 \neq 0\) iterating once we get a polynomial of the form

\[
  f(x) = x^{n+1} + a_{n-1}x^n + \cdots + a_0x - a_0a_j^{-1}x^n - \cdots - a_0x^j - a_0a_j^{-1} + a_0^{-1}
\]

Iterating once more we obtain a polynomial of degree \(n\) where the exponent of the first \(a_j\) such that \(a_j \neq 0\), has now decreased by one. Applying this pair of iterations \(2(j - 1)\) times we get,

\[
  g(x) = x^n + \cdots + (-)a_0x + a_0
\]

Another iteration yields a polynomial of degree \(n + 1\) which is divisible by \(x^2\). After iterating two more times we get a polynomial of degree \(n - 1\), which converge to 1 in at most \((n - 1)^2 + 2(n - 1)\) steps (by induction). At this point we have used \(I \leq 3 + 2(j - 1) + (n - 1)^2 + 2(n - 1) = n^2 + 2j\) steps. As \(1 \leq j \leq n\), \(I \leq n^2 + 2n\) and we are done. \(\square\)

References