Active Control of Combustion Instabilities

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Abstract
Modern gas turbines use lean premixed combustion to achieve the best compromise between emissions and efficiency. This type of combustion is very sensitive to thermoacoustic instabilities causing high pressure sound waves. One reason for these intense pressure fluctuations is the flame acting as source of sound, that is placed in an acoustic resonator accumulating acoustic energy. Due to the reflection of sound at the ends standing waves occur with very high amplitudes. The characteristics of these standing waves are determined by the design of the combustor and the speed of sound through the gas. They basically correspond to harmonics of an organ-pipe and define the instability frequency. The high acoustic pressure can lead to higher emissions and structural damage.

Suppressing the acoustic fluctuations is critical for efficient and long lasting use of a gas turbine and this thesis focuses on an active control strategy. The basic idea is to excite external sound waves in the combustion system, achieving sound canceling. A feedback control loop is applied where the controller is connected to a valve modulating the fuel to the burner. The varying fuel flow excites sound waves through the thermoacoustic coupling of heat release and acoustic pressure. By measuring the acoustic pressure in the combustor with a watercooled microphone, amplitude and phase can be registered. This pressure signal is used by the controller that determines which sound waves to be excited in the combustion chamber. The controllers were derived by using a mathematically complex control strategy called H-infinity optimization. They were optimized based on an analytical acoustic model and then tested on a combustion test-rig. The results of the optimal controllers were very encouraging since the acoustic pressure was reduced to a fraction of the non-controlled system.

Certain characteristics of the optimized controllers were observed and a simple controller overtaking only the characteristics around the instability frequency was designed. This type of controller not based on a mathematical model, with parameters obtained using a simple system identification technique at the instability frequency, also efficiently reduced the acoustic pressure without increasing any pollutants.
Active Control of
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by

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Nomenclature

\( \rho \)  
\( c \)  
\( p \)  
\( p' \)  
\( \hat{p} \)  
\( v \)  
\( v' \)  
\( \hat{v} \)  
\( \omega \)  
\( k \)  
\( M \)  
\( SISO \)  
\( LTI \)  
\( K(s) \)  
\( P(s) \)  
\( L(s) \)  
\( S_0(s) \)  
\( K_\infty(s) \)  
\( UHC \)  
\( CO \)  
\( NOx \)  
\( \mathcal{R} \)  
\( \in \)  
\( \text{Re}(\cdot) \)  
\( I_n \)  
\( A^T \)  
\( A^{-1} \)  
\( \lambda(A) \)  
\( \sigma_i(A) \)  
\( \mathcal{F}(\cdot, \cdot) \)

\( \rho \) density  
\( c \) speed of sound  
\( p \) pressure measurement  
\( p' \) pressure fluctuations  
\( \hat{p} \) Fourier Transform of \( p(t) \) (frequency domain)  
\( v \) velocity measurement  
\( v' \) velocity fluctuations  
\( \hat{v} \) Fourier Transform of \( v(t) \) (frequency domain)  
\( \omega \) Frequency (Hz)  
\( k \) Wave number  
\( M \) Mach number  
\( SISO \) Single Input Single Output System  
\( LTI \) Linear Time Invariant  
\( K(s) \) controller  
\( P(s) \) plant  
\( L(s) \) Loop transfer function  
\( S_0(s) \) Sensitivity function  
\( K_\infty(s) \) controller resulting from \( H_\infty \) optimization  
\( UHC \) Unburned Hydro Carbons  
\( CO \) Carbon Monoxide  
\( NOx \) Nitrogen Oxides  
\( \mathcal{R} \) field of real numbers  
\( \in \) belong to  
\( \text{Re}(\cdot) \) real part of a complex number  
\( I_n \) \( n \times n \) identity matrix  
\( A^T \) transpose of \( A \)  
\( A^{-1} \) inverse of \( A \)  
\( \lambda(A) \) eigenvalue of \( A \)  
\( \sigma_i(A) \) \( i \)th singular value of \( A \)  
\( \mathcal{F}(\cdot, \cdot) \) Linear Fractional Transform
Chapter 1

Introduction

The most interesting specifications for an energy producing plant are efficiency and emission levels, where the efficiency of a gas turbine increases with higher inlet temperature of the burner. But just increasing the inlet temperature also increases the levels of pollution, where the main concern is the \( NO_x \)-levels produced by the turbine. The optimal trade-off between \( NO_x \) and efficiency is achieved by using lean premixed combustion, but this has encountered a drawback concerning the thermoacoustic behaviour in the combustion chamber. Periodic vibrations in large combustion systems are undesirable and even dangerous because of the intense sound, loss of flame stability and destruction of metal parts. Usually an enclosed combustion chamber will have characteristic acoustic frequencies which are determined by its dimensions and the speed of sound. The fluctuations are caused by the flame that are placed in an acoustic resonator, which accumulates acoustic energy. Due to reflections at the ends, standing waves occur (figure 1) in the first combustion chamber. Typically the instabilities follows from exciting one frequency corresponding to a harmonic of the system.

The acoustic system can be seen as a feedback system where the standing waves are thermoacoustically coupled to the flame, producing periodic heat release. This influences the amplitude and velocity of the waves and under unfavourable conditions combustion instabilities can occur in the feedback loop, which leads to strong oscillations giving intense sound. Various types of acoustic oscillations may occur. The simplest is the lowest mode of the fundamental organ-pipe mode of a long combustion tube.

The work on suppressing sound waves are divided into passive and active control. Passive control is accomplished through re-designing the geometry of the parts belonging to the turbine, such as the chamber and the burner. Another possibility is adding acoustic dampers like Helmholtz resonators. The strategies are working well for predefined frequencies, but often causes high amplitudes at other frequencies. Active control uses a feedback strategy by measuring the pressure in the
combustor and then a controller feeds a linear fuel-valve for modulating additional fuel into the system. By varying the fuel a change in the heat release is obtained causing sound waves. These sound waves are modulated to cancel out sound waves in the combustion chamber.

The control synthesis follows from modern robust control techniques, where the plant model is important though not precise. The characteristics of the combustion process with one large characteristic main peak makes the concept of $H_\infty$-optimization interesting. This technique optimizes a controller to damp the highest magnitude of the system as much as possible.

Figure 1.1: Plan of real gasturbine.
Chapter 2

Analytical Model

In this chapter a description of how a MATLAB model for the acoustic pressure in the combustion chamber was derived. First of all the ideas behind the model and its acoustic modeling will be discussed and afterwards the acoustic interference of the different parts involved are derived and the model is set up. Due to the models size a reduction of the order is done for conveniency, which completes the chapter.

1 Acoustic Modeling

The combustion system to be modeled can be seen in figure 2.1. The part

left of the burner is called the upstream part and the right part is called the downstream part.

The acoustic fluctuations are described through the Riemann-invariants $f$ and $g$, which are mathematical definitions convenient for describing two
CHAPTER 2. ANALYTICAL MODEL

waves traveling in opposite direction. (Appendix A gives a detailed background of the Riemann-invariants.) From the $f$ and $g$ waves the acoustic pressure $p$ and velocity $v$ can be calculated through superposition of waves. The equations A.48 and A.49 in time-domain are repeated here:

\[
\frac{p_j'(t)}{\rho c} = f_j(t) + g_j(t)
\]

\[
v_j'(t) = f_j(t) - g_j(t)
\]

where the prime denotes fluctuations.

In the burner of figure 2.1, fuel injection takes place along the whole length of the burner. To model this in a realistic way, the model should be lumped in to a large number of little injection nozzles. However, for reasons of simplicity it will be assumed that all fuel injection takes place at one location: The burner exit. The flame is approximated at a fixed location in a duct. The heat release in the flame is assumed to be proportional to the amount of fuel which flows through the cross-section of the duct at the flame location.

2 Derivation of model

When deriving a model for the acoustics within a gas turbine it helps to look at the system as a network of acoustic elements\(^1\) (figure 2.2) connected through pressure and velocity or Riemann-invariants. The arrows

\[\begin{array}{c}
f_2 \\ g_2
\end{array} \quad \text{Acoustic element} \quad \begin{array}{c}
f_1 \\ g_1
\end{array} \quad \begin{array}{c}
p_2' \\ v_2'
\end{array} \quad \text{Acoustic element} \quad \begin{array}{c}
p_1' \\ v_1'
\end{array}\]

Figure 2.2: Elements relating Riemann-invariants or pressure and velocity fluctuations on both sides of the element.

in 2.2 show the direction in which the $f$ and $g$ waves travel. Equation 2.1 shows that the upstream and downstream quantities are relating Riemann-invariants, which is described as the scattering matrix. Whereas equation 2.2 relates pressure and velocity.

\[
\begin{pmatrix}
f_2 \\ g_2
\end{pmatrix} = \begin{bmatrix}
s_{11} & s_{12} \\ s_{21} & s_{22}
\end{bmatrix} \begin{pmatrix}
f_1 \\ g_1
\end{pmatrix}
\]

Equation 2.1

\(^1\)Acoustic elements are here referred to as specific parts of the combustion system that relates pressure and velocity or Riemann-invariants on both sides of the element, e.g. Straight tube, curved duct, area change, burner, flame etc.
2. **DERIVATION OF MODEL**

\[
\begin{pmatrix}
  p_1' \\
  v_2'
\end{pmatrix} =
\begin{bmatrix}
  T_{11} & T_{12} \\
  T_{21} & T_{22}
\end{bmatrix}
\begin{pmatrix}
  p_1' \\
  v_1'
\end{pmatrix}
\]  

(2.2)

These equations give relations between two quantities on each side of the element and therefore the relation is given by a 2x2 matrix. A simple example of how such a network can look for a gas turbine is shown in figure 2.3.

![Diagram of a combustion system](image)

Figure 2.3: A simple combustion system represented as network of acoustic elements.

A network is built in SIMULINK by putting the transfer matrices for each element together, giving a system of algebraic equations. The equations are then solved to obtain acoustic pressure and velocity at certain locations. The network model presented here is derived by Schuermans [1] and Gstöhl [2] and can be seen in appendix C.

**2.1 Duct**

The Duct represents the wave propagation through upstream and downstream parts of the combustion system and the model is simplified to only one dimension. Then, by using equations A.40 and A.41 the duct is described by

\[
\begin{pmatrix}
  f_d \\
  g_u
\end{pmatrix} =
\begin{bmatrix}
  e^{-ikL} & 0 \\
  0 & e^{-ikL}
\end{bmatrix}
\begin{pmatrix}
  f_u \\
  g_d
\end{pmatrix}
\]  

(2.3)

where \(k\) is the wavenumber and \(L\) the length of the duct. This is easily modeled by a time delay in MATLAB, which corresponds to the propagation of the wave.

**2.2 Burner**

The model is obtained by considering a burner without combustion and model the flame as a separate element, see section 2.3. The main principles of the \(L - \zeta\) burner model is derived by Paschereit and Polifke [12] and is based on non-steady linearized Bernouilly equations, with the result of a transfer function \(H_{burner}\) with low pass character.

The term *arearatio* is introduced because of the shape of the burner. The transfer function is measured at different reference points where the areas
Figure 2.4: Structure of submodels Duct Up (left) and Duct Down (right) in MATLAB.

Figure 2.5: Structure of submodel Burner in MATLAB.

varies, which can be seen in figure 2.6. The arearatio is easily calculated through \( \text{arearatio} = \frac{A1}{A2} \).

2.3 Flame

The flame is a very important element in the thermoacoustic network, since it is responsible for the thermoacoustic interactions rendering the system unstable. The flame contributes to the system in two essentially different ways. 1) The system acts as an independent source of sound. This source of acoustic energy is caused by the turbulent flow and due to irregularities in direction and speed of movement of the flame front. 2) The flame couples with the acoustic system. This coupling is generally a very complex mechanism involving periodic heat release caused by an interaction of mixture inhomogeneties, periodic vortex shedding and entropy waves.

A thermoacoustic model has been made of the flame for the burner type under consideration. The model relates the mixture inhomogeneities to the acoustic perturbations. It is assumed that fuel is injected at a constant rate in the burner. Acoustic fluctuations result in a fluctuating mass flow of air in the burner this will result in a periodic change of the fuel-to air ratio of the mixture. These "pockets" of fuel rich or fuel-lean mixture will be convected downstream and will arrive at the flame after a certain time-delay \( \tau \). The
mixture variation will result in periodic heat release in the flame. By substituting this relation between acoustic velocity fluctuation and heat release in the linearized Rankine-Hugoniot relations, the acoustic quantities on both sides of the flame can be related. Comparison of this model with measured transfer functions of the flame showed very good agreement ([1],[3]).

2.4 End Up and End Down

There is no accurate analytical model available for End Up and End Down and therefore they have been measured through frequency power spectras. By fitting an analog filter it is possible to translate the measurements to a state-space model. Physically the ends are nothing but reflecting elements, where End Up and End Down respectively represents a closed and an open end. The transfer functions are designated $H_{up}$ and $H_{down}$.
2.5 Rewrite

Since it is convenient to relate the duct with Riemann-invariants and the other elements with pressure and velocity it is necessary to be able to convert between the quantities. The conversion follows through the equations 2.4 and 2.5, which can be derived from A.48 and A.49:

\[
\begin{pmatrix}
  p'_1 \\
  g'_1
\end{pmatrix} =
\begin{bmatrix}
  2\rho_1 c_1 & -\rho_1 c_1 \\
  1 & -1
\end{bmatrix}
\begin{pmatrix}
  f_1 \\
  v'_1
\end{pmatrix} \tag{2.4}
\]

\[
\begin{pmatrix}
  p'_3 \\
  f'_3
\end{pmatrix} =
\begin{bmatrix}
  2\rho_3 c_3 & \rho_3 c_3 \\
  1 & 1
\end{bmatrix}
\begin{pmatrix}
  g_3 \\
  v'_3
\end{pmatrix} \tag{2.5}
\]

where \( p' \) and \( v' \) denotes the fluctuating parts of pressure and velocity respectively, whereas \( f \) and \( g \) denotes the upwards and downwards traveling Riemann-invariants. This leads to the following models in MATLAB:

\[\text{Figure 2.8: Structure of submodels Rewrite Up (left) and Rewrite Down (right) in MATLAB.}\]

3 Model reduction

For analysis and synthesis of a model or controller it is always desired to have a simple description of the characteristics. Especially since hardware often can not realize complex systems it is preferable with low order systems, but also as a time-saving measure is an order reduction often a useful tool. The analytical model contains several time-delays which are described through Padé-approximations. This enables analysis in the frequency-domain but increases the order of the system to 65. Which order it will be possible to approximate the model with is decided through the amount of loss of information that the reduction causes. In this case balanced truncation has been used for reducing the order of the model.

3.1 Balanced truncation

The idea behind this reduction method is to find out which states of the system that are dominating the behaviour of the system and just let them
3. MODEL REDUCTION

describe the model, without loosing to much information of the system. The problem is to describe the state-space realization in such a way that it becomes obvious which states that can be truncated. By looking at which states are less controllable and less observable is it possible to discern the dominating states from the non-influencing states. The state-space model is described through:

\[
G = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

When deciding whether the system is controllable or observable are the respective gramians \( P \) and \( Q \) of use, which are based on solving the Lyapunov equations:

\[
AP + PA^T + BB^T = 0, \quad P > 0 \quad (2.6)
\]

\[
A^TQ + QA + C^TC = 0, \quad Q > 0 \quad (2.7)
\]

When the gramian matrices are positive definite is the system controllable and observable. The problem is that the gramians alone cannot find the dominance of states in an input/output-behaviour and therefore balanced gramians are introduced. Transforming the state-space realization with a nonsingular matrix \( T \) to \( \tilde{x} = Tx \) yields the new realization

\[
G = \begin{bmatrix}
\tilde{A} & \tilde{B} \\
C & D
\end{bmatrix} = \begin{bmatrix}
TAT^{-1} & TB \\
CT^{-1} & D
\end{bmatrix}
\]

The gramians are then transformed to \( \tilde{P} = TPT^T \) and \( \tilde{Q} = (T^{-1})^TQT^{-1} \). The product of the gramians are transformed to \( \tilde{P}\tilde{Q} = TPQT^{-1} \), which shows that the eigenvalues of the product of the gramians are transformation invariant. It is especially interesting to consider the case when the transformation matrix consists of the eigenvectors of \( PQ \), since that transformation decomposes the system to a diagonal realization.

\[
PQ = T^{-1}\Lambda T, \quad \Lambda = \text{diag}(\lambda_1I_{s_1}, ..., \lambda_nI_{s_m}) \quad (2.8)
\]

The columns of \( T^{-1} \) are the eigenvectors of \( PQ \) and each one corresponds to an eigenvalue \( \lambda_i \) and \( s_i \) denotes the multiplicity. The eigenvectors can be chosen such that

\[
\tilde{P} = TPT^T = \Sigma \quad (2.9)
\]

\[
\tilde{Q} = (T^{-1})^TQT^{-1} = \Sigma \quad (2.10)
\]

where \( \Sigma = \text{diag}(\sigma_1I_{s_1}, ..., \sigma_nI_{s_m}) \) and \( \Sigma^2 = \Lambda \). With this transformation we have now reached a balanced realization, where the new gramians are
\( \tilde{P} = \tilde{Q} = \Sigma \). The use of this way of writing becomes clear when you realize that the Hankel singular values \( \sigma_i \) are decreasingly ordered numbers, \( \sigma_1 > \sigma_2 > \ldots > \sigma_n \). The reduction of order is now an easy task since small Hankel singular values means that those states are both less controllable and less observable than the higher values.

**Theorem 1** Suppose that \( G(s) \) is a stable and minimum realization of order \( n \), and

\[
G(s) = \begin{bmatrix}
A_{11} & A_{12} & B_1 \\
A_{21} & A_{22} & B_2 \\
C_1 & C_2 & D
\end{bmatrix}
\]

is a balanced realization with Gramian \( \Sigma = \text{diag}(\Sigma_1, \Sigma_2) \)

\[
\Sigma_1 = \text{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r})
\]

\[
\Sigma_2 = \text{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_N I_{s_N})
\]

and

\[\sigma_1 > \sigma_2 > \cdots > \sigma_r > \sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_N\]

where \( \sigma_i \) has multiplicity \( s_i, i = 1, 2, \ldots, N \) and \( s_1 + s_2 + \cdots + s_N = n \). Then the truncated system

\[
G_r(s) = \begin{bmatrix}
A_{11} & B_1 \\
C_1 & D
\end{bmatrix}
\]

is balanced and stable. Furthermore,

\[\|G(s) - G_r(s)\|_\infty \leq 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_N)\]

So, according with this theorem a good approximation in the frequency domain can be obtained if some singular values are small enough to be neglected.

4 Reduction of analytical model

The analytical model contains 65 states and is not very convinient to work with. The balanced reduction method is applied to reduce the order without losing too much information. The Hankel singular values are decreasing with almost constant steps and therefore it is difficult to reduce the model to a certain value without testing the approximations. It was seen that an approximation around 20 states, as in figure 2.9, approximates the characteristics of the main harmonics well. Further reduction causes problem when deriving controllers, since they do not stabilize the original model.
5. RELATED MATLAB COMMANDS

![Bode diagram of original model vs. reduced model](image)

Figure 2.9: Bode diagram of original model vs. reduced model

5 Related MATLAB commands

**sysb = BALREAL(sys)** returns a balanced state-space realization of the reachable, observable, stable system SYS.

**[sys,sig] = SYSBAL(sys,tol)** Finds a truncated balanced realization SYSb of the stable system state-space model SYS, with the Hankel singular values represented in SIG. Can not handle unstable systems.

**sysred = HANKMR(sys,sig,order)** Performs an optimal Hankel norm approximation on the state-space system SYS. The input system SYS must be a balanced realization with Hankel singular values SIG (i.e. SYS and SIG must be the output of SYSBAL).
Chapter 3

Robust Control

Modeling of a physical system is always afflicted with errors, which either occurs because of simplifications or of hidden dynamics in the system. Awareness that these uncertainties exists and can effect the performance of a system is often crucial. When deriving a controller for an uncertain plant it is important that the specifications for all plants within the uncertainties are met. A controller that manages to internally stabilize all the perturbed plants describing the system and meet the performance criterias are called robust. The main points in this chapter is to discuss important aspects of robust control, beginning with how to introduce uncertainties to a model and then robust stability and performance are described.

1 Model Uncertainty

The main idea behind including uncertainty of a mathematical model is to find a description that holds the information of all possible system behaviours. Two common ways of doing this is structured and unstructured uncertainties, which will be treated in this section.

1.1 Structured Uncertainty

A basic technique is to model the plant as belonging to a set $\mathcal{P}$. Such a set can either contain a number of discrete plants or it is explicitly parametrized by scalar parameters, like the second order system below:

$$
\mathcal{P} = \left\{ \frac{1}{s^2 + as + 1} : a_{\text{min}} \leq a \leq a_{\text{max}} \right\} \tag{3.1}
$$

An example of a bode diagram for some of the perturbed plants within the set $\mathcal{P}$ of the second order system in equation 3.1 with $0.4 \leq a \leq 0.8$ is plotted in figure 3.1.
Figure 3.1: Bode diagram for a second order system with structured uncertainty

The parameters of a transfer model are often not enough to satisfy our demands. It works well for systems where the uncertainty can be related to explicit variables of a system. The disadvantage with this kind of representation is the difficulties in modeling hidden dynamics of a plant, which the unstructured representation manages.

1.2 Unstructured Uncertainty

The unstructured approach uses frequency domain bounds on transfer functions to describe a set of models $\mathcal{P}$. Outgoing from a nominal model $P_0$ is every point on the frequency curve afflicted with a frequency dependent uncertainty. This uncertainty is for simplicity said to be disk-like with the center on the nominal model. The Nyquist curve in figure 3.2 shows the principal idea.

Outgoing from the nominal model can the disklike uncertainty be represented by $\Delta W$ where $W(jw)$ is a fixed stable transfer function, the weight, and $\Delta(jw)$ is a variable stable transfer function satisfying $\| \Delta(jw) \|_\infty < 1$.

The common uncertainty models are summarized below

\begin{align*}
  P &= P_0(1 + \Delta W) \\
  P &= P_0 + \Delta W \\
  P &= \frac{P_0}{1 + \Delta WP_0} \\
  P &= \frac{P_0}{1 + \Delta W}
\end{align*}

The uncertainty models have different ways of use depending on the type of uncertainty that should be described. For convenience only one model will
be discussed and the focus is on the *multiplicative uncertainty* in equation 3.2.

## 2 Robust Stability

The stability test of a stable nominal system with unstructured perturbations is based on the *small gain theorem*, Theorem 2, derived for an interconnected system as figure 3.3 shows. A system is *robustly stable* when the controller achieves internal stability for all perturbed plants.

![Figure 3.3: Loop for stability analysis](image)

Theorem 2 *Suppose that* $M(s)$ *is a real-rational stable transfer function*
2. **ROBUST STABILITY**

and let $\gamma > 0$. Then the interconnected system shown in Figure 2 is internally stable for all real-rational stable $\Delta(s)$ with

1. $\|\Delta\|_{\infty} \leq 1/\gamma$ if and only if $\|M\|_{\infty} < \gamma$

2. $\|\Delta\|_{\infty} < 1/\gamma$ if and only if $\|M\|_{\infty} \leq \gamma$

Therefore, if $M(s)$, in Figure 3.3, represents the interconnection between plant and controller. A controller designed to maximize uncertainty or perturbation tolerance while keeping stability has to minimize $\|M\|_{\infty}$.

The proof of Theorem 2 is easy when considering the Nyquist criterion. Since $\Delta$ and $M$ both are stable transfer functions with no poles in the right half-plane or on the imaginary axis, the system is stable if the frequency curve does not encircle $-1$. The limitations of the transfer function norm through $\gamma$ ensures that the magnitude is never larger then 1 and therefore $-1$ cannot be encircled.

Even though the small gain theorem uses a simple system when defining the stability-criterium, is it possible to derive powerful robust stability tests for all the uncertainty models 3.2-3.5. For instance is the stability test for the multiplicative perturbation model with $L$ representing the loop transfer function

$$\left\| \frac{W_2 L}{1 + L} \right\|_{\infty} \leq 1$$

(3.6)

which in practice means that robust stability is achieved when the input signal is damped for all frequencies, which results in a bode magnitude curve that is never larger than zero. It is easy to show the validity of this result when considering the perturbed feedback system in figure 3.4.

![Figure 3.4: Perturbed feedback system.](image-url)
The transfer function from the output of $\Delta$, $z$, to the input of $\Delta$, $w$, equals $W_2L/(1 + L)$, where $L := PK$. The system then reduces to a system of the form in figure 3.3 with $M := W_2L/(1 + L)$. Since $\|\Delta\|_\infty < 1$, the limiting constant is $\gamma = 1$ and $\|M\|_\infty$ must be $\leq 1$. Therefore the stability criterion given in 3.6 for the multiplicative model is concluded.

## 3 Robust Performance

The general notion of robust performance is that internal stability and some pre-specified performance criterion should hold for all perturbed plants in a set $\mathcal{P}$. The performance specifications are to be chosen among a lot of possibilities of the control designer, i.e. sensitivity, tracking, steady-state error etc., and the latter case will be considered here. First the performance condition of the nominal plant with a sinusoidal input with amplitude $< 1$ will be derived which then can be expanded to the robust case.

The transfer function from the reference input to the error signal is $S_0 := 1/(1 + L)$, and is designated sensitivity function. The searched nominal specification would then be a bound on $S_0$, like

$$\|S_0\|_\infty < \epsilon$$

By defining a weighting function $W_1(s) = 1/\epsilon$, the specification can be written like

$$\|S_0W_1\|_\infty < 1$$

The performance of a perturbed plant, belonging to a set $\mathcal{P}$, can be derived in a similar way. As mentioned earlier the general notion of robust performance is internal stability and some specification, in this case bounded steady-state error. The stability condition was derived in the last section to be $\|\frac{W_2L}{1+W_1L}\|_\infty \leq 1$ and the robust specification is derived by considering a multiplicative perturbation of the nominal model $P_0$ to $P = (1 + \Delta W_2)P_0$. The perturbed sensitivity function is then

$$S = \frac{1}{1 + (1 + \Delta W_2)L} = \frac{S_0}{1 + \Delta W_2 T_0}$$

where $T_0 = L/(1 + L)$ and is named complementary sensitivity function because of $T_0 = 1 - S_0$. The bound of the perturbed sensitivity function for achieving the specifications is $\|W_1S\| < 1$. The next theorem summarizes the robust performance conditions into one relation.

**Theorem 3** A necessary and sufficient condition for robust performance is

$$\|\|W_1S\| + |W_2T_0|\|_\infty < 1$$

The proof is omitted here but can be found in [13].
Chapter 4

$H_\infty$-Control

A useful concept when defining performance specifications is the size of the signal. A bounded signal is always necessary to obtain stability and the ability of limiting this upper bound is desired. Especially it is interesting to limit the size of an output signal from a given input signal by limiting the size of the transfer function. The size of transfer functions is defined as norms and the two important norms that are commonly used in control theory are the $H_2$- and $H_\infty$-norms.

The $H_2$ norm is related to energy and indicates whether the transfer function amplifies or damps the total energy of an incoming signal and is defined as

$$
\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\{G^*(j\omega)G(j\omega)\} d\omega}
$$

where $G$ is a transfer function.

The effects of a transfer function on a signal can also be determined by the norm of the highest amplifying value for different frequencies. The value corresponds to the largest peak on the Bode magnitude plot and is defined as

$$
\|G\|_\infty = \sup_{\omega} \tilde{\sigma}\{G(j\omega)\}
$$

As it will be seen later the use of the $H_\infty$-norm, surves the problem specification of damping the acoustics and therefore the $H_2$-norm will be left out from now on.

1 Problem Formulation

The system in figure 4.1 is described as a Linear Fractional Transform, see Appendix B, with frequency dependent weights, which will be discussed in the next section.
The input to the plant $P_0$ is a signal $w$, which for instance could represent a disturbance load, and the output is $z$. A controller $K$ should be derived to stabilize the plant and achieve wanted performance, which in this case would be a bounded output signal. The goal of the controller is to minimize the effects of $w$ on $z$ in respect of the norm 4.2, which is represented with the transfer function $T_{zw}$. A controller achieving this is called $H_\infty$ optimal. The problem can be formulated as:

**Given** $\gamma > 0$ **find an internally stabilizing controller** $K(s)$, **if there are any, such that** $\|T_{zw}\|_\infty < \gamma$

## 2 Weighting Functions

As seen in figure 4.1 two blocks $W_z$ and $W_u$ are present, with the object of making the performance task more tractable. The weights are frequency dependent transfer functions, which gives the designer the possibility of rejecting input signals at certain frequency range. The new input/output transfer function is from $w$ to $z_1$, where the weight $W_z$ determines which frequencies that have high priority in the optimization process of the controller. The norm to be optimized is

$$\|T_{z_1w}\|_\infty = \|T_{zw}W_z\|_\infty$$

The weights can be regarded as cost functions, where the amplitude of the weighting function is corresponding to the cost. In a bode diagram this would mean that where the amplitude is positive a cost is added to the
3. $H_\infty$ Loop Shaping Technique

This design technique incorporates the classical loop-shaping methods to obtain performance/robust stability tradeoffs, and a particular $H_\infty$ optimization problem to guarantee closed-loop stability and a level of robustness at all frequencies. The design methodology uses only basic concepts of loop-shaping methods, commonly used in classical frequency based designs like lead-lag controllers, and a robust stabilization controller for a normalized coprime factor perturbed system is used to construct the final controller.

It is important to mention that this approach, in contrast to the classical loop-shaping method, is done without explicit regard for the nominal plant phase information. That is, closed-loop stability requirements are disregarded at this stage. Therefore, the design procedure is both simple and systematic and only assumes knowledge of elementary loop-shaping principles on the part of the designer.

The basic assumption of this procedure is that the open-loop plant is a coprime factor perturbed plant $P = (N + \Delta_N)(M + \Delta_M)^{-1}$, $P_0 = NM^{-1}$, where $M$ and $N$ are stable transfer functions. Under this assumption the robust stability problem is stated as

$$\min_K \left\| \begin{bmatrix} K & I \\ \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \\ \end{bmatrix} \right\|_{\infty}$$  \hspace{1cm} (4.3)

by the Small-Gain Theorem. So, define the parameter $b_{P,K}$ as the inverse of the minimum value achieved of $\| \cdot \|_{\infty}$ after the optimization procedure, i.e.

$$b_{P,K} = \left( \left\| \begin{bmatrix} K & I \\ \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \\ \end{bmatrix} \right\|_{\infty} \right)^{-1}$$

**Design Procedure (SISO):**

1. Loop-shaping: the frequency response of the open-loop plant is shaped using a compensator $W(s)$ to give the desired open-loop shape. The
nominal plant $P$ and the shaping compensator $W$ are combined to form the shaped plant $P_s$, where $P_s = PW$. It is assumed that there is no pole-zero cancellation of unstable modes of $P$.

2. Synthesize a stabilizing controller $K_\infty$ for $P_s$, through solving (4.3) and compute $b_{P,K}$ (MATLAB: ncfsyn).

3. Check the resulting parameter $b_{P,K}$, if $b_{P,K} \ll 1$ then return to (1) and adjust $W$.

4. The final feedback controller $K$ is then constructed by combining the $H_\infty$ controller $K_\infty$ and the shaping compensator $W$, such that

$$K = K_\infty W$$

5. If it is necessary apply model reduction to the resulting controller $K$ (Balance Truncation)

For a more detailed description of this design technique and justification for $H_\infty$ Loop Shaping refer to [4].
Chapter 5

Active Control of Combustion Systems

Active control of a combustion process employs a feedback strategy according to figure 5.1. The fluctuating pressure will be measured by watercooled microphones at a certain point in the downstream part, which will be the input signal to the controller. The controller then feeds a linear moog-valve, that controls the fuel flow into the burner. The modulation of the fuel flow changes the heat release of the flame. Thus the pressure fluctuations vary and it is possible to cut off the self-excited instabilities. An appropriate way of understanding the concept is to consider the fuel flow as a source of sound, which can be modulated by the controller to achieve sound-cancellation. Since the combustion chamber excites particularly one specific frequency (figure 2.9) and because of the uncertainties of the analytical model, the features of $H_\infty$-control that minimizes the maximum amplification suits our

![Diagram](image-url)
purpose. In the next sections controllers will be derived and simulated.

1 Control Synthesis

1.1 $H_{\infty}$-optimization

The objective of the controller synthesis is minimizing the noise from the flame into the combustion chamber, which means reducing the effects of a disturbance to the output. As Figure 2.9 shows are the magnitudes from the flame mostly amplified at one specific peak frequency and therefore it is important to attenuate this peak. For this reason $H_{\infty}$-control seems promising, where the problem is to minimize $\| \cdot \|_{\infty}$ from the disturbance to the output.

Furthermore it is important to assure that the peak optimization does not excite peaks at other frequencies and that the controller signal is limited since the maximum input signal to the moog-valve is 1.8V. This can be formulated into a weighted $H_{\infty}$ problem, according to Figure 4.1, with the weights $W_z$ and $W_u$ (Figure 5.2), representing pressure output and control signal weightings respectively. The figure shows how the frequencies around the peak of the plant are punished during the optimization with the weighting function $W_z$. The frequency width of $W_z$ is much broader than the peak of the plant since avoiding other excited frequencies are crucial. The control signal is saturated at 1.8V and larger signals would mean loosing control of the plant, since the moog cannot fulfill the fuel flow intended of the controller. Therefore the weight $W_u$ damps the control signal to a level where it is no risk of saturation but still allows the controller freedom. While the combination of high frequencies and large amplitudes would reduce the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5_2.png}
\caption{Weighting functions $W_z$ and $W_u$.}
\end{figure}
lifetime of the valve considerably is the damping larger at high frequencies.

![Bode diagram of analytical model with $H_\infty$-controller.](image1)

![Simulation of analytical model with $H_\infty$-controller.](image2)

The $H_\infty$ optimization was carried out with MATLAB and the $\mu$-toolbox and it was often found that the resulting optimal controller was unstable. An unstable controller would obviously be impossible to use because of high sensitivity and lack of robustness. It is not an error during the controller computations and neither a numerical problem. The $H_\infty$ theory only guarantees the stability of the closed loop including plant and controller but does not leave any guarantees for the controller. Campos-Delgado [5] suggests an approach where the MATLAB algorithm is manipulated. When optimizing the norm of the closed loop MATLAB uses the bisection algorithm. If there exists a $\gamma_1 > 0$, for which a controller $K_1$ exists that stabilizes the closed loop, then check a new norm $\gamma_2$, such that $\gamma_2 < \gamma_1$ with a stabilizing controller $K_2$. These iterations continues until the norm cannot be further reduced. By checking the poles of the controller during the iteration it can be seen that the poles of the controller move towards the right half-plane for smaller norms. By interrupting the iteration before the controller becomes unstable a norm $\gamma_{\text{stable}}$ is achieved, where $\gamma_{\text{stable}} > \gamma_{\text{optimal}}$. The constraint of deriving a stable controller limits the performance of the system but is necessary.

To see the impact of the controller, the bode diagrams for the open- and closed-loop are shown in Figure 5.3, with the suboptimal $H_\infty$-controller $K$ in figure 5.5. As the bode diagram shows, is the controller attenuating the main harmonic well, without exciting any other frequencies. A simulation of the system in SIMULINK is shown in Figure 5.4.
1.2 $H_\infty$-loopshaping

The main problem during the synthesis of a loop-shaping controller is the design of the open-loop weights. Some performance objectives can be related to the design of the weights and most important is the closed-loop stability, which can be reached by manipulating the bandwidth of the open-loop. Therefore the weights are designed as low-pass filters with the declining part starting after the instability frequency. The computer now optimizes equation 4.3 giving a certain amount of robustness, since the procedure maximizes the stability margin. The controller (figure 5.6) can be reduced to 5th order without loosing stability or pressure damping (figure 5.7).

Figure 5.6: Bode diagram of a 5th order loop-shaping controller.

simulation (figure 5.8) of the closed-loop system also shows good suppression
of the acoustics. As it can be seen (figure 5.6), the optimized controller has

![Bode diagram for analytical model with loop-shaping controller.](image)

![Simulation of analytical model with loop-shaping controller.](image)

Figure 5.7: Bode diagram for analytical model with loop-shaping controller.

Figure 5.8: Simulation of analytical model with loop-shaping controller.

a simple structure except for a resonance peak at a lower frequency than the instability frequency. There is no obvious reason why the controller would need such a peak at another frequency than the instability of the open-loop. On the contrary it may be dangerous to amplify just one frequency. The idea leading to the next section is that the peak is caused by a degree of freedom in the optimization process and can be disregarded.

### 1.3 Derivation of low-order controller

Due to the simple characteristics of the loopshaping controller and the good robustness, the idea of trying some simple controllers was founded. The first assumption is that the main peak of the controller (figure 5.6) is a mathematical optimization phenomena and that it can be disregarded. Therefore a very simple low-pass filter function (equation 5.1) was approximated to fit the low- and high-frequency magnitudes of the loop-shaping controller. The most important design constraint was to keep the magnitude at the instability frequency the same as for the loop-shaping controller.

\[ K(s) = \frac{w_n^2}{s^2 + \omega_n \xi s + \omega_n^2} \]  
\[ s^2 + \omega_n \xi s + \omega_n^2 \]  
\[ \omega_n \]  
\[ \xi \]

The obtained result corresponded to the reasoning above and the peak attenuation was almost as good as with the loop-shaping controller.

Even though the controller worked well in simulations, low-frequency noise may cause problems in reality and therefore a damping effect at low frequencies would be appropriate. This lead to a second approach, where
the controller was defined as a second order band-pass filter (equation 5.2) with the controller peak matched to the open-loop peak and with the same magnitude as the loop-shaping controller.

\[ K(s) = \frac{s}{(\frac{s}{w_1} + 1)(\frac{s}{w_2} + 1)} \]  

(5.2)

By tuning the phase with a time-delay \( \tau \) the phase of the new controller could be tuned to the same phase as the loop-shaping controller at the instability frequency (figure 5.9).

Figure 5.9: Band-pass controller vs. loop-shaping controller.

Figure 5.10: Bode diagram of analytical model with band-pass controller

Figure 5.11: Simulation of analytical model with band-pass controller.

The band-pass controller attenuates the peak as well as the loop-shaping controller (figure 5.10) and simulations (figure 5.11) confirms this result.
2. RELATED MATLAB COMMANDS

2 Related MATLAB Commands

\[ [K,Tzw,gamma] = \text{HINFSYN}(sys) \] Returns the (sub)optimal controller K that stabilizes the plant SYS (system matrix). The closed-loop system is described in Tzw and the minimized norm is gamma.

\[ \text{norm} = \text{HINFNORM}(sys) \] Calculates the \( H_\infty \) norm of stable, SYSTEM matrices.

\[ K=\text{NCFSYN}(sysgw) \] Synthesizes a controller K to robustly stabilize a family of systems.
Chapter 6

Measured Model

This chapter has the intention of introducing the model used for describing one working condition of the combustion process. It will be explained how the model has been derived from the test rig and what characteristics it has. The model is seen as a network of acoustic elements, which all are seperately measured and then compounded to an input/output model. The experimental combustion test rig is shown in figure 6.1. Everything right of the burner is called the downstream part and everything on the left side is called the upstream part.

![Figure 6.1: Combustor Test-Rig.](image)

To be able to consider the system as linear, the acoustic field downstream of combustionzone can be described by one-dimensional convective wave equation.
1 Transfer function measurements

The derivation of the transfer functions is done by using the multi-microphone method, which is an extension of the Two Microphone Method that will be discussed here. The principles are the same except that the Multi Microphone Method determines the acoustic pressure at several axial positions and gives an overdetermined system from which the most suitable Riemann-invariants are found [1].

1.1 The Two Microphone method

The two-microphone method determines the acoustic pressure on the up- and downside of the acoustic element (figure 6.2), which can be written as two traveling \( f \) and \( g \) waves described through Riemann-invariants (Appendix A) superpositioned in that point. The sum of the invariants equals the acoustic pressure. Once the Riemann-invariants are known at one location, the equations 6.1 and 6.2 can be used to find the up and down traveling waves at different locations.

\[
f(x_2) = f(x_1)e^{-ik_{x_2}(x_2-x_1)}
\]

\[
g(x_2) = g(x_1)e^{ik_{x_2}(x_2-x_1)}
\]

So, the acoustic pressure at two locations can be written as a function of the Riemann-invariants at one location

\[
\begin{pmatrix}
\frac{p(x_1)}{p_c} \\
\frac{p(x_2)}{p_c}
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
\Phi^+ & \Phi^-
\end{pmatrix} \begin{pmatrix}
f(x_1) \\
f(x_2)
\end{pmatrix}
\]

where \( \Phi^\pm = e^{-i k_{x_2}(x_2-x_1)} \).

By inverting the matrix the Riemann-invariants at one location are found as a function of pressure at two locations, which are being recorded by the microphones.

\[
\begin{pmatrix}
f(x_1) \\
f(x_2)
\end{pmatrix} = \frac{1}{(\Phi^- - \Phi^+)p_c} \begin{pmatrix}
\Phi^+ & -1 \\
-\Phi^- & 1
\end{pmatrix} \begin{pmatrix}
p(x_1) \\
p(x_2)
\end{pmatrix}
\]

When the Riemann-invariants are known it is possible to gain amplitude- and phase-information before and after the acoustic element. The Riemann-invariants can then be used for determining the transfer matrix.

The methods for measuring the transfer function are all based on the same principle of varying the acoustic load. The measurement techniques assume that the loudspeakers are the only forcing signals but in reality the turbulent flow around the burner and the flame produces noise. Thus the Riemann-invariants are not only depending on the forcing signal. A signal...
picked up by a microphone is considered to consist of three contributions.

1) The response to the loudspeaker.
2) The response to the flame’s source term.
3) Contribution of turbulence at the microphone location (pseudo sound)

By applying sequential forcing with pure tones, and calculating the Fourier coefficients exactly at the forcing frequencies, the contributions to the signal that are not correlated to the loudspeaker signal will vanish for sufficiently long measurement time.

2 Combustor measurements

The overall model consists of a combination of four measured transfer functions. The realization of the transfer functions are accomplished by fitting curves to the frequency-dependent relation of the Riemann-invariants. Every forcing frequency results in an amplitude and a phase to which an analog filter is fitted by a least square method. Appendix D contains the fitted frequency response curves of the elements.

- Downstream ($H_{down}$)
- Upstream ($H_{up}$)
- Moog Actuator ($H_{moog}$)
- Flame($H_{source}$)

2.1 Downstream Model

1. A fully reflective plate is put at the downstream end.

2. The fuel is modulated sequentially with pure tones using the linear moog-valves.

3. At each forcing frequency the pressure Fourier coefficients are calculated for each microphone location.
4. The \( \hat{f} \) and \( \hat{g} \) waves are computed at each frequency using the Multi-Microphone method.

5. The downstream model is obtained from \( H_{\text{down}} = \hat{g} / \hat{f} \).

6. Fit transfer function to \( H_{\text{down}} \).

### 2.2 Upstream Model

1. Absorbing downstream end.

2. Now, the pressure field is modulated upstream with pure tones by loudspeakers placed downstream.

3. At each forcing frequency the pressure Fourier coefficients are calculated for each microphone location.
4. The $\hat{f}$ and $\hat{g}$ waves are computed at each frequency using the Multi-
Microphone method.

5. The upstream model is obtained from $H_{up} = \hat{f}/\hat{g}$.

6. Fit transfer function to $H_{up}$.

2.3 Moog Actuator

1. Absorbing downstream end.

2. Using the data acquired during the fuel modulation stage, the corres-
ponding Riemann-invariant $\hat{f}$ and $\hat{g}$ are extracted.

3. The following relation is used $\hat{n}_{moog} = \hat{f} - H_{up}\hat{g}$.

4. Finally, the forcing signal $\hat{u}$ is known, therefore the moog model is
given by $H_{moog} = \hat{n}_{moog}/\hat{u}$.

5. Fit transfer function to $H_{moog}$.

![Figure 6.5: Upstream Actuator Interaction](image)

2.4 Flame Model

![Figure 6.6: Upstream Flame Interaction](image)
3. EXPERIMENTAL MODEL

1. Absorbing downstream end.
2. Take the pressure measurement without forcing.
3. Calculate $f$ and $g$ from time data.
4. Transform $f$ and $g$ from time domain to frequency domain $\hat{f}$ and $\hat{g}$ (Fourier Transform).
5. Use the following relation $\hat{n}_{\text{flame}} = \hat{f} - H_{\text{up}} \hat{g}$.
6. Assuming that $w$ is white noise, fit a transfer function $H_{\text{source}}$ to the frequency data $\hat{n}_{\text{flame}}$.

Remark 1 The hypothesis that the flame can be modeled by white noise is supported by work of Klein [10].

3 Experimental model

The upstream interaction is quite complex since it consists of three blocks, Figure 6.7. Thus, it is necessary to follow the procedure in the correct sequence to obtain the correct overall model. Moreover, it is intended to control the combustor using fuel flow modulation, of particular interest is then to obtain the right description for the moog actuator.

Finally, using equation A.48 the output value is transferred from Riemann-invariants to pressure. The final model describing the complete combustor process from the input control voltage of the moog valves to the microphone measurements is given in Figure 6.9 with belonging bode diagram in Figure 6.8. Note that voltage signal to the moog valves cannot be arbitrarily large, so there is a saturation of $\pm 1.8$ V. Hence, the control actuator power is limited.

![Figure 6.7: Upstream Interaction](image-url)
Figure 6.8: Bode diagram of measured model.

Figure 6.9: Combustor Experimental Model
Chapter 7

Controller Tuning for Measured Model

As shown in the last chapter the optimized controllers worked well according to simulations over time and the main peak was attenuated in the frequency response. The sense of deriving the controllers is obviously to use it on the real plant, but the plant has different operating conditions and therefore one controller, though robust, can not stabilize plant models deviating to much from the analytical model. It has been experimentally confirmed that the frequency responses for the different operating conditions have the same characteristics as the analytical with one sharp main peak, but with a shift in frequency. At first the analytical model is compared with one well known operating condition (chapter 6) and then the controller is expanded with some tuning parameters to increase the range of use.

1 Model comparison

The measured model described in Chapter 6 is representing one operating condition and gives the opportunity to test the derived controllers and observe the possibilities or limitations of the approach with tuning parameters. From the frequency responses of the two open-loop transfer functions (figure 7.1), it can be seen that the analytical model does not correspond completely to this unique operating condition. Yet, it is easy to see the resemblance of the characteristics and especially important is that each model contains one sharp peak. The controllers derived in chapter 5 is applied in a feedback loop of the measured model but do not damp the main frequency. The need of expanding the controller to increase the range of use is obvious.
2 Tuning parameters

To solve the problem of unflexibility, a new adjustable controller is introduced. All operating conditions have one characteristic sharp peak, which are to be attenuated by the controllers. Since these peaks are some $\Delta f$ away from the analytical peak, the controller is adjusted with a parameter $\alpha$, which introduces the possibility to shift it on the frequency scale. This shift corresponds to the frequency difference of the sharp peaks, $\Delta f$, and can be tuned towards the current operating condition. Apart from the frequency shift, one gain and one delay element are inserted (Figure 7.2).

![Diagram of new controller with tuning parameters]

Figure 7.2: New controller with tuning parameters.

The delay and phase components are added for stability reasons. The shifted controller does not give any stability guarantees for the measured model, which of course is necessary. Therefore the tuning of the two extra parameters can be crucial for the system. A tuning session for the measured model
2. TUNING PARAMETERS

Figure 7.3: Bode diagram of measured model with tuned controller.

Figure 7.4: Simulation of measured model with tuned controller.

with the $H_\infty$-controller resulted in a peak-attenuation (figure 7.3) and simulations (figure 7.4) show some damping but obviously not in the amount of the analytical model. Just the indication of an improvement makes it interesting and hopefully worthwhile to apply the same methods at a real plant.
Chapter 8

Test-Rig Implementation

When carrying out the tests of the controllers a dSpace system DS1103 was used to implement the controllers. It contains a DSP data acquisition board with 20 analog inputs and 10 outputs. A convenient feature of the board is the connection to SIMULINK. By converting a model to C-code through the real-time workshop, the board, consisting of a 300 MHz processor, functions as an interface between SIMULINK and the physical world. The feature of the model is simulated on the board. By defining the state-space realization of the controller as blocks in SIMULINK the controller characteristics are overtaken by the DSP. A scheme of the test arrangement is shown in figure 8.1.

![Diagram of test arrangement]

Figure 8.1: Scheme of arrangement during tests.

As input to the DSP a watercooled microphone from Brüel & Kjaer is used to measure the pressure fluctuations in the combustion chamber at a position
20cm away from the burner. To get hold off all dynamics the sampling frequency is 6 kHz. The DSP converts the pressure to a control signal according to the controller features. The control signal modulates additional fuel to the system through the moog-valve.

Software belonging to the DSP offers the possibility to observe the measured pressure in real-time and the signal is recorded by the computer. Moreover a Hewlett-Packard spectrum analyser was used to observe the frequency peaks during the tests.

1 Active Control Performance Index

During the modeling stage, only the acoustic response of the combustor is considered. Consequently, the controller must be judged according with its capability to reduce pressure pulsations or noise in general inside of the chamber. So, the following two indexes $NRR$ (Noise Reduction Ratio) and $PA$ (Peak Attenuation) are used to measure the controller performance

$$NRR = 10 \cdot \log_{10} \left[ \frac{\text{Pressure power}_{\text{on}}}{\text{Pressure power}_{\text{off}}} \right] \text{dB}$$

$$PA = 20 \cdot \log_{10} \left[ \frac{\max_{\omega} \hat{p}_{\text{on}}(\omega)}{\max_{\omega} \hat{p}_{\text{off}}(\omega)} \right] \text{dB}$$

where $\omega$ represents frequency, $\hat{p}(\cdot)$ Fourier Transform of the pressure measurement $p(t)$, the sub indexes ‘$\text{on}$’ and ‘$\text{off}$’ test with controller on and off respectively, and

$$\text{Pressure power}_{(\cdot)} = \frac{1}{N} \sum_{\omega} |\hat{p}(\omega)|^2$$

with $N$ number of samples. It is important to notice that the definition of decibels (dB) is consistent with notation used in acoustics to define sound power level, [15].

Not only acoustic influence of the controller can be considered but also pollutants. A controller with good acoustic influence but leading to an obvious increase in pollutants is not very useful. The pollutants measured during the combustion test are $NO_x$, $CO$ and $UHC$ (Unburned Hydro Carbons). It was observed that the emissions of $UHC$ during the uncontrolled operating condition were negligible.

2 Test results

During the tests the operation condition observed in the measured model could not be reproduced and the controllers were facing new characteristics, though one main instability peak still was the case. Three different controllers were tested:
• $H_\infty$-optimal controller of order 15

• $H_\infty$-optimal loop-shaping controller of order 5

• Band-pass controller of order 2

The tuning parameters frequency shift, gain and delay are added to the $H_\infty$ controllers for compensating the different operating condition. The band-pass controller on the other hand is designed depending on the instability frequency and then tuned with a gain and delay. During the tuning of the controllers it was seen that incorrect fuel modulation could even increase the sound pressure in the combustor instead of reducing it.

First of all the loop-shaping controller was tested and after tuning the parameters good suppression was achieved. Figure 8.2 shows time- and frequency-response for off- and on-control. The controller considerably reduces the main pressure peak without exciting any other frequencies worth mentioning. The table below accounts for the results. The peak attenuation (Equation 8.2) and noise reduction ratio (Equation 8.1) are shown in $dB$ and as ratio between off- and on-control. The emissions of $NO_x$ and $CO$ are represented as ratio between off- and on-control, whereas only a significant increase in $UHC$ will be reported.

The controller achieves very good attenuation of both main peak and pres-
2. **TEST RESULTS**

<table>
<thead>
<tr>
<th>$H_\infty$ LS</th>
<th>$PA$ (dB)</th>
<th>PA</th>
<th>$NRR$ (dB)</th>
<th>NRR</th>
<th>$NO_x$</th>
<th>CO</th>
<th>UHC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_\infty$</td>
<td>-18.52</td>
<td>0.12</td>
<td>-9.70</td>
<td>0.11</td>
<td>0.67</td>
<td>0.73</td>
<td>negligible</td>
</tr>
</tbody>
</table>

sure energy, furthermore the emissions were decreased.

The $H_\infty$-optimal controller was next to be tested. It was seen that this controller was harder to tune, probably because of the high order and the complexity that follows. A small change in the tuning parameters might result in a drastic change of the controller characteristics. Figure 8.3, contains the time- and frequency-plots.

![Figure 8.3: On- vs. Off-control on test-rig with $H_\infty$-optimal controller.](image)

The controller attenuates the main peak very well but has a tendency of exciting a new instability frequency, which could cause problem. The results are summarized in a table.

<table>
<thead>
<tr>
<th>$H_\infty$</th>
<th>$PA$ (dB)</th>
<th>PA</th>
<th>$NRR$ (dB)</th>
<th>NRR</th>
<th>$NO_x$</th>
<th>CO</th>
<th>UHC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_\infty$</td>
<td>-8.72</td>
<td>0.37</td>
<td>-7.28</td>
<td>0.19</td>
<td>0.63</td>
<td>1.00</td>
<td>negligible</td>
</tr>
</tbody>
</table>

The controller reduces the pressure power to a satisfactory level and reduces the $NO_x$.

As third test the approach with a very simple band-pass controller (Equation 5.2) was realized (Figure 8.4). The controller parameters were set after...
observing the instability peak. The break-frequencies $\omega_1$ and $\omega_2$ were set so that the frequency of the maximum magnitude of the controller corresponds with the instability frequency.

![Off-control and On-control](image)

Figure 8.4: On- vs. Off-control on test-rig with band-pass controller.

<table>
<thead>
<tr>
<th></th>
<th>$PA(dB)$</th>
<th>$PA$</th>
<th>$NRR(dB)$</th>
<th>$NRR$</th>
<th>$NO_x$</th>
<th>$CO$</th>
<th>$UHC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band-pass</td>
<td>-21.34</td>
<td>0.09</td>
<td>-10.09</td>
<td>0.10</td>
<td>0.64</td>
<td>0.95</td>
<td>negligible</td>
</tr>
</tbody>
</table>

A bit astonishing it could be seen the band-pass controller was not worse than the $H_\infty$ controllers in any sense. On the contrary was the performance of the band-pass better than the $H_\infty$ optimized controller both in sence of acoustic damping and emissions.
Chapter 9

Conclusions

As it could be seen, active control through fuel modulation is proven to be an interesting strategy. The success during tests, encourages continuing research. All derived controllers reduced the acoustic pressure considerably in the combustion chamber and furthermore without generating higher pollution levels. Surprisingly it was found that the $H_\infty$ optimization might be an unnecessarily complex method, for achieving good suppression. The costly mathematical optimization process, where the designer looses overview of the course of events, might be an overkill. On the contrary a simple second-order controller that can be designed during combustion tests is performing very well. The real-time designing during tests is done by observing the main instability peak and then tune amplitude and phase at this frequency. A very nice feature of this method is that the designing of a controller on an uncertain mathematical model, no more is necessary. In fact no knowledge about the model is necessary. The approach is limited by the lack of general applicability and the extensions to other plants is uncertain, but since the combustion test-rig is built to imitate a real-size combustor the strategy might not be un conceiveble. The complexity of the acoustic systems makes it hard concluding a common use of the simple approach.

The next step in the active control approach would be to measure different operating conditions of the test-rig, with the instability peak within a certain frequency range. For each operating condition a simple stabilizing controller is designed. By applying gain scheduling on these controllers an overall controller can be developed that is insensitive to changes in operating condition. Hopefully this will be done in a near future with an encouraging outcome.
Appendix A

Basic Acoustic Relations

In this chapter basic one-dimensional acoustic relations will be derived for waves travelling through media with mean flow. The derivation is based upon work done by Schuermans [1] and Gstöh [2].

1 Conservation Equations

In order to derive equations for acoustic quantities in a gas turbine combustion system, the conservation equations will be linearised. The following assumptions were made [1]:

- Isentropic and homentropic flow: the assumption of homentropic flow has two underlying assumptions, the entropy of each particle remains constant and the entropy is the same for each particle. The first condition, isentropy, is violated in a gas turbine, because the flame adds heat to the fluid. However, if the flame front is treated as a discontinuity, the flow in front of and behind the flame may be considered to be isentropic. Homentropy is equivalent to a uniform temperature distribution on both sides of the flame. This is not true, but for a premixed, turbulent flame the temperature differences are sufficiently low.

- Perfect gas: the pressure in a gas turbine combustion chamber is significantly higher than atmospheric, ranging from 15 to 30 bar, but still low enough to maintain the relation for a perfect gas as a good approximation for the state equation.

- One dimensional flow: This assumption means that acoustic perturbations are planar waves. The wave front (defined as a surface at which all points have the same amplitude and phase of the acoustic pressure and velocity) is a plane normal to the direction of wave propagation.
- Negligible body forces: gravity forces (the only forces which could be of importance) are small enough to be neglected.

- No interaction with boundaries: the combustion chamber is assumed to be rigid walled, therefore there is no interaction between structural and fluid oscillations.

With these assumptions the conservation equations can be written as:

Conservation of mass (continuity):
\[
\frac{\partial \rho v}{\partial x} + \frac{\partial \rho}{\partial t} = 0
\] (A.1)

Conservation of momentum:
\[
\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0
\] (A.2)

Conservation of Energy (isentropic equation of state):
\[
\frac{\hat{\rho}^\gamma}{p} = \text{constant}
\] (A.3)

Equation A.3 can be expressed in a more useful way if the speed of sound is introduced. To derive an equation that includes the speed of sound, a single pressure wave through a straight tube will be considered. Consider a system boundary around the pressure wave, travelling with the same speed as the pressure wave (the speed of sound). Since the cross section of the tube is constant, the equation of continuity can be written as:

\[
\rho c = \text{constant}
\] (A.4)

so:

\[
\rho dc + cd\rho = 0
\] (A.5)

The momentum equation for this case can be written as:

\[
\rho c^2 + p = \text{constant}
\] (A.6)

or

\[
2\rho dc + c^2 dp + dp = 0
\] (A.7)

Combining these equations, the speed of sound can be expressed as:

\[
c^2 = \frac{\left( \frac{\partial p}{\partial \rho} \right)_{s}}{\rho} = \frac{\gamma p}{\rho}
\] (A.8)

where the subscript \(s\) indicates isentropic conditions.
2 Linearisation

Since acoustic fluctuations will be studied in this report, the flow variables are written as the sum of a mean quantity and a fluctuating part:

\[ p(x, t) = p_0 + p'(x, t) \]  \hspace{1cm} (A.9)

\[ v(x, t) = v_0 + v'(x, t) \]  \hspace{1cm} (A.10)

\[ \rho(x, t) = \rho_0 + \rho'(x, t) \]  \hspace{1cm} (A.11)

The subscript 0 denotes the mean value and the prime the fluctuating part. Linearising the conservation equations around the mean values of the flow variables yields:

\[ \frac{\partial p'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} + v_0 \frac{\partial p'}{\partial x} = 0 \]  \hspace{1cm} (A.12)

\[ \rho_0 \frac{\partial v'}{\partial t} + \rho_0 v_0 \frac{\partial v'}{\partial x} + \frac{\partial p'}{\partial x} = 0 \]  \hspace{1cm} (A.13)

\[ p' = \frac{\gamma \rho_0}{\rho_0} \rho' = c^2 \rho' \]  \hspace{1cm} (A.14)

After substituting \( \rho' = p'/c^2 \) and \( \rho_0 = \gamma \rho_0/c^2 \) into the conservation equations of mass and momentum (equations A.12 and A.13) and then multiplying the first equation by \( c^2 \) and dividing the second equation by \( c^2 \) yields:

\[ \frac{\partial p'}{\partial t} + \gamma \rho_0 \frac{\partial v'}{\partial x} + v_0 \frac{\partial p'}{\partial x} = 0 \]  \hspace{1cm} (A.15)

\[ \gamma \rho_0 \frac{\partial v'}{\partial t} + \gamma \rho_0 v_0 \frac{\partial v'}{\partial x} + c_2 \frac{\partial p'}{\partial x} = 0 \]  \hspace{1cm} (A.16)

After differentiating the mass equation with respect to \( t \) and the momentum equation with respect to \( x \) and subtracting both equations the mixed derivative velocity term disappears:

\[ \frac{\partial^2 p'}{\partial t^2} + \gamma \rho_0 \frac{\partial^2 v'}{\partial x \partial t} + v_0 \frac{\partial^2 p'}{\partial x \partial t} = 0 \]  \hspace{1cm} (A.17)
\[ \gamma p_0 \frac{\partial^2 v'}{\partial t \partial x} + \gamma p_0 v_0 \frac{\partial^2 v'}{\partial x^2} + c_s^2 \frac{\partial^2 p'}{\partial x^2} = 0 \]  
(A.18)

\[ \frac{\partial^2 p'}{\partial t^2} + v_0 \frac{\partial^2 p'}{\partial x \partial t} - \gamma p_0 v_0 \frac{\partial^2 v'}{\partial x^2} - c_s^2 \frac{\partial^2 p'}{\partial x^2} = 0 \]  
(A.19)

Differentiating the mass equation with respect to \( x \) and substituting this into equation A.19 removes the acoustic velocity and the mean pressure:

\[ \frac{\partial^2 p'}{\partial t \partial x} + \gamma p_0 \frac{\partial^2 v'}{\partial x^2} + v_0 \frac{\partial^2 p'}{\partial x^2} = 0 \]  
(A.20)

\[ \frac{\partial^2 p'}{\partial t^2} + 2v_0 \frac{\partial^2 p'}{\partial x \partial t} + (v_0^2 - c_s^2) \frac{\partial^2 p'}{\partial x^2} = 0 \]  
(A.21)

After factoring, the equation may be written as:

\[ \left( \frac{\partial}{\partial t} + (c + v_0) \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - (c - v_0) \frac{\partial}{\partial x} \right) p_0 = 0 \]  
(A.22)

This one dimensional homogeneous differential equation with constant coefficients can be solved using d’Alembert’s method. According to d’Alembert a new coordinate system is introduced:

\[ \xi = t - \frac{1}{c + v_0} \cdot x \]  
(A.23)

\[ \eta = t + \frac{1}{c - v_0} \cdot x \]  
(A.24)

Instead of the unknown function \( p'(x, t) \) we get the function \( \tilde{p}(\xi, \eta) \). Re-working the differentials in terms of \( \xi \) and \( \eta \) gives:

\[ \frac{\partial p'}{\partial t} = \frac{\partial \tilde{p}}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial \tilde{p}}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = \frac{\partial \tilde{p}}{\partial \xi} + \frac{\partial \tilde{p}}{\partial \eta} \]  
(A.25)

\[ \frac{\partial^2 p'}{\partial t^2} = \frac{\partial^2 \tilde{p}}{\partial \xi^2} + 2 \frac{\partial^2 \tilde{p}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{p}}{\partial \eta^2} \]  
(A.26)

\[ \frac{\partial p'}{\partial x} = \frac{\partial \tilde{p}}{\partial \xi} \cdot \frac{1}{c + v_0} + \frac{\partial \tilde{p}}{\partial \eta} \cdot \frac{1}{c - v_0} \]  
(A.27)

\[ \frac{\partial^2 p'}{\partial x^2} = \frac{\partial^2 \tilde{p}}{\partial \xi^2} \left( \frac{-1}{c + v_0} \right)^2 - \frac{\partial^2 \tilde{p}}{\partial \xi \partial \eta} \cdot \frac{2}{c^2 - v_0^2} + \frac{\partial^2 \tilde{p}}{\partial \eta^2} \left( \frac{1}{c - v_0} \right)^2 \]  
(A.28)

\[ \frac{\partial^2 p'}{\partial x \partial t} = \frac{\partial^2 \tilde{p}}{\partial \xi^2} \cdot \frac{-1}{c + v_0} - \frac{\partial^2 \tilde{p}}{\partial \xi \partial \eta} \cdot \frac{2v_0}{c^2 - v_0^2} + \frac{\partial^2 \tilde{p}}{\partial \eta^2} \cdot \frac{1}{c - v_0} \]  
(A.29)

\[ \frac{\partial^2 p'}{\partial x^2} = \frac{\partial^2 \tilde{p}}{\partial \xi^2} \cdot \frac{-1}{c + v_0} - \frac{\partial^2 \tilde{p}}{\partial \xi \partial \eta} \cdot \frac{2v_0}{c^2 - v_0^2} + \frac{\partial^2 \tilde{p}}{\partial \eta^2} \cdot \frac{1}{c - v_0} \]  
(A.30)
2. LINEARISATION

Substituting these equations into equation A.21 yields:

\[
\frac{\partial^2 p'}{\partial t^2} + 2v_0 \frac{\partial^2 p'}{\partial x \partial t} + (v_0^2 - c^2) \frac{\partial^2 p'}{\partial x^2} = 0 \quad \iff \quad \frac{\partial^2 \tilde{p}}{\partial \xi \partial \eta} = 0 \quad (A.31)
\]

Since \( \frac{\partial}{\partial \xi} \left( \frac{\partial \tilde{p}}{\partial \eta} \right) \) is equal to zero, \( \frac{\partial \tilde{p}}{\partial \eta} \) is independent of \( \xi \). This means that \( \frac{\partial \tilde{p}}{\partial \eta} \) is a function of \( \eta \) only:

\[
\frac{\partial \tilde{p}}{\partial \eta} = \tilde{G}^*(\eta) \quad (A.32)
\]

where \( \tilde{G}^* \) denotes an unknown function of \( \eta \). After integrating equation A.32 a solution for \( \tilde{p}(\xi, \eta) \) is obtained:

\[
\tilde{p}(\xi, \eta) = \tilde{G}(\eta) + \tilde{F}(\xi) \quad (A.33)
\]

Note that the constant of integration may depend on \( \xi \). Therefore a function \( \tilde{F}(\xi) \) is used as constant of integration. Thus, the general solution for wave equation with mean flow is (equation A.21):

\[
p'(x, t) = \tilde{F} \left( t - \frac{x}{c + v_0} \right) + \tilde{G} \left( t + \frac{x}{c - v_0} \right) \quad (A.34)
\]

where the so called Riemann-invariants \( \tilde{F} \) and \( \tilde{G} \) are arbitrary functions of their arguments. Looking at the arguments it can be seen that \( \tilde{F} \) represents a disturbance travelling downstream and \( \tilde{G} \) a disturbance going upstream in the flow. In a similar way the wave equation for \( u'(x, t) \) can be solved. Introducing the Mach number, \( M = v_0/c \), and replacing the Riemann-invariants by new ones which only differ by the constant factor \( \rho c \), the pressure and the velocity fluctuations can be written as:

\[
p'(x, t) = \rho c \left[ F \left( t - \frac{x}{(1 + M)c} \right) + G \left( t + \frac{x}{(1 - M)c} \right) \right] \quad (A.35)
\]

\[
u'(x, t) = F \left( t - \frac{x}{(1 + M)c} \right) - G \left( t + \frac{x}{(1 - M)c} \right) \quad (A.36)
\]

Although it would not be entirely correct physically, we could consider the Riemann-invariants as two waves going in opposite directions. Apart from the \( \rho c \) factor, the acoustic pressure is found by adding both waves, the acoustic velocity by subtracting the waves. In the following chapters it will
be seen that these easy transitions between the Riemann-invariants and the fluctuating pressure and velocity are very useful.

If time dependence is assumed to be of the exponential form \( e^{i\omega t} \), equation A.35 becomes:

\[
\frac{p'(x, t)}{\rho c} = f_0 \exp \left[ i\omega \left( t - \frac{x}{(1 + M)c} \right) \right] + g_0 \exp \left[ i\omega \left( t + \frac{x}{(1 - M)c} \right) \right]
\]  
(A.37)

where \( f_0 \) and \( g_0 \) are the Fourier transforms of the Riemann-invariants at location \( x = 0 \). Using the wave number defined as \( k_{x+} = \frac{\omega/c}{1+M} \) and \( k_{x-} = \frac{-\omega/c}{1-M} \) the pressure and the velocity can be written as:

\[
\frac{p'(x, t)}{\rho c} = f_0 \cdot e^{i(\omega t - k_{x+}x)} + g_0 \cdot e^{i(\omega t - k_{x-}x)}
\]  
(A.38)

\[
v'(x, t) = f_0 \cdot e^{i(\omega t - k_{x+}x)} - g_0 \cdot e^{i(\omega t - k_{x-}x)}
\]  
(A.39)

The values of \( f_0 \) and \( g_0 \) can be found if the boundary conditions are known. Note that in these equations:

\[
f_0 \cdot e^{i(\omega t - k_{x+}x)} = F
\]  
(A.40)

\[
g_0 \cdot e^{i(\omega t - k_{x-}x)} = G
\]  
(A.41)

Equations A.38 and A.39 can also be written as:

\[
\frac{p'(x, t)}{\rho c} = \left[ f_0 \cdot e^{-i k_{x+}x} + g_0 \cdot e^{-i k_{x-}x} \right] e^{i\omega t}
\]  
(A.42)

\[
v'(x, t) = \left[ f_0 \cdot e^{-i k_{x+}x} - g_0 \cdot e^{-i k_{x-}x} \right] e^{i\omega t}
\]  
(A.43)

or,

\[
\frac{p'(x = x_j, t)}{\rho c} = [f_j + g_j] e^{i\omega t}
\]  
(A.44)

\[
v'(x = x_j, t) = [f_j - g_j] e^{i\omega t}
\]  
(A.45)
2. **LINEARISATION**

with

\[ f_j = f_0 \cdot e^{-ikx_jx_j} \]  \hspace{1cm} (A.46)

\[ g_j = g_0 \cdot e^{-ikx_jx_j} \]  \hspace{1cm} (A.47)

Taking the Fourier transforms of equation A.44 and A.45, the following relations for the pressure and velocity fluctuations in the frequency domain are found:

\[ \hat{\bar{p}}_j(\omega) = \frac{\rho c}{\rho c} f_j + g_j \]  \hspace{1cm} (A.48)

\[ \hat{\bar{v}}_j(\omega) = f_j - g_j \]  \hspace{1cm} (A.49)

The circumflex denotes the Fourier transforms of the quantity.
Appendix B

Control Theory

1 Basics

A finite dimensional linear time invariant system can be represented by the following linear constant coefficient differential equations:

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(t_0) = x_0 \\
y &= Cx + Du
\end{align*}
\] (B.1)

The A, B, C and D are real constant matrices and describes the characteristics of the system from the input signal \(u(t)\) to the output signal \(y(t)\). A notation commonly used for a system like B.1 is:

\[
G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\] (B.2)

**Definition 1** *The dynamical system is said to be stable if all the eigenvalues of A are in the open left hand plane; that is, \(\text{Re}(\lambda) < 0\)*

The state-space realization of a system can be written in an infinite number of ways and by transforming the coordinates with a nonsingular matrix \(T\), \(\tilde{x} = Tx\), a new realization is created.

\[
G(s) = \begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}
\] (B.3)

These kinds of transformations are invariant and do not change the eigenvalues of the system. One especially interesting transformation is the one where the columns of the matrix \(T\) are the eigenvectors of the A matrix. The transformed realization is then described through a diagonalized system matrix with the eigenvalues \(\lambda_i\) as elements.
\[
\mathbf{\tilde{A}} = T \mathbf{A} T^{-1} = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_N
\end{bmatrix}
\]

It is also possible to describe the model in the frequency plane through a transfer function, which can be obtained by a Laplace transformation of B.1. The input/output relation then looks like:

\[Y(s) = G(s)U(s)\]

where \(Y(s)\) and \(U(s)\) are the Laplace transform of \(y(t)\) and \(u(t)\) respectively. The transfer function is:

\[G(s) = C(sI - A)^{-1}B + D\]  \hspace{1cm} (B.4)

When deciding whether a system is stable or not by checking the placement of the poles is it important to be sure of that no dynamics are hidden because of pole-zero cancellation, since this might result in a faulty analysis. All the poles are covered by the eigenvalues of the matrix \(A\) and if not all eigenvalues are represented by the poles, a pole-zero cancellation has taken place.

\[\text{poles}(G(s)) \subseteq \lambda(A)\]

Important concepts in modern control theory is controllability and observability. Each of them represents physical capabilities of our system linked to the state-space representation.

**Definition 2** The dynamical system (B.1) is said to be controllable if, for any initial state \(x(0) = x_0\), \(t_1 > 0\) and final state \(x_1\), there exists a input \(u(\cdot)\) such that the solution of (B.1) satisfies \(x(t_1) = x_1\).

This concept can be related to solving the following Lyapunov equation

\[AP + PA^T + BB^T = 0, \quad P > 0\]  \hspace{1cm} (B.5)

where \(P\) is called controllability Gramian and when it is positive definite is the system B.1 controllable.

**Definition 3** The dynamical system (B.1) is said to be observable if, for any \(t_1 > 0\), the initial state \(x(0) = x_0\) can be determined from the time history of the input \(u(t)\) and the output \(y(t)\) in the interval \([0,t_1]\).
Similarly, observability is achieved if and only if there exist a solution to

\[ A^T Q + Q A + C^T C = 0, \quad Q > 0 \]  \hspace{1cm} (B.6)

where \( Q \) is called observability Gramian.

A system that is both controllable and observable also guarantees that no dynamics are hidden in the transfer functions. These systems are called minimal.

**Definition 4** A state-space description \( (A, B, C, D) \) of \( G(s) \) is said to be minimal realization of \( G(s) \) if and only if the system is both controllable and observable.

## 2 Internal stability

This extended stability condition is defined because it is not always enough only to look at the input/output transfer functions. The output signal might be stable and yet an internal signal can be unbounded causing damage to the physical system. By analysing the closed-loop transfer functions from all inputs to all internal signals is it possible to decide whether the system is internally stable or not.

**Theorem 4** The feedback system is internally stable if there are no close-loop poles in \( \text{Re} \geq s \)

## 3 Linear Fractional Transform

The Linear Fractional Transforms, henceforth abbreviated LFT, offers a framework to rearrange all interconnected sysytems in a unified way and therefore can be analysed and synthesized with similar techniques. Figure B.1 describes a LFT with a coefficient matrix \( M \) and a controller \( K \).

![Diagram of LFT](image)

**Figure B.1:** Lower LFT.

The LFT is represented by the following equations:
Every system can be formulated to fit the coefficient matrix with the mapping equation

\[
\begin{pmatrix}
  z \\
y
\end{pmatrix}
= M
\begin{pmatrix}
  w \\
u
\end{pmatrix}
= \begin{pmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
  w \\
u
\end{pmatrix}
\]

\[u = Ky_1\]

\[\mathcal{F}(M, K) := M_{11} + M_{12} K \left( I - M_{22} K \right)^{-1} M_{21}\]

provided that the inverse \((I - M_{22} K)^{-1}\) exists.

The physical meaning of a mapping to a LFT is clear if we consider \(M\) as a proper transfer matrix. The LFT is then the closed loop transfer matrix from \(w \rightarrow z\), described through \(T_{zw} = \mathcal{F}(M, K)\). \(M\) could for instance be the controlled plant and \(K\) the controllers.
Appendix C

Analytical MATLAB Model

Figure C.1: Summarized MATLAB model
Appendix D

Bode Diagrams of Measured Elements

Figure D.1: Bode diagram of the downstream part of the burner.

Figure D.2: Bode diagram of the upstream part of the burner.

Figure D.3: Bode diagram of the moog valve.

Figure D.4: Bode diagram of the flame.
Bibliography


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