Abstract

**Title**  Control Variates for Monte Carlo-Pricing of Three-Asset Spread Options with Application in the Energy Markets

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**Purpose**  The purpose of this paper is to compare a collection of control variates for Monte Carlo-valuation of spread options on three assets with a view towards energy markets and to lay a foundation for continued research on control variates, e.g. combinations of control variates and adaption for quasi-Monte Carlo.

**Theoretical foundation**  The paper builds upon previous research on option pricing using Monte Carlo-simulation and closed form approximations.

**Methodology**  The use of underlying assets, underlying spread, call option, exchange options and delta hedge as control variates are tested in both a parametric study to test the impact of every input parameter, and a real-world scenario using data from the Dutch energy markets.

**Conclusion**  In the real-world scenario the exchange option outperforms the other control variates in most cases. Uneven results for the exchange option in the parametric study leads to the conclusion that the use of delta hedge as control variate is the best performing based on the test results.

**Keywords**  energy markets, quantitative finance, Monte carlo simulation, control variates, option pricing, spread options, three assets
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1 Introduction

1.1 The Energy Sector and the Energy Markets

The energy sector is an important part of a modern economy. It is also a sector that for many years has been under pressure from the public, the media and from legislators in the western world due to the relatively heavy emissions associated with the sector. During the 21st century much of this pressure has focussed on reduced emission of carbon dioxide in order to halt global warming. However, as of 2013 oil is the most important energy source globally, corresponding to almost one third of the world’s energy production, and despite the many voices pushing for reduced emissions, the increased use of solid fuels, mostly coal, has been the most prominent change in the composition of the global energy production in recent years, running up from a percentage of 22.7 in the year 2000, to 29.4% in 2013. Natural gas has also seen a significant increase in use in the time period, from 20.5% to 24.3%. Meanwhile, renewable energy sources have only seen an increase from 12.9% in 2000 to 13.4% in 2013. During 2000-2013 the world energy production increased by 36.0%, largely due to the emerging economies in primarily Asia. (European Union, 2015)

With 18.3% (2013) of the world final energy consumption, electricity is very important. The compositions of fuels used for the generation of electricity differ from that of the total energy production. Only 4.4% of the world electricity generation comes from petroleum. Solid fuels stand for 41.2% and is by far the most used fuel for generating electricity. Gas and renewable sources stand for almost 22% each, with hydro power by far the biggest renewable source. Nuclear power stands for 10.6% of the electricity generation with a decreasing share of the total production. (European Union, 2015)

Electricity, coal, gas and oil are exchange traded commodities and are, as many assets, accompanied by an ever-growing collection of derivatives, including basic forwards, futures and options as well as more complex constructions. Common applications for derivatives on these commodities are to replicate refineries and power plants. The replication of a gas powered power plant is commonly referred to as a spark spread and a coal powered power plant a dark spread. (European Union, 2015)(Carmona and Durrleman, 2003)(Green, 2015)

In order to decrease greenhouse gas emissions the EU introduced a ‘cap-and-trade’-
system, the European Union Emissions Trading Scheme (EU ETS) in 2005. The idea is to cap the volume of greenhouse gas emission allowed from aircrafts, industrial plants and power plants and trade emission allowances on the market. This implies that a power plant operator in the EU no longer has to consider the electricity and coal or gas markets only, but take the price of carbon dioxide into consideration as well. Replication of a gas powered power plant under these conditions is referred to as a clean spark spread and a coal powered power plant as a clean dark spread. The problem of pricing options on these spreads has thus moved from pricing a two asset spread option, to the slightly more complicated matter of pricing a three asset spread option. (EU ETS Handbook, n.d.)(Green, 2015)

1.2 Option Pricing

Only a few of the most basic types of options have analytic solutions for exact pricing available, the most famous solution probably is the one proposed by Black and Scholes (1973). For some options, closed form approximations are available, but for more exact pricing it is necessary to use numerical procedures. For derivatives where the owner can make decisions prior to maturity, such as American options, methods based on trees to represent asset price movements and finite difference methods are commonly used. Options on several assets and options with a pay-off depending on the history of the underlying asset prices are often priced using Monte Carlo-simulation (Hull, 2012). However, these calculations are often very demanding of computing power, depending of required precision. One advantage of Monte Carlo over other methods is that a confidence interval for the approximation can easily be provided.

1.3 Options in the Energy Markets

Spread options are an important type of derivatives in the energy markets. Ordinary spread options on two assets have been common for quite some time and much research has been done on the subject (Carmona and Durrleman, 2003). Although they lack an exact analytical solution, they are commonly priced using Kirk’s approximation (Hull, 2012) (Green, 2015). An improved approximation has been proposed by (Bjerksund and
1.4 Previous Research

Despite the importance of spread options on three assets, there has not been any excessive amount of research published on the matter. Closed form approximations for the price of three asset European spread options have been suggested by Li et al. (2006) and Green (2015). For the closely related problem of pricing basket options on any number of assets much research has been done. However, the general methods for improving performance while pricing an option on any number of assets are not necessarily the best choice for any specific option, especially when considering the effort of implementation (Glasserman, 2003). To the author’s knowledge there is not any published research comparing control variates for three asset spread options. There are however much research and literature published on the subject of Monte Carlo-simulation.

1.5 Purpose and Limits

This paper aims to use the specific properties of the three asset European spread option to increase the accuracy while pricing said option with Monte Carlo-simulation using control variates, thereby allowing faster calculations of an approximate price with required precision. This provides a steppingstone for further research in the field, e.g. combinations of control variates and adaption for quasi-Monte Carlo.

Because of the simplicity in calculating put prices from call prices and vice-versa, due to the put-call parity, this paper will almost exclusively discuss call options. We will focus solely on European options, i.e. options that can only be exercised at the time of maturity, and positive interest rates.

2 Theory

This section starts with a short resumé of basic option theory, a description of the types of options relevant for this paper and a more detailed definition of important options on the
energy markets mentioned in Section 1. Then follows the derivation of pricing formulas for the options, when available. The section ends with the necessary theory for implementation of the Monte Carlo-simulations.

2.1 Options

2.1.1 Vanilla Options

This is the classic class of options, notably ordinary call options and put options with the pay-off’s at the time of maturity;

$$\max(S_T - K, 0)$$

for call options, and

$$\max(K - S_T, 0)$$

for put options, where $T$ is the time to maturity, $K$ the strike price, and $S_T$ the asset price at the time of maturity. The notation is read as; at the maturity time, the holder of the call option receives either $S_T - K$ or 0, whichever is greatest. That is, a call option gives the buyer the right, but not the obligation, to sell the underlying asset for an agreed price in the future. Options can therefore be viewed as ”insurance contracts”. (Byström, 2014) (Hull, 2012)

Newer and more complex types of options are often referred to as ”exotic options”. This class of options includes a vast variety of different kinds of options, with new ones added occasionally. (Rubinstein and Reiner, 1992)

2.1.2 Spread Options

The pay-off at the time of maturity from a call spread option on two assets is

$$\max(S_{1T} - S_{2T} - K, 0)$$

Common two-asset spread options in the energy markets are;

Crack spreads on e.g. crude oil and gasoline, replicating a refinery. (Carmona and Durrleman, 2003) (CME Group, 2013)


### 2.1.3 Three-Asset Spread Options

The pay-off at the time of maturity from a call spread option on three assets is

\[
\max(S_{1T} - S_{2T} - S_{3T} - K, 0)
\]

In recent years, partly because of the European Union Emissions Trading Scheme (EU ETS Handbook, n.d.), interest in spread options on three assets has risen (Green, 2015). Common three-asset spread options are;

**Three-asset crack spreads** on crude oil, gasoline and heating oil. (Carmona and Durrleman, 2003) (CME Group, 2013)

**Clean spreads** involving electricity, natural gas or coal and emissions. These can be viewed as a development of spark spreads and dark spreads to accommodate to the price of emissions. Energy markets information provider Platts defines a clean spread as

*Baseload power price - commodity price / fuel efficiency factor* - *(EUA\(^1\) price * emissions intensity factor * energy conversion / fuel efficiency) \]

with the energy conversion factor 3.412141 and the emissions intensity factor 0.053942. Spreads are listed for several fuel efficiencies. Platts lists clean spark spreads for gas-powered power plants with efficiencies of 0.45, 0.5 and 0.6 for most markets.


\(^1\)European Emission Allowances
Clean spark spreads involving electricity, natural gas and emissions. These will be the focus of the tests conducted in this paper.

Clean dark spreads are very similar to clean spark spreads, but involving coal instead of natural gas. (Platts, McGraw Hill Financial, 2015)

2.1.4 Exchange Options

The pay-off from an exchange option at the time of maturity is

\[ \max(S_{1T} - S_{2T}, 0) \]

Exchange options can be viewed as a special case of a spread option, where \( K = 0 \). These options have an exact analytical solution, which we will return to later on. (Hull, 2012)

2.2 The Black-Scholes-Merton Model

2.2.1 Wiener-Processes

A Markov process is a kind of stochastic process where only the current value of a variable is required for determining future values. Stock prices are usually assumed to follow a Wiener process, a special kind of Markov process also used in physics to describe a Brownian motion, or random walk, for particles. What sets a Wiener process apart from the general Markov process is that it has a standard normal distribution with a mean change of 0 and a variance rate of 1 per time unit, \( N(0, 1) \), while a Markov process may have any standard normal distribution \( N(a, b) \). A variable \( z \) that follows a Wiener process has the following two properties;

The change \( \Delta z \) during a short period of time \( \Delta t \) is

\[ \Delta z = \epsilon \sqrt{\Delta t} \]

where \( \epsilon \) is a random variable with a standard normal distribution \( N(0, 1) \) and \( \Delta z \) for any two different intervals of time \( \Delta t \) are independent.

Further, we can define a generalised Wiener process as

\[ \Delta x = a \Delta t + b \epsilon \sqrt{\Delta t} \]
A generalised Wiener process where the parameters $a$ and $b$ are functions of $x$ and $t$ is known as an Itô process:

$$
\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}
$$

Stock prices are usually modelled by a geometric Brownian motion

$$
\frac{dS}{S} = \mu dt + \sigma dz \tag{3}
$$
or in discrete time

$$
\Delta S = \mu S\Delta t + \sigma S\epsilon\sqrt{\Delta t}
$$

where $S$ is the stock price, $\mu$ is the stock’s expected rate of return and $\sigma$ is the volatility of the stock price during $\Delta t$. This can be extended from one to two or more variables following correlated stochastic processes. (Hull, 2012)

2.2.2 Itô’s Lemma

Suppose that a variable $x$ follows an Itô process

$$
\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}
$$

Itô’s lemma shows that a function $G$ of $x$ follows the process

$$
dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz
$$

Itô’s Lemma can be derived, but not proven, with well known results in differential calculus. With the model for stocks (3) Itô’s lemma becomes

$$
dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz \tag{4}
$$

Itô’s lemma shows that the Wiener processes underlying $S$ and $G$ are the same. (Hull, 2012)
2.2.3 The Black-Scholes-Merton Differential Equation

The idea behind The Black-Scholes-Merton differential equation is a simple no-arbitrage argument. Suppose we set up a riskless portfolio consisting of a position in the derivative and a position in the stock. If there are no arbitrage opportunities, the return of the portfolio must be the risk-free interest rate \( r \).

Suppose that the price \( S \) of a stock follows the process (3) and that \( f \) is the price of a derivative on the stock \( S \). Equation (4) gives

\[
df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz
\]  

(5)

Because the underlying Wiener process is the same for \( S \) and \( f \), a portfolio of the derivative and the stock that removes the Wiener process can be constructed with -1 derivative and +1 \( \frac{\partial f}{\partial S} \) shares in the stock.

Define the value of the portfolio as

\[
\Pi = -f + \frac{\partial f}{\partial S} S
\]  

(6)

The change in value during the time interval \( \Delta t \) is

\[
\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S
\]  

(7)

Equations (5), (6) and (7) give

\[
\Delta \Pi = \left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t
\]  

(8)

Because this equation does not contain the change \( \Delta z \), the value of the portfolio will not change during \( \Delta t \). In other words, the portfolio is riskless during \( \Delta t \). The assumption of no arbitrage opportunities gives that the portfolio must then earn the risk-free rate of interest \( r \),

\[
\Delta \Pi = r \Pi \Delta t
\]  

(9)

Equations (6), (8) and (9) give
\[
\left( -\frac{\partial f}{\partial t} - \frac{1}{2} \sigma^2 S^2 \right) \Delta t = r \left( -f + \frac{\partial f}{\partial S} S \right) \Delta t
\]

This is the Black-Scholes-Merton differential equation. (Hull, 2012)

### 2.2.4 Solution for Vanilla Call Options

The Black-Scholes-Merton differential equation can be solved for call options as

\[
c = S_0 N(d_1) - Ke^{-rT} N(d_2)
\]  

(11)

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}
\]

and

\[
d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}}
\]

(Hull, 2012)

### 2.2.5 Margrabe’s Pricing Formula for Exchange Options

Margrabe’s formula for valuation option to exchange one asset for another is

\[
c = S_{10} e^{-q_1 T} N(d_1) - S_{20} e^{-q_2 T} N(d_2)
\]  

(12)

where

\[
d_1 = \frac{\ln(S_{10}/S_{20}) + (q_1 - q_2 + \sigma^2/2)T}{\sigma \sqrt{T}}
\]

and \(d_2 = d_1 - \sigma \sqrt{T}\), \(\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2}\) and \(q_1\) and \(q_2\) are the assets dividends. (Margrabe, 1978) (Hull, 2012)

### 2.3 Monte Carlo Simulation

Somewhat simplified, a Monte Carlo simulation simulates a large amount of possible outcomes and calculates the average of these outcomes to find an approximative answer.
As option pricing in a risk-neutral world results in the same price as in the real world, risk-neutral valuation is a handy tool used in Monte Carlo simulation (Hull, 2012)(Glasserman, 2003). In a risk-neutral world, the stochastic differential equation (3) used to describe the asset price in the Black-Scholes-Merton model can be solved as

$$S(T) = S(0)e^{((\hat{\mu} - \frac{\sigma^2}{2}) T + \sigma \epsilon \sqrt{T})}$$

(13)

where \(\epsilon\) is a standard normally distributed random variable, \(\sigma\) the volatility of the asset price, \(T\) the time to maturity and \(S(0)\) the asset price at time 0. \(\hat{\mu}\) is the asset’s expected return in a risk-neutral world and is equal to the risk free rate \(r\) for a non-dividend-paying asset. To illustrate the principle, an ordinary call option on e.g. a non-dividend-paying stock can be valued using the algorithm;

For every \(i = 1, \ldots, n\)

- generate \(\epsilon_i\)
- calculate the terminal asset price \(S_i(T)\) for \(\epsilon_i\) using (13).
- calculate the value of the option \(C_i = e^{-rT}\max(S_i(T) - K, 0)\)

Now, \(\hat{C} = (C_1 + \cdots + C_n)/n\) is an estimator of the option price \(c\). \(\hat{C}\) is strongly consistent

$$\hat{C} \rightarrow C \text{ with probability 1, as } n \rightarrow \infty$$

and unbiased

$$E[\hat{C}] = e^{-rT}E[\max(S_i(T) - K, 0)] \equiv C$$

This implies that when the number of trails \(n\) increases, \(\hat{C}\) gets closer to the correct value. Figures 1 and 2 illustrate this for a call option. For \(n = 50\), calculated option values for every trail is illustrated in Figure 1, along with a dotted line illustrating \(\hat{C}\) and the correct value calculated with Black-Scholes-Merton’s formula illustrated by the solid line. As \(n\) increases, the dotted line gets closer to the solid line, meaning a better estimate. Figure 2 shows the same simulation for \(n = 50000\).

(Hull, 2012) (Glasserman, 2003)
Figure 1: Valuation of a European call option using Monte Carlo with 50 trails. The dotted line shows the estimated option price. The solid line shows the correct value.

Figure 2: Valuation of a European call option using Monte Carlo with 50 000 trails. The dotted line shows the estimated option price. The solid line shows the correct value.
2.4 Variance Reduction and Control Variates

Using variance reduction is a method to improve the efficiency of Monte Carlo simulations. The lower variance makes it possible to run fewer simulations to receive the same accuracy of the result, resulting in shorter execution time. The idea behind control variates is to use errors in estimation of known quantities to reduce the error in an estimate of an unknown quantity. We illustrate this with a vanilla option as an example; Use a random number to simulate a future price of the underlying asset, and from that the option pay-off. This is the first part of an ordinary Monte Carlo-simulation. Now, use the simulated price to also calculate the value of something we already know the correct value of, in this case we use the underlying asset itself. We know that the value of the underlying asset $S$ should be $Se^{rT}$ at the time $T$ in a risk-neutral world. The simulated price is likely something different, but now we know the error of the simulation of the known quantity and it is intuitive plausible that we are able to use this to reduce the error of the simulation of the option that uses the same underlying asset. (Glasserman, 2003)

Denote the outputs of $n$ replications of a simulation as $Y_1, ..., Y_n$ and thus an unbiased estimator is the sample mean $\bar{Y} = (Y_1 + ... + Y_n)/n$. Suppose that we for every replication also calculate another output $X_i$ with known expectation $E[X]$, such that every pair $Y_i$ and $X_i$ are independent and identically distributed (i.i.d.). We can, for any fixed $b$, calculate

$$Y_i(b) = Y_i - b(X_i - E[X])$$ (14)

from observation $i$ and then calculate the sample mean

$$\bar{Y}(b) = \bar{Y} - b(\bar{X} - E[X])$$

$$\Leftrightarrow \bar{Y}(b) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - b(X_i - E[X]))$$

This is a control variate estimator.

The control variate is unbiased as an estimator of $E[Y]$;

$$E[\bar{Y}(b)] = E[\bar{Y} - b(\bar{X} - E[X])] = E[\bar{Y}] = E[Y]$$
The control variate is also consistent;

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i(b) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (Y_i - b(X_i - E[X])) = E[Y - b(X - E[X])] = E[Y]
\]

with probability 1.

The variance of each \( Y_i \) is

\[
Var[Y_i(b)] = Var[Y_i - b(X_i - E[X])] = \sigma_Y^2 - 2b\sigma_X\sigma_Y\rho_{XY} + b^2\sigma_X^2 = \sigma^2(b)
\]

The control variate estimator \( \bar{Y}(b) \) has variance \( \sigma^2/n \), while the ordinary sample mean \( \bar{Y} \) has variance \( \sigma_Y^2/n \). Consequently the control variate estimator has smaller variance than the standard estimator if \( b^2\sigma_X < 2b\sigma_Y\rho_{XY} \). The optimal coefficient \( b^* \) that minimises the variance of \( Y_i(b) \) is given by

\[
b^* = \frac{\sigma_Y}{\sigma_X} \rho_{XY} = \frac{Cov[X,Y]}{Var[X]} \tag{15}
\]

(Glasserman, 2003) (Asmussen and Glynn, 2010)

2.4.1 Multiple Control Variates

The control variate method can be generalised to accommodate multiple control variates. For \( d \) controls, each replication \( i \) of a simulation gives the outputs \( Y_i \) and \( X_i = (X_{i1}, \ldots, X_{id})^T \). We assume that the vector of expectations \( E[X] \) is known and that every pair \( (Y_i, X_i) \) are i.i.d. with the covariance matrix

\[
\begin{pmatrix}
\Sigma_X & \Sigma_{XY} \\
\Sigma_{XY}^T & \sigma_Y^2
\end{pmatrix}
\]

where \( \Sigma_X \) is \( d \times d \) and \( \Sigma_{XY} \) is \( d \times 1 \). The control variate estimator is \( \bar{Y}(b) = \bar{Y} - b^T(\bar{X} - E[X]) \) where \( \bar{X} \) is the vector of sample means of the controls. Analogue to the case of a single control variate, the variance of \( \bar{Y}(b) \) is minimised at

\[
b^* = \Sigma_X^{(-1)}\Sigma_{XY} \tag{16}
\]

(Glasserman, 2003)
2.4.2 Proposed Control Variates

**Underlying asset** $S_1$. Probably the most basic control variate. The expected value at time $T$ is $E[S_{1T}] = S_{10}e^{rT}$

**Underlying spread** $S_1 - S_2 - S_3$, with the expected value $E[S_{1T} - S_{2T} - S_{3T}] = (S_{10} - S_{20} - S_{30})e^{rT}$

**Vanilla option** $S_1 - K$. Ordinary european call option. The expected value at time $T$ is the true value obtained using Black-Scholes-Merton’s formula (11).

**Exchange option** For spread options on three assets, the third asset is in practice often CO$_2$, which has a low price and volatility compared to the other two assets and is likely to affect the price less. This makes the exchange option very similar to the three-asset spread, and a good candidate for a control variate. The expected value at time $T$ is the true value obtained using Margrabe’s formula (12). One extension is to use multiple exchange options as control variates, e.g. a combination of one exchange option on asset 1 & 2 and one on asset 1 & 3.

**Delta hedge** is a procedure to ideally eliminate the risk of a portfolio by using the ratio of change $\Delta$ of the option price relative to the asset price. The calculation of the optimal hedge ratio is identical to calculation of the optimal coefficient vector $b^*$ while using $S_1, S_2$ and $S_3$ as control variates (Glasserman, 2003). This means that we will use three control variates, each with its own expected value

$$E[S_{1T}] = S_{10}e^{rT}$$
$$E[S_{2T}] = S_{20}e^{rT}$$
$$E[S_{3T}] = S_{30}e^{rT}$$

and a vector $b^*$ with three optimal coefficients given by (16).
3 Data

The data used to create a realistic scenario to perform tests on the different control variates are historical daily data from the Dutch markets for natural gas, electricity and emissions. The data were sourced from Intercontinental Exchange (ICE, Intercontinental Exchange, n.d.) with the latest values from 9th of February 2016. Data of the same length as the maturity time of the option to be valued are used. The maximum historical data length is thus 480 days.

3.1 Estimations from Historical Data

There are many approaches for modelling volatilities and correlations from historical data. For the purpose of this paper the most basic methods are deemed to be sufficient, as the estimated volatilities and correlations only serve as realistic input parameters in the tests and whether they serve as the best models possible lies outside the scope of this paper.

3.1.1 Estimating Volatility

Volatility, the standard deviation of returns, is for the purpose of this paper approximated as \( \sigma = \frac{s}{\sqrt{\tau}} \), where \( s \) is the standard deviation of the continuously compounded return of \( S \) during day \( i \); \( u_i = \ln\frac{S_i}{S_{i-1}} \), and \( \tau \) is the length of the time interval in years. (Hull, 2012)

3.1.2 Estimating Correlation

Similar to the definition of volatility, correlations are in financial contexts often defined as the correlation between daily changes in market variables. The estimation used in this paper is \( \rho = \text{corr}(u_1, u_2) \) for the continuously compounded returns for two assets \( S_1 \) and \( S_2 \). (Hull, 2012)

4 Method

All implementation is done in Matlab (MathWorks, n.d.). To implement the Monte Carlo-simulations we need to generate correlated random numbers. Of course, real random
numbers cannot be generated by a computer, but sufficiently good algorithms to generate pseudo-random numbers are available.

4.1 Pseudo-Random Number Generation

The pseudo-random numbers are generated using the widely used method proposed by Matsumoto and Nishimura (1998). The algorithm is used as default by Matlab and its derivation lies outside the scope of this paper. The method was tested and compared to other methods in Matlab’s standard library and deemed to be sufficiently good.

4.2 Cholesky Decomposition

To make the pseudo-random numbers correlated we will use the Cholesky decomposition

\[ C = LL^T, \]

where \( C \) is the correlation matrix and \( L \) is a lower triangular matrix, the Cholesky root. Its derivation is omitted here and the algorithm is available in Matlab’s standard library. From a vector \( X \) of uncorrelated pseudo-random numbers a vector \( Z \) with correlated numbers can be generated as

\[ Z = LX \]


4.3 Two Step Simulation

The simulations involving control variates are done in two steps. First, a relative small-sample pre-simulation with \( n/100 \) trails is executed to determine the vector of optimal coefficients \( b^* \) for the control variate. Then the main simulation is run.

The full algorithm for the Monte Carlo simulations using control variates are;

Pre-simulation

For every \( i = 1,...,n/100 \)

- generate \( \epsilon_{1i}, \epsilon_{2i} \) and \( \epsilon_{3i} \)
• calculate the terminal asset prices $S_{1i}(T)$, $S_{2i}(T)$ and $S_{3i}(T)$ for $\epsilon_{1i}$, $\epsilon_{2i}$ and $\epsilon_{3i}$ using (13).
• calculate the value of the option using (1)
• calculate the value of the control variates as defined in Section 2.4.2.

Now calculate the optimal coefficient $b^*$ using (16)

Main simulation
For every $i = 1,...,n$
• generate $\epsilon_{1i}$, $\epsilon_{2i}$ and $\epsilon_{3i}$
• calculate the terminal asset prices $S_{1i}(T)$, $S_{2i}(T)$ and $S_{3i}(T)$ for $\epsilon_{1i}$, $\epsilon_{2i}$ and $\epsilon_{3i}$ using (13).
• calculate the value of the option using (1)
• calculate the value of the control variates as defined in Section 2.4.2.
• calculate $Y_i$ using (14)

Now the sample mean $\bar{Y} = (Y_1 + ... + Y_n)/n$ is an estimator of the option price $c$.

4.4 Reference Methods

In addition to the proposed control variates, a few other methods will be used to give reference and perspective to the results.

**Brute-Force Monte Carlo**  without any variance reduction. Ordinary Monte Carlo not using any methods to improve performance.

**Kirk’s approximation** is a closed-form approximation. The formula is on a form similar to Black-Scholes-Merton pricing formulas. While originally proposed by Kirk in 1995, a generalisation to three assets was proposed by Lia et al. (2010). The original two-asset version can be considered to be the standard valuation formula for two-asset spread options. The derivation is omitted as it is only used as reference. (Green, 2015)
Green’s approximation is another closed-form approximation. It is a generalisation to three assets of the Bjerksund and Stensland (2014) approximation. As with Kirk’s approximation, the derivation is omitted as it is only used as reference. (Green, 2015)

Quasi-Monte Carlo uses low discrepancy values from a Sobol sequence instead of pseudo-random numbers to greatly increase precision and thereby decrease run-time. In contrast to ordinary Monte Carlo, quasi-Monte Carlo methods do not mimic randomness. Instead, points that are too evenly distributed to be random are generated with e.g. a Sobol sequence. Matlab provides methods for picking numbers from a Sobol sequence. These numbers are used instead of pseudo-random numbers in the simulation. (Glasserman, 2003) (Hull, 2012)

Antithetic variates is another method, like control variates, to reduce variance. Antithetic variance reduction works by introducing negative dependence between pairs of random numbers. A basic implementation is to simply pair the sequence \( Z_1, Z_2, \ldots, Z_n \) of independent and identically distributed (i.i.d.) standard normal distributed \( (N(0, 1)) \) random variables with the sequence \(-Z_1, -Z_2, \ldots, -Z_n\), i.e. mirroring all the simulated paths. The antithetic method represents the generic improvements applicable on any problem. (Glasserman, 2003)

4.5 Tests and Input Parameters

Many tests of the different pricing methods are conducted. A parametric study is conducted to test the behaviour of the different methods efficiency while changing one input value at a time. Tests are also conducted in a real world scenario.

All simulations use \(10^5\) trials. To measure and compare the precision of the different proposed methods, average errors for \(10^3\) runs are used. The error for each run is calculated as \(\frac{|c - c_{\text{ref}}|}{c_{\text{ref}}}\), where \(c\) is the approximated price and \(c_{\text{ref}}\) is a reference price calculated with ordinary brute-force Monte Carlo with \(10^9\) simulations. Other possible choices of measurements include something based on the variance, e.g. standard deviation or confidence interval. These can be calculated for the Monte Carlo-methods but not for the reference
closed-form approximations. The error can be used for any method and is a more robust measurement in the sense that even badly functioning pricing methods, that might display a very low variance, gets a high error. Another possible measurement is the elapsed time to calculate an estimation of a given error. Although this will be briefly mentioned, it is not the focus of this paper due to the potentially high impact of the implementations themselves and the technical nature of these. For example, many of the algorithms used can probably be implemented more efficiently, if nothing else using a faster programming language such as C++.

4.5.1 Parametric Study

The parameters used in the parametric study are choosen to be of the same order of magnitude as might be found in energy markets as well as stock markets and to reflect possible scenarios, albeit somewhat extreme in some cases. We limit us to positive correlations as this is the most common within markets. The base values for the parametric study are

- First asset initial price $S_{10} = 100$
- Second asset initial price $S_{20} = 50$
- Third asset initial price $S_{30} = 40$
- First asset volatility $\sigma_1 = 0.5$
- Second asset volatility $\sigma_2 = 0.5$
- Third asset volatility $\sigma_3 = 0.5$
- Risk free interest rate (continuously compounded) $r = 0$
- Strike price $K = 10$ (at the money)
- Time to maturity $T = 1$ year
- Correlation asset one and two $\rho_{12} = 0.5$
- Correlation asset one and three $\rho_{13} = 0.5$
- Correlation asset two and three $\rho_{23} = 0.5$

Each input parameter will be set to a number of different values, while all other inputs are set to their respective base value.
4.5.2 Dutch Clean Spark Spread Options

The real-world scenario uses historical data from the Dutch markets for natural gas, electricity and emissions, along with the constants and definition of clean spark spreads (2) from Platts, McGraw Hill Financial (2015), described in Section 2. The present paper uses a fuel efficiency of 0.5, as it is the the middle value of the commonly listed; 0.45, 0.5 and 0.6, and and should be most representative. Option prices are calculated for 30, 90, 180, 360 and 480 days and strike prices in, at and out of the money. As we limit us to positive interest rates, and as of February 2016 3, 6 and 24 month government bond yields are negative in the Netherlands, as in large parts of Europe, the risk-free interest rate is assumed to be 0.4% i.e. $r = 0.004$ based on an average of UK and US government 1, 3, 6, 12 and 24 month bonds, as well as the Netherlands government 10 year bond. The volatilities and correlations are estimated from historical data of the length as the option to be valued (Hull, 2012). This implies that the tests with different time to maturity $T$ also use different volatilities and correlations. Table 1 shows the input parameters for the tests.

<table>
<thead>
<tr>
<th>$T$</th>
<th>1/12</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>4/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{20}$</td>
<td>12.156</td>
<td>12.156</td>
<td>12.156</td>
<td>12.156</td>
<td>12.156</td>
</tr>
<tr>
<td>$S_{30}$</td>
<td>4.9500</td>
<td>4.9500</td>
<td>4.9500</td>
<td>4.9500</td>
<td>4.9500</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.3195</td>
<td>0.2662</td>
<td>0.2603</td>
<td>0.2735</td>
<td>0.3131</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.5456</td>
<td>0.3844</td>
<td>0.2998</td>
<td>0.3074</td>
<td>0.4277</td>
</tr>
<tr>
<td>$\sigma_3$</td>
<td>0.5855</td>
<td>0.4008</td>
<td>0.3250</td>
<td>0.3585</td>
<td>0.3908</td>
</tr>
<tr>
<td>$\rho_{12}$</td>
<td>0.6545</td>
<td>0.6375</td>
<td>0.5409</td>
<td>0.6326</td>
<td>0.7370</td>
</tr>
<tr>
<td>$\rho_{13}$</td>
<td>0.4627</td>
<td>0.3974</td>
<td>0.3755</td>
<td>0.2733</td>
<td>0.1168</td>
</tr>
<tr>
<td>$\rho_{23}$</td>
<td>0.1452</td>
<td>0.2111</td>
<td>0.2031</td>
<td>0.1746</td>
<td>0.1044</td>
</tr>
</tbody>
</table>

Table 1: Inputs for the real-world scenario tests
5 Results

This section begins with the test results from the parametric study, followed by the real-world scenario tests. All figures in this section use the same colours and order to illustrate the different pricing methods used, as shown in Figure 3.

![Figure 3: Legend for all result figures.](image-url)

5.1 Parametric Study

Figures 4 and 5 show the impact of different initial asset prices for an at-the-money option. Figure 6 displays average pricing errors for options at, in and out of the money. This is followed by Figure 7 showing errors for different values on $T$ as well as on $r$. The results from the tests of different volatilities on the assets are shown in Figures 8 and 9, followed by Figures 10 and 11 with results from the tests of different correlations. Every figure is provided with the reference values from a $10^9$ trail brute force Monte Carlo-simulation.
Figure 4: Average errors in valuation for different initial prices on asset 1. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 16.7526, 16.7538, 16.7529.

Figure 5: Average errors in valuation for different initial prices on asset 2 and 3. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 16.7539, 16.7543, 16.7530, 16.7523.
Figure 6: Average errors in valuation for different strike prices. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 16.7520, 20.4766, 13.6479.

Figure 7: Average errors in valuation for different times to maturity and interest rates. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 23.6033, 8.3975, 16.7996, 17.1900.
Figure 8: Average errors in valuation for different volatilities on asset 1. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 13.6746, 16.7552, 29.1487.

Figure 9: Average errors in valuation for different volatilities on asset 2 and 3. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 16.7428, 20.1132, 16.8193, 19.0361.
Figure 10: Average errors in valuation for different correlations between assets. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are $19.3687$, $13.5277$, $18.8898$, $14.2029$.

Figure 11: Average errors in valuation for different correlations between assets. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are $15.6394$, $17.8192$, $20.4218$, $11.9607$. 
5.2 Real-World Scenario

The results from the real-world energy market scenario are shown in figures in order of
time to maturity. Every figure is provided with the reference values from a $10^9$ trail brute
force Monte Carlo-simulation. Table 2 shows average execution times for a 360 day option
using different number of trails in the simulations.

<table>
<thead>
<tr>
<th>Number of trails</th>
<th>$10^5$</th>
<th>$10^6$</th>
<th>$10^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brute force</td>
<td>0.0478</td>
<td>0.3467</td>
<td>3.3186</td>
</tr>
<tr>
<td>CV S1</td>
<td>0.0468</td>
<td>0.3521</td>
<td></td>
</tr>
<tr>
<td>CV S1 - S2 - S3</td>
<td>0.0477</td>
<td>0.3530</td>
<td></td>
</tr>
<tr>
<td>CV Call</td>
<td>0.0502</td>
<td>0.3732</td>
<td></td>
</tr>
<tr>
<td>CV Exchange</td>
<td>0.0375</td>
<td>0.3893</td>
<td></td>
</tr>
<tr>
<td>CV Exchange (S1 - S2 &amp; S1 - S3)</td>
<td>0.0402</td>
<td>0.4136</td>
<td></td>
</tr>
<tr>
<td>CV Delta hedge S1 S2 S3</td>
<td>0.0498</td>
<td>0.4048</td>
<td></td>
</tr>
<tr>
<td>Antithetic</td>
<td>0.0618</td>
<td>0.5201</td>
<td></td>
</tr>
<tr>
<td>QMC</td>
<td>0.0524</td>
<td>0.5258</td>
<td></td>
</tr>
<tr>
<td>Green</td>
<td>0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kirk</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Execution times in seconds for different number of trails in the simulations (average
of 100).
Figure 12: Average errors in valuation of 30 day clean spark option for different strike prices. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 1.4430, 1.1360, 0.8706.

Figure 13: Average errors in valuation of 90 day clean spark option for different strike prices. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 1.7455, 1.4450, 1.1774.
Figure 14: Average errors in valuation of 180 day clean spark option for different strike prices. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 2.1964, 1.9087, 1.6462.

Figure 15: Average errors in valuation of 360 day clean spark option for different strike prices. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 2.8000, 2.5155, 2.2510.
Figure 16: Average errors in valuation of 480 day clean spark option for different strike prices. The reference values from a $10^9$ trail brute force Monte Carlo-simulation are 3.5757, 3.2694, 2.9801.
6 Discussion and Conclusion

In this section we will analyse and discuss the results of the tests for every pricing method as well as provide some more general comments. This is followed by a conclusion and some suggestions on further research.

6.1 Discussion

We start on a note concerning the parametric study tests of correlations and volatilities in Figures 8-11, where the tests might be somewhat unrealistic and unreliable. Because of the relation between correlation and volatilities, changing only one on these parameters may result in unreliable results.

6.1.1 Control Variates

Underlying asset  
The underlying asset $S_1$ is the most basic of the tested control variates, and unsurprisingly it does not perform best in any test. It gives reasonably high level of variance reduction in the parametric study tests in Section 5.1 but little in the real world tests of Section 5.2. For the 30-day option the $S_1$ control variate even gives higher error than when no CV at all is used as can be seen in Figure 12. Overall the test results suggest that the method is quite insensitive to changes in conditions and parameters. One notable exception is shown in Figures 8 and 9 where the $S_1$ CV, as most of the other CVs, performs better for high volatilities on asset 1 and low on asset 2 and 3. This is probably because the pay-off (1) for the spread option varies more with $S_1$ in these cases. If $S_2$ and $S_3$ were to have volatilities so low that they are approximately constant, the pay-off would only depend on the changes in $S_1$.

Underlying spread  
The underlying spread $S_1 - S_2 - S_3$ is one of the top candidates. Despite being almost as simple as the underlying asset $S_1$ control variate, the underlying spread performs good and is the second or third best performing CV in most of the parametric tests. It performs in line with the delta hedge CV in the real world scenario, with exception for the 480 day option in Figure 16. A similar observation is made in Figure 7.
where the spread CV does not perform as well as the delta hedge for higher values of $T$. This suggests that the spread CV does not perform well for high values of $T$. Of course, all parameters are different in the different time to maturity in the real world tests, but the drop in performance with the change of $T$ is the only clearly replicated effect in the parametric tests. Another test where the spread performs inferior to the delta hedge in the parametric study is when all assets are highly correlated in Figure 11. However, this is not the case for the 480 day clean spark spread. Although the underlying spread CV’s inferior performance compared to the delta hedge CV is not surprising considering its simplicity, it seems difficult to give these particular results a simple theoretical explanation. As mentioned earlier, in the case of the highly correlated assets in the parameter study, the test may be flawed and its results somewhat misleading.

**Vanilla option** The control variate based on a vanilla call option performs similarly to the single asset $S_1$ CV. This is somewhat expected because they both only consider $S_1$. It is also a sign that the call option is seldom executed, in which case the call option CV and its correlation with the three asset spread option becomes very similar to the $S_1$ CV. One development of this CV, with possibly high potential, is to use several options, on all the assets, and choose the strike prices wisely.

**Exchange option** The results are similar for the single $(S_1 - S_2)$ and the double $(S_1 - S_2)$ and $(S_1 - S_3)$ exchange option CV. The exchange option control variates show catastrophic results where the two CVs perform very inferiorly to not using any CV at all in some of the tests in the parametric study. Notably in Figure 7 when $T$ is high and in Figures 8 and 9 for high volatilities. For tests affecting asset 3 only the dual exchange $(S_1 - S_2) & (S_1 - S_3)$ performs badly as it is the only one of the two involving $S_3$. In addition to the really bad results, the exchange CVs perform worst of all CVs when assets 1 & 2 or 1 & 2 & 3 have low correlation in Figures 10 and 11. These results are unexpected and can not be explained by the possible problems with the volatility and correlation tests in the parametric study, as the exchange options show bad results for high values on $T$ as well. The basic idea behind the use of exchange options as control variates for three asset spread
options is that the two options are similar when $S_3$ is small. As this is not the case in the parametric tests, less than excellent performance is not surprising, at least for the single exchange option.

Completely contrary to the parametric study results, the exchange option CVs outperforms all other CVs in the real world scenario, achieving much lower errors in all tests.

**Delta hedge** The delta hedge CV is one of the expected top candidates. It performs best, or very close, of all the CVs in the parametric study. In most of the real world tests it performs second best, after the exchange option CVs. The delta hedge CV performs consistently well and predictably with no surprises.

To summarise, the use of delta hedge as control variate performs best of all Monte Carlo methods (excluding quasi-Monte Carlo) in all parametric tests, followed by the spread $S_1 - S_2 - S_3$ and two exchange options. However, in the real-world scenario, the single exchange option and the two exchange options outperform all other Monte Carlo methods.

As mentioned in Section 4 the times in Table 2 are not very interesting to base conclusions on. The run-times are roughly the same for all control variates, with a little longer run-times for the control variates based on options, as these run one or more extra if-statements for every trail. Simulation using the delta hedge CV also takes a longer time, possibly due to the larger matrix operations required to determinate the optimal $b^*$, essentially using three control variates.

### 6.1.2 Reference Monte Carlo Methods

A few aspects about the antithetic and QMC methods need to be commented on. The way the tests are conducted, all Monte Carlo-methods use the same number of random numbers. In the case of the antithetic sampling method, this results in twice as many sample paths as the other methods, which is likely to give a better approximation as well as a considerably longer run-time, as seen in Table 2. A direct comparison between this method and the control variates is therefore not entirely representative. An alternative approach might have been to use half as many random numbers as for the control variates.
A similar comment can be made on the QMC-method, where the same samples from a Sobol sequence as random numbers for the other methods were used even though this is arguably not necessary. Alternative approaches include to either match the run-time with the other methods and likely arrive at a lower error for the same run-time, or measure the run-time to achieve a given error. However, a direct comparison between any of these two methods and the control variates is not very interesting due to the possibility to combine the use of control variates with either of them, which might be subject to further studies as stated at the very beginning of the paper.

**Antithetic variates**  Despite the arguably unfair advantage of using more trail than the other Monte Carlo-methods, resulting in a significantly longer runtime as seen in Table 2, the antithetic method is outperformed by control variates in all tests conducted. Notably by the delta hedge CV in all tests in the parametric study and the exchange option CVs in the real world tests. The antithetic method and the delta hedge CV perform very similar in the real world tests, with the delta hedge achieving lower error in 8 of 15 tests. The conclusion is that the exchange option is the clear winner, especially considering the run-time.

The results for the antithetic method support that problem-specific CVs perform better than generic variance reduction methods, as were mentioned in the very beginning of the paper and is supported by Glasserman (2003).

**Quasi-Monte-Carlo**  The Sobol quasi-Monte Carlo performs very good as expected.

**6.1.3 Reference Closed Form Approximations**

In the real world scenario, especially for at the money options, both Kirk’s and Green’s approximations perform well, in some cases even better than the QMC, as seen in Figures 13 and 14 as well as close to the QMC in the parameter study in Figures 8 and 11. Green’s method outperforms Kirk’s in 7 of 15 cases in the real world scenario and performs much better for options deep in- or out of the money and with short time to maturity as seen in Figure 12-14. For longer times to maturity the relationship is almost the opposite,
as seen in Figures 15 and 16 where Kirk outperforms Green for high strike-prices. Kirk outperforms Green in all of the parameter tests.

Altogether, both the closed form approximations perform very well for options with strike-prices close to at-the-money in normal conditions, but may become unreliable in more extreme cases.

Comparing error between the estimates provided by Monte Carlo methods and the closed form approximations has little meaning without also considering the runtime, as the precision of the MC methods depends on the number of trails and can be arbitrary good at the expense of longer runtimes. However, it is evident from the errors in the tests and the runtimes in Table 2 that using control variates in conjunction with ordinary Monte Carlo based on pseudo-random numbers can not compete with the closed form approximations in precision in relation to runtime. It is possible that using quasi-Monte Carlo in combination with control variates and other variance reducing methods may come close enough to the precision in relation to runtime achieved by the closed form approximations to be able to compete in practical use while generating a more reliable price in more extreme and unexpected market conditions, such as the recent heavy drop in oil prices.

6.2 Conclusion

From the results, it is somewhat difficult to give a straightforward answer to the main question; which control variate is best. In the real-world scenario, the exchange option CVs perform consistently best, but in the parametric study the same can instead be said about the delta hedge CV. The average errors from the tests of high values on $T$ and high volatilities shown in Figures 7, 8 and 9 undermine the suitability of exchange options as CV for valuing spread options on three assets. While it is tempting to draw the conclusion that the exchange options are the best control variate based on the real-world scenario, the overall best and most reliably performing control variate is the delta hedge.
6.3 Further Research

Further areas of research might be combinations of CVs and CVs combined with antithetic sampling or other methods. There might also be performance gains to be made in better choosing strike prices in the control variates based on options. One potentially interesting extension of the comparison conducted in this paper is to include the use of Kirk’s and Green’s approximations as control variates. Possibly the most interesting future research is to adapt the control variates to quasi-Monte Carlo.
References


