Multifactor Affine Term Structure with Macroeconomic Factors

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Abstract

We present a multifactor model of the affine term structure of interest rates with dynamics of macroeconomic factors following the diffusion process in the Vasicek model. Using observable series, we investigate the goodness of fit of the model and the impact of the variables on bond yields. We describe in detail the derivation of the model and the numerical techniques for estimating it. We find that the model achieves a stronger fitness for bonds of 3-month, 6-month and 1-year maturities during and post financial crisis and the inclusion of all selected macroeconomic variables enables a better-performing model.
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# Table of Contents

Introduction .......................................................................................................................... 1

I. Literature Review ............................................................................................................ 4

1.1. Macroeconomic news and bond yields................................................................. 4
1.2. Macroeconomic fundamentals and bond yield ..................................................... 5
1.3. From one-factor to multifactor affine term structure models ......................... 6
1.4. Multifactor affine term structure models ............................................................... 6
1.5. Joint modelling of interest rate with macroeconomic variables ...................... 8
1.6. Diffusion process ...................................................................................................... 9

II. The Multifactor Vasicek Model .................................................................................... 10

2.1. Selection of State Variables .................................................................................. 10
2.2. Model Development ............................................................................................... 12

III. Data and Methodology ............................................................................................... 14

3.1. Data ......................................................................................................................... 14
3.2. Descriptive Summary ............................................................................................. 15
3.3. Methodology ........................................................................................................... 16
3.3.1. GMM Estimation ............................................................................................... 16
3.3.2. Computation of Covariances ......................................................................... 18

IV. Results ......................................................................................................................... 19

4.1. Parameter Estimates ............................................................................................. 19
4.2. Model Fit for Yields ............................................................................................. 20
4.3. Effect of Macroeconomic Factors ....................................................................... 23
4.4. Discussion .............................................................................................................. 26

V. Conclusion .................................................................................................................. 30

References .......................................................................................................................... 32
**Introduction**

Bond yields have been the subject of interest for investors, policymakers and researchers alike. A vast amount of extant literature is dedicated to the pursuit of capturing what moves bond yields.

In our paper, we develop a straightforward multifactor affine term structure model extended from the Vasicek model. Based on Bolder (2001), we derive its multivariate version by expressing the instantaneous short-term interest rate as a linear combination of \( n \) correlated state variables which are the macroeconomic factors and then study the fitness of the model to the actual zero-coupon bond yields of different maturities. We depart from his work by presenting a more realistic expression of the short-term interest rate by assuming it is affected differently by each state variable. New expressions of coefficients of the state variables in affine term structure which are different from those in previous studies are established.

There are inherently many reasons to seek for underlying factors of the yield curve. The current yield curve can reveal the future state of the economy as yields of long-maturity bonds are expected values of average future short-maturity bond yields after adjusting for risk. Various authors (Fama, 1990; Mishkin, 1990; Hamilton and Kim, 2002; Ang et al., 2006) establish that yield spreads contain information about the future values of a range of economic variables such as real output and inflation. The yield curve reflects the monetary policies implemented by the central bank and the transmission mechanism to the economy at large. Utilising the yield curve by investors to formulate hedging strategies and derivative pricing that are much dependent on the path of the economy also involves the heavy use of information contained in the yield curve. A fitting yield curve model thus enables individuals, firms and authorities to come up with a more informed decision for the future.
Term structure models attempt to replicate a zero-coupon interest rate curve to a set of bond price observations. They are of special importance since they impose the cross-equation restrictions implied by the no-arbitrage condition and thus permit a close examination of the relationship between the term structure and the state variables. These models also allow yields to not follow necessarily the normal distribution and therefore allow better fitness, especially in the recent period (Piazzesi, 2010).

Affine term structure models are a special class of term structure models that expresses the bond yield as a linear combination of state variables and the evolution of each state variable is specified with a dynamic process. A major advantage of these models is that they provide tractable solutions and thus offer computational convenience. In addition, they allow for time-varying means and volatilities of the state variables as panel data is used (Dai and Singleton, 2000). Related literature first starts with single-factor models of Vasicek (1977) and Cox et al. (1985). Upon realising the inadequacy of using merely one factor, a class of multifactor models is developed, consisting of the work by Langetieg (1980), Longstaff and Schwartz (1992) and Chen (1996) etc. Authors such as Ho and Lee (1986), Black et al. (1990), Hull and White (1990), etc, suggest another category of affine models that try to fit the model’s term structure to the observed current term structure using numerical methods.

A large body of existing literature has been developing on the work of Duffe and Kan (1996) whom generalise the class of affine term structure models and thus, much of the investigation on multifactor models involve the use of latent variables extracted from observations on the yield curve. However, the latent variable specification has its limitations that it is cumbersome to carry out invariant transformations on the state variables and econometric identification of parameters is often unattainable (Dai and Singleton, 2000; Collin-Dufresne, 2004). Dai and Singleton (2000) hence propose a characterisation of affine models that provide admissible and maximal parameterisation. Nevertheless, estimation of the parameters is not always direct
and the difficult identification of maximal model hinders the process of estimation (Collin-Dufresne, 2004). Interpretation of macroeconomic fundamentals thus is rather abstract and empirical procedures are often somewhat demanding. In the paper, we exploit the advantages of using multifactor models in order to generate more accurate bond yields and we use only the observable fundamentals to allow easier and more straightforward interpretation. The popular generalised method of moments (GMM) technique is employed to estimate the model parameters as it does not require us to impose strong distributional assumptions on the dynamic processes of the variables.

The effects of macroeconomic factors on bond yields are well-substantiated by the rich extant empirical evidence (Fleming and Remolona, 1997, 1999; Balduzzi et al., 1999; Kim et al., 2004; Ludvigson and Ng, 2009). Expanding on previous literature, we study both the effects of the more typical macroeconomic fundamentals, namely Gross Domestic Product (GDP), inflation rate and unemployment rate, along with the less investigated ones, specifically risk aversion and stock market growth. The root mean squared error (RMSE) is used to indicate the measure of fit of the model and we compare the RMSEs of different specifications of the model constructed based on selections of macroeconomic factors to determine whether these factors would affect the goodness of fit. The F-test is also used to identify statistical performance of the model in fitting the yield data. Using the aforementioned methods, our model enables a clear-cut examination of the deterministic relationships between bond yields and the selected macroeconomic fundamentals and consequently, their interpretation becomes very much less elusive than the previous models.

We find that the model performs better for bonds of 3-month, 6-month and 1-year maturities during and post financial crisis indicated by the lower RMSEs. GDP growth and inflation rate are the variables with the heaviest yield weights, accounting for almost 97% of the total weight. The model is statistically better with the inclusion of all state variables as indicated by
the F-test statistics and we notice that there is a greater impact on the model performance for bonds of 3-month, 6-month and 1-year maturities ensuing the removal of these variables.

The remainder of the paper is organised as follows. Section I presents the literature review which consists of the work on the impact of macroeconomic factors on bond yields and those on affine term structure models, both of single-factor and multifactor. In Section II, we show our derived model with explicit solutions for the bond pricing and yield functions. Section III describes data used and econometric method implemented for estimating the parameters in the model. Section IV discusses the obtained empirical results and their implications. Finally, we conclude in Section V.

I. Literature Review

1.1. Macroeconomic news and bond yields

A great deal of literature demonstrates macroeconomic news releases have a significant effect on yields in the US Treasury bond market. In particular, Fleming and Remolona (1997, 1999) study price volatility and trading behaviour by using data from the secondary market for US Treasury securities. They conclude the arrival of public information in the US Treasury market sets off a two-stage adjustment process for prices, trading volume and bid-ask spreads. Especially in the first stage, the release of a major macroeconomic announcement induces a sharp and nearly instantaneous price change with a reduction in trading volume. It is consistent with the finding of Balduzzi et al. (1999, 2001) that the influence of macroeconomic information releasing on bond prices is significant in the US market and that the effects are significantly different according to maturity. In addition, Bollerslev et al. (2000) suggest a significant relationship between the returns in the US Treasury bond market and the macroeconomic news releasing through investigating 27 different macroeconomic information effects. They find that reading volume and volatility are significantly higher
following “important” economic announcements. Koski and Michaely (2000) conclude that economic information announcement has a positive effect on prices.

An alternative facet presented in the literature suggests that financial markets can produce different responses to macroeconomic news. Kim et al. (2004) divide financial markets into three markets, namely the stock market, the foreign exchange rate market and the bond market. In the foreign exchange market, unexpected announcements related to trade transactions are playing a significant role in the mean return changes. In the case of the stock market, macroeconomic news on consumer and producer pricing information play an important role in price changes, whereas in the bond market, news on internal economy have a determining impact on the bond price fluctuations. Kim et al. (2004) and Green (2004) depict that these differential analyses on “policy feedback” effects reflect that the market participants have different interpretations on the same macroeconomic information releasing. The underlying reasons are explored by Kim and Verrecchia (1994, 1997) whom suggest that information asymmetry in fact contributes to such discrepancies. Green (2004) further concludes that information asymmetry is not caused by the lack of macroeconomic information but resulted from different macroeconomic announcements interpretations made by the market participants.

1.2. Macroeconomic fundamentals and bond yield

Studies have attempted to build on how macroeconomic fundamentals affect bonds are comparatively scarce. Hilscher and Nosbusch (2007) recognize that macroeconomic fundamentals, in particular the volatility of terms of trade, have statistically and economically significant effects on bond spreads. Ludvigson and Ng (2009) use dynamic factor analysis and show that macroeconomic variables are highly influential on bond risk premium. Specifically, they find macroeconomic events have substantial predictive power on variation in excess bond returns. Huang et al. (2009) divide the macroeconomic variables into two groups,
namely real activities and monetary variables, and suggest that both “real” and “monetary”
factors significantly affect the bond return volatility. In particular, the “real” factor affects the
volatility across all maturities, while the monetary variables are significantly related to the
volatility of short-term bonds and weakly related to the volatility of medium-term bonds.

1.3. From one-factor to multifactor affine term structure models

As the objective of the models is to fit the observed term structure of interest rates, using only
one factor is seemingly inadequate. A major shortcoming in these models is that interest rates
with different maturities and thus all bond prices are instantaneously perfectly correlated.
Under the one-factor assumption, possible forms of the term structure of interest rates that can
be generated are restricted and therefore can become susceptible to an oversimplification of
the true stochastic behaviour of the interest rate movement which in fact is much more
complexed than what a single source of uncertainty can summarise. There is a considerable
amount of evidence suggesting that the use of a single state variable or factor is insufficient.
Chan et al. (1992) find that one-factor models often do not perform well in capturing the
dynamics of the interest rate. Canabarro (1995) suggests that the pricing and hedging of
interest rate derivatives involves more than merely one factor and McManus and Watt (1999)
mention that one-factor models appear to be inadequate in explaining the evolution of the
Canadian commercial paper rate and thus casting doubt on their usefulness.

1.4. Multifactor affine term structure models

In lieu of the limitation imposed by using a single factor, a number of authors have extended
the models to include multiple factors in an affine structure. Langetieg (1980) first attempts to
develop a multifactor case of the Vasicek model. Cox et al. (1985) have already included a
multivariate form of their model and authors such as Longstaff and Schwartz (1992) and Chen
and Scott (1993) have proposed variations to the multifactor version of the CIR model. A
significant development is made by Duffie and Kan (1996) who characterise the general class of multifactor affine models which can be viewed as a mixture of two elementary units, the Vasicek and CIR models. Building upon them, Dai and Singleton (2000) classify the affine models into subfamilies according to the number of factors or state variables that enter the conditional volatility of yields. $A_m(n)$ is defined as a model such that $N$ refers to the number of factors and $m$ is the number of processes that affect the volatility. For example, a three-factor Vasicek model is nested within the family of $A_0(3)$ models and a three-factor CIR model is within the family of $A_3(3)$ models.

An extensive literature has explored the empirical fit of these models to actual data. Litterman and Scheinkman (1991) find that changes in the US Treasury bond prices can be effectively explained with three factors constructed from bond yields or returns using the principal components approach. Duffee (2002) examines the forecasting ability of the affine term structure models and demonstrates that the essentially affine $A_0(3)$ model which allows greater flexibility in fitting variations in the interest rate risk premium is able to produce higher forecast accuracy for future short-term interest rates and excess return than a random walk benchmark model. Dai and Singleton (2002) try to match the bond data with a large number of different affine term structure models and find that the multifactor Gaussian models are able to match the risk premium dynamics. Brandt and Chapman (2005) produce similar result with the three-factor Gaussian-quadratic model being the most successful in capturing the features of the US Treasury yields.

In their paper, Duffie and Kan (1996) suggest to explain yields with a latent set of state variables which can be extracted from observations on the yield curve. In light of the above, much of the literature has dedicated efforts to construct affine term structure models of latent variables. There are two broad types of labels associated with latent factors. The first type refers to statistical properties of the short rate. The simpler versions consist of stochastic mean
models and stochastic volatility models. Balduzzi et al. (1998) propose a $A_0(2)$ model with a normally distributed stochastic mean and Chen (1996) suggests a $A_1(2)$ model with the stochastic mean following a square-root process. Longstaff and Schwartz (1992) develop a $A_1(2)$ model with the volatility specified to a square-root process. In addition, many three-factor models such as the $A_1(3)$ model of Balduzzi et al. (1997) and the $A_2(3)$ model of Chen (1996) have incorporated the short rate, a stochastic mean and stochastic volatility of the short rate.

The second type of labels refers to those based on fundamentals. The idea arises from the fact that the underlying fundamentals have implications on the state variables that affect the yield curves in general equilibrium models. In these applications, the variables from fundamentals often carry empty labels as their dynamics “have little to do with their historical behaviour” (Piazzesi, 2010). For instance, Pearson and Sun (1994) infer the unobservable state variables from the observed market prices and use an extended two-factor CIR model. They perform the empirical investigation using an exogenously specified variable on expected inflation rate without using any actual price level (CPI) data but only the yields data. One problem with these models is that latent factors are identifiable only if the parameter vector is known. Two approaches have been developed by Dai and Singleton (2000) and Duffie and Kan (1996) separately to deal with the issue. However, they are either difficult to implement or the economic interpretation of the factors are limited (Collin-Dufresne, et al., 2008).

1.5. **Joint modelling of interest rate with macroeconomic variables**

Some literature has focused their attention to incorporating the effects of macroeconomic variables on yields. Ang and Piazzesi (2003) compare the models that include solely latent variables with the models that combine macroeconomic factors and latent factors under no-arbitrage assumptions. They find the macroeconomic factors indeed play a significant role in
the movements at the short and middle part of the yield curve and including macroeconomic factors in the term structure model improves the forecasting accuracy of bond prices. D’Amico et al. (2004) explore the existence of liquidity premium in the Treasury Inflation-Protected Securities yields using affine-Gaussian no-arbitrage term structure models, incorporating actual inflation. They suggest that liquidity premiums and inflation risk premiums could impact movement in TIPS breakeven inflation. Rudebusch and Wu (2004) combine the affine term structure with standard macroeconomic aggregate relationships for GDP growth and inflation and find that the fundamentals could facilitate interpretation of how the yield curve behaves. Dewachter and Lyrio (2006) provide a macroeconomic interpretation to forecast bond yields in the long run through introducing inflation and output gap. They find linkages between these macroeconomic factors and latent variables “level”, “slope” and “curvature”. Specifically, the inflation expectation is closely related to “level”; “slope” can be regarded as an accumulation of business cycle conditions; and the monetary policy from central bank influences “curvature”. Ang et al. (2007) include the same two macroeconomic variables and a latent variable which can be interpreted as policy shock to model the short rate. They find that inflation and GDP growth account for more than half of the time-variation of time-varying excess bond returns and almost all of the movements in the term spread are due to inflation.

1.6. Diffusion process

The dynamics of stochastic processes of macroeconomics factors are extensively represented by diffusion processes. For instance, Cairns (2000) models the consumer price index (CPI) as a diffusion process in determining index-linked bond prices. Longstaff and Piazzesi (2004) assume aggregate consumption follows the exponential-affine jump-diffusion process when they examine the relation between corporate cash flows and economic shocks. Posch (2007, 2009) models output according to a diffusion process with each state variable also driven by a

II. The Multifactor Vasicek Model

2.1. Selection of State Variables

The paper explores the effects of the macroeconomic variables, namely GDP growth, inflation rate, change in unemployment rate, risk aversion and stock market growth.

An extensive amount of literature has indicated apparent effects of either macroeconomic news or fundamentals associated with GDP growth, CPI and unemployment rate on the bond market.

**GDP growth**: the impact of economic growth on bond yields is direct and apparent. With strong growth, the demand for money becomes higher and drives up inflation. The Fed then increases interest rates to counteract inflationary pressure. Consequently, Treasury bond prices fall and yields rise. On the contrary, a slow economic growth reduces the demand for money as individuals and firms prefer to save, resulting in a lower interest rate. In addition, as Treasury bonds offer a less risky investment alternative, its demand increases when the prospects are poor. Thus, bond prices rise and yields fall. Huang and Kong (2005) show that the Treasury yield curve slope produces a considerable reaction towards announcement surprises about GDP while the spread changes of the AA-AAA rated corporate bonds are significantly negatively correlated with surprises about the GDP growth rate.

**Inflation rate**: the relationship between inflation and Treasury yields is perspicuous. A higher inflation reduces the demand for bonds and lead to higher bond yields as inflation erodes the
real purchasing power of their future cash flows. Its own uncertainty can also affect interest rates through risk premiums. Thus, a higher yield is needed to compensate for the inflation risk. Becker et al. (1996) find the Treasury bond market reacts strongly to releases of news on CPI and the fifteen-minute returns on bond futures are greatly affected. Balduzzi et al. (1999) discover that the announcements which describe the inflationary process, namely those on CPI, significantly affect the bond prices of at least three maturities. Ederington and Lee (1993) explore the impacts of 19 different announcements on the futures markets of the Treasury bonds. Announcements are responsible for most of the observed time-of-day and day-of-the week volatility patterns in these markets. CPI is one of the announcements which can cause significant price changes through impact on interest rates.

**Change in unemployment rate:** one of the key drivers for inflation expectations is wage development. A higher unemployment rate typically leads to decreasing wages and subsequently affects the bond yields through a poor inflation expectation. Jones et al. (1998) estimate the reaction of the US Treasury bond prices to the releases of news on employment rate and they find that bond market volatility and bond earnings are considerably higher on the days of these macroeconomic news releases. Veredas and Durenard (2002) find surprise releases of news on unemployment rate are among the most statistically significant variables that stimulate market reaction. Goeij and Marquering (2006) recognise that the bond market incorporates the implications of macroeconomic announcements faster than other information and after distinguishing between types of macroeconomic announcements, releases of the employment situation is especially influential at the intermediate and long end of the yield curve.

Risk aversion and stock market growth are relatively less investigated variables in previous research, at least not explicitly. Including them in our model enables us to gain new insights on how they might affect bond yields.
Risk aversion: a market dominated by risk-averse investors implies that riskier securities must have higher expected rates of returns. A higher risk aversion means a higher demand for Treasury securities and thus prices increase and yields decrease. Ananchotikul and Zhang (2014) find that global risk aversion has a significant impact on the volatility of bond prices.

Stock market growth: a burgeoning equity market would divert investment from the bond market since equities generally offer a higher rate of return. This would reduce demand for bonds and thus prices of bonds would gravitate downwards and yields would otherwise increase. The literature on how the Treasury yield curve can influence the stock market movements is profuse but not contrariwise. Chen et al. (1986) find a positive relationship between changes in term structure and the US stock returns. Choe et al. (1993) approximate the decision on stock issuance using the Treasury bond yields and their results indicate their changes have a significant explanatory power in equity offering. Mukherjee and Naka (1995) find that long term government bond rate is cointegrated with the Japanese stock market and Mayasmai and Koh (2000) extend the study to the Singapore context and find similar results. Stivers and Sun (2002) study on the association of stock market returns and Treasury bonds returns. They find that the stock and bond returns show a negative relation during times of high stock market uncertainty and vice versa.

2.2. Model Development

The Vasicek model is one of the most common one-factor short rate models and describes the interest rate movement as driven by only one source of market risk.

In Bolder (2001), he expresses the instantaneous short-term interest rate as a linear combination of \( n \) correlated state variables or factors, which are denoted as \( y_1, y_2, \ldots, y_n \), so

\[
r = \sum_{i=1}^{n} y_i.
\]
We extend on his work by assuming state variables would have different degrees of effect on the short-term interest rate and add coefficient to each of the variables. Therefore, we have the following expression:

\[ r = a + \sum_{i=1}^{n} b_i y_i. \]

Considering the Ornstein-Uhlenbeck process in the Vasicek model, the dynamic process for each state variable is described by the following stochastic differential equation:

\[
dy_1(t) = \kappa_1 (\bar{\theta}_1 - y_1(t))dt + \sum_{j=1}^{n} \sigma_{1j} dW_j(t),
\]

\[ \vdots \]

\[
dy_n(t) = \kappa_n (\bar{\theta}_n - y_n(t))dt + \sum_{j=1}^{n} \sigma_{nj} dW_j(t),
\]

where \( W_1, W_2, \ldots, W_n \) are standard scalar Wiener processes (i.e. with independent components) defined on a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\) \(^1\) for \( i = 1, 2, \ldots, n \), and \( \kappa_i, \bar{\theta}_i \) and \( \sigma_{ij} \) are positive constants. \( \kappa_i \) is the speed of mean reversion and \( \bar{\theta}_i \) is the long-run equilibrium. The \( \kappa_i (\bar{\theta}_i - y_i(t)) \) terms represent the individual drift coefficient for each process and the \( \sigma_{ij} \) terms represent the diffusion coefficients which capture the covariance between the different sources of uncertainty as multiple state variables are incorporated.

The drift term is mean-reverting with parameters \( \kappa_i \) and \( \bar{\theta}_i \) to be estimated while the diffusion term has volatility parameter \( \sigma_{ij} \) for estimation.

For simplicity, we suppress the time argument on the state variables (i.e. \( y \equiv y(t) \)).

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\(^1\) \( \Omega \) is the set of all possible outcomes, \( \mathcal{F} \) specifies the set of all events (subsets of \( \Omega \)), to which probability numbers will be assigned, and \( \mathbb{P} \) is a probability measure operating on \( \mathcal{F} \).
We claim that the bond price is a function of a combination of \( n \) state variables and thus the price function of a pure discount bond for maturity, \( T \), has the following form:

\[
P(t, T) = P(t, T, y_1, y_2, \ldots, y_n).
\]

Following Bolder (2001), we develop the \( n \)-factor Vasicek model with the following bond pricing equation:

\[
P(\tau, y_1, y_2, \ldots, y_n) = e^{A(\tau) - \sum_{i=1}^{n} B_i(\tau)y_i},
\]

where

\[
B_i(\tau) = \frac{b_i}{\kappa_i}(1 - e^{-\kappa_i \tau}),
\]

\[
A(\tau) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_{ij}}{2\kappa_i \kappa_j} \left( b_i b_j \tau - b_j B_i(\tau) - b_i B_j(\tau) + \frac{b_i b_j}{\kappa_i + \kappa_j} \left( 1 - e^{-\left(\kappa_i + \kappa_j\right) \tau} \right) \right)
- \sum_{i=1}^{n} \tilde{\theta}_i (b_i \tau - B_i(\tau)) - \alpha \tau.
\]

We know that

\[
YTM(\tau) = \frac{\ln P(\tau, y_1, y_2, \ldots, y_n)}{\tau},
\]

and therefore

\[
YTM(\tau) = \frac{A(\tau)}{\tau} - \frac{\sum_{i=1}^{n} B_i(\tau)y_i}{\tau}.
\]

The detailed derivation steps can be found in Appendix B.

### III. Data and Methodology

#### 3.1. Data

We obtain 20-year data on zero-coupon bond yields of maturities 3 and 6 months, and 1, 3, 5, 10 years spanning from January 1995 to December 2014 from DataStream. All bond yields are US Treasury yields adjusted to constant maturities. Data on common macroeconomic factors including quarterly real GDP, monthly CPI, monthly unemployment rate, and monthly
stock market, proxied using data on the NASDAQ Stock Market, are all retrievable from the Federal Reserve Bank of St. Louis. Risk aversion is proxied by the monthly VIX index and obtained from the database of the Chicago Board Options Exchange. The VIX measures the market expectations of near-term volatility conveyed by S&P 500 Index option prices and is a popular indicator of risk aversion in financial markets.

The monthly GDP growth is calculated using the below function:

\[
\text{Monthly GDP growth} = \left( \frac{GDP_{t+3}}{GDP_t} \right)^{\frac{1}{3}} - 1
\]

The inflation process is proxied using monthly CPI. We compute the monthly inflation rate using the following function:

\[
\pi_{t+1} = \ln P_{t+1} - \ln P_t
\]

Monthly change in unemployment rate and stock market growth is calculated as follow:

\[
\text{Monthly change} = \frac{y_t - y_{t-1}}{y_{t-1}}
\]

The variables are tested for non-stationarity using the Augmented Dickey-Fuller test and the test statistics in Table 2 indicate all are stationary processes.

3.2. Descriptive Summary

The descriptive statistics are presented in Table 3 and correlation between the state variables in Table 4. The mean values of almost all state variables approximate at 0 except for risk aversion which is 20.86. During the 20-year period, volatility is generally low. We observe the highest variance for risk aversion and the lowest for GDP growth. Risk aversion also shows the largest range at 49.47. The correlations between the state variables are generally not high. The most positively correlated variables are unemployment rate and risk aversion at
0.2981 while the most negatively correlated ones are GDP growth and risk aversion at -0.3911. The least correlated ones are inflation rate and unemployment rate change at -0.0729.

3.3. Methodology

3.3.1. GMM Estimation

For easier and more convenient expression, we rewrite the diffusion process of state variables into the form adopted in the CKLS classification of short-term interest rate dynamics:

\[ dy_i(t) = (\alpha_i + \beta_i y_i(t))dt + \sigma_i y_i(t)\gamma dW(t), \]

where \( \alpha_i, \beta_i, \sigma_i \) and \( \gamma \) are model parameters. \( \sigma_i \) is the \( i \)th row of the diffusion matrix process \( \Sigma \) where

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \cdots & \sigma_{1n} \\
\vdots & \ddots & \vdots \\
\sigma_{n1} & \cdots & \sigma_{nn}
\end{pmatrix},
\]

and \( W \) is assumed to be a \( n \)-dimensional standard vector Wiener processes such that

\[
W = \begin{pmatrix}
W_1 \\
\vdots \\
W_n
\end{pmatrix}.
\]

For the diffusion process in the Vasicek model, \( \gamma = 0 \) so we have

\[ dy_i(t) = (\alpha_i + \beta_i y_i(t))dt + \sigma_i dW(t). \]

We can also write the dynamic processes as

\[ dy_i(t) = (\alpha_i + \beta_i y_i(t))dt + ||\sigma_i||d\bar{W}_i(t), \]

where
\[ \alpha_i = \kappa_i \bar{\theta}_i, \beta_i = -\kappa_i \| \sigma_i \| = \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2}, \]

and \( \bar{W}_i, i = 1, 2, \ldots, n \) are correlated Wiener processes with the local correlation matrix \( \rho \) where \( \rho = \delta \delta^T \) and \( \delta_i = \| \sigma_i \| \). \( \beta_i < 0 \) is the speed of mean reversion, \( \alpha_i \) is the product of \( \kappa_i \) and the long-run equilibrium and \( \| \sigma_i \| \) refers to the variance level and is a scaling parameter.

We rewrite \( W_i \)’s as \( \bar{W}_i \)’s since we know independent \( W_1, W_2, \ldots W_n \) may equivalently be written as correlated \( \bar{W}_1, \bar{W}_2, \ldots, \bar{W}_n \) (Björk, 1998).

The continuous-time process for each state variable \( i \) is discretised into the following expressions:

\[ y_{t+1} - y_t = \alpha + \beta y_t + \varepsilon_{t+1}, \]

\[ \varepsilon_{t+1} \equiv \sigma^2 y_t^\gamma N(0,1), \]

where \( N(0,1) \) is a random shock with zero mean and unit variance.

We can observe that

\[ E[\varepsilon_{t+1}] = 0, \]

\[ E[\varepsilon_{t+1}^2] = \sigma^2 y_t^{2\gamma}. \]

A major advantage of GMM over maximum-likelihood and Bayesian estimation is that it is not necessary to assume the error term is normally distributed. The distribution of changes in the state variable only needs to be stationary and ergodic. An additional advantage is that the GMM estimator and its standard errors are consistent in the case of conditionally heteroscedastic disturbances.

The moment conditions need to fulfil the condition:
\[ E[f_t(\theta)] = 0, \]

and they are constructed as:

\[
f_t(\theta) = \begin{bmatrix}
\varepsilon_{t+1} \\
\varepsilon_{t+1}^2 - \sigma^2 y_t^{2\gamma} \\
\varepsilon_{t+1} y_t \\
(\varepsilon_{t+1}^2 - \sigma^2 y_t^{2\gamma}) y_t
\end{bmatrix},
\]

where \(\theta \equiv [\alpha \quad \beta \quad \sigma \quad \gamma]'\).

The first two moments follow from the mean and variance of the error term and the next two reflect the orthogonality condition. Since \(\gamma = 0\) in the Vasicek model, the number of parameters to be estimated for each diffusion process is reduced to three. Four moment conditions allow an over-identified system.

For more efficient parameter estimates, we perform a joint estimation by collating the moment conditions of each state variable in the set of moments.

### 3.3.2. Computation of Covariances

Björk (1998) proposes that the connections between \(W'_t\)'s and \(\tilde{W}'_t\)'s are given by the following expressions:

\[
\begin{align*}
\tilde{W}_i &= \frac{1}{\|\sigma_i\|}\sigma_i W, \\
\delta_i &= \|\sigma_i\|, \\
p_{ij} &= \frac{\sigma_i \sigma_j^T}{\|\sigma_i\| \|\sigma_j\|}.
\end{align*}
\]

where

\[
p_{ij} = E[d\tilde{W}_i d\tilde{W}_j] = \frac{1}{\|\sigma_i\| \|\sigma_j\|} \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk} = \frac{\sigma_i \sigma_j^T}{\|\sigma_i\| \|\sigma_j\|}.
\]
for $i, j = 1, 2, ..., n$. Using this proposition, we can then calculate each $\sigma_{ij}$ after obtaining $\|\sigma_i\|$ from GMM estimation.

The following equations can be then derived:

$$\delta_i = \|\sigma_i\| = \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2},$$

$$\rho_{ij} \|\sigma_i\| \|\sigma_j\| = \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk}.$$ 

Without loss of generality, we assume that the matrix $\Sigma$ is lower triangular (Dai and Singleton, 2000; Duffee, 2002; Ang and Piazzesi, 2003; Cairns and Garcia Rosas, 2004; Halberstadt and Stapf, 2012) and thus we have an exactly determined system of linear equations.

**IV. Results**

The 3-month Treasury bill yields are used to proxy the short rate. Note that we do not use the 1-month rate as Duffee (1996) reveals that Treasury bills with maturities less than 3 months behave differently from other short-term yields, for instance Eurodollar rates or Fed funds rates, and also longer-term Treasuries. Geyer and Pichler (1999) point out that in their models, the large error variance for the 1-month rate seems to imply the inappropriateness to use it as a proxy for the short rate.

**4.1. Parameter Estimates**

Table 6 shows the parameter estimates of the model and their GMM standard errors and t statistics. All parameter estimates are significantly from zero except for $\alpha_4$, $\alpha_5$, $\beta_4$, $\beta_5$, $\delta_3$, $\delta_4$ and $\delta_5$. Table 7 reports the implied estimates of model parameters so as to reflect the mean reversion process. There is evidence of statistically significant mean reversion for state
variables of GDP growth, inflation rate and unemployment rate. However, even for the
significant mean reversion processes, the size of of $\kappa_i$’s are quite small with numerical values
approaching zero, indicating their slow speed of reversion.

4.2. Model Fit for Yields

Using the parameter estimates for the linear equation of short-term interest rate (Table 5) and
diffusion processes of the state variables (Table 6, Table 7, Table 8), we model a set of zero-
coupon bond yields for each term to maturity. Figure 1 to Figure 12 show the actual and
model-implied bond yields of different maturities. The actual bond yields of all maturities
show an overall decreasing trend through the 20-year period with two humps in 1998-2003
and in 2003-2008 respectively while the model-implied yields are very much flatter and the
model seems to significantly achieve a better fit towards the end of the period.

We also construct the model-implied yield curve (Figure 14) for January 2015 and compare it
with the actual curve. The term structure follows a similar upward sloping trend as maturity
increases and also shows relatively little variations from the 3-month bond to the 1-year bond.
While the actual yield curve sees a rather gradual increment, we observe a large raise in yield
from the 5-year bond to the 10-year bond for the model-implied yield curve.

Mean Square Error (MSE) and RMSE are used to measure effectiveness of the model in
replicating bond yields. We observe large MSEs and standard deviations for all maturities,
suggesting that the model might not be able to fit the bond yields for the entire 20-year span.
These statistics therefore propel us to divide the data into 5-year periods and further test
goodness of fit for each sub-period for a closer examination of the model.

The 20-year period is divided into four sub-periods: Jan. 1995 - Dec. 1999 (period 1); Jan.
(period 4). In addition, the period during and post financial crisis, from Dec. 2008 to Dec.
2014 (period 5), is also considered as we realise that there is a huge decrease in aggregate levels of bond yields from period 3 to period 4 which may affect the efficiency of RMSE in providing information on goodness of fit of the model.

The MSEs capture the difference between the observed bond yields and model-implied bond yields. The MSE and the RMSE are calculated as follow:

\[
MSE = \frac{1}{n} \sum_{t=1}^{n} (y_t - \hat{y}_t)^2,
\]

\[
RMSE = \sqrt{\frac{\sum_{t=1}^{n} (y_t - \hat{y}_t)^2}{n}}.
\]

Looking at the overall fitness of our model for different maturities (Table 10), we observe the greatest RMSE is 376 basis points for the 3-year bond yields whereas the smallest RMSE is 149 basis points for the 10-year bond yields. The RMSEs for the 3-month, 6-month and 5-year bond yields are approximately 350 basis points but they are all smaller than the RMSE for 1-year bond yields (3.7386).

For the 3-month bond yields, the largest RMSE is 525 basis points in period 1 whereas the smallest RMSE is 18 basis points in period 4. The RMSE in period 3 (3.4668) is larger than that in period 2 (3.4066). We notice the same observation that the largest RMSE is in period 1 and the smallest RMSE is in period 4 for maturities of 6-month, 1-year, 3-year and 5-year. The RMSEs for 6-month bond yields and 1-year bond yields in period 2 are smaller than those in period 3, vice versa for the RMSEs for 3-year and 5-year bond. A distinction from bonds of other maturities is noted for the 10-year bond yields that the smallest RMSE (0.6728) is in period 3. However, the largest RMSE (2.1687) is still in period 1 and the RMSE in period 2 (1.1189) is significantly smaller than that in period 4 (1.5892).
The period from Dec. 2008 to Dec. 2014 (period 5) is singled out because we observe a large difference in volatility between period 3 and 4 which may result in inconsistency in calculation of RMSEs for these two sub-periods. From Table 10, the largest RMSE is 149 basis points for the 10-year bond yields while the smallest is 19 basis points for the 3-month bond yields. The RMSEs for the 6-month (0.2506) and 1-year (0.3272) bonds yields are both significantly smaller than those for the 3-year (0.7171) and 5-year (0.8439).

Table 11 reports the volatility of bond yields, represented by standard deviation of the yields. We observe an increasing trend for volatility of the 3-month actual bond yields from period 1 to period 2 and a drastic drop from period 2 to 4, indicated by the sharp decrease in standard deviation of bond yields between these two periods. A similar trend is observed for other maturities including the 6-month, 1-year, 3-year, 5-year and 10-year actual bond yields. Volatilities of the model-implied yields of bonds of all maturities exhibit an increasing trend from period 1 to 3 and then experience a drop in period 4. We further find that volatilities of model-implied bond yields for all maturities in period 4 are greater than those in period 1. However, they appear to be the smallest in period 4 for actual bond yields across all sub-periods. This results in much smaller differences between the actual and model-implied yield volatilities in period 4 (Table 12).

We expect that there would be a smaller RMSE associated with a smaller difference in volatilities as it could demonstrate how effective the model can predict the yield volatility. Table 12 reports the differences in volatilities and we indeed notice that for bonds of all maturities, RMSEs decrease with decreasing difference values from period 2 to 4. We do however mark that with the significantly reduced volatility differences in period 4, all bonds have much smaller RMSEs in comparison with the ones for previous sub-periods.
Table 13 reports $B_t'$ yield weight percentages for the model during the 20-year period. Specifically, we observe that GDP growth contributes approximately 77% to bond yields of all maturities, followed by inflation rate, accounting for 20% for each sub-period. Unemployment rate takes up less than 2% while risk aversion and stock market growth both bear very little influence on bond yields regardless of terms to maturity.

4.3. Effect of Macroeconomic Factors

From the basic specification for which five state variables are included, we observe that GDP and inflation rate alone account for almost 97% of the total weight (Table 13). The other variables are minor contributors to bond yields across all sub-periods. We are thus interested to further investigate how model fitness would change when the variables, especially GDP growth and inflation rate, are ruled out separately from the basic specification and thereby deduce if they are indeed appropriate variables to be included in the model given their heavy aggregate yield weights. M2, M3, M4, M5 and M6 refer to the specifications when the variables of GDP growth, inflation rate, unemployment rate, risk aversion and stock market growth are removed from the basic specification M1 respectively. The RMSEs for each model specification are reported in Table 15.

We find that RMSEs (Table 15) of M2 increase across all maturities in period 2, period 4 and period 5 but we cannot observe the same pattern in other sub-periods. RMSEs of M2 are larger than those of M1 for all maturities over the whole period, implying that the fitness of M2 is much worse than that of M1. Differences in RMSEs between M1 and M2 in period 4 to 5 are higher than those in the first three sub-periods for bonds of shorter-term maturities of 3-month and 1-year. For bonds of longer-term maturities, differences in RMSEs between M1 and M2 are in period 1 to 3 are higher than those in the latter two sub-periods. It could suggest the goodness of fit of the model fitness for bonds of shorter-term maturities in from period 4 to 5 gets increasingly worse, vice versa for bonds of longer-term maturities. Through
the F-test (Table 17), we find that null hypothesis can be rejected for the 3-month, 6-month and 1-year bond yields over the whole period at the 10% significance level whereas we cannot do so for the 3-year, 5-year and 10-year bond yields. It implies that the model performs significantly worse for shorter-term bonds than longer-term bonds when GDP growth is excluded.

We notice that RMSEs of M3 increase across all maturities during each sub-period. Comparing RMSEs of M1 and M3, RMSEs of M3 are larger than those of M1 for all maturities across the overall period such that M3 fits worse than the basic specification. All RMSEs of M3 (Table 15) in period 4 and 5 are smaller than those in the first three sub-periods and differences in RMSEs of longer-term maturities exhibit a similar pattern as M2 such that it could also imply that the goodness of fit of M3 deteriorates. More specified results can be inferred from the F-test. The test statistics show that the null hypothesis can be rejected for the 3-month, 6-month and 1-year bonds at the 10% significance level. However, it cannot be rejected for longer-term bonds, i.e. the 3-year, 5-year and 10-year, indicating that the model performs significantly worse for shorter-term bonds than longer-term bonds when inflation rate is excluded.

RMSEs of M4 increase across all maturities during each sub-period except from the 5-year bond yield to 10-year bond yield in period 1. RMSEs of M4 for all maturities in period 4 and 5 are smaller than those in the first three sub-periods. RMSEs of M4 are larger than those in M1 for all maturities in each sub-period. Differences in RMSEs between M1 and M4 (Table 16) for bonds of shorter-term maturities up to 1-year in period 4 and 5 are larger than those in the first three sub-periods. For bonds of longer-term maturities, differences in RMSEs in period 4 and 5 are smaller than those in the first three periods. It could suggest that model fitness for bonds of shorter-term maturities in latter sub-periods gets increasingly worse, vice versa for bonds of longer-term maturities. The F-test statistics (Table 17) point out to a result
that the null hypothesis can only be rejected for the 3-month, 6-month bonds at the 10% significance level, suggesting that the model is statistically better when the unemployment rate variable is included for shorter-term bonds.

We observe that RMSEs of M5 increase across bonds of shorter-term maturities in each sub-period, namely from the 3-month to 1-year, except for the 3-month to 6-month in period 4 and they are smaller in period 4 and 5 than the first three sub-periods for the same bonds. RMSEs of M5 are larger than those of M1 for all maturities in each sub-period but we note there are smaller values in some sub-periods. A similar pattern is observed for the differences in RMSEs between M1 and M5 as with those between M1 and M4 for shorter-term maturities so that we can deduce that model fitness gets increasingly worse for those bonds in period 4 and period 5. The F-test statistics (Table 17) shows that the null hypothesis can be rejected for bonds of maturities up to 1-year at the 10% significance level so that we can infer that the model is better when the risk aversion variable is incorporated at least for shorter-term bonds.

RMSEs of M6 are larger than those of M1 over the entire period and for shorter-term bonds, they increase across maturities up to 1-year in period 1, 2 and 3. For longer-term bonds, RMSEs exhibit an increasing trend across maturities from 3-year to 10-year for each sub-period except from the 5-year to 10-year in period 1. Differences in RMSEs between M1 and M6 of all maturities bonds in period 1 are larger than those in other sub-periods except for the 3-month and 1-year bond yields, and RMSEs of longer-term bonds in period 4 are smaller than those in other periods. The F-test statistics (Table 17) yet again indicate that the model performs statistically better when stock market growth is included for shorter-term bonds with the null hypothesis rejected at the 5% significance level for bonds of maturities up to 1-year.

Based on the above results, we derive that excluding each of the variables results in a worse fit compared to the basic specification, and that excluding them has greater impact on shorter-
term maturities bond yields, i.e. the 3-month, 6-month and 1-year, than longer-term bond yields including the 3-year, 5-year and 10-year. It demonstrates that these state variables are necessary to be used in the model to determine bond yields of different maturities, especially for the shorter-term bonds. We also deduce that they should be included for modelling bond yields during and post crisis.

4.4. Discussion

While the model generates yield movements that generally adhere to the actual ones after 2009, the conspicuous deviation of the model-implied yields from the actual yields for period before 2009 is baffling. We nevertheless find some elucidation in tracking the Fed’s actions as the bond yields essentially reflect its monetary policies. The massive Fed interventions during the financial crisis brought the yields to unprecedented low levels and it gives very little room for them to show huge fluctuations. The non-traditional monetary stimulus used by the Fed could foster the traditional transmission mechanism connecting monetary policies to the broader economy whose indicators happened to involve some of the selected variables. We also note that the volatilities of GDP growth and inflation rate which were the two factors with the highest yield weights decreased by at least 20% after 2009 (Table 14). It is highly possible that the restriction on adjusting the Fed Funds rate caused by sustaining the zero lower bound combined with the highly stimulated yet less volatile macroeconomic variables enables the model to perform significantly better as it becomes more sensitive to even small changes in the Fed’s positions.

The circumstance was however rather different before 2009 in that the Fed was able to adjust the Fed Funds rate to quite a degree and as it happened, it indeed did so for easing or tightening the monetary policy. The incoherent timelines of the transmission mechanism might also contribute to the failure of the model in tracking the effects of the Fed’s monetary policies on bond yields. We deduce that the model could have fallen short in reflecting the
changing policy stances as the selected macroeconomic factors constantly varied according to the Fed’s differing targets.

Smaller RMSEs for period 4 can be attributed to the fact that the actual bond yields become less volatile and experience a drastic drop in level, thus the model provides a more accurate fitting for this particular period. For bonds of shorter-term maturities, specifically the 3-month, 6-month and 1-year, we observe a pattern of decreasing RMSEs with decreasing difference in yield volatility and also significantly smaller RMSEs than those for bonds of longer-term maturities in period 4. It could suggest that our model is more appropriate for yield estimation of bonds of shorter-term maturities in period 4.

One disadvantage of using the Vasicek model is that it could produce negative interest rates which might not be realistic and are not economically meaningful. However, the deteriorating financial conditions resulted from a series of recessions and slowdowns have propelled a few countries such as Japan, Sweden and Denmark to adopt negative interest rates. For US Treasury securities, negative yields have also been observed as stated by the US Department of the Treasury (U.S. Department of the Treasury, 2015):

> Current financial market conditions, in conjunction with the low levels of interest rates during the recent financial crisis have resulted in negative yields for some Treasury securities trading in the secondary market.

The need to avoid certainty of positive interest rates in term structure modelling in this particular period is no longer necessary. Therefore, the Vasicek model becomes more pertinent and the fitness of our extended model greatly improved in period 4. The results from period 5 which indicates the time after a substantial decrease in bond yields is observed reports also significantly smaller RMSEs for bonds of all maturities except the 10-year. It
suggests that indeed the model is more effective in the unique circumstance of adopting negative interest rates.

One reason to why the model fits better for shorter-term bonds during and post crisis may be due to factors, for example liquidity issues and government debt, that have been revealed to be among the major long-run determinants of bond yields are not covered in the study. The variables with heavier yield weights in the model, i.e. GDP growth and inflation rate, correspond to the ones that would have short-run effects on the yield curve as Poghosyan (2012) discovers that short-run changes in government bond yields respond to changes in these factors. He also mentions that as the long-run relationship between bond yields and their macroeconomic determinants has weakened during the crisis, some factors such as GDP growth and inflation should be regressed as short-run variables which may suggest that these determinants would become more pertinent to modelling of short-term bond yields. The significant F-test statistics indicate that the model fitness deteriorates for shorter-term bonds when the variables of GDP growth and inflation rate are removed respectively from the basic specification, implying that they should be included if we attempt to model these bond yields in the short-run. Hence, the model not only gives a better fit for the shorter-term bonds but also achieves so during the period of crisis, as shown by the much lower RMSEs for period 5.

We find that for the shorter-term bonds, the F-test statistics are more significant when the variable of inflation is removed than when the variable of GDP growth is removed. Even though the test statistics demonstrate that GDP growth is an apt variable, the model produces a better fit with inflation rate. We could attribute the greater use of inflation rate in the model to the fact that monetary policy has an important influence on inflation especially in the short run while effects of these policy measures take longer time to transmit to production and consumption. A lower federal fund rate pushes up demand for goods and services, directly driving prices up. The transmission effect of output related activities however does not
permeate the bond yields immediately or at least as fast as price levels and thus the usefulness of adding GDP growth into the model seems not to show up as strongly as inflation rate.

The linkage between monetary policy and direct inflation could also account for the model performing better during and post crisis shown by the lower RMSEs in period 5. It is possible that although the yield weight of inflation rate is the second heaviest for bonds of all maturities, the monetary measures implemented during the crisis period which were unconventional both in terms of size and scope have triggered more effects on the shorter-term bonds as several empirical studies have established that monetary policy plays a much more significant role in short-term bond volatility than in long-term bond volatility (Evans and Marshall, 1998; Goeij and Marquering, 2006; Huang et al., 2009).

Surprisingly, we find that the model fitness deteriorates significantly when the unemployment rate variable is removed comparing the differences in RMSEs between model specifications despite the little weight it has on bond yields. In addition, the F-test statistics are significant for the model without the unemployment rate variable for shorter-term bonds even though unemployment rate typically lags behind interest rate changes and the transmission effect should be very slow. We expect that unemployment rate would affect the model fitness more for longer-term bonds as inflation expectations play a large role in setting long-term bond yields and the concomitant effects on employment can be rather significant through wage and price changes. The US long-run inflation expectations however hardly changed during the financial crisis (Trehan & Zorrilla, 2012) and thus, the model fitness does not decline as badly as we would expect in period 4 and 5.

Similar with the unemployment rate variable, we surprisingly find that even with extremely minute yield weights, RMSEs increase quite considerably when the variables of risk aversion and stock market growth are removed.
The F-test statistics for the specification without the stock market growth variable are commensurate with our expectation that in the short run, changes in stock values would impact on bond yields as they compete for investment resources. This relation also could explain why the removal of the stock market growth variable would cause the model fitness to decrease more during the financial crisis for the short-term bonds reflected by the higher increases in RMSEs. The investors might have become more cautious of market fluctuations when they allocate financial resources and the increase in uncertainty about the economic outlook that has caused the correlation between equity prices and bond yields to rise during the financial crisis (Rankin & Idil, 2014) can further lead to the increasingly weak model fitness in period 4 and 5.

V. Conclusion

The paper intends to investigate the fitness of our derived multifactor Vasicek model to the actual Treasury bond yields and the relevance of the selected macroeconomic variables for a statistically better model. We establish a simplified yet straightforward affine term structure model extended on the work of Bolder (2001) by expressing the instantaneous short-term interest rate as a linear combination of five observable state variables, including GDP growth, inflation rate, unemployment rate, risk aversion and stock market growth, in order to estimate more accurately the bond yields during a 20-year period from January 1995 to December 2014. The GMM technique is adopted to obtain the parameters in the diffusion processes of the state variables in the Vasicek model and we separate the 20-year span into five sub-periods in order to generate a more thorough and insightful analysis on the model fitness. Through interpreting the RMSEs, we find that our model achieves stronger goodness of fit for shorter-term bonds of 3-month, 6-month and 1-year maturities during and post financial crisis. However, for longer-term bonds of 3-year, 5-year and 10-year, the goodness of fit is typically less than ideal throughout the 20-year span. We also introduce other specifications of the
model by removing the macroeconomic factors respectively. The RMSEs and the F-test statistics indicate that all of the macroeconomic variables are necessary to be incorporated for a better model, including risk aversion and stock market growth which are rarely used in term structure modelling.

We note that our model is not able to produce a strong fit to the actual Treasury bond yields especially in the earlier part of the investigated period but the fit increases significantly for shorter-term bonds during and post financial crisis. It suggests that the selected macroeconomic factors are more pertinent in modelling shorter-term bond yields in this unique period. This paper provides also evidence on the feasibility of the Vasicek model during period of negative interest rate and clues to the choices of macroeconomic factors to be included for a more effective term structure modelling and the facets to be further explored for bonds of different maturities.
References


Huang, J. & Kong, W., 2005. Macroeconomic News Announcements and Corporate Bond Credit Spreads. *Available at SSRN 693341*.


[Accessed 5 May 2016].

[Accessed 23 April 2016].

Appendix A

Table 1. Definition of State Variables

<table>
<thead>
<tr>
<th>State Variables $y_i$</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>GDP Growth</td>
</tr>
<tr>
<td>$y_2$</td>
<td>Inflation Rate</td>
</tr>
<tr>
<td>$y_3$</td>
<td>Unemployment Rate Change</td>
</tr>
<tr>
<td>$y_4$</td>
<td>Risk Aversion</td>
</tr>
<tr>
<td>$y_5$</td>
<td>Stock Market Growth</td>
</tr>
</tbody>
</table>

Table 2. ADF Test –Unit root test

<table>
<thead>
<tr>
<th>$y_i$</th>
<th>ADF t-statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>-3.2073</td>
</tr>
<tr>
<td>$y_2$</td>
<td>-10.3672</td>
</tr>
<tr>
<td>$y_3$</td>
<td>-2.9189</td>
</tr>
<tr>
<td>$y_4$</td>
<td>-4.5155</td>
</tr>
<tr>
<td>$y_5$</td>
<td>-11.3940</td>
</tr>
</tbody>
</table>

Critical Values

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>-3.4580</td>
</tr>
<tr>
<td>5%</td>
<td>-2.8736</td>
</tr>
<tr>
<td>10%</td>
<td>-2.5733</td>
</tr>
</tbody>
</table>

Note: ADF test is used to test if the variables are stationary. The null hypothesis is $y_i$ has a unit root. If the ADF t-statistic value is smaller than the critical value (one-side) at the specified significance level, we can reject the null hypothesis. The test statistics indicate all $y_i$’s are stationary processes at least at the 10% significance level.
Table 3. Descriptive Summary

<table>
<thead>
<tr>
<th>$y_i$</th>
<th>Mean</th>
<th>Variance</th>
<th>Median</th>
<th>Range</th>
<th>Maximum</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.0036</td>
<td>5.3137E-06</td>
<td>0.0039</td>
<td>0.0148</td>
<td>0.0082</td>
<td>-0.0066</td>
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<tr>
<td>$y_2$</td>
<td>0.0019</td>
<td>7.9554E-06</td>
<td>0.0019</td>
<td>0.0315</td>
<td>0.0137</td>
<td>-0.0179</td>
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<tr>
<td>$y_3$</td>
<td>0.0005</td>
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<td>1E-09</td>
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<tr>
<td>$y_4$</td>
<td>20.8583</td>
<td>6.3436E+01</td>
<td>19.4700</td>
<td>49.4700</td>
<td>59.8900</td>
<td>10.4200</td>
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<tr>
<td>$y_5$</td>
<td>0.0095</td>
<td>3.3392E-03</td>
<td>0.0175</td>
<td>0.3843</td>
<td>0.1690</td>
<td>-0.2153</td>
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</tbody>
</table>

Note: Table 3 reports the summary statistics for the state variables. The number of observations is 240.

Table 4. Correlation Table

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>$y_5$</th>
</tr>
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<tr>
<td>$y_1$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_2$</td>
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<td></td>
<td></td>
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<td>$y_3$</td>
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<td>-0.0729</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_4$</td>
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<td>-0.2702</td>
<td>0.2981</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$y_5$</td>
<td>0.1970</td>
<td>0.1337</td>
<td>-0.0869</td>
<td>-0.3014</td>
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</tr>
</tbody>
</table>

Table 5. Coefficient Estimates for Short-Term Interest Rate

<table>
<thead>
<tr>
<th>a</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$b_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0190</td>
<td>2.8175</td>
<td>0.7381</td>
<td>0.0704</td>
<td>-0.0002</td>
<td>-0.0269</td>
</tr>
<tr>
<td>(0.0058)*</td>
<td>(0.6875)*</td>
<td>(0.5182)</td>
<td>(0.0549)</td>
<td>(0.0002)</td>
<td>(0.0255)</td>
</tr>
</tbody>
</table>

Note: $r = a + \sum_{i=1}^{n} b_i y_i$. $r$ is proxied using the 3-month Treasury bill yields. Coefficients are estimated by ordinary least squares. The standard errors significant at the 1% significance level are marked with *
Table 6. GMM Parameter Estimates for State Variable Diffusion Process

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Variance</th>
<th>Standard Error</th>
<th>t statistics</th>
<th>p value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td>1E-05</td>
<td>7.4479E-09</td>
<td>5.5707E-06</td>
<td>1.7951</td>
<td>0.0739</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>-0.0001</td>
<td>3.9333E-11</td>
<td>4.0483E-07</td>
<td>-247.0182</td>
<td>7.4E-290</td>
</tr>
<tr>
<td>( \delta_1 )</td>
<td>0.0002</td>
<td>1.1870E-10</td>
<td>7.0327E-07</td>
<td>284.3867</td>
<td>2E-304</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.0004</td>
<td>3.9658E-08</td>
<td>1.2855E-05</td>
<td>34.0172</td>
<td>1.42E-93</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>-0.0001</td>
<td>6.9543E-10</td>
<td>1.7022E-06</td>
<td>-58.7459</td>
<td>4.8E-144</td>
</tr>
<tr>
<td>( \delta_2 )</td>
<td>0.0003</td>
<td>1.1829E-09</td>
<td>2.2201E-06</td>
<td>112.6075</td>
<td>4.4E-209</td>
</tr>
<tr>
<td>( \alpha_3 )</td>
<td>0.0029</td>
<td>6.0267E-06</td>
<td>1.5847E-04</td>
<td>18.0902</td>
<td>1.16E-46</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>-0.0001</td>
<td>8.0764E-07</td>
<td>5.8010E-05</td>
<td>-1.7238</td>
<td>0.0860</td>
</tr>
<tr>
<td>( \delta_3 )</td>
<td>0.0023</td>
<td>1.1618E-03</td>
<td>2.2002E-03</td>
<td>1.0454</td>
<td>0.2969</td>
</tr>
<tr>
<td>( \alpha_4 )</td>
<td>1E-05</td>
<td>2.1994E-01</td>
<td>3.0272E-02</td>
<td>0.0003</td>
<td>0.9997</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>-0.0001</td>
<td>1.2178E-04</td>
<td>7.1233E-04</td>
<td>-0.1404</td>
<td>0.8885</td>
</tr>
<tr>
<td>( \delta_4 )</td>
<td>0.7000</td>
<td>7.2578E+03</td>
<td>5.4992E+00</td>
<td>0.1273</td>
<td>0.8988</td>
</tr>
<tr>
<td>( \alpha_5 )</td>
<td>1E-05</td>
<td>1.9609E-05</td>
<td>2.8584E-04</td>
<td>0.0350</td>
<td>0.9721</td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>-0.0001</td>
<td>5.5520E-06</td>
<td>1.5210E-04</td>
<td>-0.5175</td>
<td>0.5115</td>
</tr>
<tr>
<td>( \delta_5 )</td>
<td>0.0040</td>
<td>1.0370E-02</td>
<td>6.5732E-03</td>
<td>0.6085</td>
<td>0.5434</td>
</tr>
</tbody>
</table>

Note: Table 6 reports the GMM estimates of model parameters, variance, standard errors and t statistics for the five state variables in CKLS form, \( dy_t(t) = (\alpha_t + \beta_t y_t(t))dt + \delta_t d\tilde{W}_t(t) \) with \( \tilde{W}_t \)'s being correlated Wiener processes. GMM estimation employs the moment conditions listed in Section III. A joint estimation is performed for all \( y_t \)'s. Standard errors are computed based on the asymptotic normal distribution of the GMM estimate: \( \sqrt{T}(\hat{\theta} - \theta) \rightarrow N[0,(\hat{\theta}^{-1}H^{-1})^{-1}] \) where \( \hat{\theta} \) refers to the set of parameter estimates, \( \hat{\theta} \) is defined as \( \hat{\theta} = \frac{\partial E(f(x_t,\theta))}{\partial \theta} \) and \( \hat{\theta} \) is defined as \( \hat{\theta} = \hat{\theta} [f(x_t,\hat{\theta})f(x_t,\hat{\theta})]' \). The sample period is 1995:01 to 2014:12.
Table 7. Implied Parameter Estimates

<table>
<thead>
<tr>
<th>Implied Parameter</th>
<th>Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>0.1000</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>4.3728</td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>28.6667</td>
</tr>
<tr>
<td>$\kappa_3$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>0.1000</td>
</tr>
<tr>
<td>$\kappa_4$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>0.1000</td>
</tr>
<tr>
<td>$\kappa_5$</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Note: Table 7 reports the parameter estimates for state variables in Vasicek form, $dy_i(t) = \kappa_i(\theta_i - y_i(t))dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)$ with $W_i$'s being independent Wiener processes.

Table 8. The Diffusion Matrix Process $\Sigma$

$$
\Sigma = \begin{pmatrix}
0.0002 & 0 & 0 & 0 & 0 \\
-4.4105E-11 & 0.00025 & 0 & 0 & 0 \\
-1.2054E-08 & -1.9E-09 & 0.0023 & 0 & 0 \\
-0.0001 & -0.0006 & -0.0033 & 0.7000 & 0 \\
-3.2649E-08 & 8.4E-08 & 6.35E-08 & -0.0004 & 0.0040
\end{pmatrix}
$$

Note: The matrix is restricted to be lower triangular and captures the covariance between the different sources of uncertainty for independent Wiener processes, $W_i$'s.
Table 9. Parameter Estimates for Yield to Maturity

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$A'_i$</th>
<th>$B'_1$</th>
<th>$B'_2$</th>
<th>$B'_3$</th>
<th>$B'_4$</th>
<th>$B'_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 Month</td>
<td>-0.2780</td>
<td>8.4512</td>
<td>2.2141</td>
<td>0.2112</td>
<td>-0.0005</td>
<td>-0.0807</td>
</tr>
<tr>
<td>6 Month</td>
<td>-0.5140</td>
<td>16.8998</td>
<td>4.4274</td>
<td>0.4223</td>
<td>-0.0010</td>
<td>-0.1614</td>
</tr>
<tr>
<td>1 Year</td>
<td>-0.6726</td>
<td>33.7894</td>
<td>8.8522</td>
<td>0.8443</td>
<td>-0.0019</td>
<td>-0.3227</td>
</tr>
<tr>
<td>3 Year</td>
<td>9.7273</td>
<td>101.2468</td>
<td>26.5249</td>
<td>2.5299</td>
<td>-0.0058</td>
<td>-0.9669</td>
</tr>
<tr>
<td>5 Year</td>
<td>55.6512</td>
<td>168.5425</td>
<td>44.1551</td>
<td>4.2114</td>
<td>-0.0096</td>
<td>-1.6096</td>
</tr>
<tr>
<td>10 Year</td>
<td>480.9046</td>
<td>336.0767</td>
<td>88.0461</td>
<td>8.3976</td>
<td>-0.0192</td>
<td>-3.2095</td>
</tr>
</tbody>
</table>

Note: The equation $\tau YTM(\tau) = A'(\tau) - \sum_{i=1}^{n} B'_i(\tau)y_i$ is estimated by ordinary least squares. YTM are the observed monthly yields from 1995:01 to 2014:12.
Table 10. Measurement Error Statistics

<table>
<thead>
<tr>
<th></th>
<th>3m</th>
<th>6m</th>
<th>1yr</th>
<th>3yr</th>
<th>5yr</th>
<th>10yr</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1995.01-2014.12 (Overall)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>2.7832</td>
<td>2.8983</td>
<td>2.9756</td>
<td>3.1192</td>
<td>2.8616</td>
<td>0.3776</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>2.2564</td>
<td>2.2891</td>
<td>2.2680</td>
<td>2.1006</td>
<td>1.8491</td>
<td>1.4490</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.1048</td>
<td>0.1263</td>
<td>0.1566</td>
<td>0.0621</td>
<td>-0.3052</td>
<td>-2.4752</td>
</tr>
<tr>
<td>RMSE</td>
<td>3.5800</td>
<td>3.6903</td>
<td>3.7386</td>
<td>3.7581</td>
<td>3.4050</td>
<td>1.4944</td>
</tr>
<tr>
<td><strong>1995.01-1999.12 (period 1)</strong></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Mean</td>
<td>5.2409</td>
<td>5.3885</td>
<td>5.5104</td>
<td>5.5357</td>
<td>4.9790</td>
<td>2.0571</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.3828</td>
<td>0.4275</td>
<td>0.5119</td>
<td>0.6450</td>
<td>0.6904</td>
<td>0.6924</td>
</tr>
<tr>
<td>Minimum</td>
<td>4.1775</td>
<td>4.3005</td>
<td>4.1908</td>
<td>3.9246</td>
<td>3.2672</td>
<td>0.5372</td>
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<tr>
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<td>29.2160</td>
<td>30.6224</td>
<td>31.0525</td>
<td>25.2591</td>
<td>4.7031</td>
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<tr>
<td>RMSE</td>
<td>5.2546</td>
<td>5.4052</td>
<td>5.5337</td>
<td>5.5725</td>
<td>5.0258</td>
<td>2.1687</td>
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<td><strong>2000.01-2004.12 (period 2)</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>2.8072</td>
<td>2.8989</td>
<td>3.0102</td>
<td>3.3965</td>
<td>3.2655</td>
<td>0.7916</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1.9462</td>
<td>1.9550</td>
<td>1.8248</td>
<td>1.5238</td>
<td>1.2075</td>
<td>0.7974</td>
</tr>
<tr>
<td>Minimum</td>
<td>1.0048</td>
<td>1.0446</td>
<td>1.0850</td>
<td>1.2587</td>
<td>1.3614</td>
<td>-0.6587</td>
</tr>
<tr>
<td>Maximum</td>
<td>6.4546</td>
<td>6.4864</td>
<td>6.3968</td>
<td>6.5105</td>
<td>5.7738</td>
<td>2.6585</td>
</tr>
<tr>
<td>RMSE</td>
<td>3.4066</td>
<td>3.4874</td>
<td>3.5123</td>
<td>3.7174</td>
<td>3.4781</td>
<td>1.1189</td>
</tr>
<tr>
<td>Period</td>
<td>3m</td>
<td>6m</td>
<td>1yr</td>
<td>3yr</td>
<td>5yr</td>
<td>10yr</td>
</tr>
<tr>
<td>--------------------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>2005.01-2009.12</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>2.9132</td>
<td>3.0989</td>
<td>3.1346</td>
<td>3.0740</td>
<td>2.7189</td>
<td>0.1247</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1.8952</td>
<td>1.8723</td>
<td>1.7627</td>
<td>1.3643</td>
<td>1.0609</td>
<td>0.6667</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.1048</td>
<td>0.2406</td>
<td>0.3710</td>
<td>0.7820</td>
<td>0.5747</td>
<td>-1.6053</td>
</tr>
<tr>
<td>Maximum</td>
<td>5.2636</td>
<td>5.3668</td>
<td>5.2872</td>
<td>4.8282</td>
<td>4.1509</td>
<td>1.1108</td>
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<tr>
<td>MSE</td>
<td>12.0190</td>
<td>13.0502</td>
<td>12.8813</td>
<td>11.2799</td>
<td>8.4991</td>
<td>0.4527</td>
</tr>
<tr>
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<td>3.4668</td>
<td>3.6125</td>
<td>3.5891</td>
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<td>0.6728</td>
</tr>
<tr>
<td>2010.01-2014.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.1716</td>
<td>0.2069</td>
<td>0.2473</td>
<td>0.4707</td>
<td>0.4830</td>
<td>-1.4630</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0458</td>
<td>0.0543</td>
<td>0.0846</td>
<td>0.3502</td>
<td>0.5395</td>
<td>0.6259</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.1090</td>
<td>0.1263</td>
<td>0.1566</td>
<td>0.0621</td>
<td>-0.3052</td>
<td>-2.4752</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.2612</td>
<td>0.3342</td>
<td>0.5146</td>
<td>1.3783</td>
<td>1.6610</td>
<td>-0.1491</td>
</tr>
<tr>
<td>MSE</td>
<td>0.0315</td>
<td>0.0457</td>
<td>0.0682</td>
<td>0.3421</td>
<td>0.5196</td>
<td>2.5256</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.1775</td>
<td>0.2138</td>
<td>0.2611</td>
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<td>0.7208</td>
<td>1.5892</td>
</tr>
<tr>
<td>Mean</td>
<td>0.1826</td>
<td>0.2352</td>
<td>0.2972</td>
<td>0.5878</td>
<td>0.6127</td>
<td>-1.3478</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0574</td>
<td>0.0870</td>
<td>0.1378</td>
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<td>0.5843</td>
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</tr>
<tr>
<td>Minimum</td>
<td>0.1048</td>
<td>0.1263</td>
<td>0.1566</td>
<td>0.0621</td>
<td>-0.3052</td>
<td>-2.4752</td>
</tr>
<tr>
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<td>1.7864</td>
<td>-0.1491</td>
</tr>
<tr>
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<td>0.0628</td>
<td>0.1071</td>
<td>0.5142</td>
<td>0.7122</td>
<td>2.2259</td>
</tr>
<tr>
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<td>0.2506</td>
<td>0.3272</td>
<td>0.7171</td>
<td>0.8439</td>
<td>1.4919</td>
</tr>
</tbody>
</table>

Note: Measurement error is the difference between actual yields and model-implied yields for different maturities. Table 10 reports mean, standard deviation, minimum and maximum of the measurement error. MSEs and RMSEs of the model for the overall period and sub-periods are calculated as $RMSE = \sqrt{\frac{\sum_{t=1}^{n}(y_t - \hat{y}_t)^2}{n}}$ and $MSE = \frac{1}{n} \sum_{t=1}^{n}(y_t - \hat{y}_t)^2$. 
Table 11. Volatility of Yields

<table>
<thead>
<tr>
<th>Actual yields</th>
<th>3m</th>
<th>6m</th>
<th>1yr</th>
<th>3yr</th>
<th>5yr</th>
<th>10yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>overall period</td>
<td>2.2494</td>
<td>2.2821</td>
<td>2.2610</td>
<td>2.0938</td>
<td>1.8428</td>
<td>1.4436</td>
</tr>
<tr>
<td>period 1</td>
<td>0.3802</td>
<td>0.4244</td>
<td>0.5080</td>
<td>0.6397</td>
<td>0.6846</td>
<td>0.6864</td>
</tr>
<tr>
<td>period 2</td>
<td>1.9305</td>
<td>1.9391</td>
<td>1.8099</td>
<td>1.5114</td>
<td>1.1979</td>
<td>0.7913</td>
</tr>
<tr>
<td>period 3</td>
<td>1.8737</td>
<td>1.8512</td>
<td>1.7426</td>
<td>1.3469</td>
<td>1.0460</td>
<td>0.6554</td>
</tr>
<tr>
<td>period 4</td>
<td>0.0456</td>
<td>0.0541</td>
<td>0.0838</td>
<td>0.3465</td>
<td>0.5341</td>
<td>0.6195</td>
</tr>
<tr>
<td>period 5</td>
<td>0.0581</td>
<td>0.0889</td>
<td>0.1391</td>
<td>0.4112</td>
<td>0.5800</td>
<td>0.6391</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model-implied yields</th>
<th>3m</th>
<th>6m</th>
<th>1yr</th>
<th>3yr</th>
<th>5yr</th>
<th>10yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>overall period</td>
<td>0.0072</td>
<td>0.0072</td>
<td>0.0072</td>
<td>0.0072</td>
<td>0.0071</td>
<td>0.0071</td>
</tr>
<tr>
<td>period 1</td>
<td>0.0045</td>
<td>0.0045</td>
<td>0.0045</td>
<td>0.0045</td>
<td>0.0045</td>
<td>0.0045</td>
</tr>
<tr>
<td>period 2</td>
<td>0.0055</td>
<td>0.0055</td>
<td>0.0055</td>
<td>0.0055</td>
<td>0.0054</td>
<td>0.0054</td>
</tr>
<tr>
<td>period 3</td>
<td>0.0105</td>
<td>0.0105</td>
<td>0.0105</td>
<td>0.0104</td>
<td>0.0104</td>
<td>0.0104</td>
</tr>
<tr>
<td>period 4</td>
<td>0.0050</td>
<td>0.0050</td>
<td>0.0050</td>
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<td>0.0050</td>
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<tr>
<td>period 5</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
</tr>
</tbody>
</table>

Note: Volatilities of yields are estimated by standard deviations of yield series. Period 1 is from 1995:01 to 1999:12; period 2 is from 2000:01 to 2004:12; period 3 is from 2005:01 to 2009:12; period 4 is from 2010:01 to 2014:12; period 5 is from 2008:12 to 2014:12.
Table 12. Difference of Volatility

<table>
<thead>
<tr>
<th>Difference of volatility</th>
<th>3m</th>
<th>6m</th>
<th>1yr</th>
<th>3yr</th>
<th>5yr</th>
<th>10yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>0.3757</td>
<td>0.4199</td>
<td>0.5035</td>
<td>0.6351</td>
<td>0.6801</td>
<td>0.6819</td>
</tr>
<tr>
<td>Period 2</td>
<td>1.9250</td>
<td>1.9337</td>
<td>1.8045</td>
<td>1.5060</td>
<td>1.1925</td>
<td>0.7859</td>
</tr>
<tr>
<td>Period 3</td>
<td>1.8633</td>
<td>1.8407</td>
<td>1.7321</td>
<td>1.3364</td>
<td>1.0356</td>
<td>0.6450</td>
</tr>
<tr>
<td>Period 4</td>
<td>0.0406</td>
<td>0.0491</td>
<td>0.0788</td>
<td>0.3415</td>
<td>0.5291</td>
<td>0.6145</td>
</tr>
<tr>
<td>Period 5</td>
<td>0.0519</td>
<td>0.0827</td>
<td>0.1329</td>
<td>0.4050</td>
<td>0.5738</td>
<td>0.6329</td>
</tr>
</tbody>
</table>

Note: The difference of volatility is calculated by the volatility of actual yields minus the volatility of model-implied yields of different maturities for each sub-period. Period 1 is from 1995:01 to 1999:12; period 2 is from 2000:01 to 2004:12; period 3 is from 2005:01 to 2009:12; period 4 is from 2010:01 to 2014:12; period 5 is from 2008:12 to 2014:12.

Table 13. Yield Weight Percentage

<table>
<thead>
<tr>
<th></th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 Month</td>
<td>0.77126</td>
<td>0.20206</td>
<td>0.01927</td>
<td>0.00004</td>
<td>0.00737</td>
</tr>
<tr>
<td>6 Month</td>
<td>0.77126</td>
<td>0.20206</td>
<td>0.01927</td>
<td>0.00004</td>
<td>0.00737</td>
</tr>
<tr>
<td>1 Year</td>
<td>0.77126</td>
<td>0.20206</td>
<td>0.01927</td>
<td>0.00004</td>
<td>0.00737</td>
</tr>
<tr>
<td>3 Year</td>
<td>0.77126</td>
<td>0.20206</td>
<td>0.01927</td>
<td>0.00004</td>
<td>0.00737</td>
</tr>
<tr>
<td>5 Year</td>
<td>0.77126</td>
<td>0.20206</td>
<td>0.01927</td>
<td>0.00004</td>
<td>0.00737</td>
</tr>
<tr>
<td>10 Year</td>
<td>0.77126</td>
<td>0.20206</td>
<td>0.01927</td>
<td>0.00004</td>
<td>0.00737</td>
</tr>
</tbody>
</table>

Note: Yield weight percentage $= \frac{B_i}{\sum(B_1 + B_2 + B_3 + B_4 + B_5)}$, $i = 1, 2, ..., 5$. Period 1 is from 1995:01 to 1999:12; period 2 is from 2000:01 to 2004:12; period 3 is from 2005:01 to 2009:12; period 4 is from 2010:01 to 2014:12; period 5 is from 2008:12 to 2014:12.
Table 14. Volatility of State Variables

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>$y_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before 2009</td>
<td>0.0024</td>
<td>0.0031</td>
<td>0.0285</td>
<td>7.8775</td>
<td>0.0642</td>
</tr>
<tr>
<td>After 2009</td>
<td>0.0019</td>
<td>0.0021</td>
<td>0.0247</td>
<td>8.1053</td>
<td>0.0375</td>
</tr>
<tr>
<td>% change</td>
<td>0.2182</td>
<td>0.3210</td>
<td>0.1359</td>
<td>-0.0289</td>
<td>0.4155</td>
</tr>
</tbody>
</table>

Note: Volatility is calculated by standard deviation of the series. Before 2009 refers to period from 1995:01 to 2008:12. After 2009 refers to the period from 2009:01 to 2014:12.
Table 15. RMSEs for Different Model Specifications

<table>
<thead>
<tr>
<th></th>
<th>3 Month</th>
<th>6 Month</th>
<th>1 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M1</td>
<td>M2</td>
<td>M3</td>
</tr>
<tr>
<td>period 1</td>
<td>5.2546</td>
<td>5.2493</td>
<td>5.1823</td>
</tr>
<tr>
<td>period 4</td>
<td>0.1775</td>
<td>0.1770</td>
<td>0.1114</td>
</tr>
<tr>
<td>period 5</td>
<td>0.1913</td>
<td>0.1912</td>
<td>0.1268</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3 Year</th>
<th>5 Year</th>
<th>10 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>period 1</td>
<td>5.5725</td>
<td>5.8834</td>
<td>5.8623</td>
</tr>
<tr>
<td>period 4</td>
<td>0.5849</td>
<td>0.8611</td>
<td>0.8417</td>
</tr>
<tr>
<td>period 5</td>
<td>0.7171</td>
<td>0.9943</td>
<td>0.9759</td>
</tr>
</tbody>
</table>

Note: M1 refers to the basic specification for which all five variables are included, namely GDP, inflation rate, unemployment rate, risk aversion and stock market growth. M2 refers to specification of M1 excluding the variable GDP. M3 refers to specification of M1 excluding the variable inflation rate. M4 refers to specification of M1 excluding the variable Unemployment rate. M5 refers to specification of M1 excluding the variable risk aversion. M6 refers to specification of M1 excluding the variable stock market growth. Period 1 is from 1995:01 to 1999:12; period 2 is from 2000:01 to 2004:12; period 3 is from 2005:01 to 2009:12; period 4 is from 2010:01 to 2014:12; period 5 is from 2008:12 to 2014:12.
<table>
<thead>
<tr>
<th></th>
<th>3 Month</th>
<th>6 Month</th>
<th>1 Year</th>
<th>3 Year</th>
<th>5 Year</th>
<th>10 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>period 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2-M1</td>
<td>-0.0053</td>
<td>-0.0723</td>
<td>0.0663</td>
<td>0.1517</td>
<td>-0.0649</td>
<td>0.0159</td>
</tr>
<tr>
<td>M3-M1</td>
<td>-0.0034</td>
<td>-0.0588</td>
<td>0.0372</td>
<td>-0.0022</td>
<td>0.0827</td>
<td>0.0415</td>
</tr>
<tr>
<td>M4-M1</td>
<td>-0.0009</td>
<td>-0.0595</td>
<td>0.0188</td>
<td>-0.0042</td>
<td>0.1504</td>
<td>-0.0536</td>
</tr>
<tr>
<td>M5-M1</td>
<td>-0.0005</td>
<td>-0.0661</td>
<td>0.0118</td>
<td>0.0461</td>
<td>0.0654</td>
<td>0.0420</td>
</tr>
<tr>
<td>M6-M1</td>
<td>-0.0001</td>
<td>-0.0645</td>
<td>0.0122</td>
<td>0.0487</td>
<td>0.0597</td>
<td>0.0654</td>
</tr>
<tr>
<td><strong>period 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2-M1</td>
<td>0.3109</td>
<td>0.2898</td>
<td>0.3494</td>
<td>0.2731</td>
<td>0.8773</td>
<td>0.9390</td>
</tr>
<tr>
<td>M3-M1</td>
<td>0.2887</td>
<td>0.2699</td>
<td>0.3253</td>
<td>0.2538</td>
<td>-0.7059</td>
<td>0.9024</td>
</tr>
<tr>
<td>M4-M1</td>
<td>0.2928</td>
<td>0.2727</td>
<td>0.3280</td>
<td>0.2566</td>
<td>-1.2676</td>
<td>0.9026</td>
</tr>
<tr>
<td>M5-M1</td>
<td>0.2761</td>
<td>0.2568</td>
<td>0.3116</td>
<td>0.2928</td>
<td>0.2407</td>
<td>-3.5534</td>
</tr>
<tr>
<td>M6-M1</td>
<td>0.2772</td>
<td>0.2588</td>
<td>0.3132</td>
<td>0.2954</td>
<td>0.2434</td>
<td>-3.4158</td>
</tr>
<tr>
<td><strong>period 3</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2-M1</td>
<td>0.3109</td>
<td>0.2898</td>
<td>0.3494</td>
<td>0.2731</td>
<td>0.8773</td>
<td>0.9390</td>
</tr>
<tr>
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<td>0.2538</td>
<td>-0.7059</td>
<td>0.9024</td>
</tr>
<tr>
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<td>0.2727</td>
<td>0.3280</td>
<td>0.2566</td>
<td>-1.2676</td>
<td>0.9026</td>
</tr>
<tr>
<td>M5-M1</td>
<td>0.2761</td>
<td>0.2568</td>
<td>0.3116</td>
<td>0.2928</td>
<td>0.2407</td>
<td>-3.5534</td>
</tr>
<tr>
<td>M6-M1</td>
<td>0.2772</td>
<td>0.2588</td>
<td>0.3132</td>
<td>0.2954</td>
<td>0.2434</td>
<td>-3.4158</td>
</tr>
<tr>
<td><strong>period 4</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2-M1</td>
<td>0.3109</td>
<td>0.2898</td>
<td>0.3494</td>
<td>0.2731</td>
<td>0.8773</td>
<td>0.9390</td>
</tr>
<tr>
<td>M3-M1</td>
<td>0.2887</td>
<td>0.2699</td>
<td>0.3253</td>
<td>0.2538</td>
<td>-0.7059</td>
<td>0.9024</td>
</tr>
<tr>
<td>M4-M1</td>
<td>0.2928</td>
<td>0.2727</td>
<td>0.3280</td>
<td>0.2566</td>
<td>-1.2676</td>
<td>0.9026</td>
</tr>
<tr>
<td>M5-M1</td>
<td>0.2761</td>
<td>0.2568</td>
<td>0.3116</td>
<td>0.2928</td>
<td>0.2407</td>
<td>-3.5534</td>
</tr>
<tr>
<td>M6-M1</td>
<td>0.2772</td>
<td>0.2588</td>
<td>0.3132</td>
<td>0.2954</td>
<td>0.2434</td>
<td>-3.4158</td>
</tr>
<tr>
<td><strong>period 5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2-M1</td>
<td>0.3109</td>
<td>0.2898</td>
<td>0.3494</td>
<td>0.2731</td>
<td>0.8773</td>
<td>0.9390</td>
</tr>
<tr>
<td>M3-M1</td>
<td>0.2887</td>
<td>0.2699</td>
<td>0.3253</td>
<td>0.2538</td>
<td>-0.7059</td>
<td>0.9024</td>
</tr>
<tr>
<td>M4-M1</td>
<td>0.2928</td>
<td>0.2727</td>
<td>0.3280</td>
<td>0.2566</td>
<td>-1.2676</td>
<td>0.9026</td>
</tr>
<tr>
<td>M5-M1</td>
<td>0.2761</td>
<td>0.2568</td>
<td>0.3116</td>
<td>0.2928</td>
<td>0.2407</td>
<td>-3.5534</td>
</tr>
<tr>
<td>M6-M1</td>
<td>0.2772</td>
<td>0.2588</td>
<td>0.3132</td>
<td>0.2954</td>
<td>0.2434</td>
<td>-3.4158</td>
</tr>
</tbody>
</table>

Note: Table 16 reports the difference in RMSEs between M1 and other model specifications. M1 refers to the basic specification for which all five variables are included, namely GDP, inflation rate, unemployment rate, risk aversion and stock market growth. M2 refers to specification of M1 excluding the variable GDP. M3 refers to specification of M1 excluding the variable inflation rate. M4 refers to specification of M1 excluding the variable Unemployment rate. M5 refers to specification of M1 excluding the variable risk aversion. M6 refers to specification of M1 excluding the variable stock market growth. Period 1 is from 1995:01 to 1999:12; period 2 is from 2000:01 to 2004:12; period 3 is from 2005:01 to 2009:12; period 4 is from 2010:01 to 2014:12; period 5 is from 2008:12 to 2014:12.
Table 17. F-test statistics

<table>
<thead>
<tr>
<th></th>
<th>3 Month</th>
<th>6 Month</th>
<th>1 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M2</td>
<td>M3</td>
<td>M4</td>
</tr>
<tr>
<td>RRSS</td>
<td>3076.80</td>
<td>3076.15</td>
<td>3089.13</td>
</tr>
<tr>
<td>URRS</td>
<td>3075.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F statistic</td>
<td>0.0003**</td>
<td>0.0001*</td>
<td>0.0043***</td>
</tr>
<tr>
<td>p-value</td>
<td>0.0130</td>
<td>0.0059</td>
<td>0.0521</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3 Year</th>
<th>5 Year</th>
<th>10 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M2</td>
<td>M3</td>
<td>M4</td>
</tr>
<tr>
<td>RRSS</td>
<td>3884.22</td>
<td>3849.59</td>
<td>3948.44</td>
</tr>
<tr>
<td>URRS</td>
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<td></td>
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<tr>
<td>F statistic</td>
<td>0.1459</td>
<td>0.1357</td>
<td>0.1649</td>
</tr>
<tr>
<td>p-value</td>
<td>0.2972</td>
<td>0.2871</td>
<td>0.3149</td>
</tr>
</tbody>
</table>

Note: M2 refers to specification excluding the variable GDP. M3 refers to specification excluding the variable inflation rate. M4 refers to specification excluding the variable Unemployment rate. M5 refers to specification excluding the variable risk aversion. M6 refers to specification excluding the variable stock market growth. The F-test statistic is test statistic = (RRSS − URRS) / URSS × (T − k) / m. RRSS equals to the residuals sum of squared from restricted regression. URRS represents the residuals sum of squared from unrestricted regression. The number of total observations (T) is 240. The number of restrictions (m) is 1. The number of regressors in unrestricted regression including constant (k) is 6. Two degrees of freedom are 1 and 234. The sample period is from 1995:01 to 2014:12. The null hypothesis is the coefficients of each state variable are jointly zero. The alternative hypothesis is that at least one coefficient of state variable is not zero. The F-test statistics marked with * (**) (***) indicate that the null hypothesis can be rejected for short maturities up to 1 year at the 1% (5%) (10%) significance level except M4 for the 1-year bond.
Appendix B

Derivation of the Multifactor Vasicek Model

The differential dynamics of the zero-coupon bond price is determined using Itô’s Lemma:

\[ dP = P_t dt + \sum_{i=1}^{n} P_{y_i} dy_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} P_{y_iy_j} d(y_i, y_j)(t), \]

where

\[ P_{y_i} = \frac{\partial P}{\partial y_i}, \]

and

\[ P_{y_iy_j} = \frac{\partial P}{\partial y_i \partial y_j}. \]

Following the steps in Bolder (2001), we derive a partial differential equation for arbitrary maturity \( t \):

\[ P_t + \sum_{i=1}^{n} \kappa_i (\bar{\theta}_i - y_i) P_{y_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} P_{y_iy_j} - r P(t, T) = 0. \]

The instantaneous short-term rate enters the equation in the last term, \( r = a + \sum_{i=1}^{n} b_i y_i \), and thus we write:

\[ P_t + \sum_{i=1}^{n} \kappa_i (\bar{\theta}_i - y_i) P_{y_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} P_{y_iy_j} - (a + \sum_{i=1}^{n} b_i y_i) P(t, T) = 0. \]

Given the boundary condition \( P(T, T, y_1, y_2, \ldots, y_n) = 1 \), for the bond price function. An affine term-structure is maintained even though multiple state variables are added into the model. Denoting term to maturity \( T - t = \tau \), the solution has the general form

\[ P(\tau, y_1, y_2, \ldots, y_n) = e^{A(\tau) - \sum_{i=1}^{n} B_i(\tau) y_i}. \]

The partial derivatives are computed as follows:

\[ P_t = \left( -A'(\tau) + \sum_{i=1}^{n} B_i'(\tau) y_i \right) e^{A(\tau) - \sum_{i=1}^{n} B_i(\tau) y_i} = \left( -A'(\tau) + \sum_{i=1}^{n} B_i'(\tau) y_i \right) P(\tau), \]
\[ P_{y_{i}} = -B_{i}(\tau)e^{A(\tau)-\sum_{i=1}^{n}b_{i}(\tau)y_{i}} = -B_{i}(\tau)P(\tau), \]
\[ P_{y_{j}y_{j}} = B_{i}(\tau)B_{j}(\tau)e^{A(\tau)-\sum_{i=1}^{n}b_{i}(\tau)y_{i}} = B_{i}(\tau)B_{j}(\tau)P(\tau), \]
for \( i, j = 1, 2, \ldots, n. \)

Substitute them into the partial differential equation, we have

\[
P_{t} + \sum_{i=1}^{n} \kappa_{i}(\bar{\theta}_{i} - y_{i})P_{y_{i}} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}P_{y_{i}y_{j}} - (a + \sum_{i=1}^{n} b_{i}y_{i})P(t, T) = 0
\]

\[
\Rightarrow \left( -A'(\tau) + \sum_{i=1}^{n} B'_{i}(\tau)y_{i} \right)P(\tau) - \sum_{i=1}^{n} \kappa_{i}(\bar{\theta}_{i} - y_{i})B_{i}(\tau)P(\tau) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_{ij}}{2}B_{i}(\tau)B_{j}(\tau)P(\tau)
\]

\[
- \left( a + \sum_{i=1}^{n} b_{i}y_{i} \right)P(\tau) = 0
\]

\[
\Rightarrow -A'(\tau) + \sum_{i=1}^{n} B'_{i}(\tau)y_{i} - \sum_{i=1}^{n} B_{i}(\tau)\kappa_{i}(\bar{\theta}_{i} - y_{i}) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_{ij}}{2}B_{i}(\tau)B_{j}(\tau) - \left( a + \sum_{i=1}^{n} b_{i}y_{i} \right)
\]

\[
= 0
\]

\[
\Rightarrow -A'(\tau) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_{ij}}{2}B_{i}(\tau)B_{j}(\tau) - \sum_{i=1}^{n} \kappa_{i}\bar{\theta}_{i}B_{i}(\tau) - \sum_{i=1}^{n} \left( b_{i} - B'_{i}(\tau) - \kappa_{i}B_{i}(\tau) \right)y_{i} - a
\]

\[
= 0.
\]

We thus obtain two ordinary differential equations as follows:

\[ B'_{i}(\tau) + \kappa_{i}B_{i}(\tau) = b_{i}, \]
\[ -A'(\tau) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_{ij}}{2}B_{i}(\tau)B_{j}(\tau) - \sum_{i=1}^{n} \kappa_{i}\bar{\theta}_{i}B_{i}(\tau) = a, \]
for \( i, j = 1, 2, \ldots, n. \) The boundary conditions are defined as

\[ A(0) = B_{1}(0) = \cdots = B_{n}(0) = 0. \]

We solve for the first ODE as:

\[ B_{i}(\tau) = \frac{b_{i}}{\kappa_{i}}(1 - e^{-\kappa_{i}\tau}). \]
To find $A(\tau)$, we first integrate the second ODE:

$$\int_t^T (a + A'(T - s)) ds = \int_t^T \left( \sum_{i=1}^n \sum_{j=1}^n \frac{\sigma_{ij}}{2} B_i(T - s) B_j(T - s) - \sum_{i=1}^n \kappa_i \tilde{\theta}_i B_i(T - s) \right) ds$$

$$\Rightarrow a(T - t) + A(T - t) - A(0)$$

$$= \int_t^T \sum_{i=1}^n \sum_{j=1}^n \frac{\sigma_{ij}}{2} B_i(T - s) B_j(T - s) ds - \int_t^T \sum_{i=1}^n \kappa_i \tilde{\theta}_i B_i(T - s) ds$$

$$\Rightarrow \alpha T + A(\tau) = \sum_{i=1}^n \sum_{j=1}^n \frac{\sigma_{ij}}{2} \int_t^T B_i(T - s) B_j(T - s) ds - \sum_{i=1}^n \kappa_i \tilde{\theta}_i \int_t^T B_i(T - s) ds.$$

To solve the first integral:

$$\frac{\sigma_{ij}}{2} \int_t^T B_i(T - s) B_j(T - s) ds$$

$$= \frac{\sigma_{ij}}{2} \int_t^T \frac{b_i}{\kappa_i} (1 - e^{-\kappa_i(T-s)}) \frac{b_j}{\kappa_j} (1 - e^{-\kappa_j(T-s)}) ds$$

$$= \frac{\sigma_{ij} b_i b_j}{2 \kappa_i \kappa_j} \int_t^T (1 - e^{-\kappa_i(T-s)})(1 - e^{-\kappa_j(T-s)}) ds$$

$$= \frac{\sigma_{ij} b_i b_j}{2 \kappa_i \kappa_j} \int_t^T (1 - e^{-\kappa_i(T-s)} - e^{-\kappa_j(T-s)} + e^{-(\kappa_i + \kappa_j)(T-s)}) ds$$

$$= \frac{\sigma_{ij} b_i b_j}{2 \kappa_i \kappa_j} \left( (T - t) - \frac{1}{\kappa_i} (1 - e^{-\kappa_i(T-t)}) - \frac{1}{\kappa_j} (1 - e^{-\kappa_j(T-t)}) \right)$$

$$+ \frac{1}{\kappa_i + \kappa_j} \left( 1 - e^{-(\kappa_i + \kappa_j)(T-t)} \right)$$

$$= \frac{\sigma_{ij}}{2 \kappa_i \kappa_j} \left( b_i b_j (T - t) - b_j B_i(\tau) - b_i B_j(\tau) + \frac{b_i b_j}{\kappa_i \kappa_j} \left( 1 - e^{-(\kappa_i + \kappa_j)\tau} \right) \right).$$

To solve the second integral:
\[ \kappa_i \bar{\theta}_i \int_t^T B_i(T - s) \, ds \]

\[ = \kappa_i \bar{\theta}_i \int_t^T \frac{b_i}{\kappa_i} \left( 1 - e^{-\kappa_i(T - s)} \right) \, ds \]

\[ = \bar{\theta}_i b_i \int_t^T \left( 1 - e^{-\kappa_i(T - s)} \right) \, ds \]

\[ = \bar{\theta}_i b_i \left[ s - \frac{e^{-\kappa_i(T - s)}}{\kappa_i} \right]_t^T \]

\[ = \bar{\theta}_i b_i \left( T - t \right) - \frac{1}{\kappa_i} \left( 1 - e^{-\kappa_i(T - t)} \right) \]

\[ = \bar{\theta}_i \left( b_i \tau - B_i(\tau) \right). \]

We then add the two integrals to find \( A(\tau) \), and thus:

\[ A(\tau) = \sum_{i=1}^n \sum_{j=1}^n \frac{a_{ij}}{2 \kappa_i \kappa_j} \left( b_i b_j \tau - b_j B_i(\tau) - b_i B_j(\tau) + \frac{b_i b_j}{\kappa_i + \kappa_j} \left( 1 - e^{-\left(\kappa_i + \kappa_j\right)\tau} \right) \right) \]

\[ - \sum_{i=1}^n \bar{\theta}_i \left( b_i \tau - B_i(\tau) \right) - \alpha \tau. \]
Appendix C

Figure 1. 3-month actual Treasury yields and model-implied Treasury yields

Figure 2. 3-month model-implied Treasury yields
Figure 3. 6-month actual Treasury yields and model-implied Treasury yields

Figure 4. 6-month model-implied Treasury yields
Figure 5. 1-year actual Treasury yields and model-implied Treasury yields

Figure 6. 1-year model-implied Treasury yields
Figure 7. 3-year actual Treasury yields and model-implied Treasury yields

Figure 8. 3-year model-implied Treasury yields
Figure 9. 5-year actual Treasury yields and model-implied Treasury yields

Figure 10. 5-year model-implied Treasury yields
Figure 11. 10-year actual Treasury yields and model-implied Treasury yields

Figure 12. 10-year model-implied Treasury yields
Figure 13. Yield Weights

![Yield Weights Diagram]

Figure 14. Yield Curve

![Yield Curve Diagram]