A Floating Currency Macro Term Structure Model

Evidence of unspanned latent exchange rate effects in the US T-bill term structure

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ABSTRACT
During the last decade there has been many advances in the field of research focusing on term structure models that include macroeconomic risks. The fact that such risks adds to the predictive power of risk premia is evident. However, there is no such models that includes exchange rate dynamics and accounts for these potentially latent effects on the yield curve.

This thesis presents a discussion on term structure models. A concept for pricing bonds on the entire range of maturities. Specifically, we look at the family of term structure models called macro-finance term structure models (MTSM), which takes the standard framework of the standard term structure models and adds sources of macroeconomic risks. Our discussion focuses on the role of the exchange rate dynamics, motivating a formulation that can include it, and investigating to see if it adds any information in describing the bond risk premium. Our vantage point comes from that of the unspanned MTSM, and subsequently modifying it to accommodate for exchange rate effects. We are able to present regression evidence supporting our idea of a latent exchange rate effect in the bond term structure.

Keywords: Macro-finance, term structure models, exchange rate risks.
JEL classification: E43, E44, G12.

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Contents

I Introduction ................................. 1
   A Background ............................................. 1
   B Previous Evidence ..................................... 2
   C Objective & Structure ................................ 3

II Theoretical Framework ....................... 5
   A Economic & Financial Framework .................... 5
   B Interest Rate Models & the Affine Term Structure .................. 8
   C The Eigendecomposition .................................. 10

III General components of the term structure model .......... 12
   A Pricing Kernels ........................................... 12
   B Term Structure Models and Risk Factors .................. 13

IV Term Structure Models and the Macroeconomy .............. 15
   A New Keynesian Macro Model .................. 15
   B Macro-Finance Term Structure Models ............ 16

V Macro Term Structures with Floating Currency ............. 19
   A Exchange rate dynamics & inflation ................. 19
   B Affine short-rate model ............................... 20
   C State dynamics ........................................... 20
   D Regressing the excess bond return .................. 21

VI Empirical Analysis .............................. 22
   A Excess Return Regression .................. 22
   B Model Formulation, Data Illustration & Estimation ............. 23
   C Data ................................................. 26
   D Results .................................................. 27
   E Discussion .............................................. 28

VII Conclusion .................................... 31
   A The Meta-Theorem .................................... 36
   B Itô’s Lemma ............................................. 36
List of Figures

1. **Three Yield Curves**: An illustration of the yield level on ten different maturities in February 1989 (blue, top), March 1997 (green, middle), and December 2014 (red, bottom). ................................................................. 1

2. **Historical Yield**: five long- to short-term yields spanning from 1985 to 2017. The lower the curve, the lower the maturity, for example, the purple one (top) is the ten-year, and dark blue one (bottom) is the one-year maturity. Economic crashes occurred year 1987, 2001 and 2008. Greenspan conundrum occurred in 2003. ........................................... 6

3. **Level, Slope & Curvature**: five factor loadings plotted ................................................. 24

4. **Level, Slope & Curvature**: the three first principal components of the term structure ................................................................................................................. 25

5. **Prediction Comparison**: a plotted comparison of the performance of the resulting prediction model. On top we see a predictive attempt with only the macro variables, trying to predict risk premiums. In the middle, we see the standard TSM trying to predict risk premiums, much more successfully than only macro risks. On the bottom we see the full modified MTSM, with by far the most accurate prediction compared to the standard TSM ......................................................... 29

6. **Prediction of excess bond returns**: Red line represents the actual excess bond return defined as in equation (49). The blue line is the prediction model using Growth and the Monte Carlo simulated model for the inflation measure ........................................... 30
I. Introduction

A. Background

A major task for economists, since the inception of the field, has been to understand how debt markets work. Debt has been the fuel for which modern economies operate and grow, both closed as well as open ones. During the last centuries, the most central traded asset within these debt markets has been government debt. This is a type of asset that has for a long time been viewed as the most safe investment of all, contingent on the fact that the borrowing government is in general conceived as stable. Therefore, sovereign government debt has been a central benchmark, and subsequently subject to, a lot of research in the financial and macroeconomic research communities.

The key indicator agents observe in the market related to sovereign government debt is the yield of the debt. This is of course natural as it is paramount for any rational investor to expect some return for lending capital. Therefore, because of the fact that this yield will determine the market’s attitude towards lending some government their money, it has become an indicator that is not only important for people directly interested in actually buying debt, as it has become an indicator to use for gauging the economic health of the country, in which the relevant government operates. One of the most interesting viewpoints on this has therefore become to look at the differences between the yields at the different maturities, how the yield changes over the different maturities, or final date of payment, at which the government borrows. The natural assumption would be of course that one will, at longer maturities, require a higher yield, but as we see below, that is not always the case. In figure 1 we see three different yield levels ‘curve’ along the different maturities. These plots

Figure 1. Three Yield Curves An illustration of the yield level on ten different maturities in February 1989 (blue, top), March 1997 (green, middle), and December 2014 (red, bottom).
are therefore called the yield curves. In this example we are looking at the yield curve in February 1989 as blue, March 1997 in green, and December 2014 in red.

There are therefore quite a few important implications to gather from the slope of the yield curve, as this holds information about the state of the economy. For example, an upwards sloping yield curve is indicative of the fact that people are expecting a higher yield in the future, which according to various experts can be explained by a wide range of economic and financial effects. However, the most prominent explanation, is the story of a higher expected economic activity, which has a positive effect on the majority of interest rate setting rules for future expected interest rates. Inversely, a flat yield curve is indicative of investors not expecting higher interest rates in the future, which has had the historical interpretation that the economy is facing a lower productivity or even a recession, which forces the central bank to lower the rates. In research, pricing debt assets by accounting for the difference in the yield at different maturities is called to model the term structure of interest rates. This goes beyond the standard task of pricing a bond related to one specific yield and potential coupon, and further creates some model to account for the all the information contained in the term structure.

In this thesis we will construct a model of term structures such that we revisit the increasingly popular concept of adding macroeconomic ideas in the state vectors of the factors in affine term structure models, by reconsidering some assumptions on the economy in which the debtor operates. What we are aiming to do is as follows; based upon the conclusions of Taylor (2001), that the exchange rate of an economy most likely has some latent effect on the interest rates. Thereafter, alongside with the general reasoning of Cochrane and Piazzesi (2005), Singleton (2006) and Bekaert et al. (2010), on how and why to include macroeconomic variables in term structure models, with the inclusion of spanning conditions on the macroeconomic factors, in line with Joslin et al. (2014) and Bauer and Rudebusch (2016). We create a Macro Finance Term Structure model (MTSMS), which includes the dynamics of the underlying exchange rate. To investigate whether this will to a further extent than previous research, capture the dynamics of the risk in interest rate forward premiums, in order to create a framework to price bond forwards more efficiently.

\[\text{B. Previous Evidence}\]

During the last three or so decades, a long line of literature has been produced to model the term structure of the yield curve. Fama (1984) found that the one- to six-month forward rate of the US Treasury bill predicts the direction of the changes in the short term interest rates. From the resulting equations of the Expectations Theory of interest rate, Campbell and Shiller (1991) further studies the broader spectrum of the yields, and finds that there is useful information in the term structure about the future dynamics of yields. From these insights, Duffie and Kan (1996) derives an affine multi-factor model of the term structure, in which each individual bond yield becomes a factor in the equation. Duffie and Singleton (1999) goes further on these ideas and also motivates the

\[\text{1 GDP change (Growth) and fluctuations from target inflation are typically the main ingredients of interest rate setting rules, see Taylor (1993).}\]
extension on one of the more famous interest rate models by [Vasicek (1977) and Cox et al. (1985)], by the fact that they do not capture negative correlation in an optimal manner by construction. Given these insights, [Dai and Singleton (2000)] formulates a standard framework for the canonical representation of affine term structure models, in which they describe the factors of the yields by the level, slope and ‘butterfly’ (for which we will use the term curve or curvature).

Although the strong theoretically appealing nature of the standard term structure framework, they have been found to not successfully capture the entire dynamics in the forward premiums. A risk factor that has been thought of coming from the markets perceived risks. This uncertainty has somewhat been thought of to come from latent macroeconomic risk sources. Therefore, from the ideas of the macroeconomic impact in bond yields by [Estrella and Hardouvelis (1991)], and the fact that we can subsequently model these ideas as a multi-factor vector autoregressive model ([Estrella and Mishkin 1997]). [Ang and Piazzesi (2003)] shows that we can model the dynamic term structure with these latent macroeconomic variables, and therefore forms the idea of the Macro-Finance Term Structure Model (MTSM). These ideas are also discussed by [Bekaert et al. (2010)], who formulates a rigorous macroeconomic derivation of a VAR from a system of equation based upon a Taylor rule and the Phillips curve.

However, there exists a structural issue with this idea, in which [Joslin et al. (2014)] illuminates the issue of spanning in MTSMs. This implies that due to the explanatory richness in the principal components of the yield curves, macro-variables such as in the spanned models, can be replicated by a portfolio of yields. [Joslin et al. (2014)] proposes a solution to this with a spanning condition that is put on the model formulation. In their formulation, they focus on using the growth rate and inflation as their macroeconomic factors.

C. Objective & Structure

The objective of this thesis is to form an unspanned MTSMs in line with [Joslin et al. (2014)], but with a similar derivation of the macro-factors as [Bekaert et al. (2010)]. This is to find an interest rate term structure model that will be able to capture the dynamics of future yields in order to better model interest rate forward prices and the relevant risk premiums. Our contribution to this line of research will be the fact that we include the ideas of [Taylor (2001)], who say that the exchange rate of an economy has some latent effect on the interest rates, and derive a modified inflation proxy factor, which will encompass the exchange rate effects on the economy. We will then estimate this model and investigate its fitting properties to the forward premium in order to see if there is any difference when the exchange rate dynamics are included, and compare these results to the standards yield-only term structure model. In this investigation we will use data from USA, namely ten different treasury bills with maturities ranging from three months to twenty year. The horizon subject to investigation is between 1985 and 2017. We will use macroeconomic data such as inflation and growth rate measured as US industrial production, also ranging from 1985 to 2017. In our estimation, we use OLS with GMM standard errors to estimate our MTS factor model, which produces a fit on the excess bond return, which is our measure for the forward premia. We find that
our modified model did indeed improve the level of fit when compared with the standard models, and thereafter satisfies the fitting properties for a canonical unspanned MTSM. This thesis will thenceforth be structured as follows: firstly, a theoretical overview on the related economic and financial concepts of bond-markets, term structure models, as well as the relevant mathematical concepts needed to understand the mechanics of the subsequent analysis. Secondly, we will motivate mathematically our model from the insights of the previous literature. Lastly, we will provide an overview of our empirical implementation of this model and review its results. We will limit this thesis to the extent that we will not try to empirically estimate the likelihood function, as it have been done previously by, Ang and Piazzesi (2003), Ang et al. (2008), Joslin et al. (2014) and more. Nor to spend too much time reviewing forward premium accounting, as this would be too time consuming in regards to the limited time we have to write this master’s thesis. We will purely focus on fitting a regression of the term structure macro model that we develop, and compare statistical results with the standard yield-only term structure model.
II. Theoretical Framework

In the following section we will spend some time on the central concepts that will be important for progressing further with this thesis. It is important to understand how and why we can create a model with the mathematical tools that we use. The methods are not based on straight forward analysis of simple time series. We use a spectral decomposition – or eigendecomposition as we will refer to it – of the term structure and macro factors. We will therefore need to present a discussion clarifying the relationship between the interest rate and its maturities, what information we can extract from these relationships, and present the method for how we are able to do this mathematically. Because it is a subject that is not so very straight forward upon the first encounter. This section will be structured as follows; firstly, we explain some of the economic ideas and frameworks driving the interest rate market at the different maturities. Secondly, we discuss the general financial mathematics of interest term structure pricing, as well as how to derive this into a model of affine form. Lastly, we discuss the concept of eigendecomposition as it will be a key tool for creating the equations that will tell us information on the bond forward premium.

A. Economic & Financial Framework

In this thesis we are investigating term structure models of interest rates. Why this is such an important area of research is because of the fact that sovereign debt markets are one of the largest markets by capitalization\(^2\), dwarfing the equity markets. Agents acting in these markets are usually trading using forwards or other derivatives, based upon said forwards, meaning that the risk premium is an important pricing component.

The term structure of interest rates has, as previously explained, been thought of as some form of function on future expectations on economic growth. There are two main forces who are actually driving the bond market, as it is still nothing more than an actual market, trading derivatives on interest rates. The first driving force is the central banks, which has been the dominating force in setting short-term interest rates, depending on their assessment of what will be the best for the economy. The second force determining bond markets are the market actors, i.e. mutual funds, hedge funds, banks, other governments, very wealthy individuals, and other market actors who are interested in a relatively safe stream of fixed income. Their assessments and market actions are not as rule based and transparent as the central banks. They are acting more out of selfish interest (as they should), which introduces a more stochastic determination of bond prices and bond yields.

In figure 2 we see three interest rates evolve through time from the mid eighties to present day. Considering that idea, we should see the three lines converge every time when the market has expected lower future returns, i.e. during periods of economic stagnation. The most impacting economic crisis’s in our period range are found in 1987, 2000, and 2008. And comfortingly, we can see some evidence of this, however, if we also remember that in 2005, the Chairman of the federal

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\(^2\)See the post "What are the differences between debt and equity markets?" on the website of the Federal Reserve Bank of San Francisco for comparison between the US equity market and the US Debt market
Figure 2. Historical Yield: five long- to short-term yields spanning from 1985 to 2017. The lower the curve, the lower the maturity, for example, the purple one (top) is the ten-year, and dark blue one (bottom) is the one-year maturity. Economic crashes occurred year 1987, 2001 and 2008. Greenspan conundrum occurred in 2003.

reserve Alan Greenspan raised federal funds rate by 150 basis-points, and as can be seen, only the short term rates increase in yield, while the ten-year bond does not follow suit. This was during a time when the US economy was booming. However, it is actually not a unique condition of those specific years. One can see that this has also occurred around 1987, where we see a sharp increase in the short rate, but not much adaptation in the long rates. This concept has later received the infamous name ‘the Greenspan conundrum’, and is only one of several interesting interest rate effects that has puzzled economists (Thornton, 2012).

Because of this, in the following section, we will explain in further detail some of the economic ideas and concepts regarding interest rates, most notably the expectations hypothesis of interest rates. As this is considered one of the most fundamental ideas that governs the bond rate market, and in a very basic setting, is the condition assumed for the bond market to be arbitrage free. We will also dive into some mathematical properties of this idea, and then go on and explain the ideas of interest rate modelling and term structures, followed by the mathematical framework, in which we will operate.

A.1. The Expectations Hypothesis

The Expectations Hypothesis is one of the older and most widely discussed concepts in financial and macro economics. In short, it connects the idea of the different maturities of bonds to an idea of the expected future yield. The main implication of this hypothesis is then of course that there is no arbitrage opportunities in the bond market for investors to seize in between the decision of
investing in a long term bond and a short term bond, which can be illustrated by equation (1)

\[(1 + i_l)^n = \Pi_j^n (1 + i_s)^j\]  \hspace{1cm} (1)

where \(n\) is the length of the long term maturity, and \(s\) is the short term maturity. Equation (1) says therefore that the return from a long period bond, holding it for \(n\) years, is equal to the product of overturning short period bonds \(n\) times. Meaning that there is no difference in the return on invested capital between investing in one long term bond, or to simply just reinvesting in the short term bonds until one reaches that same time, as their expected values are the same.

More formally, within the discussion of forward rate processes. If we call \(f(t, T)\) the estimated future short rate \(r(T)\), we need – in order for the Expectations Hypothesis to hold – the following equivalence in equation (2)

\[f(t, T) = E[r(T) | \mathcal{F}] \]  \hspace{1cm} (2)

to hold. Meaning that if we consider the ‘efficient market’ to imply that it is an ‘arbitrage free’ market, then we can apply continuous time dynamics to further analyze this relationship. If one then goes on to formulate this expectation under a martingale \(Q\)-measure, we can according to Björk (2009) prove that the Expectation Hypothesis is false under the \(Q\)-measure. (Björk, 2009, For proof, see Lemma 26.10 pp 405-406). Empirical refutations of the Expectation Hypothesis have been provided manifold in research, most notably by Fama (1984).

However, these refutations of the Expectations Hypothesis are not entirely as mathematically apocalyptic as they might seem. As both Singleton (2006) and Björk (2009) provides probabilistic explanations for how one can model term structures without arbitrage, which comes from the idea of matching the sources of randomness and number of assets. We will not spend any time deriving it or discussing how it works, but the general idea is summarized in the meta-theorem (see appendix).

### A.2. Macroeconomic Forces

In the standard MTSM framework it is commonplace to assume that there is a link between macroeconomic factors and the the direction of the yield curve. This is at least to some extent very clear. As central banks, the entity who in general sets the short term rate, and therefore future short term interest rates, abides by known macroeconomic rules and transparent targets, from which we know are calculated with macroeconomic data. We also know that investors risk appetite usually follows some aversion towards increasing risks, which bond markets will suffer from contingent on the occurrence of macroeconomic shocks. As it might affect the value of the amount of money to be paid, from inflationary effects, or even the governments ability to pay back their debt at all. We can therefore assume that there must be some, at least latent, relation between the macroeconomy and the yield curve.

Following this logic, it is not too hard to argue that further macroeconomic variables might

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3See the non-monotonicity in expectations of T-Bill returns by Fama (1984)
be important to properly describe some of the states within the equations of interest rate rules. As we, in recent years, have been able to observe, world markets are becoming more open, and more interconnected than ever before. International trade barriers are fewer and lower than ever, domestic producers are importing components for production, and in order to compete fairly in any developed market, companies needs often to export goods more now than ever before. Also, during the last decade of the twentieth century, many countries made the switch to flexible exchange rates due to the increasing number of ‘attacks’ from speculative market actors (Sorensen and Whitta-Jacobsen, 2010). Hence, due to all this interconnectivity, we consider the theoretical contribution of a country’s currency to be even more of an important factor now than when Taylor (2001) first proposed the idea of possible latent exchange rate effect on domestic interest rates, and subsequent policy.

B. Interest Rate Models & the Affine Term Structure

A concept frequently recurring in this thesis is that of the term structure model. We have provided heuristics on how and why it is important to phrase a predictive model for bond yields. However, we will now provide a brief explanation for their mathematical and financial heuristics.

Suppose we have a model for the short rate under an objective probability measure $\mathbb{P}$. From this, we will have a solution to a Stochastic Differential Equation (SDE), of the following

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

where $\mu(t, r(t))$ is a function for the drift in the process, $\sigma(t, r(t))$ is a term inducing the diffusion, and $dW(t)$ is a so-called innovation term, following some known stochastic process. The drift and diffusion are functions dependent on both time, and in this case, the interest rate. The interest rate is a time dependent stochastic function of the short rate. From equation (3), we can determine that the following price process is defined by the dynamics in equation (4)

$$dB(t) = r(t)B(t)dt$$

where we interpret this as a model for a bank account with the stochastic short rate $r(t)$. If we assume that there exists one exogenous risk-free asset, where the price is denoted by $B(t)$, as well as a market with zero coupon bonds for every maturity $T$. Then, from the perspective of bonds being derivative assets on interest rates that follows the dynamics on $r(t)$ in (3), we therefore know that bond prices are not uniquely determined by this short rate dynamic. The reason is, the arbitrage valuation is the valuation of a derivative in relation to some underlying asset. In our case, we do not have enough underlying assets to find a unique price of one specific bond, for a specific value

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4The mathematical motivations are gathered from Björk (2009), which I encourage the interested reader to look at for further insight in this subject. We have provided some important insights regarding stochastic differential equations, Itô’s Lemma, and the Meta-theorem in the appendix.

5A probability measure is the real valued function defined on a set of events in a probability space, i.e. the probabilities assigned to each possible event in the entire set of possible events.
of $T$. Therefore, if we take one particular bond and its price as a benchmark, then all other bonds will be uniquely determined in the terms of this benchmark and the properties of the dynamics of the benchmark (Björk, 2009). This is the first important realization on why we can do what we do. Because, even though the arbitrage-free condition can be proven to fail if we set that condition to be contingent on the expectations hypothesis, our derivative can be proven to be arbitrage-free and complete if it abides by the meta-theorem, which this model will do. This system will then hold, in accordance to all the theoretical assumptions, in a term structure constellation, if the existence of term structure equations in affine form can be shown.

Therefore, assuming that there exists a market for all maturities of the zero coupon bond, and a smooth three-variable function $F$ such that we can denote the price of a bond by equation (5)

$$p(t, T) = F(t, r(t); T)$$

with the boundary condition $F(T, r; T) = 1, \forall r$. In order to apply the concepts mentioned above, we can apply the Itô-formula\(^6\) to (5) as well as its boundary condition, and find the so called term structure equation

$$F_t + \{\mu - \lambda \sigma\}F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0$$

$$F(T, r) = 1$$

where $\lambda$ denotes the market price of risk. The subscript $(t, r, rr)$ is in this instance a notation for the partial derivatives of $F(t, r(t); T)$ in relation to the three dependent variables. This follows with the slightly more practical, or at least easy to read, notation of Shreve (2008).

To enforce an affine structure in this framework, we need to make some assumptions. If we assume there are two deterministic functions, $A$ and $B$, then our process $F$ has an affine term structure if the following form can be proven to hold:

$$F(t, r(t); T) = \exp\{A(t, T) - B(t, T)r\}. \tag{7}$$

This implies that the, stochastic discount process, or pricing kernel, follows an exponential function determined by the deterministic functions $A$ and $B$. This is our second important insight. Equation (7) is the result of price derivation, which holds an affine form, meaning that we have found a way of expressing the price of the bond as an affine function of interest rates. From this, we can use (7) to compute the partial derivatives of $F$, which in it self must solve equation (6), we are able to obtain the following partial differential equation

\(^6\)The Itô-formula is a consequence of Itô’s lemma which gives us the differential equations of a time dependent stochastic function. See Itô’s lemma in appendix.
\begin{equation}
A_t(t, T) - \{1 + B(t, T)\} r - \mu(t, r) B(t, T) + \frac{1}{2} \sigma^2 B^2(t, T) = 0, \tag{8}
\end{equation}

\[ A(T, T) = 0 \]

\[ B(T, T) = 0. \]

If we then assume that \( \mu \) and \( \sigma^2 \) are functions on \( r(t) \), which possess this affine property, meaning that they are linear with a constant, formulated as

\begin{equation}
\mu(t, r) = \alpha(t) r + \beta(t) \tag{9}
\end{equation}

\begin{equation}
\sigma^2 = \gamma(t) r + \delta(t), \tag{10}
\end{equation}

then the model submits to the form of the affine term structure. After inserting (9) into (8), assuming the same boundary condition as before, and fixating the choice of \( t \) and \( T \), we can formulate the final equation as

\[ B(t, T) + \alpha(t) B(t, T) - \frac{1}{2} \gamma(t) B^2(t, T) = -1 \tag{10} \]

\[ B(T, T) = 0 \]

where if we solve \( B(t, T) \) and plug this into \( A(t, T) \), we get the following

\[ A(t, T) = \beta(t) B(t, T) - \frac{1}{2} \delta(t) B^2(t, T) \tag{11} \]

\[ A(T, T) = 0, \]

proving that our price process is an affine term structure model. Note here as well that equation (10) and (11) are Ricatti equations. The beauty of the affine term structure form is that we have an easy to interpret, and easy to structure set of equations on which we can more easily estimate the parameters. Later in this thesis we will make assumptions on the nature of the bond term structure discount processes, which are contingent on this existence of the affine form.

\subsection{The Eigendecomposition}

As we have now covered the fundamental financial and mathematical idea on why we can formulate a pricing formula on the term structure of interest rates as we want, we can now shift the focus to some of the mathematics behind one of the tools that will frequently be used when building equations that we can actually estimate. This concept is called the eigendecomposition, and is a very popular tool for mathematicians to use when one needs to decompose information contained in square matrices.

Consider a standard linear equation such as \( Ax = b \), which are derived from some steady state, the solution to the derivatives are easy to find, and therefore the dynamics more so. However,

\[ \text{In short, a Ricatti equation is an ordinary differential equation that also is quadratic in an unknown term.} \]
eigenvalues and eigendecompositions are very useful tools for solving more complex functions, where a derivative might be time dependent, which cannot be found with classical calculus tools (Strang, 2009).

Hence, consider the non-zero vector $v$ as an eigenvector on the square matrix $A$

$$Av = \lambda v$$

(12)

for the scalar $\lambda$, which is the eigenvalue of $A$. In practice, this only means that we are able to explain one thing, multiplied with a scalar, as linear combinations of the other. This is the general idea of the eigenvalue, i.e. to solve problems based on the principle seen in equation (12).

From this fundamental property\(^8\) when we consider the idea of equation (12) in matrix form, we are able to express the eigendecomposition, or diagonalization of said square matrix (Strang, 2009). Namely, for the square matrix $A$, we can write,

$$A = QAQ',$$

(13)

where $Q = Q' = Q^{-1}$ implies $QQ' = QQ^{-1} = I$, which are some fundamental properties of $Q$, and are subsequently frequently used in evaluation of complex dynamic systems. It is also one of the key principles used in Principal Component Analysis (PCA).

This is very helpful notation to utilize when looking at square matrices, such as a covariance matrix. This means that if we can express the covariance matrix in a canonical form, such as in equation (13), then describe the entire variability in the variables by the different states, the $q_i$ columns of $Q$. This is subsequently the third important insight of this sections. The columns of $Q$ from a decomposed term structure covariance matrix, acts as the factor loadings, or weights, in the standard setting of the term structure model.

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\(^8\) As well as the important property of the results of the determinant $|A - \lambda I| = 0$, which is called the characteristic polynomial
III. General components of the term structure model

In the following section we will expand on the theory that we started to encounter in Section II.B. We will focus on the components of the term structure model, such that all mechanics behind our motivations later are understood.

In section II.B we focused on the concept of deriving an equation for the term structure of interest rates, in which we came to the conclusion that that would be determined via the short rate equation. The direct consequence of this is that we can then define a discount process. In short, a discount process on the short rate $r(t)$ can be defined as

$$P(t) = \exp\left(-\int_t^T r(u)du\right)$$

(14)

where we can see that the solution to the integral simply is the continuously compounded interest rate on the between a starting time and the maturity $T$ (Shreve, 2008). This relationship is essential for the survival of any interest rate model, as anything but equality in equation (14), directly implies arbitrage opportunities in the interest rate market (Neftci, 2000, pp 414-415). If the short rates payoffs are determined by a stochastic factor $m(t)$, then we get the following, so called, stochastic discount process

$$P(t) = E^P[m(t+1)x(t+1)]$$

(15)

where the expectation is performed under some risk-neutral measure $P$. This will henceforth be referred to as the pricing kernel, as $m(t)$, is in this context, a kernel function (Cochrane, 2001, pp 9-10). However, in order to work with models with several probability measures at play, we need to explain further in some detail where the pricing kernel comes from.

A. Pricing Kernels

We will now present a more formal discussion on the framework behind the stochastic discount factor, or the pricing kernel. Later in this thesis, we will make assumptions on how we are able to structure the interest rate equations, which comes from the validity of the following assumptions.

Consider the following, for an information set $\mathcal{A}$ and pricing kernel $M$, we can define the payoff space

$$\mathcal{P}^+_{t+1} = \{ M_{t+1} \in \mathcal{A}_{t+1} : E[M_{t+1}^2 | \mathcal{A}_t] < \infty \},$$

(16)

meaning that $\mathcal{P}^+_{t+1}$ represents the set of random variables in the information set $\mathcal{A}_{t+1}$, which when conditioned on $\mathcal{A}_t$, will be a well behaved function, meaning that it will have finite second moments. We will subsequently interpret this as a set of payoffs contingent on realizations of variables in the information set (Singleton, 2006, pp 195-197).

Now, suppose that we have the following process $Y(t)$

$$dY(t) = \mu_Y(Y,t)dt + \sigma_Y(Y,t)dW(t)$$

(17)
in which $\mu_Y$ is a vector of drifts determined by a probability measure $\mathbb{P}$, and $\sigma_Y$, the diffusion parameter, is some state-dependent volatility matrix. This then implies that we can write a state-dependent pricing kernel $M$ as

$$dM_t = -r_t dt - \Lambda_t^T dW(t)$$

where $\Lambda_t^T$ is the transposed vector of market prices of risk, and $r_t$ as the risk-free rate. With a dividend at $h(Y(t), t)$ and payoff at $g(Y(T))$, the price can be evaluated as

$$P(Y(t), t) = \mathbb{E}_t \left[ \int_t^T \frac{M(u)}{M(t)} h(Y(u), u) du \right] + \mathbb{E}_t \left[ \frac{M(T)}{M(t)} g(Y(T)) \right]$$

which poses an issue when computing the expectations (Singleton, 2006, pp 205-207). The standard approach proposed in the literature, which changes the numéraire of the historical information in $\mathbb{P}$ to the risk-neutral $\mathbb{Q}$ measure, takes the logarithm of equation (18), in order to obtain the pricing kernel as the following stochastic process

$$d\log M_t = -r_t - \frac{1}{2} \Lambda_t^T(t)\Lambda(t) dt - \Lambda_t^T(t) dW(t),$$

where the market prices of risk $\Lambda$ is defined as

$$\Lambda_t = \Lambda_0 + \Lambda_1 x_t,$$

which gives us the final affine term structure model in the pricing kernel. For example, the level of the parameter representing the market price of risk, is a direct consequence of that the market actually in fact not risk free, and investors will demand risk premiums $^9$. This issue, of not being risk-neutral, can be avoided in the model by using the assumptions on the measure in the randomness affecting the diffusion (Singleton, 2006). This discussion is however important to have in mind so that one are able to follow how the theory connects to the models, which will attempt to predict forward risk premiums, in our case defined as the expected excess return on bond investments.

**B. Term Structure Models and Risk Factors**

In order to follow what we will do in the next section, we will spend some time investigating the derivation of the Gaussian Dynamic Term Structure Model (DTSM), derived by Joslin et al. (2011).

The initial framework that we need to set up is as follows. We define a pricing kernel similar to (20), where we assume that the price of the bond is determined by the following

$$M_{zt+1} = \exp \left( -r_t - \frac{1}{2} \Lambda_{zt}^T \Lambda_zt - \Lambda_{zt}^t \eta_{t+1}^p \right)$$

where $Z_t$ is the state-space representation of the $R \times 1$ vector that encapsulates all risk in the economy.

---

$^9$ For further reading on risk premium accounting, see Joslin et al. (2014) as well as earlier paper by Kenneth Singleton
which we assume will be determined by the following Gaussian process

$$Z_t = K_{0,Z}^P + K_{1,Z}^P Z_{t-1} + \sqrt{\Sigma} Z_{\eta_t}^P$$

(23)

where the market prices of risk $\Lambda_Z$ of $\eta_t^P \sim N(0, I)$ risks in $\eta_{t+1}^P$, are affine functions of $Z_t$ such that a one-period bond $r_t$ will be defined as

$$r_t = \rho_{0,Z} + \rho_{1,Z} Z_t,$$

(24)

which is the affine function of $Z_t$. To make this concept more manageable, [Dai and Singleton (2003)] provides a proof for an invariant transformation on a state-vector such as equation (23), which according to [Joslin et al. (2011)], allows us to replace $Z_t$ with some observable $P_t$, which gives us

$$r_t = \rho_{0,P} + \rho_{1,P} P_t,$$

(25)

where we can see that this is a central observation for the derivation of an interest rate equation derived from a model that includes both yield structure information, as well as information from some other source. This source, can for example be some type of macroeconomic factor. Meaning that we have motivated mathematically an affine model of the term structure of interest rates, which can account for some macroeconomic source of risk. This leads us to the next topic of discussion. How can we in this model, derive a model that includes such macroeconomic sources in $P_t$, and still be risk- and arbitrage-free, as well as unspanned by the term structure of the yields. Also, how can we motivate which macroeconomic sources to use in such a model.
IV. Term Structure Models and the Macroeconomy

The following section will provide a brief explanation on notations that will be frequently used in the sections after. This will cover some of the probabilistic assumptions and derivations performed by Singleton (2006), as well as the macroeconomic motivations by Ang and Piazzesi (2003), and Bekaert et al. (2010) regarding why macro-factors has a natural place in the term structure framework, and how these are implemented in the pricing kernel and the predictive regression for the risk premia.

A. New Keynesian Macro Model

Ever since Ang and Piazzesi (2003) proposed their idea on how to incorporate macroeconomic factors in a term structure equation. There have been presented more and more ideas on how to further these types of ideas, which have then wrought the concept of a MTSM that follows the principles of the New Keynesian School of thought. In a general sense, what is done in the earlier forms of the MTSM, is to motivate a macroeconomic rule for the interest rate. This is typically done by using some form of the Taylor rule for interest rate setting, and the deriving a formula for the short rate, as we provided in some detail in the previous section.

A more thorough focus on the macroeconomic input in these types of model are found in Singleton (2006) as well as Bekaert et al. (2010). They provides detailed discussions on the relationship between the macroeconomic IS-LM and AS-AD framework, with the term structure model and its pricing kernel. In brief, they take the standard Phillips Curve equation as

$$\pi_t = \delta E_t \pi_{t+1} + (1 - \delta) \pi_{t-1} + \kappa(y_t - y^*_t) + \epsilon_{AS_t}$$

with \(\pi_t\) as inflation and \(y^*_t\) as the natural rate of output, in which the difference \((y_t - y^*_t)\) represents the gap and the \(\epsilon_{AS_t}\) represents the exogenous supply shocks. In order to account for expectation on how monetary authorities sets interest rates, Bekaert et al. (2010) assumes a combination of a standard forward looking Taylor rule, as can be seen in any standard macroeconomic literature, and forms a linear combination with this with a short-term rate defined as the weighted average of rate targets

$$i_t = \alpha_{MP} + \rho i_{t-1} + (1 - \rho)(\beta(E \pi_{t+1} - \pi^*_t) + \gamma(y_t - y^*_t)) + \epsilon_{MP}$$

where \(\alpha_{MP}\) is the smoothed short-term rate \(i_{short-term}\), which follows in the linear combination being multiplied by \((1 - \rho)\). \(\epsilon_{MP}\) is here as well a exogenous shock.

In order to take these new Keynesian ideas into a state space vector model we write the full model as the following system of five variable equations,

$$\pi_t = \delta E_t \pi_{t+1} + (1 - \delta) \pi_{t-1} + \kappa(y_t - y^*_t) + \epsilon_{AS_t}$$

$$y_t = \alpha_{IS} + \mu E_t y_{t+1} + (1 - \mu)y_{t-1} - \phi(i_t - E_t \pi_{t+1}) + \epsilon_{IS,t}$$

$$i_t = \alpha_{MP} + \rho i_{t-1} + (1 - \rho)(\beta(E \pi_{t+1} - \pi^*_t) + \gamma(y_t - y^*_t)) + \epsilon_{MP}$$

where \(\alpha_{IS}\) is the smoothed short-term rate \(i_{short-term}\), which follows in the linear combination being multiplied by \((1 - \rho)\). \(\epsilon_{IS}\) is here as well a exogenous shock.
\[ y^n_t = \lambda y^n_{t-1} + \epsilon_{y,t} \]

\[ \pi^*_t = \phi E_t \pi^*_t + \phi_2 \pi^*_{t-1} + \phi_3 \pi_t + \epsilon_{\pi^*,t}, \]

which can be expressed more easily in matrix form as

\[ Bx_t = \alpha + AE_t x_{t+1} + Jx_{t-1} + C\epsilon_t \]

where \( x_t \) is the vector of all macroeconomic variables gathered from the IS-LM framework, e.g. \( x_t = [\pi_t, y_t, i_t, y^n_t, \pi^*_t] \) \cite[p.36-40]{Bekaert et al.}. We can then formulate the rational expectations equilibrium as a first-order VAR as

\[ x_t = c + \Omega x_{t-1} + \Gamma \epsilon_t \]

The \( x_t \) is then used to form an equation for interest rate, giving us the tools required to structure the differential equations and finding the pricing kernel of an affine Term Structure model with factors from the macroeconomy \cite{Piazzesi}. This approach is interesting from a macroeconomic perspective, as it successfully incorporates a huge theoretical macroeconomic framework directly in the term structure equation, via some state-space representation \cite{Mergner}. However, as we will see, there are some issues with this. By using too many sources of macroeconomic risks, we are in danger of overparameterizing the distribution. This is an issue addressed by \cite{Joslin et al.}, which we will now discuss.

### B. Macro-Finance Term Structure Models

In this section, we discuss the concept of the MTSM, as it was proposed by \cite{Joslin et al.} and later investigated by \cite{Bauer and Rudebusch}. It differs from the thorough approach, in a macroeconomic sense, by \cite{Bekaert et al.}, in which that it does not assume as much latent effects from the entire IS-LM framework. This is because \cite{Joslin et al.} notes that there has been much evidence of MTSMs suffering from spanning of macro variables in the interest rate term structure, as well as the fact that too many macroeconomic sources overparameterize \( Z_t \) in equation \( (23) \), i.e. the risk-neutral distribution that underpins the probabilistic structure on which we built this model.

The MTSM is as we have seen, not something unique to \cite{Joslin et al.}. It has been a popular tool subject to mathematical and economic analysis in research during the last few decades. The beauty in which lies the fact that, in both reduced- and equilibrium-form, it can be used to investigate the dynamics between macro variables and interest rates at different maturities. All this and still retain desirable mathematical and financial properties, such that it becomes a very useful tool for pricing and estimating risks in the market \cite{Bauer and Rudebusch}. However, recent findings have indicated that the source of macroeconomic risks in the model is spanned by the yield term structure. The implication of which is that, with a portfolio of bond yields, one can replicate these macro variables leading to the shattering conclusion that macro variable bare
no information on either excess returns, i.e. risk premiums, nor future values of $M$ (Joslin et al., 2014).

As this might feel as a rather anti-climactic result from the analysis of a theoretically appealing model, Joslin et al. (2014) propose two central conceptual assumptions for motivating a MTSM that solves, or at least circumvents, this spanning puzzle. Firstly, investors in the market use a pricing kernel that depends on a set of priced macroeconomic risks. Secondly, the short rate equation is determined by an affine function of coupled bond yields of $n$ different maturities, which we discussed in previous sections.

In order to further understand the pricing mechanism of the macro finance term structure model, we need to investigate the pricing kernel $M$. Joslin et al. (2014) suggests that the kernel can be interpreted as the projection of $M_Z$ on a portfolio of risks $\mathcal{P}$ as,

$$M_{\mathcal{P},t+1} \overset{\text{def}}{=} \text{Proj}[M_{Z,t+1} | \mathcal{P}, Z_t] = \exp\left(-r_t - \frac{1}{2} \Lambda_P^T \Lambda_P - \Lambda_P^T \epsilon_{\mathcal{P},t+1} \right)$$

(35)

in which $\mathcal{P}$ in equation (35) denotes the portfolio of risks, and $\Lambda_P$ denotes the market price of risk. Even if (35) is a kernel similar to the previous model, which are supposed to possess spanning in the macro factors. The term $\epsilon_{\mathcal{P},t+1}^P$ in (35) comes from the noise in

$$\begin{bmatrix} P_t \\ M_t \end{bmatrix} = \begin{bmatrix} K_{0,P} \\ K_{0,M} \end{bmatrix} + \begin{bmatrix} K_{PP} & K_{PM} \\ K_{MP} & K_{MM} \end{bmatrix} \begin{bmatrix} P_{t-1} \\ M_{t-1} \end{bmatrix} + \Sigma_{Z,\epsilon^P},$$

(36)

which according to Joslin et al. (2014), implies that $M_t$ is not spanned by $P_t$. Assuming that $P_t$ follows autonomous Gaussian VAR, the risk-neutral pricing will follow

$$P_t = K_{0,P} + K_{PP}P_{t-1} + \Sigma_{PP,\epsilon^P},$$

(37)

from which we can come to the conclusion that the yield of an $n$-period bond, has the affine form on $P_t$,

$$y^n_t = A_p(n) + B_p(n)P_t$$

(38)

where the $A$ and $B$ are loadings based on known functions of the parameters steering the $Q$ distribution. Giving a yield equation in the affine form that we want in order to construct our model.

From the previous critique on the spanning puzzle of macro finance term structure, Joslin et al. (2011), Duffee (2012), and Joslin et al. (2014) suggests that $M_t$ is related to $P_t$ by a so called macro-spanning condition

$$M_t = \gamma_0 + \gamma_1P_t.$$ 

(39)

However, the framework from (35) implies there is a residual term $v_t$ in the linear projection of $M_t$ as

$$M_T = \gamma_0 + \gamma_1P_t + v_t$$

(40)

where this residual should be informative, i.e. non-zero, of the shocks relating the macro-economy.
This implies that it is informative of the risk premiums and subsequently the future of bond yields (Joslin et al., 2014).

This is also related to one of the regression based results of Joslin et al. (2014), which they refer to as the three fitting properties. The first property states that the number of sources of risk should be small. The second property states that macro risks are not contained within the span of the bond yields. The third property says that components of $M_t$ are unspanned if they have predictive content in the risk premiums. In our model, which we will define in the next section, we will formulate a model that is structured such that we believe that it abides by these properties. In the section after that, we will test the regressions, and subsequently investigate the predictive properties of our model, and provide a discussion on our models relation to these three fitting properties in practice.
V. Macro Term Structures with Floating Currency

We will now attempt to form our model, which is drawn from conclusion of previously shown results in this thesis. We will ease the accuracy in notation in comparison to the models developed by [Joslin et al. (2014)], which in itself is not damaging for our conclusions as our accuracy in notation will still be in line with other authors such as [Ang et al. (2008); Piazzesi (2010), and Bauer and Rudebusch (2016)]. The main objective in creating this model, is to formulate it such that it follows the three fitting properties of [Joslin et al. (2014)], as well as include exchange rate dynamics implicitly. The discussion on whether or not that this is in violation of the first fitting property will be allocated to the discussion section in the end of the empirical analysis of the model implementation.

This section will be structured as follows; a motivation for an inflation proxy measure that can be inserted in our factor of macroeconomic risk sources, followed by an extension of the discussion of the model formulation, and how the model will look. Lastly, we will quickly discuss how to define excess bond return in such a manner so that we can perform the regression of our model on that.

A. Exchange rate dynamics & inflation

In the following section we will focus on expanding the ideas on modelling the macroeconomy in section [IV]. As previously mentioned, there has historically been grave consequences from missing out on how to model the economy without incorporating a currency factor, as a majority of all the open economies in the world, are vastly more economically interconnected now than in comparison to as early as twenty or thirty years ago ([Sorensen and Whitta-Jacobsen (2010)]. Because of this, we consider, instead of a standard policy rule as the ones used by [Ang and Piazzesi (2003)], a modified one in accordance to the one presented by [Sorensen and Whitta-Jacobsen (2010)]. This is so that we can fit this formulation such that it does not divert from the rules of the three fitting properties of [Joslin et al. (2014)]. Therefore, we make the initial assumption that inflation, as proven by [Joslin et al. (2014)], is not spanned by the bond yields. And according to [Sorensen and Whitta-Jacobsen (2010)], we can create an expression for inflation

\[ \pi_t = \beta_0 y_t + \beta_1 (S_t - S_{t-1}) + \epsilon_t \]  

(41)

with floating currency, where \( y_t \) is a variable representing the economic activity, and \( S_t \) is the spot value of the dollar index. This means that we estimate an inflation proxy measure as a linear combination of exchange rate and economic development. The equation used by [Bekaert et al. (2010)] involves lagged and predicted values in the inflation equation. However, due to reasons of parsimony and importance of focus for this thesis, we will only use the formulation as (41). In our empirical application, we will estimate \( \beta_0 \) and \( \beta_1 \) by both an OLS regression, as well as with a moment matching Monte Carlo simulation, and compare the results.
B. **Affine short-rate model**

Now, to get an expression for the short-rate dynamics, we first look at the Taylor expression for the interest rate

\[ r_t = c + \pi_t + \alpha y_t + \gamma(\pi - \pi^*) \]  

(42)

where \( c \) is some constant and \( \pi^* \) is the target inflation, which in the case of Taylor (1993), are both set to 2. We can from this in a very logical way express this as vector equations and directly obtain an affine term structure (linear term plus constant). So the rate equation in terms of latent factors can be written as the first equation in (43), which needs to be complemented with a general factor model on the term forwards, which gives us the two equation system (43)

\[
\begin{align*}
r_f^t &= c_0 + c_1' X_f^t \\
r_m^t &= b_0 + b_1' X_m^t
\end{align*}
\]

(43)

where \( X_m^t \) represents the latent macro factors, and \( X_f^t \) represents the yield forward factors. These two equations gives us a now two affine forms for describing the term structure. We can subsequently combine them to get the equation

\[ r_t = \delta_0 + \delta_{11} X_m^t + \delta_{12} X_f^t. \]  

(44)

This has now yielded an equation for the interest rate, which is now dependent on both the term structure of bond yields, as well as the macroeconomic risk sources of growth and inflation, measured as a proxy based on a linear combination from the macroeconomic motivation previously discussed. We will now briefly discuss the state dynamics of this model such that our empirical formulation will make sense.

C. **State dynamics**

In order to estimate the forward dynamics of the modified model we formulate, in accordance with Ang and Piazzesi (2003), Ang et al. (2008) and Joslin et al. (2014), a structural VAR equation with our macro factors and forward factors.

\[
\begin{bmatrix}
F_t \\
M_t
\end{bmatrix} =
\begin{bmatrix}
K_{0,F} & K_{F,F} & K_{F,M} \\
K_{0,M} & K_{M,F} & K_{M,M}
\end{bmatrix}
\begin{bmatrix}
F_{t-1} \\
M_{t-1}
\end{bmatrix} +
\begin{bmatrix}
\sigma_{F,t} \\
\sigma_{M,t}
\end{bmatrix}
\]

(45)

where all factor loading \( K \) comes from a historical measure \( \mathbb{P} \). We assume here as well that \( F_t \) follows a autonomous Gaussian VAR under the risk neutral measure \( \mathbb{Q} \)

\[ F_t = K_{0,F} + K_{F,F} F_{t-1} + \sqrt{\Sigma_{F,F}} \epsilon. \]  

(46)
We have now defined the basic model framework in which we will operate. The model should be in line with the three fitting properties stated by [Joslin et al. (2014)], which we discussed in the previous section. This implies that we can now estimate a predictive regression on the bond premia that comes from an unspanned, complete, as well as arbitrage- and risk-free framework. To finish this section, we need to define something upon which we can regress such a model. As the objective of this thesis is to investigate if there is any additional predictive power in the exchange rate dynamics on bond risk premia, leads us of course to the fact that we need to regress this on bond risk premia. To do this, we need to properly define how we are able to calculate this.

\subsection*{D. Regressing the excess bond return}

In most financial literature, risk premia is defined by the expected excess return. Following that logic, we define the bond risk premia to be the excess bond return. Finding a way of measuring this, such that we can perform our analysis, we look to the definition proposed by Cochrane and Piazzesi (2005). Writing the forward rate as

\[ f_t^{(n)} \overset{\text{def}}{=} p_{t(n-1)} - p_{t(n)} \]  

at \( t \), subtracting the prices between time \( t + n - 1 \) and \( t + 1 \). We use the same concept of the notation here as [Cochrane and Piazzesi (2005)], meaning that the superscript within the parenthesis indicates the maturity of the bond. This is so that it is not confused with some exponent style notation. In order to define excess bond returns, consider the simple example of buying a two-year bond today, and selling it as a one-year bond, in one year. The resulting balance we will have after this transaction will subsequently be the return on our investment. In a more formal sense, we define the return during a holding period as buying an \( n \)-period maturity bond at \( t \), which we will then sell as a \( n - 1 \) maturity at \( t + 1 \), which gives us the return formulation

\[ r_t^{(n)} \overset{\text{def}}{=} p_{t+1(n-1)} - p_{t(n)} \], \hspace{1cm} (48) \]

which then allows us to define excess return as

\[ r_{x_{t+1}} \overset{\text{def}}{=} r_t^{(n)} - y_t^{(1)}, \hspace{1cm} (49) \]

where \( y_t^{(1)} \) is the one-period yield. Meaning that we subtract the profit of holding the one-period yield, by the one-period ahead bond market gamble. This gives us an easy to test dependent variable, on which we can formulate the following regression

\[ r_{x_{t+1}} = \delta_0 + \delta_1 X_t^{m} + \delta_2 X_t^f. \hspace{1cm} (50) \]

We are now able to perform a regression style analysis on equation (50) to test if the imposed exchange rate dynamics affect the results when forecasting the forwards.
VI. Empirical Analysis

In order to fully explain all of the parameters in all aspects of the models and their consequences, one employs a maximum likelihood estimation via a Kalman filter. The Kalman Filter technique is a technique for evaluating a dynamic prediction state space model. This is the approach for estimation by authors such as Ang and Piazzesi (2003), Dai and Singleton (2003) and Joslin et al. (2014). We will however only be looking for indicative evidence, to see if our model will be in line with the Joslin et al. (2014) fitting properties, and limit our research objective to that level. This is in line with previous investigations, except for the specific discussion on the fitting properties, from authors such as Fama and Bliss (1987), Chang et al. (1992) and Cochrane and Piazzesi (2005).

More specifically, the regression will be performed in a vector auto-regressive manner for parameter estimation. This means that we need to specify the construction of a model of the interest rate term structure, by decomposing the covariance matrix. Motivate the definition of a term structure bond forward, and how to express this in vector form. To complete the model such that is goes from being a standard TSM to a MTSM, we need to explain how we create the macro-variable matrix. We will then be able to employ an OLS regression model with GMM standard errors.

This section will be structured as follows; firstly, a review on the regression formulation that will be utilized. Secondly, a discussion on the data used and motivations for manipulating the data in the manner performed in this investigation. Thirdly, provide results from our regressions. Lastly, we will provide a discussion on the topic such that we will compare with previous results and see if our modification of the MTSM has been able to add anything useful.

A. Excess Return Regression

As we are looking for potential evidence of validity on our formulation, we will in line with Fama and Bliss (1987) and Cochrane and Piazzesi (2005), formulate a predictive regression on expected excess returns on (44) of the form

$$E[r_{t+1}X_{t+1}] = \delta_0 + \delta_1 X_{ty}^y + \delta_2 X_{tm}^m + \Sigma_t$$

(51)

where \(r_{t+1}\) is the risk premium defined as the excess bond return in (49), and regress these factors, both yield as macro, on the excess return on the bonds. Specifically, each \(X\) in (51) are composed as follows

$$X_{ty}^y = \begin{bmatrix} \text{Level} \\ \text{Slope} \\ \text{Curvature} \end{bmatrix}, X_{tm}^m = \begin{bmatrix} y_t \\ \pi(y_t, S_t) \end{bmatrix}$$

(52)

where the individuals in \(X_{ty}^y\) (Level, Slope & Curvature), are the first three principal components of the term structure. \(y_t\) is a factor for economic growth, and \(\pi(y_t, S_t)\) is constructed inflation measure, which is dependent on economic activity and the exchange rate of the economy. \(\delta_i\) for \(i = 0, 1, 2\) are the factor loadings, or weights, which determine the effect of the factors in relation to the expected excess return.
There are of course some critique that can follow these types of simplified methods. However, as we are only looking for some form of indicative proof, we do not consider this critique fatal for our purpose.

B. Model Formulation, Data Illustration & Estimation

B.1. Standard Term Structure Framework

In order to estimate the model, we needed to evaluate all the parameters that would subsequently be used. However, there are a few steps to how the term structure equation is derived empirically. Firstly, we direct our focus to the eigendecomposition of the term structure. This means that we want to find some structure in the variation of the compounded data. Essentially, what we want to do is that we want to create a model that takes a form of

\[
\begin{bmatrix}
  y^{(1)} \\
  \vdots \\
  y^{(n)}
\end{bmatrix} = \begin{bmatrix}
  \gamma^{(1)}_1 \\
  \vdots \\
  \gamma^{(n)}_1
\end{bmatrix} \text{FACTOR}_1 + \begin{bmatrix}
  \gamma^{(1)}_2 \\
  \vdots \\
  \gamma^{(n)}_2
\end{bmatrix} \text{FACTOR}_2 + \ldots + \begin{bmatrix}
  \gamma^{(1)}_k \\
  \vdots \\
  \gamma^{(n)}_k
\end{bmatrix} \text{FACTOR}_k + \epsilon
\]  

(53)

where we have taken away the time index for readability, meaning that \( \gamma^{(j)}_i \) represents the factor loadings, or weights. This is the parameter that will determine how the vector of bond yields will move, depending on how the various factors tells it to move. To obtain these factors, we are required to chose the weights and loadings by a study of the structure of the covariance matrix. We are mathematically able to do this if we Eigendecompose it to further study its structure and to find the principal components.

To find the principal components of our term structure we first calculate the covariance of the term structure. Suppose that \( Y \) is the KxN matrix of yields with K different maturities over the time interval \( t = 1, \ldots, N \), then

\[
\text{Cov}(Y) = \Sigma = \begin{bmatrix}
  \sigma_{1,1} & \cdots & \sigma_{1,k} \\
  \vdots & \ddots & \vdots \\
  \sigma_{k,1} & \cdots & \sigma_{k,k}
\end{bmatrix},
\]  

(54)

gives us the square KxK covariance matrix. Because this is square, we can utilize the Eigendecomposition idea such that

\[
\Sigma = \begin{bmatrix}
  \sigma_{1,1} & \cdots & \sigma_{1,k} \\
  \vdots & \ddots & \vdots \\
  \sigma_{k,1} & \cdots & \sigma_{k,k}
\end{bmatrix} = Q\Lambda Q^T = \begin{bmatrix}
  q_1 & \cdots & q_k \\
  \vdots & \ddots & \vdots \\
  q_1 & \cdots & q_k
\end{bmatrix} \begin{bmatrix}
  \lambda_1 & 0 & \cdots \\
  0 & \ddots & \vdots \\
  0 & \cdots & \lambda_k
\end{bmatrix} \begin{bmatrix}
  q_1 & \cdots & q_k
\end{bmatrix}^T
\]  

(55)

where \( \Lambda \) is the diagonal of eigenvalues. If we then want to construct an equation of the factors as \( X_t = Q^T Y \) we get that \( \text{Cov}(X_t, X_{t-1}) = Q^T \Sigma Q = Q^T Q \Lambda Q^T Q = \Lambda \), which is the standard result of the Q-matricies of eigenvectos shown in section II.C and implies that our factor loadings are
uncorrelated, meaning that we can do the regressions that we want to perform.

Suppose that we form the following equation

\[ Y = QX = \begin{bmatrix} q_1 \\ \vdots \\ q_k \end{bmatrix} X_1 + \ldots + \begin{bmatrix} q_k \\ \vdots \\ q_k \end{bmatrix} X_k \]

(56)

where we see now how the columns of \( Q \) are working directly as the loadings, or weights, on each individual in \( X \). With this, we can now truncate some of the factors, as we are able to obtain enough explanatory power by just a few, and leaving the rest for the error term to capture. This is intuitively illustrated in figure 3 where we see the clear purpose of the movement in the first three factors, while going onward to the fourth and fifth factor, the economic intuition becomes more and more noisy, which is part of the reason of our three factor truncation. The information contained in each of these factor components are then accumulatively able to explain more and more of the variance in the covariance matrix. In fact, in this example, our first three components can explain more than 95% of the variation in the term structure. In figure 4 we can see the first three principal components plotted against each other, but now they are plotted over time.

Looking at figure 3 and figure 4 it is more clear why they are given their names; Level, Slope and Curvature. We are able to see where this information is coming from, and why are able to use it, and also why we truncate at three.

Figure 3. Level, Slope & Curvature: five factor loadings plotted.
Figure 4. Level, Slope & Curvature: the three first principal components of the term structure.

B.2. Term Structure Formulation with Macro Variables

As we have seen, modifying a term structure equation to fit the regression equation, which is the goal for testing the objective of this thesis, is not too hard going on from here. As we have formulated our concept of the modified inflation measure in equation (41), and as we motivated in section IV.C, utilize a straight forward measure of real industrial production (IP), as our function of economic activity. In order to estimate our inflation measure as we defined it to be in equation (41), we performed an OLS regression on

\[ \pi_{t+1} = \beta_0 y_t + \beta_1 \Delta S_t + \epsilon_t \]  

where \( \pi_{t+1} \) is the one-month forward inflation rate, \( y_t \) is a scaled IP, and \( \Delta S_t \) is the month-on-month change in the dollar index. The estimation got \( \beta_0 = 0.97 \) and \( \beta_1 = 0.00862 \). This gives us an easy way to use the equation and the resulting data as our Inflation factor, and the straight forward real IP Growth Rate as our growth factor. In table [1] we printed out all of the descriptive statistics of the data used. We noticed then that our regression based inflation measure, however significant in the regression, did not capture the statistical properties of inflation. This can partly be expected as actual measured inflation does not directly contain the dynamics of the exchange rate or economic activity. However, we figured that it would be interesting if one employed a moment matching Monte Carlo simulation in order to obtain coefficients such that it forces our inflation proxy to obtain statistical characteristics closer to that of actual inflation. We did this via a random sampling of coefficients in an algorithms, which updated the parameters whenever we got closer to creating a proxy measure with sample moments closer to actual inflation. When doing this, we got the coefficients for equation (57) as \( \beta_0 = 3.5897 \) and \( \beta_1 = 0.1032 \).
We can now formulate the modified equation (52) and formulate a regression for evaluating the predictive properties on our factors as

\[
rx_{t+1} = \begin{bmatrix} \gamma_0 \\ \gamma_1,1 \\ \gamma_1,2 \\ \gamma_1,3 \end{bmatrix} + \begin{bmatrix} Level_t \\ Slope_t \\ Curve_t \end{bmatrix} \begin{bmatrix} \gamma_2,1 \\ \gamma_2,2 \end{bmatrix} \begin{bmatrix} Growth_t \\ Inflation_t \end{bmatrix} + \Sigma_t
\]  

(58)

where we are now able to perform a OLS regression. We will in our implementation use a GMM, and use Newey-West weighing standard errors in line with the implementation suggestion of Cochrane and Piazzesi (2005), Ludvigson and Ng (2009) and Duffee (2011). We will however due to time constraints, limit our analysis of the regression specifications to this extent.

C. Data

In order to perform our empirical investigation of this relationship between the term structure and our modified hypothesis on the exchange rate effect we had to gather some data. In table I we see an overview of the statistical moments as well as the largest and smallest values of each member in the dataset that was used.

For creating the yield term structure, we used historical values of the US Treasury Constant Maturity Rate on the maturities three-month, six-month, one through three, five, seven, ten, twenty and thirty years of maturity bonds. To include the economic growth factor, we used real industrial production as a measure instead of GDP as per the motivation of Chinn and Kucko (2015), who argues that, although GDP is a broader indicator, Industrial Production will capture the dynamics more timely, and is in general a more reliable measure for economic activity. Also, as per the motivation of Hamilton (2017), we decided not to apply the Hodrick-Prescott filter to the data, as it can produce spurious dynamics, which have no relation to the underlying process itself. This is also done in the implementation of the ideas proposed by Bekaert et al. (2010) and Kaplan et al. (2016), performed by Cochrane (2017).

Regarding the exchange rate effect, we found ourselves in relatively unprecedented territory. As a majority of exchange rates are typically measured as pairs against another country’s currency, it was difficult to find appropriate data. There exists a generally acknowledged index based on the dollar, measured against a basked of currencies such as the Euro, Chinese Yuan, Canadian Dollar, Swedish Krona and more. We therefore found that as an appropriate proxy to obtain the dynamics of the US Dollar. All data was obtained via the database of the Federal Reserve Bank of St. Louis. As some data mismatched in time before 1985, we chose to omit all previous observations and therefore only perform our investigation on data spanning between January 1985 and February 2017.

The data for testing the dependent variable in equation (58) was obtained in a data set provided by John H. Cochrane, coming from the Fama and Bliss (1987) CRSP dataset on bond prices, which is also the same data used by Joslin et al. (2014), albeit with a range between 1985 to 2013. This is the reason for why we are not able to investigate the actual regression fitting results past this year as we can see in figure 5 and 6.
Descriptive Statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
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<tbody>
<tr>
<td>3-Month</td>
<td>3.4285</td>
<td>6.7311</td>
<td>0.0501</td>
<td>1.7411</td>
<td>0.0100</td>
<td>8.8200</td>
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<tr>
<td>6-Month</td>
<td>3.5319</td>
<td>6.7340</td>
<td>0.0262</td>
<td>1.7280</td>
<td>0.0400</td>
<td>8.9000</td>
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<tr>
<td>1-Year</td>
<td>3.8401</td>
<td>7.5923</td>
<td>0.0612</td>
<td>1.7821</td>
<td>0.1000</td>
<td>9.8600</td>
</tr>
<tr>
<td>2-Year</td>
<td>4.1913</td>
<td>7.7821</td>
<td>0.0733</td>
<td>1.8438</td>
<td>0.2100</td>
<td>10.7100</td>
</tr>
<tr>
<td>3-Year</td>
<td>4.4201</td>
<td>7.4550</td>
<td>0.0927</td>
<td>1.9218</td>
<td>0.3300</td>
<td>11.0500</td>
</tr>
<tr>
<td>5-Year</td>
<td>4.8162</td>
<td>6.6411</td>
<td>0.1417</td>
<td>2.0850</td>
<td>0.6200</td>
<td>11.5200</td>
</tr>
<tr>
<td>7-Year</td>
<td>5.1274</td>
<td>6.0992</td>
<td>0.2093</td>
<td>2.2146</td>
<td>0.9800</td>
<td>11.8200</td>
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<tr>
<td>10-Year</td>
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<td>5.4465</td>
<td>0.2872</td>
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<td>20-Year</td>
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<td>12.0600</td>
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<tr>
<td>30-Year</td>
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<td>4.3566</td>
<td>0.4172</td>
<td>2.4531</td>
<td>2.2300</td>
<td>11.8100</td>
</tr>
<tr>
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<td>12.5995</td>
<td>-4.2000</td>
<td>0.9000</td>
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<tr>
<td>Inflation Regression</td>
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<td>Inflation Monte Carlo</td>
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<td>-0.4925</td>
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<tr>
<td>Actual Inflation</td>
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<td>0.8341</td>
<td>0.9261</td>
<td>2.8516</td>
<td>1.0644</td>
<td>5.1207</td>
</tr>
</tbody>
</table>

Table I. A table of the four first moments as well as the smallest and largest value in the data set used for investigation, i.e. the yields at ten different maturities ranging from 3-months to 30 years.

D. Results

The following section will focus on presenting the results of the discussed regression analysis from equation (58). We will compare this to a standard term structure regression, i.e. perform a regression on equation (56), truncated at three factors and including a vector of constant of ones so that it will account for the potential existence of an intercept.

We use Matlab for our estimations. We managed to gather resources from a Matlab implementation by John H. Cochrane. From this we modified the main data management tools, so that the regressions would work with our data, as well as methods for creating the loadings matrices and formulating the expression for our exchange rate model of the inflation measure.

In table II, we see the regression results for our best formulation in the aspect to $R^2$. All p-values where significant at the 1\% level, which is the reason for us to not write it out directly in the table. We also ran regression with the Monte-Carlo estimated inflation proxy, in regards to the parameters, there were not in general much change, no too unexpectedly, only the parameter governing the inflation proxy factor changed, with $\gamma_5 = 9.06$. The coefficients otherwise where...
identical. However, the $R^2$ in that case ended up at 0.49 on average month and 0.44 on January to January regressions.

### Regression Results

<table>
<thead>
<tr>
<th></th>
<th>Three-PC TSM</th>
<th>Floating FX MTSM</th>
</tr>
</thead>
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<tr>
<td></td>
<td>Level</td>
<td>Slope</td>
</tr>
<tr>
<td>Avg. Month</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.16</td>
<td>-1.46</td>
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<tr>
<td>t-stat</td>
<td>1.66</td>
<td>-2.18</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.22</td>
<td></td>
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<tr>
<td>January</td>
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<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.17</td>
<td>-1.26</td>
</tr>
<tr>
<td>t-stat</td>
<td>2.28</td>
<td>-2.06</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.18</td>
<td></td>
</tr>
</tbody>
</table>

**Table II.** Regression results from Term Structure model using only the first three factors, compared with the Floating FX Macro Term Structure model. We also compare non-overlapping January to January forecasts.

### E. Discussion

We have now performed the relevant computations of the variables and performed the regressions. In figure 5, we see a comparison in the predictive capabilities between a standard term structure model, excess bond return by only using macro factors, and lastly, the fitted model using both the growth and the inflation proxy, as well as the three principal components of the bond yield term structure. Comparing the $R^2$ between the two implementations of the model as well as the simple plotting of the predicted times series against the expected excess bond return. We see that there is a large increase in the level of fit. Looking at the final model prediction of the expected excess bond return, one can see how our model successfully follows the motions of the peaks and troughs. We came to similar results independently on whether or not we used the OLS estimated inflation proxy measure or the Monte Carlo estimated proxy.

These results therefore indicate that the possible latent effect of changes in the exchange rate of an economy as hypothesized in [Taylor (2001)](https://www.taylor.com), is most likely existent. This means that the dynamics of the exchange rate, in which the bond is denoted, likely has a latent effect on the pricing of the yield forward premiums. However, as this area of empirical research is relatively young, meaning that there is not many straight forward implementation techniques available, one need to spend many hours programing these modelling routines by hand. We have also noted that the regression based estimation of (41), does not capture the actual statistical dynamics of inflation,
Figure 5. **Prediction Comparison:** a plotted comparison of the performance of the resulting prediction model. On top we see a predictive attempt with only the macro variables, which clearly does not predict risk premiums. In the middle, we see the standard TSM trying to predict risk premiums, much more successfully than only macro risks. On the bottom we see the full modified MTSM, with by far the most accurate prediction compared to the standard TSM.

which can partly be explained by our very primitive way of estimating the parameters. This is a task better handled by some Monte Carlo Markov Chain or Kalman Filtering approach. However, as the empirical thoroughness was not they main objective of this thesis, we suggest that to be a topic of future research. We therefore note that, even though our proxy for the inflation measure is not very optimal, there are clearly some indications of the use of the information that exists in our measure, which clearly improves the fit of the prediction of excess bond returns.

In figure 6, we show the resulting prediction model, with the Monte Carlo simulated parameters on the inflation measure, next to excess bond return. Where this previously discussed evidence, is very much apparent. One explanation for this exchange rate effect in debt markets might be explained by two factors. Firstly, the fact that there is a larger international flow of capital, as it is easier now than ever to execute such trades. It is easier for international market actors to both load and unload bonds onto their portfolios whenever there is some new information regarding central banking decision, or other economic indications that effects the future of the debt market.

Connecting these results to theory, we will look back at the fitting properties discussed by Joslin.
Figure 6. Prediction of excess bond returns: Red line represents the actual excess bond return defined as in equation (49). The blue line is the prediction model using Growth and the Monte Carlo simulated model for the inflation measure.

The first fitting property, which dictates that the sources of risks in a MTSM should be small. We recognize the fact that we are directly introducing two new sources of risk by using our inflation proxy measure as a combination of two variables. As we can see that the statistical properties of our inflation proxy does not properly match that of actual inflation, it is hard to motivate that we have the exact same amount of sources of risks as in the MTSM by Joslin et al. (2014). We do want to argue that, as the inflation measure is partly determined by a variable of economic activity, as our growth variable, the net amount of risk sources will only be one more than in the JPS-MTSM. This we do not believe will overparameterize the distribution.

The second fitting property, which states that macro risks are not contained within the span of the bond yields. Is assumed to hold. As this property is contingent on the result of testing the third fitting property. Therefore, if the macro components of our MTSM can be indicative to have some added predictive content – in relation to a standard TSM – in the risk premium, then they are said to be unspanned by the bond yields. Hence, we can therefore say that the second property holds as we provide indicative proof of the third property via our results provided in table II as well as shown in figure 5 and 6.
VII. Conclusion

This thesis has investigated term structure models. Specifically, the family of term structure models called Macro-Finance Term Structure models. We have in relation to this investigation proposed a modification of the assumption on the construction of the inflation variable, which is assumed to be a latent effect on the interest rates. Our modification entails the idea of Taylor (2001), that there might be a latent relation between the exchange rate and the interest rates.

In order to investigate this concept we have used concepts from previous formulations on a gaussian canonical arbitrage free macro term structure model with unspanned macroeconomic variables in the state vector. The model that we have had as vantage point has been the MTSM formulated by Joslin et al. (2014), as this has dealt with the previous issues regarding the spanning puzzle. We then make a formulation of a new macro-factor in line with the reasoning of Cochrane and Piazzesi (2005), Singleton (2006), and Bekaert et al. (2010), replacing the previous inflation factor of the JPS-MTSM. We subsequently made an approach to implement an excess bond return prediction model in line with the methods utilized by Chang et al. (1992), Ludvigson and Ng (2009), Duffee (2011), Cochrane (2017).

We discuss our model and the empirical results from the perspective of the Joslin et al. (2014) fitting properties. From this we conclude that our model does indeed hold such properties and can therefore be considered to be a canonical unspanned MTSM. Our results indicate that there does indeed exists a latent link between the term structure of interest rates and the exchange rate of the country, in which the interest rates are denoted. However, as we do acknowledge that there is much more to do within the empirical framework to obtain robust results, these results raises the validity of the question for future research.

This thesis has provided indicative proof of the importance of exchange rate dynamics in term structure models, however, we suggest that future research can further develop two key components of this model. Firstly, to find a better method of modelling the exchange rate dependent inflation function. Secondly, to perform the analysis on the likelihood function in order to further understand all of the components of the model, such as the pricing kernel and the market price of risk.
References


Appendix A. The Meta-Theorem

Completeness is a subject of whether or not we can define a contingent claim, or a derivative in the first place. If we have an asset \( X \), which we can replicate in some manner, we say that its hedgeable. If it is hedgeable it is complete. If we broaden our framework, if every contingent claim is hedgeable, then the underlying market is complete.

This is an important note to have in mind when working with asset pricing mechanics such as the term structure model, in which you find a unique price for several assets simultaneously, via one benchmark, which rests upon the reliability of this concept.

\( M \) denotes the number of traded assets underlying in the model. This excludes the risk free asset. \( R \) denotes the number of random sources. Then we state the following:

1. The model is free of arbitrage \( \iff M \leq R \)
2. The model is complete \( \iff M \geq R \)
3. The model is complete and arbitrage free \( \iff M = R \)

For example, in the Black-Scholes framework; the model is only driven by one underlying asset, risk-free asset (not included), and only one driving Wiener process. Therefore, \( M = R \) in that framework. (Björk, 2009, pp 118-122)

Appendix B. Itô’s Lemma

Itô’s Lemma is an important result of Itô’s calculus for finding solution to stochastic integrals. The Lemma is summarized in [1]

LEMMA 1: Let \( F(S(t),t) \) be some twice differential function of \( t \), as well as the random function \( S(t) \) for all non-negative values of \( t \).

\[
dS(t) = \mu(t)dt + \sigma(t)dW(t) \tag{B1}
\]

assuming that we have well behaved functions for drift and diffusion, \( \mu(t) \) and \( \sigma(t) \). By this we need them to be not too irregular. For example, if the integral of the square of the function in absolute value is finite, then it would satisfy this condition. Then we would have the differential equation on \( F(t) \), such that

\[
dF(t) = \frac{\partial F}{\partial S(t)}dS(t) + \frac{\partial F}{\partial t}dt + \frac{1}{2} \frac{\partial^2 F}{\partial S^2(t)} \sigma^2(t)dt \tag{B2}
\]

Substituting \( dS(t) \)

\[
dF(t) = \left[ \frac{\partial F}{\partial S(t)}\mu(t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2(t)} \sigma^2(t) \right]dt + \frac{\partial F}{\partial S(t)} \sigma(t)dW(t) \tag{B3}
\]

(Björk 2009, pp 51-52)