# A Fourier approach to valuating derivative assets

Masters thesis

Center for Mathematical Sciences, Lund University
Start date: March 22 2018
End date: August 31 2018

Oskar Rasmusson

Contact:
nilsoskarrasmusson@gmail.com

# Abstract

This paper valuates two different financial contracts, the European Call and the Spread option using the Fourier transform. In the European Call case the underlying asset is modelled by the geometric Brownian motion stochastic differential equation. All necessary conditions in order for the transform to exists are examined and it turns out that the payoff needs to be scaled by an exponential factor which includes a constant $a$ where $a < 0$. Later an optimization problem is defined in order to find the $a$ which yields the best numeric integration. At the end the Fourier method is compared against the Black Scholes formula yielding a difference with $10^{-15}$ in magnitude.

In the Spread option case the underlying assets are modelled by a two-dimensional Heston model with three volatilities, one for each asset and one for how they effect each other. Here the payoff need to be scaled by two different exponential factors each including one constant, call them $a$ and $b$ where $a < 0$ and $b < 0$. Again an optimization problem is defined in order to find the $a, b$ which yields the best numeric integration. The Fourier method for this case is compared against a Monte Carlo simulation with and without a control variate.

# Acknowledgements

I'm grateful for my family, for being supportive during my years at LTH. They have encouraged me to always do my best and to never give up. Without you I would never been able to accomplish this thesis.

I'm thankful for having Magnus Wiktorsson as a supervisor and tutor during my courses and throughout this thesis. He has been supportive and very helpful.

At last I'd like to thank LTH for providing me with the knowledge I have today. And for being supportive to their students. A special thanks to the people responsible for the LTH math forum, thanks for taking your time and answering the question I've had during my years as a student.

# Contents


**1 A first approach** **5**

1.1 The model . . . 5
1.2 The Fourier transform . . . 7
1.3 Calculating the Fourier transform and setting up the inverse Fourier transform . . . 8
1.4 The real part of the integrand is an even function . . . 10
1.5 Calculation of the inverse Fourier transform . . . 11
1.6 How to choose $a$? . . . 12
1.7 Implementation . . . 13

**2 Spread option** **15**

2.1 The Fourier transform of the spread option . . . 15
2.2 The inverse Fourier transform of the spread option . . . 17
2.2.1 The moment generating function . . . 20
2.3 Derivation of the moment generating function . . . 21
2.4 Simulation of the Heston model . . . 29
2.5 Simulation scheme . . . 29
2.6 The value of a spread option according to simulation . . . 31
2.6.1 Correlations and exposure . . . 32
2.6.2 Kappas and Thetas . . . 32
2.6.3 Initial values, Strike and interest rate . . . 33
2.7 How to choose $a$ and $b$? . . . 34
2.7.1 Can we find a minima for $h(a,b)|_{w_1=0,w_2=0}$? . . . 34
2.7.2 Can we restrict ourselves to a smaller set when searching for the min? . . . 35
2.7.3 The set where the marginal Moment generating functions are defined is not arbitrary . . . 42
2.7.4 Running an optimizer to find the values of $a$ and $b$ . . . 43
2.8 Valuation of the spread option . . . 44
2.9 Monte Carlo vs Fourier Method . . . 47

**References** **50**

# 1 A first approach

The section ”A first approach” derives and presents a result that is already known in the field of valuating derivative assets using the Fourier method. The section should be seen as a theory section where both the author and the reader get familiar with the Fourier approach. So that later when approaching a more advanced financial contract, the experience gained from the first section, will make the second part easier to read and work through. All calculations have been done by the author and thus, hopefully will come across as easy to follow.

## 1.1 The model

This subsection starts by defining what kind of financial contract will be valuated. After a model is chosen, which determines the behaviour of the stock in the financial contract. In order to understand the model a few definitions are mentioned. At the end it is explained how the value of the financial contract is obtained.

The financial contract that will be used in section is known by the name European Call. A European Call is an option, giving the buyer the right to purchase an asset at a certain price on a certain date in the future. We see $S$ as a stochastic process, i.e a family of random variables $S_t$ which are all defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. One could imagine that the value of the stock $S_t$ depends on time and some randomness, e.g. 100 SEK is worth more today than 100 SEK one year later due to inflation, on the other hand, the cost of energy might raise which leads to less profit and less dividends to stockholders and the value decreases. There fore we assume that the value of $S_t$ can be described by some function $S_t = f(t, W_t)$ where $W_t$ is a Brownian motion, i.e we assume that $S_t$ is an Itô process.

**Definition 1.** *An Itô process is defined to be an adapted stochastic process that can be expressed as the sum of an integral with respect to Brownian motion and an integral with respect to time*

$$S_t = S_{t_0} + \int_{t_0}^{t} \sigma(s, W_s) dW_s + \int_{t_0}^{t} \mu(s, W_s) ds. \tag{1}$$

This is of course, a bit heavy notation, so one usually write

$$dS_t = \mu_t dt + \sigma_t dW_t$$

The above (1) is called a Stochastic differential equation (SDE), and it's used to model many things, where one of them is the value of a stock. A natural question to ask is, what if we instead have a process $S_t = f(t, X_t)$, where $X_t$ itself may be another process driven by time and a Brownian motion. How would the dynamics look for $S_t$? The famous Itôs lemma answers this question.

**Definition 2.** *In its simplest form, Itô's lemma states the following: for an Itô drift-diffusion process*

$$dX_t = \mu_t dt + \sigma_t dW_t$$

*and any twice differentiable scalar function $f(t, x)$ of two real variables $t$ and $x$, one has*

$$dS_t = df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t.$$

In this paper the value of a stock will be modelled by different SDE:s where the first one we consider is called a geometric Brownian motion (GBM).

$$dS_t = rS_t dn + \sigma S_t dW_t \qquad \text{GBM} \quad (2)$$

$$S_{t_0} = s_t$$
$$r, \sigma \in \mathbb{R}$$

It is a well known result that the solution to (2) is $S_t = s_t e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}$ [1]. Given a European Call contract we have the following payoff function $\phi(S_T) = (S_T - K)^+$ where $K$ is the strike price. We now wish to derive that arbitrage free price of this contract using the Fourier method.

In the book written by Björk, the author argues on page 103 that the fair price is given by $P_t = e^{-r(T-t_0)}\,\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+|\mathcal{F}_t^S\right]$ [2]. Where $\mathbb{Q}$ represent the risk-neutral measure. The book presents the mathematical details of why this must be true, here we will aim for the intuition why it must be true. One could simply argue that in a fair world, the price of a financial contract at some date, should be the discounted payoff received at that date. E.g if you were to receive 100 SEK with certainty today, it would be fair to sell this certainty at a price equal to your payoff, i.e 100 SEK. But if you were to receive 100 SEK with certainty in one year, and wanted to sell that certainty today. Then the fair price would be 100 SEK discounted to todays value. But of course most contracts does not give a specific amount of money with certainty. Thus in pricing of a contract, we need to look at the payoff the holder *expects* to receive, and discount that value to today. The risk-neutral measure $\mathbb{Q}$ is a probability measure such that each share price is equal to the discounted expected future payoff, i.e exactly what we want. One can prove that such a measure exists if and only if the market is free of arbitrage [2].

Now by letting $f$ be the log distribution of $S_T$ given $S_t$ and $s = \log(S_T)$, further more let $k = \log(K)$ $\tau = (T - t_0)$, and $P_t$ be the value/price, then we obtain

$$\begin{aligned} P_t = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+|\mathcal{F}_t^S\right] = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}\left[(e^{\ln(S_T)} - e^k)^+|\mathcal{F}_t^S\right] = \\ \int_{\mathbb{R}} e^{-r\tau}\left(e^s - e^k\right)^+ f(s)ds = \int_k^\infty e^{-r\tau}(e^s - e^k)f(s)ds = g(k). \end{aligned} \quad (3)$$

## 1.2 The Fourier transform

In this subsection the Fourier transform is defined and the condition necessary for the transform to exist is presented. At the end it is investigate if the value of the financial contract, above defined as $g(k)$ can be transformed.

**Definition 3.** *For $f \in L^1(\mathbb{R})$ its Fourier transform, is defined as*

$$\tilde{f}(w) = \mathcal{F}(f)(w) = \int_{\mathbb{R}} e^{-iwx} f(x)dx$$
$$\mathcal{F} : L^1(\mathbb{R}) \to C_b(\mathbb{R}) = \{\textit{Set of bounded and continuous functions on } \mathbb{R}\}$$

Given $f \in L^1(\mathbb{R})$ then

$$|\tilde{f}(w)| = |\mathcal{F}(f)(w)| = \left|\int_{\mathbb{R}} e^{-iwx} f(x)dx\right| \leq \int_{\mathbb{R}} |e^{-iwx} f(x)|dx = \int_{\mathbb{R}} |f(x)|dx < \infty.$$

That is, $\tilde{f}$ is bounded. And considering

$$|\tilde{f}(w+h) - \tilde{f}(w)| = \left|\int_{\mathbb{R}} f(x)(e^{-i(w+h)x} - e^{-iwx})dx\right| \leq \int_{\mathbb{R}} \left|f(x)(e^{-i(w+h)x} - e^{-iwx})\right| dx =$$
$$\int_{\mathbb{R}} |f(x)||e^{-iwx}||(e^{-iwh} - 1)|dx \leq \int_{\mathbb{R}} |f(x)||(e^{-iwh} - 1)|dx = \int_{\mathbb{R}} |f(x)e^{-iwh} - f(x)|dx$$

Now $f(x)e^{-iwh}$ converges point wise to $f(x)$ and $|f(x)e^{-iwh}| \leq f(x) \in L^1(\mathbb{R})$ then by the dominated convergence theorem we have

$$|\tilde{f}(w+h) - \tilde{f}(w)| \leq \int_{\mathbb{R}} |f(x)e^{-iwh} - f(x)|dx \to 0 \quad (h \to 0).$$

That is, $\tilde{f}$ is uniformly continuous and thus, continuous, and the Fourier transform is well defined.
By plugging in what we know we can investigate the integrability of (3).

$$\begin{aligned} \|g(k)\|_1 = \int_{\mathbb{R}} \left|\int_k^{\infty} e^{-r\tau}(e^s - e^k)f(s)ds\right| dk = \\ \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} \int_k^{\infty} \underbrace{(s_0 e^{-\frac{\tau\sigma^2}{2} + \sqrt{\tau}\sigma s} - e^{-r\tau}e^k)e^{-s^2/2}}_{\geq 0} ds \right| dk = \\ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_k^{\infty} e^{-\frac{(s-\sqrt{\tau}\sigma)^2}{2}} - e^{k-r\tau-s^2/2} ds dk = \\ \int_{\mathbb{R}} \underbrace{s_0\Phi(-k+\sqrt{\tau}\sigma)}_{(*)} - \underbrace{e^{k-r\tau}\Phi(-k)}_{(**)} dk \end{aligned} \tag{4}$$

Where $(*)$ does not converge, while $(**)$ is finite.

## 1.3 Calculating the Fourier transform and setting up the inverse Fourier transform

In this subsection we start by tacking care of the problem of the non-converging integral. This is followed up by calculating the transform. Then the inverse Fourier transform is defined. In our case the inverse transform represents the value that we seek. Again we verify the necessary conditions that must be fulfilled in order for the inverse transform to exists. At the end we express the value of the European call as an inverse Fourier transform.

Earlier we saw that $(*)$ does not converge, while $(**)$ is finite. Thus we conclude that $g \notin L^1(\mathbb{R})$. On the other hand, we also learned that the factor $e^k$ made $(**)$ converge, this becomes useful.
The trick is now to define $p_t \equiv e^{-ak}P_t,\ a < 0$ then $p_t \in L^1(\mathbb{R})\ \forall k$ thus the Fourier transform exists. If $\tilde{p}_t(w)$ denotes the Fourier transform of $p_t$ then we obtain

$$\begin{aligned}\tilde{p}_t(w) = \int_{\mathbb{R}} e^{-iwk}e^{-ak}P_t(k)dk = \int_{\mathbb{R}} e^{-iwk}e^{-ak}\int_k^{\infty} e^{-r\tau}(e^s - e^k)f(s)dsdk = \\ \int_{\mathbb{R}} e^{-r\tau}f(s)\int_{-\infty}^{s} e^{-iwk}(e^{s-ak} - e^{k-ak})dkds = \\ \int_{\mathbb{R}} e^{-r\tau}f(s)e^{s(-(a+iw))}\left(\frac{e^s}{a-1+iw} - \frac{e^s}{a+iw}\right)ds = \\ \frac{e^{-r\tau}}{(a+iw)(a+iw-1)}\int_{\mathbb{R}} f(s)e^{s(1-a-iw)}ds.\end{aligned} \tag{5}$$

**Definition 4.** *For a random variable $X$ the characteristic function $\phi$ is defined as the expected value of $e^{itX}$*

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx}f_X(x)dx.$$

Using the definition of characteristic function, we notice that (5) becomes

$$\tilde{p}_t(w) = \frac{e^{-r\tau}\phi_{\log(S_T)}(-w-(1-a)i)}{(a+iw)(a+iw-1)} = \tag{6}$$

Where $\log(S_T) \sim \mathcal{N}\left(\ln(s_t) + (r - \frac{\sigma^2}{2}\tau), \sigma^2\tau\right) = \mathcal{N}(\mu, \tilde{\sigma}^2)$.

**Theorem 1.1.** *If $X$ is normal distributed, $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\phi_X(t) = e^{it\mu - \frac{1}{2}\sigma^2t^2}$*

*Proof.* Set $Y \sim \mathcal{N}(0,1)$ then $\sigma Y + \mu \sim \mathcal{N}(\mu, \sigma^2)$, and

$$\phi_Y(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \underbrace{e^{ity}}_{\cos(ty)+i\sin(ty}e^{-\frac{y^2}{2}}dy = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\cos(ty)e^{-\frac{y^2}{2}}dy.$$

Since $y \mapsto \sin(ty)e^{-\frac{y^2}{2}}$ is odd

Differentiating with respect to $t$ yields

$$\phi'_Y(t) = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\sin(ty)ye^{-\frac{y^2}{2}}dy \text{ integration by parts yields}$$

$$\phi'_Y(t) = \frac{1}{\sqrt{2\pi}}\left(\underbrace{\left[\sin(ty)e^{-\frac{y^2}{2}}\right]_{-\infty}^{\infty}}_{=0\ (odd)} - \int_{\mathbb{R}} t\cos(ty)e^{-\frac{y^2}{2}}dy\right) =$$

$$-t\phi_Y(t) \Rightarrow$$

$(\ln(\phi_Y(t)))' = -t$ and with $\phi_Y(0) = 1$ we conclude that

$$\phi_Y(t) = e^{-\frac{t^2}{2}}. \text{ Thus}$$

$$\phi_X(t) = \phi_{\sigma Y+\mu} = \mathbb{E}[e^{it(\sigma Y+\mu)}] = e^{it\mu}\,\mathbb{E}[e^{it\sigma Y}] = e^{it\mu}\phi_Y(\sigma t) = e^{it\mu}e^{-\frac{\sigma^2t^2}{2}}$$

□

And thus, $\phi_{\log(S_T)}(\gamma) = e^{i\gamma\mu}e^{-\frac{\tilde{\sigma}^2\gamma^2}{2}}$.

**Definition 5.** *For $\tilde{f} \in L^1(\mathbb{R})$ its inverse Fourier transform, is defined as*

$$f(x) = \mathcal{F}^{-1}(\tilde{f})(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iwx}\tilde{f}(w)dw$$

$$\mathcal{F}^{-1} : L^1(\mathbb{R}) \to C_b(\mathbb{R}) = \{\textit{Set of bounded and continuous functions on } \mathbb{R}\}$$

To show that the definition of the inverse Fourier transform is well defined is analogue to showing that the Fourier transform is well defined.
Now we want to calculate the inverse Fourier transform of (6). Thus we must first verify that $\tilde{p}_t \in L^1(\mathbb{R})$. We have

$$\left|\frac{e^{-r\tau}\phi_{\log(S_T)}(-w-(1-a)i)}{(a+iw)(a+iw-1)}\right| = \left|\frac{e^{-r\tau}e^{i\mu(-w-(1-a)i)-\frac{1}{2}\sigma^2\tau(-w-(1-a)i)^2}}{(a+iw)(a+iw-1)}\right| =$$

$$\frac{e^{-r\tau}\left|e^{(1-a)\mu-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)+iw((1-a)\sigma^2\tau-\mu)}\right|}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}} =$$

$$\frac{e^{-r\tau+(1-a)\mu-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)}\left|e^{iw((1-a)\sigma^2\tau-\mu)}\right|}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}} =$$

$$\frac{e^{-r\tau+(1-a)\mu-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)}}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}}$$

We are interested in determining the integrability (in $w$) thus the interesting part in this case is $\frac{e^{-\frac{1}{2}(w^2-(1-a)^2)}}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}} = g(w)$. We have that $g$ is an even function, continuous and doesn't have any singularities. By Weierstrass the maximum value of $g$ exists on any compact set, and thus we can conclude that $g$ is integrable by

$$\int_{\mathbb{R}} g(w)dw = 2\int_0^\infty \frac{e^{-\frac{1}{2}(w^2-(1-a)^2)}}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}}dw =$$

$$\int_0^M \frac{e^{-\frac{1}{2}(w^2-(1-a)^2)}}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}}dw + \int_M^\infty \frac{e^{-\frac{1}{2}(w^2-(1-a)^2)}}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}}dw <$$

$$M \max_{w\in[0,M]} g(w) + \int_M^\infty \frac{e^{-\frac{1}{2}(w^2-(1-a)^2)}}{\sqrt{a^2+w^2}\sqrt{(a-1)^2+w^2}}dw \sim$$

$$M \max_{w\in[0,M]} g(w) + \int_M^\infty \frac{e^{-\frac{1}{2}w^2}}{w^2}dw < \infty$$

Therefore we conclude that $\tilde{p}(w) \in L^1(\mathbb{R})$ and its inverse is

$$p_t(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iwk}\tilde{p}_t(w)dw.$$

Using that $p_t \equiv e^{ak}P_t$, and that the price is a real number, we obtain

$$\begin{aligned} P_t &= \Re\left(\frac{e^{ak}}{2\pi}\int_{\mathbb{R}} e^{iwk}\tilde{p}_t(w)dw\right) = \\ P_t &= \frac{e^{ak}}{2\pi}\int_{\mathbb{R}} \Re\left(e^{iwk}\tilde{p}_t(w)\right)dw. \end{aligned} \tag{7}$$

## 1.4 The real part of the integrand is an even function

This section proves that the integrand in the inverse transform i.e the value of the European call is an even function. This result comes handy later when numerically calculating the inverse.

Focusing on the product $e^{iwk}\tilde{p}_t(w)$ we obtain

$$
e^{iwk}\tilde{p}_t(w) = e^{iwk}\frac{e^{-rT}\phi_{\log(S_T)}(-w-(1-a)i)}{\underbrace{(a+iw)(a+iw-1)}_{x+iy}} =
$$

$$
\frac{e^{-r\tau}}{|x+iy|^2}(x+iy)e^{iwk+i\mu(-w-(1-a)i)-\frac{1}{2}\sigma^2\tau(-w-\sigma^2(1-a)i)^2} =
$$

$$
\frac{e^{-r\tau+\mu(1-a)-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)}}{|x+iy|^2}(x+iy)e^{i\overbrace{w(-\mu+\sigma^2\tau(1-a)+k)}^{q}} =
$$

$$
\frac{e^{-r\tau+\mu(1-a)-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)}}{|x+iy|^2}(x\cos(q)+ix\sin(q)+iy\cos(q)-y\sin(q))
$$

By taking $\Re$ we obtain

$$
\Re\left(e^{iwk}\tilde{p}_t(w)\right) =
$$

$$
\Re\left(\frac{e^{-r\tau+\mu(1-a)-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)}}{|x+iy|^2}(x\cos(q)+ix\sin(q)+iy\cos(q)-y\sin(q))\right) =
$$

$$
\frac{e^{-r\tau+\mu(1-a)-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)}}{|x+iy|^2}(x\cos(q)-y\sin(q))
$$

with

$$
x = a^2-a-w^2 \text{ even in } w
$$

$$
y = 2wa-w \text{ odd in } w
$$

$$
|x+iy|^2 = (a^2-a-w^2)^2+(2aw-w)^2 \text{ even in } w
$$

$$
e^{-r\tau+\mu(1-a)-\frac{1}{2}\sigma^2\tau(w^2-(1-a)^2)} \text{ even in } w \Rightarrow
$$

$$
\Re\left(e^{iwk}\tilde{p}_t(w)\right) \text{ even in } w
$$

## 1.5 Calculation of the inverse Fourier transform

Here we conclude that the inverse transformation is not possible to calculate analytically and thus we present on how we approximate the integral in order to calculate it numerically.

The price is even in real numbers and therefore (7) is equal to

$$
\begin{gathered}
P_t = \frac{e^{ak}}{\pi}\int_0^\infty \Re\left(e^{iwk}\tilde{p}_t(w)\right)dw = \\
\frac{e^{ak}}{\pi}\int_0^\infty \Re\left(e^{iwk}\frac{e^{-r\tau}\phi_{\log(S_T)}(-w-(1-a)i)}{(a+iw)(a+iw-1)}\right)dw = \\
\frac{e^{-r\tau}}{\pi}\int_0^\infty \Re\left(\frac{e^{k(iw+a)}\phi_{\log(S_T)}(-w-(1-a)i)}{(a+iw)(a+iw-1)}\right)dw
\end{gathered} \tag{8}
$$

It is not possible to calculate (8) analytically, which implies that we must calculate it numerically. We approximate in the following way.

$$\frac{e^{-r\tau}}{\pi}\int_0^\infty \Re\left(\frac{e^{k(iw+a)}\phi_{\log(S_T)}(-w-(1-a)i)}{(a+iw)(a+iw-1)}\right)dw \approx \frac{e^{-r\tau}}{\pi}\sum_{k\in\mathbb{N}} w_k^{(n)}\Re\left(\frac{e^{k(ix_k^{(n)}+a)}\phi_{\log(S_T)}(-x_k^{(n)}-(1-a)i)}{(a+ix_k^{(n)})(a+ix_k^{(n)}-1)}\right) \tag{9}$$

## 1.6 How to choose $a$?

In the section ”Calculating the Fourier transform and setting up the inverse Fourier transform” we said that we can make the transform exists as long as $a < 0$. In this section we determine the actual value of $a$, i.e how to find an $a$ that suits the numerical inverse integration *the best*.

We have not commented on how to choose $a$ more than that $a < 0$. We want to choose $a$ in such a way, that the integrand behaves as ”nice” as possible, to improve the behaviour of the numerical integration. Here ”nice” means that the integrand oscillates as little as possible. First of all since the integrand is holomorphic, the value of $a$ will not change the value of the integral (in theory). Set

$$h(a) = \Re\left(\frac{e^{k(iw+a)}\phi_{\log(S_T)}(-w-(1-a)i)}{(a+iw)(a+iw-1)}\right),$$

$$\text{remember that } \mu = \ln(s_t) + (r - \frac{\sigma^2}{2})\tau \Rightarrow$$

$$h(a) = \frac{e^{k(iw+a)}e^{i(\ln(s_t)+(r-\frac{\sigma^2}{2})\tau)(-w-(1-a)i)-\frac{1}{2}(\sigma^2\tau^2(-w-(1-a)i)^2)}}{(a+iw)(a+iw-1)} = \frac{e^{k(iw+a)+(\ln(s_t)+(r-\frac{\sigma^2}{2})\tau)(-iw+(1-a))-\frac{1}{2}(\sigma^2\tau^2(-w-(1-a)i)^2)}}{(a+iw)(a+iw-1)}.$$

In the paper by M.Wiktorsson, see [3], Wiktorsson argues that that in order for the inverse transform to exists. We can choose any $a$ such that $a \in A^+_{S_T}$, where

$$A^+_{S_T} = \{x > 0 : \mathbb{E}^{\mathbb{Q}}[S_T^{1+x}] < \infty\}.$$

Furthermore he draws the conclusion that the oscillations are proportional to $|h(a)|$, and that the oscillations will be more pronounced when the modulus of $w$ is small. From there he draws the conclusion that the oscillations will only be significant when $|w|$ is small. There after he conclude that $|h(a)| \leq h(a)|_{w=0}$, where $h(a)|_{w=0}$ is given by

$$h(a)|_{w=0} = \frac{e^{ka+(\ln(s_t)+(r-\frac{\sigma^2}{2})\tau)(1-a)-\frac{1}{2}(\sigma^2\tau((1-a)i)^2)}}{a(a+1)} = \frac{e^{ka+(\ln(s_t)+(r-\frac{\sigma^2}{2})\tau)(1-a)+\frac{1}{2}\sigma^2\tau(1-a)^2}}{a(a+1)}.$$

Then he states that the function $h(a)|_{w=0}$ is convex in $a$ on the set $A^+_{S_T}$, and therefore has a unique minimum in $A^+_{S_T}$. Wiktorsson says that as a rule of thumb, $a$ is chosen such that $a = a_{\min} = \min_a h(a)|_{w=0}$. Then he states that it turns out that $a = a_{\min}$ and $w = 0$ is a saddle point for the function $h(a)$.
All in all, if we run an optimizer, finding an $a$ which is a local min for $h(a)|_{w=0}$, then convexity tells us that it is a global minimizer. Thus, we can run a simple routine, in this case, the golden-section method on $(-\infty, 0)$ to find $a$.

## 1.7 Implementation

In this section the Fourier method is implemented in MATLAB and tested against another method of pricing an European call. At the end the results are compared.

We are now interested to see how the Fourier approach compares to the standard way of valuating a European Call. In both [1] p.105 and [2] p.153 the Black-Scholes formula is mentioned and derived as an exact way of pricing the European Call. Where the price is given by

$$C(S_t, t) = \mathcal{N}(d_1)s_t - \mathcal{N}(d_2)Ke^{-r\tau}$$
$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{s_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right)$$
$$d_2 = d_1 - \sigma\sqrt{\tau}$$

With the following values taken from [4] example 21.3

Table 1: Values of constants needed to determine the price.

| | |
|---|---|
| $K =$ | \$ 5.976 |
| $r =$ | 0.01 |
| $t =$ | 0 |
| $T =$ | $\frac{148}{365}$ |
| $s_t =$ | 5.03 |
| $\sigma =$ | 0.65. |

With table 1 we then obtain

Table 2: The price of a European Call, with values from table 1 for the two different methods.

| | |
|---|---|
| Fourier method price | \$ 0.511475842046357 |
| Black Scholes formula price | \$ 0.511475842046358 |
| Error | \$ $-1.998401444325282 \cdot 10^{-15}$. |

In table 2 we can see that the Fourier method is highly accurate. Below follows an image showing the value of the European Call for different strikes $K = 2, 4, 6, 8, \ldots, 20$ using both the Fourier method and the Black Scholes formula. Running the golden section method yields $a = -3.470177$.

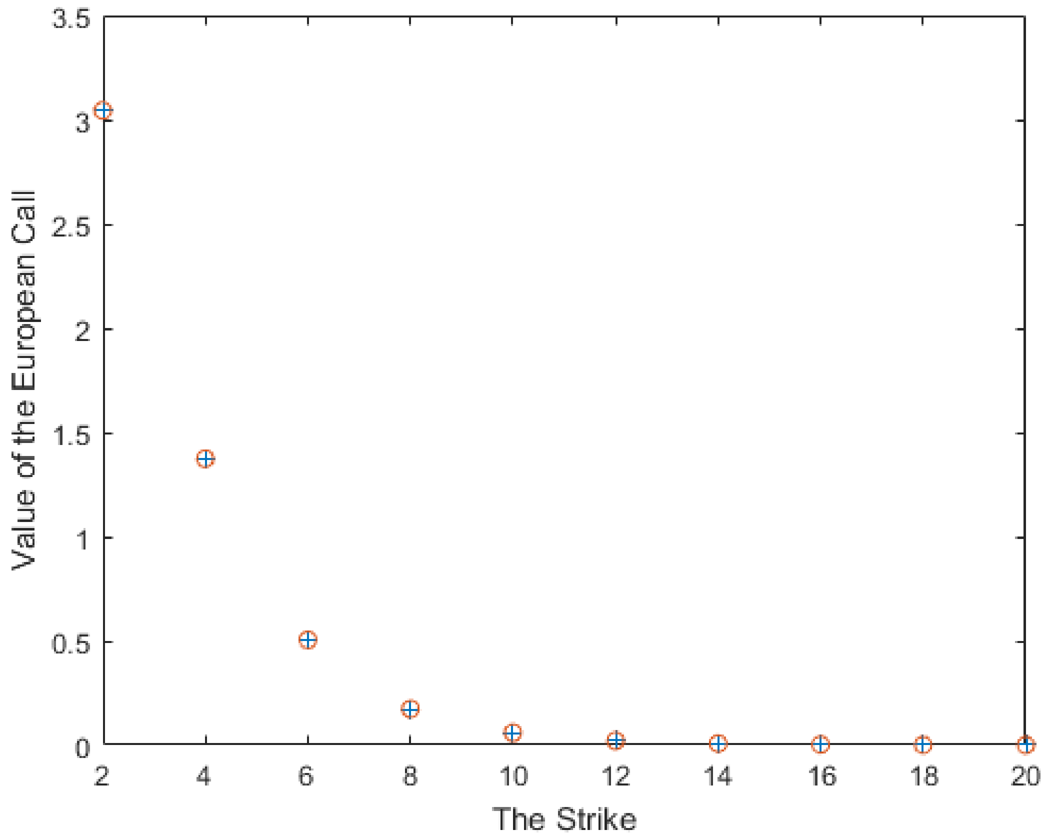


Figure 1: Here we see the value of the European Call using both the Fourier method '+' and the Black Scholes formula 'o' for different strikes.

# 2 Spread option

In this section the spread option is valuated under the assumptions that the assets follows a Heston model. The idea is to follow the same recipe as in the case of the European call. A Fourier approach to the spread option has been done before, but with other models for the assets. Thus, the combination of the spread option and a two dimensional (assets), and three dimensional (volatilities) Heston model is the contribution of this paper.

A spread option is a type of option where the payoff is based on the difference in price between to underlying assets. Here we will consider what is called the spread call, which has a payoff defined as $(S_1(T)-S_2(T)-K)^+$. With arguments as before, the price of this contract is given by $P_t(K) = e^{-r(T-t_0)}\,\mathbb{E}^{\mathbb{Q}}\left[(S_1(T)-S_2(T)-K)^+|\mathcal{F}_t^S\right]$. Since we have two underlying assets we have to use a two-dimensional Fourier transform. Here we are going to first transform in $K$ and then later in $S_2$. Let $k=\log(K)$, $x=\log(S_1(T))$, $y=\log(S_2(T))$, and let $\tilde{P}_t$ be the Fourier transform of the payoff.

## 2.1 The Fourier transform of the spread option

In this subsection we set up the Fourier transform and investigate whether it fulfills the necessary conditions for existence. Just as in the case of the European call it turns out that the transform doesn't exist but using again an exponential factor solves the problem.

We are interested in calculating

$$\begin{gathered}\Pi = e^{-r(T-t_0)}\,\mathbb{E}^{\mathbb{Q}}\left[(S_1(T)-S_2(T)-K)^+|\mathcal{F}_t^S\right] = \\ e^{-r(T-t_0)}\,\mathbb{E}^{\mathbb{Q}}\left[(e^x-e^y-e^k)^+|\mathcal{F}_t^S\right] = \\ e^{-r(T-t_0)}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{iw_2y}e^{iw_1k}\tilde{P}_t dw_1 dw_2|\mathcal{F}_t^S\right].\end{gathered} \tag{10}$$

Where $\tilde{P}_t$ denotes the Fourier transform of the spread option, and the integral in the expectation the inverse Fourier transform.
Before we can start transforming in $k$ we need to make sure that the integrand is integrable. By letting $f(k)=(e^x-e^y-e^k)^+$, we continue by

$$\|f\cdot\mathbb{1}_{\{-\infty<k<\log(S_1-S_2)\}}\|_1 = \int_{\mathbb{R}} f(k)\mathbb{1}_{\{-\infty<k<\log(S_1-S_2)\}}dk = \int_{-\infty}^{\log(S_1-S_2)} e^x-e^y-e^k dk. \tag{11}$$

Which is directly a problem, $f\notin L^1(\mathbb{R})$. We will use same trick as before multiply with

$e^{-ak}$, then $\|e^{-ak}f\| \in L^1(\mathbb{R}), \forall a < 0$.

$$\begin{gathered}\|fe^{-ak}\mathbb{1}_{\{-\infty<k<\log(S_1-S_2)\}}\|_1 = \int_{\mathbb{R}} f(k)e^{-ak}\mathbb{1}_{\{-\infty<k<\log(S_1-S_2)\}}dk = \\ \int_{-\infty}^{\log(S_1-S_2)} e^{-ak}(e^x - e^y - e^k)dk = \int_{-\infty}^{\log(S_1-S_2)} e^{x-ak} - e^{y-ak} - e^{k(-a+1)}dk < \infty\end{gathered} \tag{12}$$

Since $e^{-ak}f \in L^1(\mathbb{R}),\ a < 0$ we can now calculate its Fourier transform in $k$.

$$\begin{gathered}\int_{\mathbb{R}} (e^x - e^y - e^k)^+ e^{-ak-iw_1k}dk = \Big/ -a - iw_1 = z_1 \Big/ = \int_{-\infty}^{\log(S_1-S_2)} (e^x - e^y - e^k)e^{z_1k}dk = \\ \frac{(e^x - e^y)^{z_1+1}}{z_1(z_1+1)}\end{gathered} \tag{13}$$

Since $a < 0$ the integral vanish at $-\infty$. Now we wish to compute the transform in $y$, which means that first we must investigate the integrability.

$$\begin{gathered}\|\frac{(e^x - e^y)^{z_1+1}}{z_1(z_1+1)}\mathbb{1}_{\{x>y\}}\|_1 = \int_{\mathbb{R}} \left|\frac{(e^x - e^y)^{z_1+1}}{z_1(z_1+1)}\right| \mathbb{1}_{\{x>y\}}dy = \\ \frac{1}{|z_1(z_1+1)|}\int_{-\infty}^{x}(e^x - e^y)^{z_1+1}dy = \Big/ \ v = e^y \ \Big/ \\ = \frac{1}{|z_1(z_1+1)|}\int_0^{e^x}(e^x - v)^{z_1+1}\frac{dv}{v} = \\ \Big/ \ v = pe^x \ \Big/ = \frac{1}{|z_1(z_1+1)|}\int_0^1 (e^x - pe^x)^{z_1+1}\frac{dpe^x}{pe^x} = \\ \frac{1}{|z_1(z_1+1)|}\int_0^1 e^{x(z_1+1)}(1-p)^{z_1+1}\frac{1}{p}dp = \frac{e^{x(z_1+1)}}{|z_1(z_1+1)|}\int_0^1 p^{-1}(1-p)^{z_1+1}dp\end{gathered} \tag{14}$$

Where (14) doesn't converge, which is easily seen by investigating the integrand in a neighbourhood around the boundary. We investigate what happens if we use our normal trick and multiply by $e^{-by}$.

$$\begin{aligned}
&\|e^{-by}\frac{(e^x-e^y)^{z_1+1}}{z_1(z_1+1)}\mathbb{1}_{\{x>y\}}\|_1 = \int_{\mathbb{R}}\left|e^{-by}\frac{(e^x-e^y)^{z_1+1}}{z_1(z_1+1)}\right|\mathbb{1}_{\{x>y\}}dy = \\
&\frac{1}{|z_1(z_1+1)|}\int_{-\infty}^{x} e^{-by}(e^x-e^y)^{z_1+1}dy = \Big/\ e^y = v\ \Big/ = \\
&\frac{1}{|z_1(z_1+1)|}\int_{0}^{e^x} v^{-b}(e^x-v)^{z_1+1}\frac{dv}{v} = \\
&\Big/\ v = pe^x\ \Big/ = \frac{1}{|z_1(z_1+1)|}\int_0^1 p^{-b}e^{-xb}(e^x-pe^x)^{z_1+1}\frac{dpe^x}{pe^x} = \\
&\frac{e^{x(z_1+1-b)}}{|z_1(z_1+1)|}\int_0^1 p^{-b-1}(1-p)^{z_1+1}dp = \frac{e^{x(z_1+1-b)}}{|z_1(z_1+1)|}\beta(-b, z_1+2) = \\
&\frac{e^{x(z_1+1-b)}}{|z_1(z_1+1)|}\frac{\Gamma(-b)\Gamma(z_1+2)}{\Gamma(-b+z_1+2)}
\end{aligned} \tag{15}$$

We see that in (15) we must have $b<0$ in order to have integrability (remember $a<0$). Thus the Fourier transform becomes

$$\begin{aligned}
&\int_{-\infty}^{x}\frac{(e^x-e^y)^{z_1+1}}{z_1(z_1+1)}e^{-by}e^{-iw_2y}dy = \int_{-\infty}^{x}\frac{(e^x-e^y)^{z_1+1}}{z_1(z_1+1)}e^{y(-b-iw_2)}dy = \Big/\ -b-iw_2 = z_2\ \Big/ = \\
&\int_{-\infty}^{x}\frac{(e^x-e^y)^{z_1+1}}{z_1(z_1+1)}e^{yz_2}dy = \Big/\ e^y = v\ \Big/ = \int_0^{e^x}\frac{(e^x-v)^{z_1+1}}{z_1(z_1+1)}v^{z_2-1}dv = \Big/\ v = pe^x\ \Big/ = \\
&\int_0^1\frac{(e^x-pe^x)^{z_1+1}}{z_1(z_1+1)}(pe^x)^{z_2-1}e^x dp = \int_0^1\frac{e^{x(z_1+1)}(1-p)^{z_1+1}}{z_1(z_1+1)}p^{z_2-1}e^{x(z_2-1)}e^x dp = \\
&\frac{e^{x(z_1+1+z_2)}}{z_1(z_1+1)}\int_0^1 p^{z_2-1}(1-p)^{(z_1+2)-1}dp = \frac{e^{x(z_1+1+z_2)}}{z_1(z_1+1)}\beta(z_2, z_1+2) \\
&\frac{e^{x(z_1+1+z_2)}}{z_1(z_1+1)}\frac{\Gamma(z_2)\Gamma(z_1+2)}{\Gamma(z_1+z_2+2)} = \frac{e^{x(z_1+1+z_2)}}{z_1}\frac{\Gamma(z_2)\Gamma(z_1+1)}{\Gamma(z_1+z_2+2)} = \\
&e^{x(z_1+1+z_2)}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)} = \tilde{p}_t(w_1, w_2).
\end{aligned} \tag{16}$$

Where $\Re(z_1)>0,\ \Re(z_2)>0$. With $p(t)=e^{-ak}e^{-by}P(t)$ we have that (10) becomes

$$\frac{1}{(2\pi)^2}e^{-r(T-t_0)}e^{ak}e^{by}\,\mathbb{E}^{\mathbb{Q}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}e^{iw_2y}e^{iw_1k}\tilde{p}_t(w_1,w_2)dw_1dw_2|\mathcal{F}_t^S\right]. \tag{17}$$

## 2.2 The inverse Fourier transform of the spread option

In this subsection we verify that the transform fulfills the necessary conditions in order for the inverse Fourier transform to exists. Then the inverse is calculated.

Now in order to continue, we must verify that the inverse Fourier transform exist, i.e that $\tilde{p}_t(w_1, w_2) \in L^1(\mathbb{R}^2)$. The gamma function is an analytical function in the right half-plane and has no zeros, which implies that $\tilde{p}_t(w_1, w_2)$ is continuous and integrable at every compact subset. Thus, the only problem to consider is how $\tilde{p}_t(w_1, w_2)$ behaves for large values of $w_1$ and $w_2$ i.e what happens at the limits of $\int_{\mathbb{R}} \int_{\mathbb{R}} |\tilde{p}_t(w_1, w_2)| \, dw_1 dw_2$. While investigating if $\tilde{p}_t(w_1, w_2) \in L^1(\mathbb{R}^{\not{2}})$ the factor $e^{x(z_1+1+z_2)}$ is not of interest as it the imaginary part becomes 1 while taking the absolute value, the real part doesn't depend on $w_1, w_1$ and thus doesn't affect integrability. The interesting part is what happens with the gamma functions when the absolute value of $w_1$ and $w_2$ becomes large.

**Theorem 2.1.** ***Stirlings formula*** *For $|z| \to \infty$, $\delta > 0$ and $|\arg z| < \pi - \delta$ it holds that*

$$\log \Gamma(z) \sim \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \log\left(1 + O\left(\frac{1}{z}\right)\right)$$

See [5] p.257 Eq 6.1.41 for a reference. Then we can see that

$$\begin{aligned}
\log \Gamma(z+x) &\sim \left(z + a - \frac{1}{2}\right) \log(z+x) - z - a + \frac{1}{2}\log(2\pi) + \log\left(1 + O\left(\frac{1}{z}\right)\right) \\
&\sim \left(z + a - \frac{1}{2}\right)\log(z) + \left(z + a - \frac{1}{2}\right)\log\left(1 + \frac{a}{z}\right) - z - a + \frac{1}{2}\log(2\pi) + \log\left(1 + O\left(\frac{1}{z}\right)\right) \\
&\sim \left(a + z - \frac{1}{2}\right)\log(z) + z\log(1 + \frac{a}{z}) - z - a + \frac{1}{2}\log(2\pi) + \log\left(1 + O\left(\frac{1}{z}\right)\right) \\
&\sim \left(z + a - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \log\left(1 + O\left(\frac{1}{z}\right)\right)
\end{aligned}$$

Since $e^{\log \Gamma(z)} = \Gamma(z)$ we obtain

$$g_x(w) = |\Gamma(x + iw)| = \sqrt{2\pi} e^{-\frac{\pi}{2}|w|} |w|^{a - \frac{1}{2}} + \left(1 + O\left(\frac{1}{|w|}\right)\right)$$

under the condition that $x$ is fixed and that $w \in \mathbb{R}$. Since $g_x$ is continuous and doesn't have any zeros it follows that there exists constants $0 \leq c_x \leq C_x$ such that

$$c_x e^{-\frac{\pi}{2}|w|} \sqrt{1 + w^2} \leq g_x(w) \leq C_x e^{-\frac{\pi}{2}|w|} \sqrt{1 + w^2}$$

By using these approximations on the quotient of Gamma functions in $\tilde{p}(w_1, w_2)$ (16) we obtain that

$$|\tilde{p}(w_1, w_2)| \leq C \frac{\sqrt{1 + w_1^2}^{-a-\frac{1}{2}} \sqrt{1 + w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1 + (w_1 + w_2)^2}^{-a-b+\frac{3}{2}}} e^{-\frac{\pi}{2}(|w_1| + |w_2| - |w_1 + w_2|)}$$

Let $0 < \delta < 1$ and

$$M_\delta = \{(w_1, w_2) \in \mathbb{R}^2 : \delta \cdot (|w_1| + |w_2|) > (|w_1| + |w_2|) - |w_1 + w_2| \geq 0\}.$$

Then for $(w_1, w_2) \in \mathbb{R}^2 \backslash M_\delta$,

$$|\tilde{p}(w_1,w_2)| \leq C\frac{\sqrt{1+w_1^2}^{-a-\frac{1}{2}}\sqrt{1+w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1+(w_1+w_2)^2}^{-a-b+\frac{3}{2}}}e^{-\frac{\pi}{2}(|w_1|+|w_2|-|w_1+w_2|)} \leq$$
$$C_1\frac{\sqrt{1+w_1^2}^{-a-\frac{1}{2}}\sqrt{1+w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1+(w_1+w_2)^2}^{-a-b+\frac{3}{2}}}e^{-\delta\frac{\pi}{2}(|w_1|+|w_2|)} \leq$$
$$C_1\sqrt{1+w_1^2}^{-a-\frac{1}{2}}e^{-\delta\frac{\pi}{2}|w_1|}\cdot\sqrt{1+w_2^2}^{-b-\frac{1}{2}}e^{-\delta\frac{\pi}{2}|w_2|} \in L^1(\mathbb{R}^2)$$

On the set $M_\delta$ we instead have the following upper bound

$$|\tilde{p}(w_1,w_2)| \leq C\frac{\sqrt{1+w_1^2}^{-a-\frac{1}{2}}\sqrt{1+w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1+(w_1+w_2)^2}^{-a-b+\frac{3}{2}}}e^{-\frac{\pi}{2}(|w_1|+|w_2|-|w_1+w_2|)} \leq$$
$$C_2\frac{\sqrt{1+w_1^2}^{-a-\frac{1}{2}}\sqrt{1+w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1+(w_1+w_2)^2}^{-a-b+\frac{3}{2}}}$$

Since $\sqrt{1+t^2}$ is an even function of $t$, and with the property that $\sqrt{1+\delta^2 t^2} \geq \delta\sqrt{1+t^2}$ it holds on $M_\delta$ that $\sqrt{1+(w_1+w_2)^2} \geq \sqrt{1+\delta^2 w_j^2} \geq \delta\sqrt{1+w_j^2}$ with $j = 1 \vee 2$. This implies that

$$|\tilde{p}(w_1,w_2)| C_2\frac{\sqrt{1+w_1^2}^{-a-\frac{1}{2}}\sqrt{1+w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1+(w_1+w_2)^2}^{-a-b+\frac{3}{2}}} \leq$$
$$\frac{C_2}{\delta^{-a-b+\frac{3}{2}}}\frac{\sqrt{1+w_1^2}^{-a-\frac{1}{2}}\sqrt{1+w_2^2}^{-b-\frac{1}{2}}}{\sqrt{1+w_1^2}^{-a-\frac{3}{4}}\sqrt{1+w_2^2}^{-b+\frac{3}{4}}} = C_\delta\sqrt{1+w_1^2}^{-\frac{5}{4}}\sqrt{1+w_2^2}^{-\frac{5}{4}} \in L^1(\mathbb{R}^2).$$

The final conclusion is then that

$$\|\tilde{p}(w_1,w_2)\|_1 = \int_{\mathbb{R}}\int_{\mathbb{R}}|\tilde{p}(w_1,w_2)|dw_1dw_2 =$$
$$\underbrace{\int\int_{M_\delta}|\tilde{p}(w_1,w_2)|dw_1dw_2 + \int\int_{\mathbb{R}^2\setminus M_\delta}|\tilde{p}(w_1,w_2)|dw_1dw_2}_{\in L^1(\mathbb{R}^2)} < \infty \Rightarrow$$
$$\tilde{p}(w_1,w_2) \in L^1(\mathbb{R}^2)$$

The idea on how to prove that $\tilde{p}(w_1,w_2) \in L^1(\mathbb{R}^2)$ was developed under communication with Anders Holst at the mathematical department LTH.

### 2.2.1 The moment generating function

Since its now clear that the inverse Fourier transformation exists, the next step is to calculate it. This is done in this sub subsection, and it turns out that the moment generating function appears.

Due to Fubinis theorem (17) can be rewritten as

$$\begin{gathered}\frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}\left[e^{(iw_1k+ak)+(iw_2y+by)}\tilde{p}_t(w_1,w_2)|\mathcal{F}_t^S\right]dw_1dw_2 = \\ \frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}\left[e^{-z_1k-z_2y}\tilde{p}_t(w_1,w_2)|\mathcal{F}_t^S\right]dw_1dw_2 = \\ \frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}\left[e^{-z_1k-z_2y}e^{x(z_1+1+z_2)}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}|\mathcal{F}_t^S\right]dw_1dw_2 = \\ \frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-z_1k}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}\mathbb{E}^{\mathbb{Q}}\left[e^{-z_2y+x(z_1+1+z_2)}|\mathcal{F}_t^S\right]dw_1dw_2,\end{gathered} \tag{18}$$

remember that $x$ and $y$ are the random variables. Taking use of the following definition:

**Definition 6.** *The moment generating function of a multivariate $n$-dimensional random variable $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is*

$$M_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}\left[e^{\mathbf{t}^T\mathbf{X}}\right] = \int_{\mathbb{R}^n} e^{t^T\mathbf{X}} f_{\mathbf{X}}(x)dx_1\dots dx_n.$$

And then defining the conditional moment generating function with respect to the conditional probability density function as

$$M_{\mathbf{X}|\mathbf{Y}}(\mathbf{t}) = \int_{\mathbb{R}^n} e^{t^T\mathbf{X}} f_{\mathbf{X}|\mathbf{Y}}(x)dx_1\dots dx_n.$$

We can rewrite (18) as

$$\frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-z_1k}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}M_{X_1,X_2}(z_1+1+z_2,-z_2)dw_1dw_2 = \Pi \tag{19}$$

Where $M_{X_1,X_2}$ is to be understood as the conditional moment generating function for $(X_1, X_2) = (\log(S_1(T)), \log(S_2(T)))$, and will here after be refereed as the moment generating function.

Above in (19) we have an expression for the inverse transform. If we knew the moment generating function, the only thing left would be to evaluate the integral. Which leads to the next part.

## 2.3 Derivation of the moment generating function

Earlier we saw that if we only knew the moment generating function we could valuate the inverse transformation. The idea here is to derive the moment generating function. To be able to do this we must determine on how to model our two different assets in the spread option. With our model at hand it is possible to derive a partial differential equation that the moment generating function must fulfill. By solving the differential equation, the function is then derived.

Up to this moment we have not said anything about the structure for the underlying assets, we have just derived a formula for the price for an arbitrary structure. Therefore we are going to assume the following model for the underlying assets.

$$\begin{aligned}
dS_1 &= rS_1dt + S_1\left(\sqrt{V_1}dW_{11} + \sigma_{1m}\sqrt{V_m}dW_{12}\right) \\
dS_2 &= rS_2dt + S_2\left(\sqrt{V_2}dW_{21} + \sigma_{2m}\sqrt{V_m}dW_{22}\right) \\
dV_1 &= \kappa_1(\theta_1 - V_1)dt + \sqrt{V_1}\sigma_1 dW_{V_1} \\
dV_2 &= \kappa_2(\theta_2 - V_2)dt + \sqrt{V_2}\sigma_2 dW_{V_2} \\
dV_m &= \kappa_m(\theta_m - V_m)dt + \sqrt{V_m}\sigma_m dW_{V_m}
\end{aligned} \tag{20}$$

Where $dW_{11}, dW_{21}$ are idiosyncratic risks and $dW_{12}, dW_{22}$ are common risk. The factors $\sigma_{1m}, \sigma_{2m}$ determines on how much the assets are exposed to the common risk.

Further more we assume the following non zero correlations.

$$\begin{aligned}
dW_{12} \cdot dW_{V_m} &= \rho_{1m}dt \\
dW_{22} \cdot dW_{V_m} &= \rho_{2m}dt \\
dW_{11} \cdot dW_{V_1} &= \rho_{11}dt \\
dW_{21} \cdot dW_{V_2} &= \rho_{22}dt \\
dW_{12} \cdot dW_{22} &= \rho_{12m}dt
\end{aligned} \tag{21}$$

We wish to derive the moment generating function for $(\log(S_1(T)), \log(S_2(T)))$, thus we set $X_1 = \log(S_1(T))$ and $X_2 = \log(S_2(T))$ and calculate the dynamics for them respectively, according to (20). By Itô, definition 2, setting $X_i = \log(S_i(T))$ implies that

$$dX_i = \left(0 + \frac{1}{2}\left(-\frac{1}{S_i^2}dS_i^2\right)\right) + \frac{1}{S_i}dS_i =$$
$$\left(r - \frac{V_i}{2} - V_m\frac{\sigma_{im}^2}{2}\right)dt + \left(\sqrt{V_i}dW_{i1} + \sigma_{im}\sqrt{V_m}dW_{i2}\right)$$

and thus we obtain the following system of equations for $X_1, X_2$

$$\begin{aligned} dX_1 &= \left(r - \frac{V_1}{2} - V_m\frac{\sigma_{1m}^2}{2}\right)dt + \left(\sqrt{V_1}dW_{11} + \sigma_{1m}\sqrt{V_m}dW_{12}\right) \\ dX_2 &= \left(r - \frac{V_2}{2} - V_m\frac{\sigma_{2m}^2}{2}\right)dt + \left(\sqrt{V_2}dW_{21} + \sigma_{2m}\sqrt{V_m}dW_{22}\right) \\ dV_1 &= \kappa_1(\theta_1 - V_1)dt + \sqrt{V_1}\sigma_1 dW_{V_1} \\ dV_2 &= \kappa_2(\theta_2 - V_2)dt + \sqrt{V_2}\sigma_2 dW_{V_2} \\ dV_m &= \kappa_m(\theta_m - V_m)dt + \sqrt{V_m}\sigma_m dW_{V_m}. \end{aligned} \tag{22}$$

Going back do the definition of the moment generating function and using general arguments we have that

$$m(t) = M_{X_1,X_2}(z_1, z_2) = \mathbb{E}[e^{z_1X_1+z_2X_2}|\mathcal{F}_t^{S_1,S_2,V_1,V_2,V_m}] \tag{23}$$

Which implies for $t < u < T$

$$m(t) = \mathbb{E}^{\mathbb{Q}}[e^{z_1X_1+z_2X_2}|\mathcal{F}_t^{S_1,S_2,V_1,V_2,V_m}] =$$
$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{z_1X_1+z_2X_2}|\mathcal{F}_u^{S_1,S_2,V_1,V_2,V_m}\right]|\mathcal{F}_t^{S_1,S_2,V_1,V_2,V_m}\right] = \mathbb{E}^{\mathbb{Q}}\left[m(u)|\mathcal{F}_t^{S_1,S_2,V_1,V_2,V_m}\right]$$

Thus, $m(t)$ is a martingale which further implies that when applying Itôs lemma to $m(t)$ the drift is identical equal to zero. By definition and by using that $(X_1, X_2, V_1, V_2, V_m)$ is a Markov chain, we obtain $m(t) = M_{X_1,X_2}(z_1, z_2) =$ $\mathbb{E}^{\mathbb{Q}}[f(t, X_1(t), X_2(t), V_1(t), V_2(t), V_m(t))$ $|(X_1(0), X_2(0), V_1(0), V_2(0), V_m(0)) = (x_1, x_2, v_1, v_2, v_m)]$. While restricting our self to the set where $M_{X_1,X_2}(z_1, z_2)$ is well-defined, we can apply Itôs lemma directly to $f$ and thus obtain the following.

$$\begin{gathered} \partial_t f dt + \partial_{x_1} f dX_1 + \partial_{x_2} f dX_2 + \partial_{v_1} f dV_1 + \partial_{v_2} dV_2 + \partial_{V_m} f dV_m + \\ \frac{1}{2}\left(\partial_{tt}^2 f dt^2 + \partial_{x_1x_1}^2 f dX_1^2 + \partial_{x_2x_2}^2 f dX_2^2 + \partial_{v_1v_1}^2 f dV_1^2 + \partial_{v_2v_2}^2 f dV_2^2 + \partial_{v_mv_m}^2 f dV_m^2\right) + \\ \partial_{tx_1}^2 f dt dX_1 + \partial_{tx_2}^2 f dt dX_2 + \partial_{tv_1}^2 f dt dV_1 + \partial_{tv_2}^2 f dt dV_2 + \partial_{tv_m}^2 dt dV_m + \\ \partial_{x_1x_2}^2 f dX_1 dX_2 + \partial_{x_1v_1}^2 f dX_1 dV_1 + \partial_{x_1v_2}^2 f dX_1 dV_2 + \partial_{x_1v_m}^2 f dX_1 dV_m + \\ \partial_{x_2v_1}^2 f dX_2 dV_1 + \partial_{x_2v_2}^2 f dX_2 dV_2 + \partial_{x_2v_m}^2 f dX_2 dV_m + \\ \partial_{v_1v_2}^2 f dV_1 dV_2 + \partial_{v_1v_m}^2 f dV_1 dV_m + \\ \partial_{v_2v_m}^2 dV_2 dV_1. \end{gathered} \tag{24}$$

Using the box algebra and the correlations from (21) we have that

$$\begin{aligned}
dt^2 &= 0\\
dX_1^2 &= V_1 dt + \sigma_{1m}^2 V_m dt\\
dX_2^2 &= V_2 dt + \sigma_{2m}^2 V_m dt\\
dV_1^2 &= V_1\sigma_1^2 dt\\
dV_2^2 &= V_2\sigma_2^2 dt\\
dV_m^2 &= V_m\sigma_m^2 dt\\
dtdX_1 &= dtdX_2 = dtdV_1 = dtdV_2 = dtdV_m = 0\\
dX_1 dX_2 &= \sigma_{1m}\sigma_{2m}\rho_{12m} V_m dt\\
dX_1 dV_1 &= \sigma_1 V_1 \rho_{11} dt\\
dX_1 dV_2 &= 0\\
dX_1 dV_m &= \sigma_{1m}\sigma_m V_m \rho_{1m} dt\\
dX_2 dV_1 &= 0\\
dX_2 dV_2 &= \sigma_2 V_2 \rho_{22} dt\\
dX_2 dV_m &= \sigma_{2m}\sigma_m V_m \rho_{2m} dt\\
dV_1 dV_2 &= dV_1 dV_m = dV_2 dV_m = 0.
\end{aligned}$$

Inserting all the differentials into (24) we obtain

$$\begin{gathered}
\partial_t f dt +\\
\partial_{x_1} f \left(\left(r - \frac{V_1}{2} - V_m \frac{\sigma_{1m}^2}{2}\right) dt + \left(\sqrt{V_1} dW_{11} + \sigma_{1m}\sqrt{V_m} dW_{12}\right)\right) +\\
\partial_{x_2} f \left(\left(r - \frac{V_2}{2} - V_m \frac{\sigma_{2m}^2}{2}\right) dt + \left(\sqrt{V_2} dW_{21} + \sigma_{2m}\sqrt{V_m} dW_{22}\right)\right) +\\
\partial_{v_1} f \left(\kappa_1(\theta_1 - V_1) dt + \sqrt{V_1}\sigma_1 dW_{V_1}\right) +\\
\partial_{v_2} \left(\kappa_2(\theta_2 - V_2) dt + \sqrt{V_2}\sigma_2 dW_{V_2}\right) +\\
\partial_{V_m} f \left(\kappa_m(\theta_m - V_m) dt + \sqrt{V_m}\sigma_m dW_{V_m}\right) +\\
\frac{1}{2}\Bigg(\partial_{x_1x_1}^2 f (V_1 + \sigma_{1m}^2 V_m) + \partial_{x_2x_2}^2 f (V_2 + \sigma_{2m}^2 V_m) +\\
\partial_{v_1v_1}^2 f V_1 \sigma_1^2 + \partial_{v_2v_2}^2 f V_2 \sigma_2^2 + \partial_{v_mv_m}^2 f V_m \sigma_m^2\Bigg) dt +\\
\partial_{x_1x_2}^2 f \sigma_{1m}\sigma_{2m}\rho_{12m} V_m dt +\\
\partial_{x_1v_1}^2 f \sigma_1 V_1 \rho_{11} dt +\\
\partial_{x_1v_m}^2 f \sigma_{1m}\sigma_m V_m \rho_{1m} dt +\\
\partial_{x_2v_2}^2 f \sigma_2 V_2 \rho_{22} dt +\\
\partial_{x_2v_m}^2 f \sigma_{2m}\sigma_m V_m \rho_{2m} dt.
\end{gathered}$$

As mentioned earlier, $m(t)$ is a martingale, and thus the drift must be zero. Thus, by collecting all contributions "in" $dt$ those term must equal zero, i.e

$$
\begin{gathered}
\partial_t f dt + \\
\partial_{x_1} f \left( r - \frac{V_1}{2} - V_m \frac{\sigma_{1m}^2}{2} \right) dt + \\
\partial_{x_2} f \left( r - \frac{V_2}{2} - V_m \frac{\sigma_{2m}^2}{2} \right) dt + \\
\partial_{v_1} f \kappa_1 (\theta_1 - V_1) dt + \\
\partial_{v_2} f \kappa_2 (\theta_2 - V_2) dt + \\
\partial_{V_m} f \kappa_m (\theta_m - V_m) dt + \\
\frac{1}{2} \Big( \partial^2_{x_1 x_1} f (V_1 + \sigma_{1m}^2 V_m) + \partial^2_{x_2 x_2} f (V_2 + \sigma_{2m}^2 V_m) + \\
\partial^2_{v_1 v_1} f V_1 \sigma_1^2 + \partial^2_{v_2 v_2} f V_2 \sigma_2^2 + \partial^2_{v_m v_m} f V_m \sigma_m^2 \Big) dt + \\
\partial^2_{x_1 x_2} f \sigma_{1m} \sigma_{2m} \rho_{12m} V_m dt + \\
\partial^2_{x_1 v_1} f \sigma_1 V_1 \rho_{11} dt + \\
\partial^2_{x_1 v_m} f \sigma_{1m} \sigma_m V_m \rho_{1m} dt + \\
\partial^2_{x_2 v_2} f \sigma_2 V_2 \rho_{22} dt + \\
\partial^2_{x_2 v_m} f \sigma_{2m} \sigma_m V_m \rho_{2m} dt = 0.
\end{gathered}
$$

Which can be rewritten as the following PDE

$$
\begin{gathered}
\partial_t f + \\
\left( r - \frac{V_1}{2} - V_m \frac{\sigma_{1m}^2}{2} \right) \partial_{x_1} f + \sigma_1 V_1 \rho_{11} \partial^2_{x_1 v_1} f + \sigma_{1m} \sigma_m V_m \rho_{1m} \partial^2_{x_1 v_m} f + \frac{V_1 + \sigma_{1m}^2 V_m}{2} \partial^2_{x_1 x_1} f + \\
\left( r - \frac{V_2}{2} - V_m \frac{\sigma_{2m}^2}{2} \right) \partial_{x_2} f + \sigma_2 V_2 \rho_{22} \partial^2_{x_2 v_2} f + \sigma_{2m} \sigma_m V_m \rho_{2m} \partial^2_{x_2 v_m} f + \frac{V_2 + \sigma_{2m}^2 V_m}{2} \partial^2_{x_2 x_2} f + \\
\sigma_{1m} \sigma_{2m} \rho_{12m} V_m \partial^2_{x_1 x_2} f + \\
\kappa_1 (\theta_1 - V_1) \partial_{v_1} f + \frac{\sigma_1^2}{2} V_1 \partial^2_{v_1 v_1} f + \\
\kappa_2 (\theta_2 - V_2) \partial_{v_2} f + \frac{\sigma_2^2}{2} V_2 \partial^2_{v_2 v_2} f + \\
\kappa_m (\theta_m - V_m) \partial_{v_m} f + \frac{\sigma_m^2}{2} V_m \partial^2_{v_m v_m} f = 0.
\end{gathered} \tag{25}
$$

Going back to the definition of $m(t)$ (23), we see that

$$
\begin{gathered}
m(T) = \mathbb{E}[e^{z_1 X_1 + z_2 X_2} | \mathcal{F}_T^{S_1, S_2, V_1, V_2, V_m}] \text{ we know all information} \Rightarrow \\
m(T) = e^{z_1 X_1 + z_2 X_2} = f(T, X_1, X_2, V_1, V_2, V_m)
\end{gathered}
$$

That is, $f(T, X_1, X_2, V_1, V_2, V_m) = e^{z_1 X_1 + z_2 X_2}$ is a boundary condition to (25).

Now by making the ansatz that $f(\tau, X_1, X_2, V_1, V_2, V_m) = e^{A(\tau)+z_1X_1+z_2X_2+B_1(\tau)V_1+B_2(\tau)V_2+B_m(\tau)V_m}$, $\tau = T - t$ and by setting $\tau = 0$ the above boundary condition implies

$$f(0, X_1, X_2, V_1, V_2, V_m) = e^{A(0)+z_1X_1+z_2X_2+B_1(0)V_1+B_2(0)V_2+B_m(0)V_m} = e^{z_1X_1+z_2X_2}$$

$$\Rightarrow$$

$$A(0) = 0$$
$$B_1(0) = 0$$
$$B_2(0) = 0$$
$$B_m(0) = 0.$$

The ansatz for $f$ might appear as magic, but it can be justified. There are no correlations between $V_i, V_j$ $i \neq j$ i.e $\rho(V_i, V_j) = 0$ $i \neq j$. Thus, there will be no cross-terms in $V_i, V_j$ $i \neq j$ i.e they can be linearly separated. There is correlation between $V_i$ and $X_j$, $\rho(V_i, X_j) = \rho_{ji}dt$ and $dX_j \cdot dV_i = V_i\sigma_i\rho_{ji}dt$ which is captured as term in the function $B_i$. Also $dX_1 \cdot dX_2 = \sigma_{1m}\sigma_{2m}V_m\rho_{12m}dt$ which is captured in the function $B_m$. Thus the moment generating function can be linearly separated in the exponent in terms of $X_1, X_2, V_1, V_2$ and $V_m$.

From here and throughout this subsection, $X_i$ will be thought of as $X_i(0)$ and $V_i$ as $V_i(0)$ in order to improve readability in the coming somewhat messy calculations. With our ansatz we can determine $A, B_1, B_2$ and $B_m$ by plugging $f$ into (25). Now remember that $\tau = T - t$ which means that $d\tau = -dt$ and we obtain

$$\begin{gathered}
0 = -(A' + B_1'V_1 + B_2'V_2 + B_m'V_m)f + \\
\left(r - \frac{V_1}{2} - V_m\frac{\sigma_{1m}^2}{2}\right)z_1f + \sigma_1V_1\rho_{11}z_1B_1f + \sigma_{1m}\sigma_mV_m\rho_{1m}z_1B_mf + \frac{V_1 + \sigma_{1m}^2V_m}{2}z_1^2f + \\
\left(r - \frac{V_2}{2} - V_m\frac{\sigma_{2m}^2}{2}\right)z_2f + \sigma_2V_2\rho_{22}z_2B_2f + \sigma_{2m}\sigma_mV_m\rho_{2m}z_2B_mf + \frac{V_2 + \sigma_{2m}^2V_m}{2}z_2^2f + \\
\sigma_{1m}\sigma_{2m}\rho_{12m}V_mz_1z_2f + \\
\kappa_1(\theta_1 - V_1)B_1f + \frac{\sigma_1^2}{2}V_1B_1^2f + \\
\kappa_2(\theta_2 - V_2)B_2f + \frac{\sigma_2^2}{2}V_2B_2^2f + \\
\kappa_m(\theta_m - V_m)B_mf + \frac{\sigma_m^2}{2}V_mB_m^2f
\end{gathered}$$

We regroup in the following way

$$0 = f\Bigg( \underbrace{-A' + r(z_1+z_2) + \kappa_1\theta_1 B_1 + \kappa_2\theta_2 B_2 + \kappa_m\theta_m B_m}_{e_1}$$
$$\underbrace{V_1\left(-B_1' + \sigma_1\rho_{11}z_1B_1 + \frac{\sigma_1^2}{2}B_1^2 - \kappa_1 B_1 + \frac{z_1^2}{2} - \frac{z_1}{2}\right)}_{e_2} +$$
$$\underbrace{V_2\left(-B_2' + \sigma_2\rho_{22}z_2B_2 + \frac{\sigma_2^2}{2}B_2^2 - \kappa_2 B_2 + \frac{z_2^2}{2} - \frac{z_2}{2}\right)}_{e_3} +$$
$$V_m\Bigg( - B_m' + z_1\left(\sigma_{1m}\sigma_m\rho_{1m}B_m - \frac{\sigma_{1m}^2}{2}\right) + z_2\left(\sigma_{2m}\sigma_m\rho_{2m}z_2B_m - \frac{\sigma_{2m}^2}{2}\right) +$$
$$\frac{\sigma_{1m}^2}{2}z_1^2 + \frac{\sigma_{2m}^2}{2}z_2^2 + \sigma_{1m}\sigma_{2m}\rho_{12m}z_1z_2 - \kappa_m B_m + \frac{\sigma_m^2}{2}B_m^2\Bigg)\Bigg) =$$
$$e_1 + e_2 + e_3 + e_4.$$

We now claim that to obtain $A, B_1, B_2$ and $B_m$ we can solve $e_i = 0$ for $i = 1, 2, 3, 4$ as a system of equations. As in earlier when we justified the ansatz for the moment generating function. We argued that $V_1, V_2$ and $V_m$ could be linearly separated due to the correlation structure. This then leads to that $e_2, e_3$ and $e_4$ can be thought of as, having nothing to do with each other. Thus, there can never be combination of the three yielding zero. Since all of those three must individually be zero, logic tells us that $e_1$ must be individually zero.

None of $f, V_1, V_2$ and $V_m$ are identically zero thus, we obtain

$$-A' + r(z_1+z_2) + \kappa_1\theta_1 B_1 + \kappa_2\theta_2 B_2 + \kappa_m\theta_m B_m = 0 \tag{26}$$
$$-B_1' + \sigma_1\rho_{11}z_1B_1 + \frac{\sigma_1^2}{2}B_1^2 - \kappa_1 B_1 + \frac{z_1^2}{2} - \frac{z_1}{2} = 0 \tag{27}$$
$$-B_2' + \sigma_2\rho_{22}z_2B_2 + \frac{\sigma_2^2}{2}B_2^2 - \kappa_2 B_2 + \frac{z_2^2}{2} - \frac{z_2}{2} = 0 \tag{28}$$
$$\begin{aligned}&-B_m' + z_1\left(\sigma_{1m}\sigma_m\rho_{1m}B_m - \frac{\sigma_{1m}^2}{2}\right) + z_2\left(\sigma_{2m}\sigma_m\rho_{2m}B_m - \frac{\sigma_{2m}^2}{2}\right) +\\ &\frac{\sigma_{1m}^2}{2}z_1^2 + \frac{\sigma_{2m}^2}{2}z_2^2 + \sigma_{1m}\sigma_{2m}\rho_{12m}z_1z_2 - \kappa_m B_m + \frac{\sigma_m^2}{2}B_m^2 = 0\end{aligned} \tag{29}$$

We notice that $(27), (28)$ and $(29)$ are separable differential equations, and we start by solving (27).

$$0 = -B_1' + \sigma_1\rho_{11}z_1B_1 + \frac{\sigma_1^2}{2}B_1^2 - \kappa_1 B_1 + \frac{z_1^2}{2} - \frac{z_1}{2} =$$
$$-B_1' + B_1\underbrace{(\sigma_1\rho_{11}z_1 - \kappa_1)}_{y_1} + \underbrace{\frac{\sigma_1^2}{2}}_{y_2}B_1^2 + \underbrace{\frac{z_1^2}{2} - \frac{z_1}{2}}_{y_3} \Leftrightarrow$$
$$\frac{dB_1}{d\tau} = B_1y_1 + y_2B_1^2 + y_3 \Leftrightarrow$$
$$\frac{dB_1}{B_1y_1 + y_2B_1^2 + y_3} = d\tau \Leftrightarrow$$
$$\frac{2\arctan\left(\frac{y_1+2y_2B_1}{\sqrt{4y_2y_3-y_1^2}}\right)}{\sqrt{4y_2y_3 - y_1^2}} = \tau + C_1 \Leftrightarrow$$
$$B_1 = -\frac{y_1}{2y_2} + \frac{\sqrt{4y_2y_3 - y_1^2}}{2y_2}\tan\left(\frac{\sqrt{4y_2y_3 - y_1^2}(\tau + C_1)}{2}\right) =$$

This looks awful, but be patient, set $id_1 = \sqrt{4y_2y_3 - y_1^2}$ then

$$B_1(\tau) = -\frac{y_1}{2y_2} + \frac{id_1}{2y_2}\tan\left(\frac{id_1\tau}{2} + id_1\frac{\arctan\left(\frac{y_1}{id_1}\right)}{id_1}\right) =$$
$$-\frac{y_1}{2y_2} - \frac{d_1}{2y_2}\left(\frac{\sinh(\frac{d_1\tau}{2}) - \frac{y_1}{d_1}\cosh(\frac{d_1\tau}{2})}{\cosh(\frac{d_1\tau}{2}) - \frac{y_1}{d_1}\sinh(\frac{d_1\tau}{2})}\right) =$$
$$\frac{\left(1 - e^{-\tau d_1}\right)(z_1^2 - z_1)}{(\kappa_1 - \sigma_1\rho_{11}z_1)(1 - e^{-\tau d_1}) + d_1(1 + e^{-\tau d_1})},$$
$$d_1 = \sqrt{(\kappa_1 - \sigma_1\rho_{11}z_1)^2 + \sigma_1^2(z_1 - z_1^2)}$$

Due to the similarities between (27) and (28) the solution to (28) will be the same as for (27), just changing $_1$ to $_2$.

$$B_2(\tau) = \frac{\left(1 - e^{-\tau d_2}\right)(z_2^2 - z_2)}{(\kappa_2 - \sigma_2\rho_{22}z_2)(1 - e^{-\tau d_2}) + d_2(1 + e^{-\tau d_2})},$$
$$d_2 = \sqrt{(\kappa_2 - \sigma_2\rho_{22}z_2)^2 + \sigma_2^2(z_2 - z_2^2)}$$

For (29) we do the following

$$
\begin{gathered}
-B_m' + z_1\left(\sigma_{1m}\sigma_m\rho_{1m}B_m - \frac{\sigma_{1m}^2}{2}\right) + z_2\left(\sigma_{2m}\sigma_m\rho_{2m}B_m - \frac{\sigma_{2m}^2}{2}\right) + \\
\frac{\sigma_{1m}^2}{2}z_1^2 + \frac{\sigma_{2m}^2}{2}z_2^2 + \sigma_{1m}\sigma_{2m}\rho_{12m}z_1z_2 - \kappa_m B_m + \frac{\sigma_m^2}{2}B_m^2 = \\
-B_m' + B_m \underbrace{\left(z_1\sigma_{1m}\sigma_m\rho_{1m} + z_2\sigma_{2m}\sigma_m\rho_{2m} - \kappa_m\right)}_{y_1} + \\
\underbrace{\frac{\sigma_m^2}{2}}_{y_2}B_m^2 + \underbrace{\frac{\sigma_{1m}^2}{2}(z_1^2 - z_1) + \frac{\sigma_{2m}^2}{2}(z_2^2 - z_2) + \sigma_{1m}\sigma_{2m}\rho_{12m}z_1z_2}_{y_3} \\
B_m = -\frac{y_1}{2y_2} + \frac{\sqrt{4y_2y_3 - y_1^2}}{2y_2}\tan\left(\frac{\sqrt{4y_2y_3 - y_1^2}(\tau + C_m)}{2}\right)
\end{gathered}
$$

Setting $id_3 = \sqrt{4y_2y_3 - y_1^2}$ we apply the same strategy as before, and hence the solution becomes

$$
B_m(\tau) = \frac{\left(1 - e^{-\tau d_m}\right)\left(\sigma_{1m}^2(z_1 - z_1^2) + \sigma_{2m}^2(z_2 - z_2^2) - 2\sigma_{1m}\sigma_{2m}\rho_{12m}z_1z_2\right)}{\left(\kappa_m - z_1\sigma_{1m}\sigma_m\rho_{1m} - z_2\sigma_{2m}\sigma_m\rho_{2m}\right)\left(1 - e^{-\tau d_m}\right) + d_m\left(1 + e^{-\tau d_m}\right)},
$$

$$
d_3 = \sqrt{(\kappa_m - z_1\sigma_m\sigma_{1m}\rho_{1m} - z_2\sigma_m\sigma_{2m}\rho_{2m}))^2 + \sigma_m^2\left(\sigma_{1m}^2(z_1 - z_1^2) + \sigma_{2m}^2(z_2 - z_2^2) - 2\sigma_{1m}\sigma_{2m}\rho_{12m}z_1z_2\right)}
$$

Then we can solve for $A$ in

$$
\begin{gathered}
-A' + r(z_1 + z_2) + \kappa_1\theta_1B_1 + \kappa_2\theta_2B_2 + \kappa_m\theta_mB_m = 0 \Rightarrow \\
A(\tau) = \tau r(z_1 + z_2) + \int_0^\tau \kappa_1\theta_1B_1(\gamma) + \kappa_2\theta_2B_2(\gamma) + \kappa_m\theta_mB_m(\gamma)d\gamma
\end{gathered}
$$

Thus we need to integrate $B_1, B_2$ and $B_m$. For simplicity we rewrite $B_i$ $i = 1, 2, m$ in a way such that anything which not contains $\tau$ will be collected in a constant. Thus we can write $B_i$ $i = 1, 2, m$ as

$$
B_i(\tau) = \frac{\left(1 - e^{-\tau d_i}\right)2y_{3,i}}{-y_{1,i}\left(1 - e^{-\tau d_i}\right) + d_i\left(1 + e^{-\tau d_i}\right)}
$$

Where

$$
\begin{gathered}
\int_0^\tau B_i(\gamma)d\gamma = \int_0^\tau \frac{\left(1 - e^{-\gamma d_i}\right)2y_{3,i}}{-y_{1,i}\left(1 - e^{-\gamma d_i}\right) + d_i\left(1 + e^{-\gamma d_i}\right)}d\gamma = \\
\left[-\frac{2y_{3,i}(d_i\gamma(y_{1,i} - d_i) + 2d_i\log(y_{1,i}(e^{\gamma d_i} - 1) - d_i(e^{\gamma d_i} + 1)))}{d_i(y_{1,i} - d_i)(y_{1,i} + d_i)}\right]_0^\tau = \\
\frac{1}{\sigma_i^2}\left((-y_{1,i} + d_i)\tau - 2\log\left(\frac{-y_{1,i}(e^{\tau d_i} - 1) + d_i(e^{\tau d_i} + 1)}{2d_i}\right)\right)
\end{gathered}
$$

Which then gives us that

$$A(\tau) = \tau r(z1 + z2)+$$
$$\frac{\kappa_1\theta_1}{\sigma_2^2}\left((-\sigma_1\rho_{11}z_1 + \kappa_1 + d_1)\tau - 2\log\left(\frac{(-\sigma_1\rho_{11}z_1 + \kappa_1)(e^{\tau d_1} - 1) + d_2(e^{\tau d_1} + 1)}{2d_1}\right)\right)+$$
$$\frac{\kappa_2\theta_2}{\sigma_2^2}\left((-\sigma_2\rho_{22}z_2 + \kappa_2 + d_2)\tau - 2\log\left(\frac{(-\sigma_2\rho_{22}z_2 + \kappa_2)(e^{\tau d_2} - 1) + d_2(e^{\tau d_2} + 1)}{2d_2}\right)\right)+$$
$$\frac{\kappa_m\theta_m}{\sigma_m^2}\left((-z_1\sigma_{1m}\sigma_m\rho_{1m} - z_2\sigma_{2m}\sigma_m\rho_{2m} + \kappa_m + d_m)\tau -\right.$$
$$\left.2\log\left(\frac{(-z_1\sigma_{1m}\sigma_m\rho_{1m} - z_2\sigma_{2m}\sigma_m\rho_{2m} + \kappa_m)(e^{\tau d_m} - 1) + d_m(e^{\tau d_m} + 1)}{2d_m}\right)\right)$$

## 2.4 Simulation of the Heston model

In the case of the European Call, we presented another method for valuating the value of the financial contract. Here we wish to do the same, to somewhat have a justification for our calculations. In the case here, there is no answer sheet to look at the end. Thus, we will provide a Monte Carlo pricing strategy and compare the answer against our Fourier Method. We argue that if the value from the Monte Carlo simulation corresponds to the value with the Fourier method, then we can be convinced that our calculations are correct. Since the probability that two independent methods would yield the same incorrect value seems very low.

We wish to simulate the value of the assets $S_1$ and $S_2$. The details of the simulation will not be presented since, it's not the main focus of this paper. But the simulation that will be used is the one presented in [6].

## 2.5 Simulation scheme

The scheme for the Monte Carlo simulation is presented below.

**Initialization**

$$
\begin{aligned}
\hat{V}_{1,0} &= V_1(0),\ \hat{V}_{2,0} = V_2(0),\ \hat{V}_{m,0} = V_m(0),\\
\hat{S}_{1,0} &= S_1(0),\ \hat{S}_{2,0} = S_2(0),\ h = \frac{T}{\text{nsteps}}\\
d_1 &= \frac{4\theta_1\kappa_1}{\sigma_1^2},\ \lambda_1 = \frac{4\kappa_1 e^{-h\kappa_1}}{\sigma_1^2(1-e^{-h\kappa_1})},\ C_1 = \frac{\sigma_1^2(1-e^{-h\kappa_1})}{4\kappa_1},\\
d_2 &= \frac{4\theta_2\kappa_2}{\sigma_2^2},\ \lambda_2 = \frac{4\kappa_2 e^{-h\kappa_2}}{\sigma_2^2(1-e^{-h\kappa_2})},\ C_2 = \frac{\sigma_2^2(1-e^{-h\kappa_2})}{4\kappa_2},\\
d_m &= \frac{4\theta_m\kappa_m}{\sigma_m^2},\ \lambda_m = \frac{4\kappa_2 e^{-h\kappa_m}}{\sigma_m^2(1-e^{-h\kappa_m})},\ C_m = \frac{\sigma_m^2(1-e^{-h\kappa_m})}{4\kappa_m},\\
\rho &= \frac{\rho_{12m}-\rho_{1m}\rho_{2m}}{\sqrt{1-\rho_{1m}^2}\sqrt{1-\rho_{2m}^2}}.
\end{aligned}
$$

**for k=1 to nsteps do**

$$
\begin{aligned}
\hat{V}_{1,k} &= C_1 \times \texttt{ncx2rnd}(d_1, \hat{V}_{1,k-1}\lambda_1) (\text{MATLAB's non-central-}\chi^2 \text{ random number})\\
\hat{V}_{2,k} &= C_2 \times \texttt{ncx2rnd}(d_2, \hat{V}_{2,k-1}\lambda_2)\\
\hat{V}_{m,k} &= C_m \times \texttt{ncx2rnd}(d_m, \hat{V}_{m,k-1}\lambda_m)\\
\hat{S}_{1_k} &= \hat{S}_{1,k-1} \exp\Bigg(h\left(\left(r - \frac{\rho_{11}\kappa_1\theta_1}{\sigma_1}\right) + \frac{\hat{V}_{1,k}+\hat{V}_{1,k-1}}{2}\left(\frac{\kappa_1\rho_{11}}{\sigma_1} - \frac{1}{2}\right)\right)\\
&\quad + h\left(\left(-\frac{\sigma_{1m}\rho_{1m}\kappa_m\theta_m}{\sigma_m}\right) + \frac{\hat{V}_{m,k}+\hat{V}_{m,k-1}}{2}\left(\frac{\sigma_{1m}\kappa_m\rho_{1m}}{\sigma_m} - \frac{\sigma_{1m}^2}{2}\right)\right)\\
&\quad + \frac{\rho_{11}}{\sigma_1}(\hat{V}_{1,k} - \hat{V}_{1,k-1}) + \sqrt{h\frac{\hat{V}_{1,k}+\hat{V}_{1,k-1}}{2}(1-\rho_{11}^2)}G_{1k}\\
&\quad + \frac{\sigma_{1m}\rho_{1m}}{\sigma_m}(\hat{V}_{m,k} - \hat{V}_{m,k-1}) + \sigma_{1m}\sqrt{h\frac{\hat{V}_{m,k}+\hat{V}_{m,k-1}}{2}(1-\rho_{1m}^2)}G_{12k}\Bigg)\\
\hat{S}_{2,k} &= \hat{S}_{2,k-1} \exp\Bigg(h\left(\left(r - \frac{\rho_{22}\kappa_2\theta_2}{\sigma_2}\right) + \frac{\hat{V}_{2,k}+\hat{V}_{2,k-1}}{2}\left(\frac{\kappa_2\rho_{22}}{\sigma_2} - \frac{1}{2}\right)\right)\\
&\quad + h\left(\left(-\frac{\sigma_{2m}\rho_{2m}\kappa_m\theta_m}{\sigma_m}\right) + \frac{\hat{V}_{m,k}+\hat{V}_{m,k-1}}{2}\left(\frac{\sigma_{2m}\kappa_m\rho_{2m}}{\sigma_m} - \frac{\sigma_{2m}^2}{2}\right)\right)\\
&\quad + \frac{\rho_{22}}{\sigma_2}(\hat{V}_{2,k} - \hat{V}_{2,k-1}) + \sqrt{h\frac{\hat{V}_{2,k}+\hat{V}_{2,k-1}}{2}(1-\rho_{22}^2)}G_{2k}\\
&\quad + \frac{\sigma_{2m}\rho_{2m}}{\sigma_m}(\hat{V}_{m,k} - \hat{V}_{m,k-1}) + \sigma_{2m}\sqrt{h\frac{\hat{V}_{m,k}+\hat{V}_{m,k-1}}{2}(1-\rho_{2m}^2)}G_{22k}\Bigg)\\
\begin{bmatrix} G_{1k}\\ G_{2k}\\ G_{12k}\\ G_{22k}\end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & \rho & \sqrt{1-\rho^2}\end{bmatrix}\begin{bmatrix} X_{1k}\\ X_{2k}\\ X_{3k}\\ X_{4k}\end{bmatrix}
\end{aligned}
$$

**end**

where $\{X_{1k}, X_{2k}, X_{3k}, X_{4k}\}_{k=1}^{n}$ are iid standard Gaussian r.v.

## 2.6 The value of a spread option according to simulation

The implementation of the scheme was done in a way, that for every time step, a vectorized simulation of $M$ number of assets were done. That is, at every step, $M$ numbers of $S_1$ assets and $M$ number of $S_2$ assets were simulated. This then meant that at the last step, we had $M$ simulations of $S_1(T)$ and $S_2(T)$. With these two, we were able to calculate the payoff according to $(S_1(T) - S_2(T) - K)^+$ and then take the mean, discounting the mean and the value was obtained. At this time the standard deviation was as well calculated. Thus being able to construct at confidence interval for the discounted payoff.

After running a few simulations, it was noticed that the variance was higher than wanted. By adding a control variate, the variance was reduced. Thus, by recalculating the discounted payoff with the same simulations, but with a control variate. We obtained an approximation for the value of the spread option with lower variance and a narrower confidence interval.

The following values was used in the simulation:

### 2.6.1 Correlations and exposure

Table 3: The correlations used in the simulation of the Heston model

| | | |
|---|---|---|
| $\rho_{1m}$ | = | -0.5 |
| $\rho_{2m}$ | = | -0.6 |
| $\rho_{11}$ | = | -0.6 |
| $\rho_{22}$ | = | -0.7 |
| $\rho_{12m}$ | = | 0.7 |

Table 4: The different exposure constants used in the Heston model

| | | |
|---|---|---|
| $\sigma_{1m}$ | = | 0.3 |
| $\sigma_{2m}$ | = | 0.35 |
| $\sigma_1$ | = | 0.2 |
| $\sigma_2$ | = | 0.3 |
| $\sigma_m$ | = | 0.25 |

### 2.6.2 Kappas and Thetas

Table 5: The constants kappa and theta

| | |
|---|---|
| $\kappa_1 =$ | 2 |
| $\kappa_2 =$ | 3 |
| $\kappa_m =$ | 2.5 |
| $\theta_1 =$ | 0.02 |
| $\theta_2 =$ | 0.01 |
| $\theta_m =$ | 0.015 |

### 2.6.3 Initial values, Strike and interest rate

Table 6: Initial values of the assets and volatilities

$$
\begin{aligned}
\hat{S}_{1,0} &= 80\\
\hat{S}_{2,0} &= 60\\
\hat{V}_{1,0} &= 0.01\\
\hat{V}_{2,0} &= 0.01\\
\hat{V}_{m,0} &= 0.01
\end{aligned}
$$

Table 7: Strike, rate, time to maturity, starting time, steps, number of assets per time step

$$
\begin{aligned}
K &= 20\\
r &= 0.05\\
T &= 0.5\\
t &= 0\\
\text{nstep} &= 1000\\
M &= 10^5
\end{aligned}
$$

This lead to a value of 3.302221 for the spread option and a confidence interval $[3.2728, 3.3317]$. By using a control variate, $\beta = \frac{\mathbb{E}\left[e^{-r\tau}(S_1(T)-S_2(T)-K)^{+}e^{-r\tau}(S_1(T)-S_2(T)-K)-(S_1(0)-S_2(0)-e^{-r\tau}K)\right]}{\mathbb{E}\left[(e^{-r\tau}(S_1(T)-S_2(T)-K)-S_1(0)-S_2(0)-e^{-r\tau}K)^2\right]}$, a value of 3.305545 , a standard deviation of 0.0076 and a confidence interval of $[3.2903, 3.3208]$ was obtained. Now the idea is that, being confident enough that the simulation is done correct, if the value of the inverse Fourier transform lies in the confidence interval created using a control variate. Then we can be confident that the calculations of the inverse Fourier transform are correct.

The simulation was tested against different kind of strikes $K = 10, 15, 20, 25$, and the result is presented in the image below.

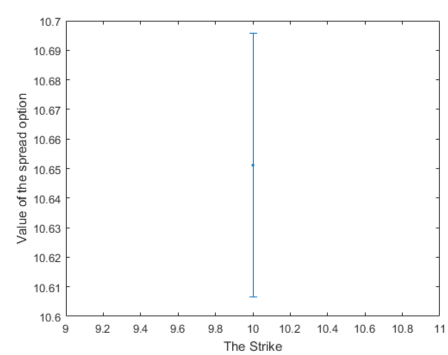


Figure 2: A confidence interval of the value of the spread option for the strike $K = 10$. The blue dot represents the mean.

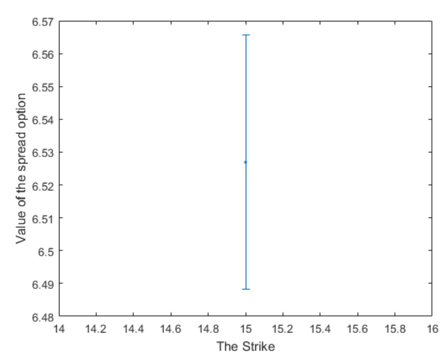


Figure 3: A confidence interval of the value of the spread option for the strike $K = 15$. The blue dot represents the mean.

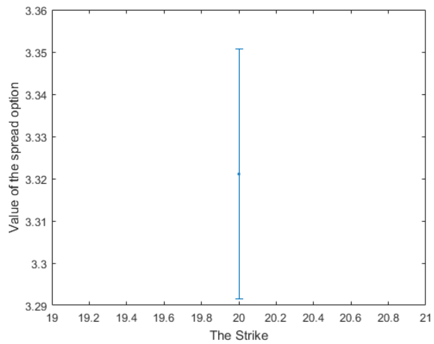


Figure 4: A confidence interval of the value of the spread option for the strike $K = 20$. The blue dot represents the mean.

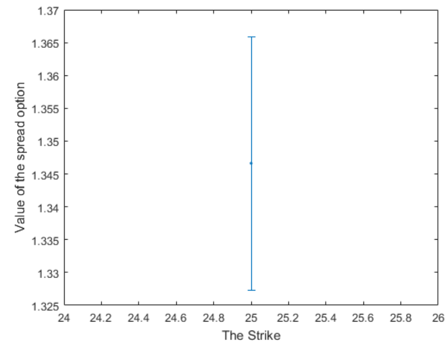


Figure 5: A confidence interval of the value of the spread option for the strike $K = 25$. The blue dot represents the mean.

## 2.7 How to choose $a$ and $b$?

Up until now we have concluded that in order for the Fourier transform to exist we must choose $a$ and $b$ such that $a < 0$ and $b < 0$. In theory, the exact value of $(a, b)$ doesn't matter, as long as they fulfill their constraints. However in practice the values of $(a, b)$ are critical. Just as in section 1.6 there exist optimal values of $(a, b)$, call the optimal values $(a_{\min}, b_{\min})$, which gives the highest precision of the numerical integration. Going back to (19) we wish to calculate

$$e^{-r(T-t_0)} \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{e^{-z_1 k} \frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1 + z_2 + 2)} M_{X_1,X_2}(z_1 + 1 + z_2, -z_2)}_{h(a,b)} dw_1 dw_2 = \Pi$$

Just as in section 1.6, the numerical integration of $h(a, b)$ will be precise when the oscillation are low, and they will only be significant when $|w_1|, |w_1|$ are small. Furthermore the oscillations of $h(a, b)$ are proportional to $|h(a, b)|$. One also has that $|h(a, b)| \leq h(a, b)|_{w_1=0,w_2=0}$. From here it makes sense to restrict one self to search for $(a_{\min}, b_{\min})$ by solving

$$\min_{(s,t)\in\{\mathbb{R}_{<0}\times\mathbb{R}_{<0}\}} h(a, b)|_{w_1=0,w_2=0}. \tag{30}$$

### 2.7.1 Can we find a minima for $h(a, b)|_{w_1=0,w_2=0}$?

We wish to find a minima for $h(a, b)|_{w_1=0,w_2=0}$. First we note that $h(a, b)|_{w_1=0,w_2=0}$ is defined on the set $\{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+\}$ which is a convex set. Thus, if we could prove that $h(a, b)|_{w_1=0,w_2=0}$ is convex, it would be sufficient to find a local min for $h(a, b)|_{w_1=0,w_2=0}$ which would imply a global min. Lets study the log of $h(a, b)|_{w_1=0,w_2=0}$.
Remember that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-z_2 \log(S_2(T))+(z_1+1+z_2)\log(S_1(T))}\right] = M_{X_1,X_2}(z_1 + 1 + z_2, -z_2) \tag{31}$$

We define

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{A(\tau)+B_1(\tau)V_1+B_2(\tau)V_2+B_m(\tau)V_m} \text{ where}$$

$$\mathbb{E}^*[X] = \int_\Omega X(w)d\mathbb{P}^*(w)$$

Then (31) can be rewritten as $\mathbb{E}^*[e^{-aX_1+bX_2}]$. Thus we can rewrite $h(a,b)|_{w_1=0,w_2=0}$ as

$$e^{ka}\frac{\Gamma(-a)\Gamma(-b)}{\Gamma(-a-b+2)}\,\mathbb{E}^*[e^{-aX_1+bX_2}]$$

Taking the log yields

$$\log\left(h(a,b)|_{w_1=0,w_2=0}\right) = \log(\Gamma(-a)) + \log(\Gamma(-b)) - \log(\Gamma(-a-b+2)) + ka + \log\left(\mathbb{E}^*\left[e^{-aX_1+bX_2}\right]\right) \tag{32}$$

**Theorem 2.2.** *Let $M_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}\left[e^{\mathbf{t}^T\mathbf{X}}\right]$ be the moment generating function for $X$. Then $\log(M(t))$ is convex.*

*Proof.* Hölders inequality says that

**Theorem 2.3.** *Let $(S, \Sigma, \mu)$ be a measure space and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, for all measureable real- or complex-valued functions $f$ and $g$ on $S$,*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Put $f = e^{(1-\theta)t_0^T X}$, $g = e^{\theta t_1^T X}$, $p = \frac{1}{1-\theta}$ and $q = \frac{1}{\theta}$ for any $0 < \theta < 1$. Taking log on both sides we obtain

$$\log\left(\mathbb{E}\left[e^{((1-\theta)t_0^T+\theta t_1^T)X}\right]\right) \leq (1-\theta)\log\left(\mathbb{E}\left[e^{t_0^T X}\right]\right) + \theta\log\left(\mathbb{E}\left[e^{t_1^t X}\right]\right)$$

□

By the Bohr-Mollerup theorem the $\log(\Gamma(z))$ function is convex on the positive reals. Thus, $\log\left(h(a,b)|_{w_1=0,w_2=0}\right)$ is convex which implies that $h(a,b)|_{w_1=0,w_2=0}$ is convex!

### 2.7.2 Can we restrict ourselves to a smaller set when searching for the min?

We concluded that $h(a,b)|_{w_1=0,w_2=0}$ is convex and if we can find a local min, we know that it is a global min. But to search numerically for a local min on the entire positive reals seems unnecessary, if we can find a smaller set which includes a local min. While we know that $M_{X_1,X_2}(z_1 + 1 + z_2, -z_2)$ is well-defined for $a < 0$, $b < 0$ we can use this information to construct our desired smaller set. Actually, in this case, trying to calculate the set where $M_{X_1,X_2}(z_1 + 1 + z_2, -z_2)$ is well-defined is to complicated, which is another reason for why we are pursuing the strategy of constructing another set.

Focusing at the moment on $M_{X_1,X_2}(z_1,z_2)$ and applying the above theorem by Hölder with $q=\frac{p}{p-1}$, one gets

$$M_{X_1,X_2}(z_1,z_2)=\mathbb{E}[e^{z_1X_1+z_2X_2}]\leq\left(\mathbb{E}[e^{pz_1X_1}]\right)^{\frac{1}{p}}\left(\mathbb{E}[e^{\frac{p}{1-p}z_2X_2}]\right)^{\frac{p-1}{p}}=$$
$$M_{X_1}\left(pz_1\right)^{\frac{1}{p}}\cdot M_{X_2}\left(\frac{p}{p-1}z_2\right)^{\frac{p-1}{p}}$$

Which implies that whenever $M_{X_1}$ and $M_{X_2}$ are well-defined, then $M_{X_1,X_2}(z_1,z_2)$ is well-defined. Now for a moment, imagine $M_{X_1}\left(z_1\right)$ being well-defined on $(c_1,d_1)$, i.e whenever $c_1<\Re(z_1)<d_1$ and $M_{X_2}\left(z_2\right)$ being well-defined on $(c_2,d_2)$, i.e whenever $c_2<\Re(z_2)<d_2$. Below in figure 6 the set of suitable $\Re(z_1),\Re(z_2)$ is visualized.

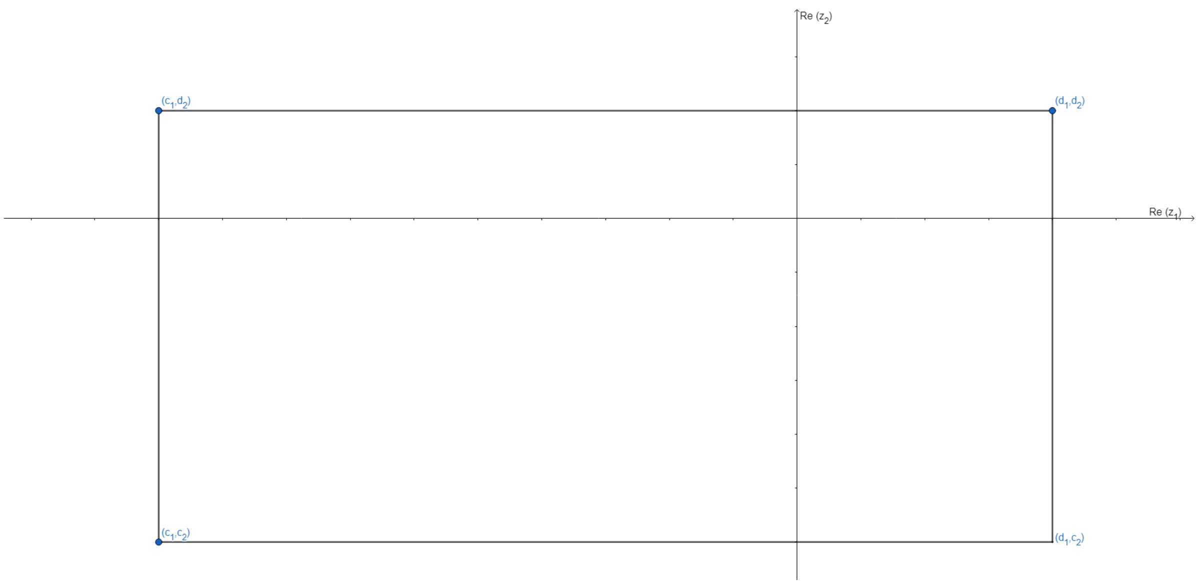


Figure 6: By making the assumption that $M_{X_1}\left(z_1\right)$ being well-defined on $(c_1,d_1)$ and that $M_{X_2}\left(z_2\right)$ being well-defined on $(c_2,d_2)$. Then this is the set of suitable $\Re(z_1),\Re(z_2)$.

Now since $M_{X_1}\left(z_1\right)$ is defined on $(c_1,d_1)$ and since $p\in[1,\infty]$ we have that $M_{X_1}$ is well-defined on $(\frac{c_1}{p},\frac{d_1}{p})$. By similar argument we have that $M_{X_2}$ is well-defined on $\left(c_2\left(1-\frac{1}{p}\right),d_2\left(1-\frac{1}{p}\right)\right)$. Now remember that any $p\in[1,\infty]$ is acceptable, which means that for *any* point in *any* of the sets $(\frac{c_1}{p},\frac{d_1}{p})\times\left(c_2\left(1-\frac{1}{p}\right),d_2\left(1-\frac{1}{p}\right)\right)$ we have that $M_{X_1,X_2}(z_1,z_2)$ is well-defined. Thus, the set we are looking for, is the one constructed by taking the unions over all $p$, i.e $\cup_{p\in[1,\infty)}(\frac{c_1}{p},\frac{d_1}{p})\times\left(c_2\left(1-\frac{1}{p}\right),d_2\left(1-\frac{1}{p}\right)\right)$. The natural question now is, how can we visualize the set $\cup_{p\in[1,\infty)}(\frac{c_1}{p},\frac{d_1}{p})\times\left(c_2\left(1-\frac{1}{p}\right),d_2\left(1-\frac{1}{p}\right)\right)$? Well, setting $p=\infty$ gives us $\{0\}\times(c_2,d_2)$, below in figure 7 the points included in this set is visualized by a green line.

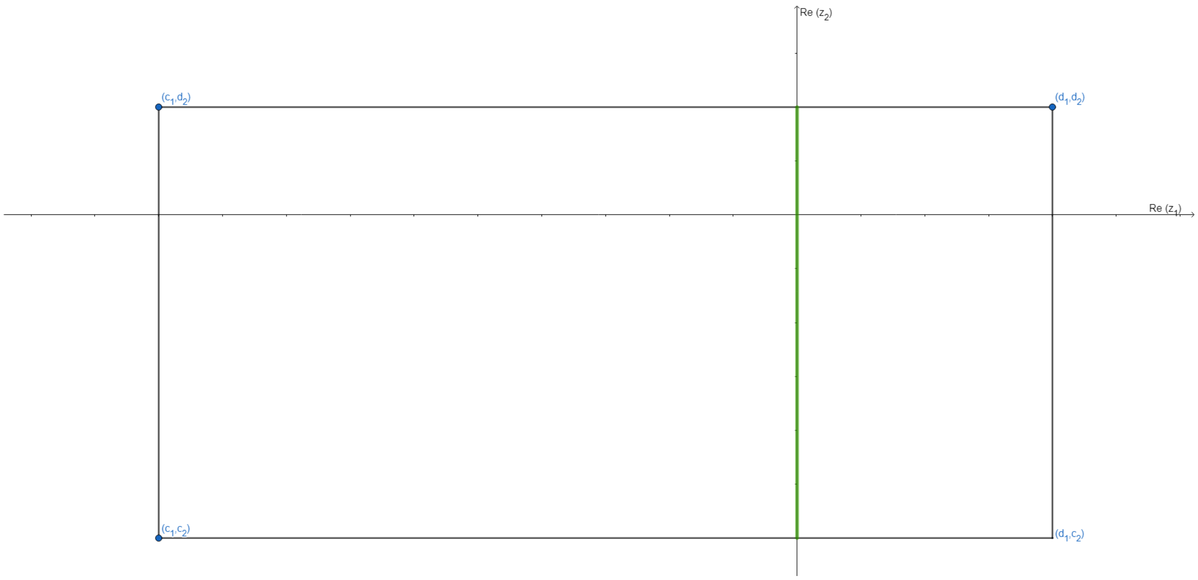


Figure 7: By setting $p = \infty$ we obtain the set $\{0\} \times (c_2, d_2)$ which is visualized by the green line.

Setting $p = 1$ gives us $(c_1, d_1) \times \{0\}$ below in figure 8 the points included in this set is visualized by a blue line.

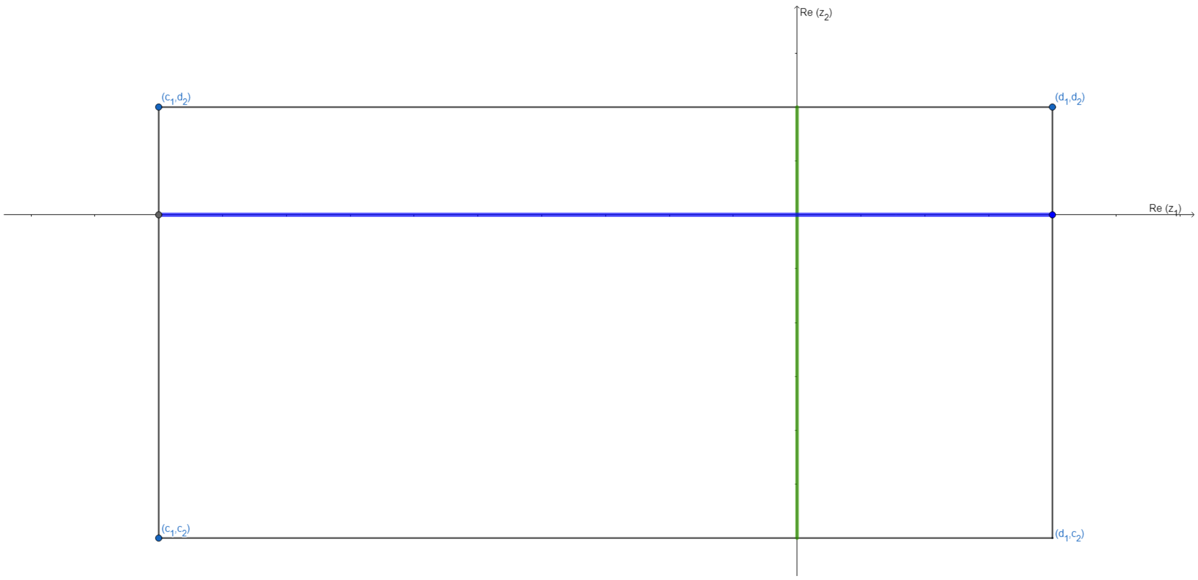


Figure 8: By setting $p = 1$ we obtain the set $(c_1, d_1) \times \{0\}$ which is visualized by the blue line.

Going back to $(\frac{c_1}{p}, \frac{d_1}{p}) \times \left(c_2\left(1-\frac{1}{p}\right), d_2\left(1-\frac{1}{p}\right)\right)$ we can visualize this as rectangle, where $(\frac{c_1}{p}, \frac{d_1}{p})$ determines the width and $\left(c_2\left(1-\frac{1}{p}\right), d_2\left(1-\frac{1}{p}\right)\right)$ determines the height of the rectangle. The union of all these rectangles will give use the desired set. Consider an arbitrary $p$ and we want to consider where the upper left corner of the rectangle will lie. Well in the expression $(\frac{c_1}{p}, \frac{d_1}{p}) \times \left(c_2\left(1-\frac{1}{p}\right), d_2\left(1-\frac{1}{p}\right)\right)$ the upper left corner is given by $\left(\frac{c_1}{p}, c_2\left(1-\frac{1}{p}\right)\right)$. But this is exactly a parameterization of a linear curve! And we already know two points which must lie on this curve, the two points $(c_1, 0)$ and $(0, d_2)$ which we obtained with our blue and green lines. Since the parameterization is a linear curve and we know two points which must lie on the line. We know the upper left corner for any $p$. Below in figure 9 a red line visualize all the possible points for which the upper left corner can lie.

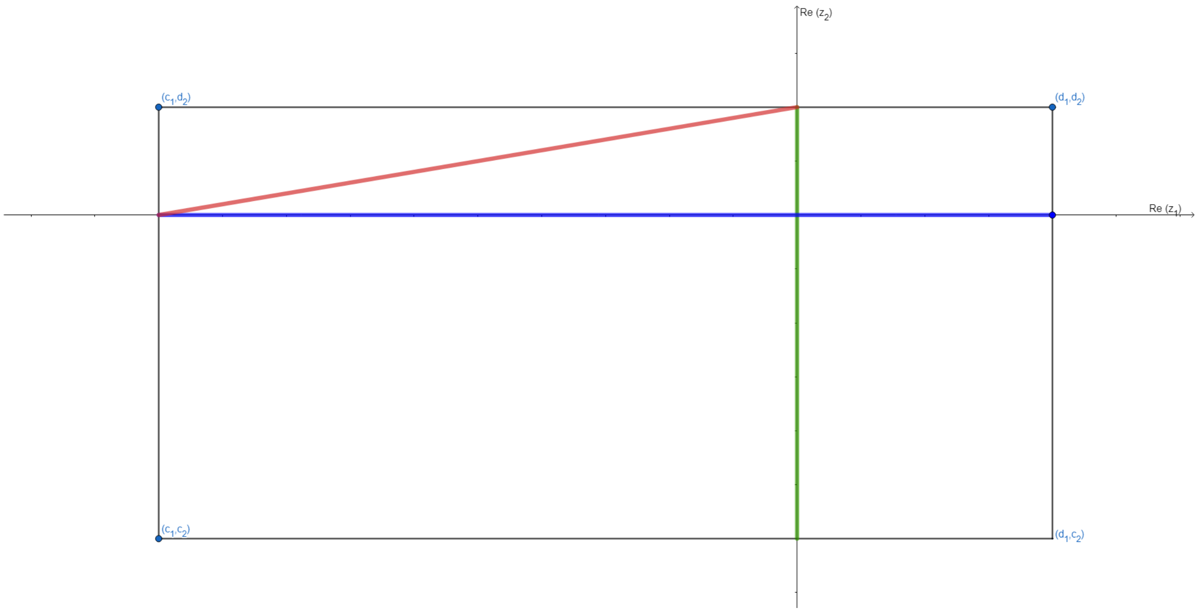


Figure 9: Visualization of the where the upper left corner of the rectangle can lie for arbitrary $p$. Any point on the red line corresponds to a upper left corner of a rectangle for some $p$.

This argument can of course be repeated for any corner of the rectangle, and thus we can obtain three other lines, each for any of the other corners, which is seen in figure 10 below.

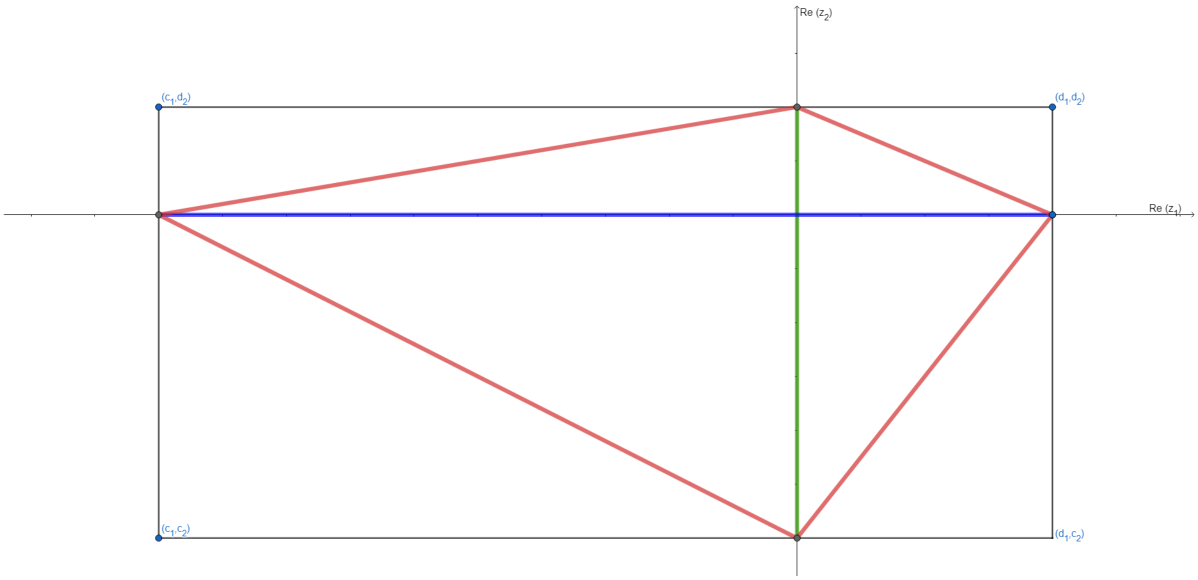


Figure 10: The red lines indicate where any of the four cornes of the rectangle may lie, for arbirtrary $p$. The upper left line indicates the upper left corner. Upper right line indicates upper right corner. Lower right line indicates lower right corner. Lower left line indicates lower left corner. Any of the rectangles constructed by putting corners on the line corresponds to some $p$. Any any of those rectangles is included in the union $\cup_{p\in[1,\infty)}(\frac{c_1}{p}, \frac{d_1}{p}) \times \left(c_2\left(1-\frac{1}{p}\right), d_2\left(1-\frac{1}{p}\right)\right)$.

But this means that we have found our desired set $\cup_{p\in[1,\infty)}(\frac{c_1}{p}, \frac{d_1}{p}) \times \left(c_2\left(1-\frac{1}{p}\right), d_2\left(1-\frac{1}{p}\right)\right)$. The final image of the entire set can be found below in figure 11.
The set above in 11 is only dependent on the values of the parameters in the Heston model, the initial volatilities, the correlations and the time to maturity. That is, the set will be same independent of the initial values of $S_1, S_2$ and $K$. Thus, the result can reused over and over for valuating the spread option with different initial values on the assets and the strike.

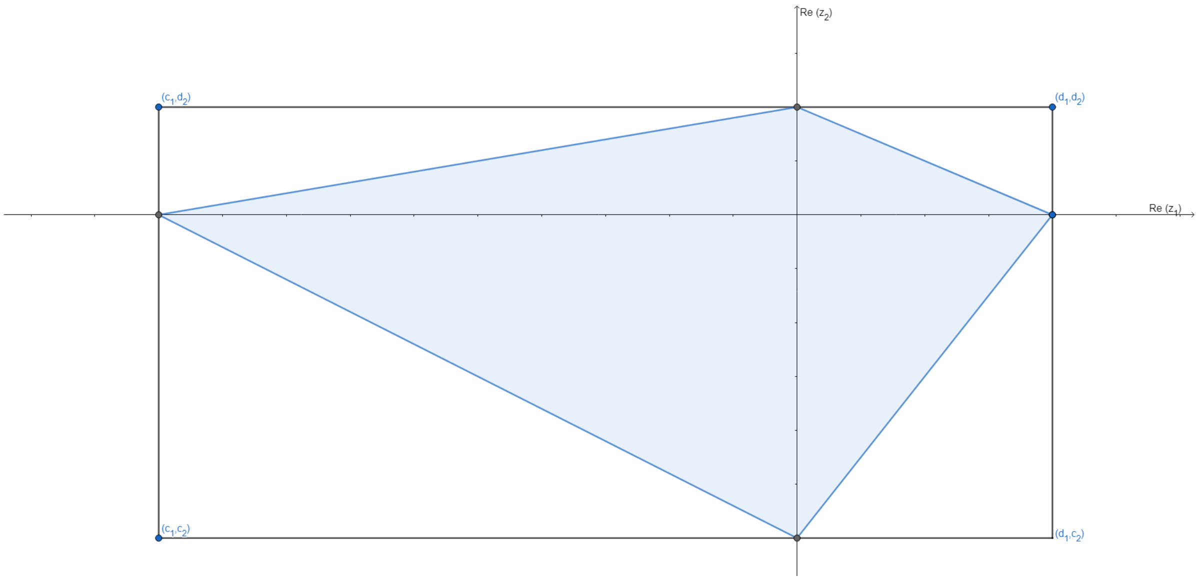


Figure 11: The set of all $\Re(z_1)$ and $\Re(z_2)$ where $M_{X_1}$ and $M_{X_2}$ are both well-defined. Given the assumption that $M_{X_1}(z_1)$ being well-defined on $(c_1, d_1)$ and $M_{X_2}(z_2)$ being defined on $(c_2, d_2)$.

The above picture can be characterized by the following inequalities

$$\begin{aligned}
\Re(z_2) &\leq -\frac{d_2}{d_1}\Re(z_1) + d_2 \\
\Re(z_2) &\leq -\frac{d_2}{c_1}\Re(z_1) + d_2 \\
\Re(z_2) &\geq -\frac{c_2}{c_1}\Re(z_1) + c_2 \\
\Re(z_2) &\geq -\frac{c_2}{d_1}\Re(z_1) + c_2.
\end{aligned}$$

This is all in all good, but the set we have constructed is the one in consideration for $M_{X_1,X_2}(z_1, z_2)$, we want the one for $M_{X_1,X_2}(z_1 + 1 + z_2, -z_2)$. Now we are going to be a bit clever and put a $\sim$ on $z_1$ and $z_2$ in $M_{X_1,X_2}(z_1, z_2)$ i.e

$$\begin{aligned}
\Re(\tilde{z}_2) &\leq -\frac{d_2}{d_1}\Re(\tilde{z}_1) + d_2 \\
\Re(\tilde{z}_2) &\leq -\frac{d_2}{c_1}\Re(\tilde{z}_1) + d_2 \\
\Re(\tilde{z}_2) &\geq -\frac{c_2}{c_1}\Re(\tilde{z}_1) + c_2 \\
\Re(\tilde{z}_2) &\geq -\frac{c_2}{d_1}\Re(\tilde{z}_1) + c_2.
\end{aligned}$$

Now set $\tilde{z}_1 = z_1 + 1 + z_2$ and $\tilde{z}_2 = -z_2$, and we obtain

$$\begin{aligned}
\Re(-z_2) &\leq -\frac{d_2}{d_1}\Re(z_1 + 1 + z_2) + d_2 \\
\Re(-z_2) &\leq -\frac{d_2}{c_1}\Re(z_1 + 1 + z_2) + d_2 \\
\Re(-z_2) &\geq -\frac{c_2}{c_1}\Re(z_1 + 1 + z_2) + c_2 \\
\Re(-z_2) &\geq -\frac{c_2}{d_1}\Re(z_1 + 1 + z_2) + c_2.
\end{aligned}$$

We have that $\Re(z_1) = -a$ and $\Re(z_2) = -b$, which gives us

$$\begin{aligned}
b &\leq -\frac{d_2}{d_1}(-a + 1 - b) + d_2 \\
b &\leq -\frac{d_2}{c_1}(-a + 1 - b) + d_2 \\
b &\geq -\frac{c_2}{c_1}(-a + 1 - b) + c_2 \\
b &\geq -\frac{c_2}{d_1}(-a + 1 - b) + c_2.
\end{aligned}$$

Remember that we must still guarantee that the Fourier transform to exist, so we must add $-a > 0$ and $-b > 0$, and finally we obtain

$$D = \begin{cases} -a > 0 \\ -b > 0 \\ b \leq -\frac{d_2}{d_1} - a + 1 - b + d_2 \\ b \leq -\frac{d_2}{c_1} - a + 1 - b + d_2 \\ b \geq -\frac{c_2}{c_1} - a + 1 - b + c_2 \\ b \geq -\frac{c_2}{d_1} - a + 1 - b + c_2. \end{cases} \tag{33}$$

Now by going back to (30) our problem is reduced to search for a min in a smaller set, i.e

$$\min_{(s,t)\in D} h(a,b)|_{w_1=0,w_2=0}$$

### 2.7.3 The set where the marginal Moment generating functions are defined is not arbitrary

Earlier we assumed that $M_{X_1}(z_1)$ was well-defined on $(c_1, d_1)$ and $M_{X_2}(z_2)$ was well-defined on $(c_2, d_2)$. These sets are of course not arbitrary, they are unique sets that we can calculate. Since we know the structure for both assets $S_1$ and $S_2$ (20) and the correlations (21). Actually we also need to know the time to maturity. The sets are obtained by solving a third degree polynomial, obtaining the zeros. Doing that, with $\tau = 0.5$ (year) and the same values as presented in 3, 4, 8, 6 and 7 we obtain

$$|M_{X_1}(z_1)| < \infty \; \forall \Re(z_1) \in \{(-25.458563, 61.182879) \cap \Re(z_1) > 0\}$$
$$|M_{X_2}(z_2)| < \infty \; \forall \Re(z_2) \in \{(-17.381963, 58.169061) \cap \Re(z_2) > 0\}.$$

The set $D$ i.e. (33) can thus be rewritten as

$$D = \begin{cases} -a > 0 \\ -b > 0 \\ 0.05185656b \leq a + 60.1855 \\ 1.43766b \leq 26.4581 - a \\ 0.464558b \geq a - 26.4569 \\ 4.51989b \geq -a - 60.1827. \end{cases} \tag{34}$$

All these inequalities looks messy, and it can actually be simplified. By drawing the set in GeoGebra, we obtain the following picture.

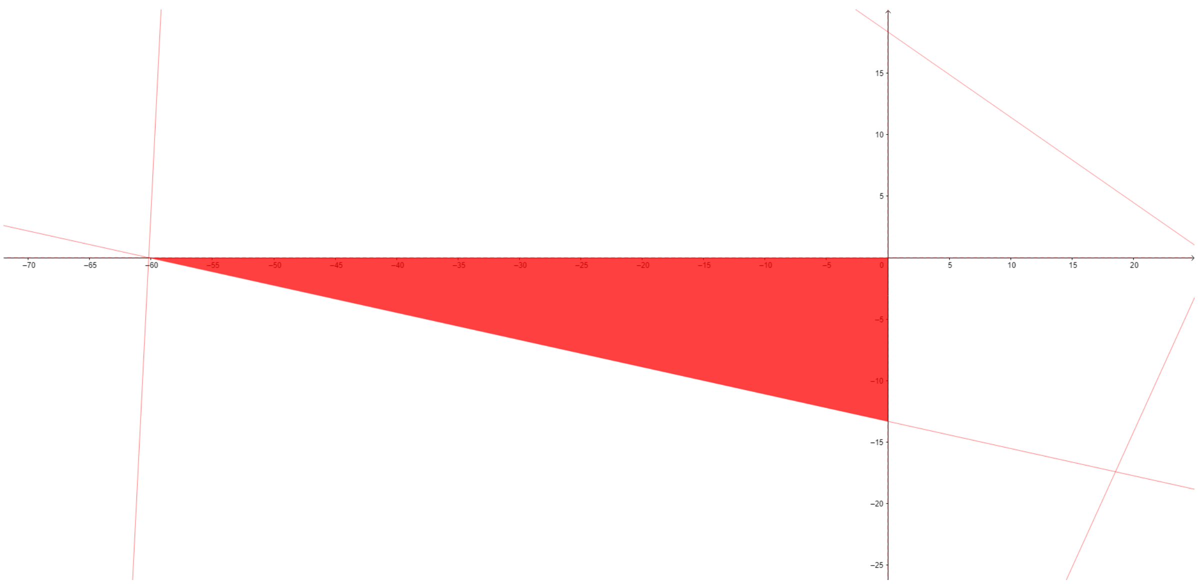


Figure 12: Here we see the set $D$, as the red filled in part in the above plane.

Thus, we realize that the set $D$ can be described using only

$$D = \begin{cases} -a > 0 \\ -b > 0 \\ 4.51989b \geq -a - 60.1827. \end{cases} \tag{35}$$

#### 2.7.4 Running an optimizer to find the values of $a$ and $b$

By using MATLABs built in function *fmincon* the solution to

$$\min_{(s,t)\in D} h(a,b)|_{w_1=0,w_2=0}$$

where

$$h(a,b)|_{w_1=0,w_2=0} = e^{-ak}\frac{\Gamma(-b)\Gamma(-a)}{\Gamma(-a-b+2)}M_{X_1,X_2}(-a+1-b,b)$$

and

$$D = \begin{cases} -a > 0 \\ -b > 0 \\ 4.51989b \geq -a - 60.1827. \end{cases} \tag{36}$$

In order to improve numerical calculations and reduce the risk for overflow. We are going to ask MATLAB to minimize the function $\log\left(h(a,b)|_{w_1=0,w_2=0}\right)$ instead. Giving the advantage to use the function *gammaln* with its clever implementation which directly calculates the log of the gamma function, instead of first calculating the gamma function, then taking the log. By finding the values of $(a,b)$ that minimizes $\log\left(h(a,b)|_{w_1=0,w_2=0}\right)$ and using the logic that if $x_1 < x_2$ then $e^{x_1} < e^{x_2}$ we obtain the minima.
With the same values of all the constants in our model as in the simulation of the Heston model, the minima of $h(a,b)|_{w_1=0,w_2=0}$ is obtained at
$(a_{\min}, b_{\min)} = (-3.975063; -9.947600) \in D\backslash\partial D$. It is important that the minima doesn't lie on the boundary. Because if it did, we couldn't be certain that what we had found actually was a local minima. We have focused on looking for a minima in a smaller set than the actual set. A potential minima on the boundary could mean that there is a local minima outside our set but the method can't find it, since it's not allowed to search there. If that were the case, we would have had to find another way to approximate the complicated set where $M_{X_1,X_2}$ is well-defined.

## 2.8 Valuation of the spread option

Now going all the way back (19) we want to evaluate

$$\frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-z_1k}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}M_{X_1,X_2}(z_1+1+z_2,-z_2)dw_1dw_2 = \Pi$$

This can not be done analytically, so we are going to calculate $\Pi$ numerically. We know that the value of $\Pi$ is real, thus by taking $\Re(\Pi)$ wont change anything. Doing this, we will use the following approximation

$$\Re\left(\frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-z_1k}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}M_{X_1,X_2}(z_1+1+z_2,-z_2)dw_1dw_2\right)=$$
$$\frac{1}{(2\pi)^2}e^{-r(T-t_0)}\int_{\mathbb{R}}\int_{\mathbb{R}}\Re\left(e^{-z_1k}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}M_{X_1,X_2}(z_1+1+z_2,-z_2)\right)dw_1dw_2\approx$$
$$\frac{e^{-r(T-t_0)}}{(2\pi)^2}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}w_{1,k}^{(n)}w_{2,j}^{(n)}\Re\left(e^{-z_1k}\frac{\Gamma(z_2)\Gamma(z_1)}{\Gamma(z_1+z_2+2)}M_{X_1,X_2}(z_1+1+z_2,-z_2)\right)\Big|_{w_1=x_{1,k}^{(n)},w_2=x_{2,l}^{(n)}}+$$
$$\frac{e^{-r(T-t_0)}}{(2\pi)^2}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}w_{1,k}^{(n)}w_{2,j}^{(n)}\Re\left(e^{-\bar{z}_1k}\frac{\Gamma(z_2)\Gamma(\bar{z}_1)}{\Gamma(\bar{z}_1+z_2+2)}M_{X_1,X_2}(\bar{z}_1+1+z_2,-z_2)\right)\Big|_{w_1=x_{1,k}^{(n)},w_2=x_{2,l}^{(n)}}+$$
$$\frac{e^{-r(T-t_0)}}{(2\pi)^2}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}w_{1,k}^{(n)}w_{2,j}^{(n)}\Re\left(e^{-z_1k}\frac{\Gamma(\bar{z}_2)\Gamma(z_1)}{\Gamma(z_1+\bar{z}_2+2)}M_{X_1,X_2}(z_1+1+\bar{z}_2,-\bar{z}_2)\right)\Big|_{w_1=x_{1,k}^{(n)},w_2=x_{2,l}^{(n)}}+$$
$$\frac{e^{-r(T-t_0)}}{(2\pi)^2}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}w_{1,k}^{(n)}w_{2,j}^{(n)}\Re\left(e^{-\bar{z}_1k}\frac{\Gamma(\bar{z}_2)\Gamma(\bar{z}_1)}{\Gamma(\bar{z}_1+\bar{z}_2+2)}M_{X_1,X_2}(\bar{z}_1+1+\bar{z}_2,-\bar{z}_2)\right)\Big|_{w_1=x_{1,k}^{(n)},w_2=x_{2,l}^{(n)}}$$

Using the same values as in the simulation, a value using the Fourier method is obtained and $\Pi = 3.3175$. While valuating the spread option using the Fourier method the number of weights were varied. Below follows an image showing the convergence as a function of the number of weights.

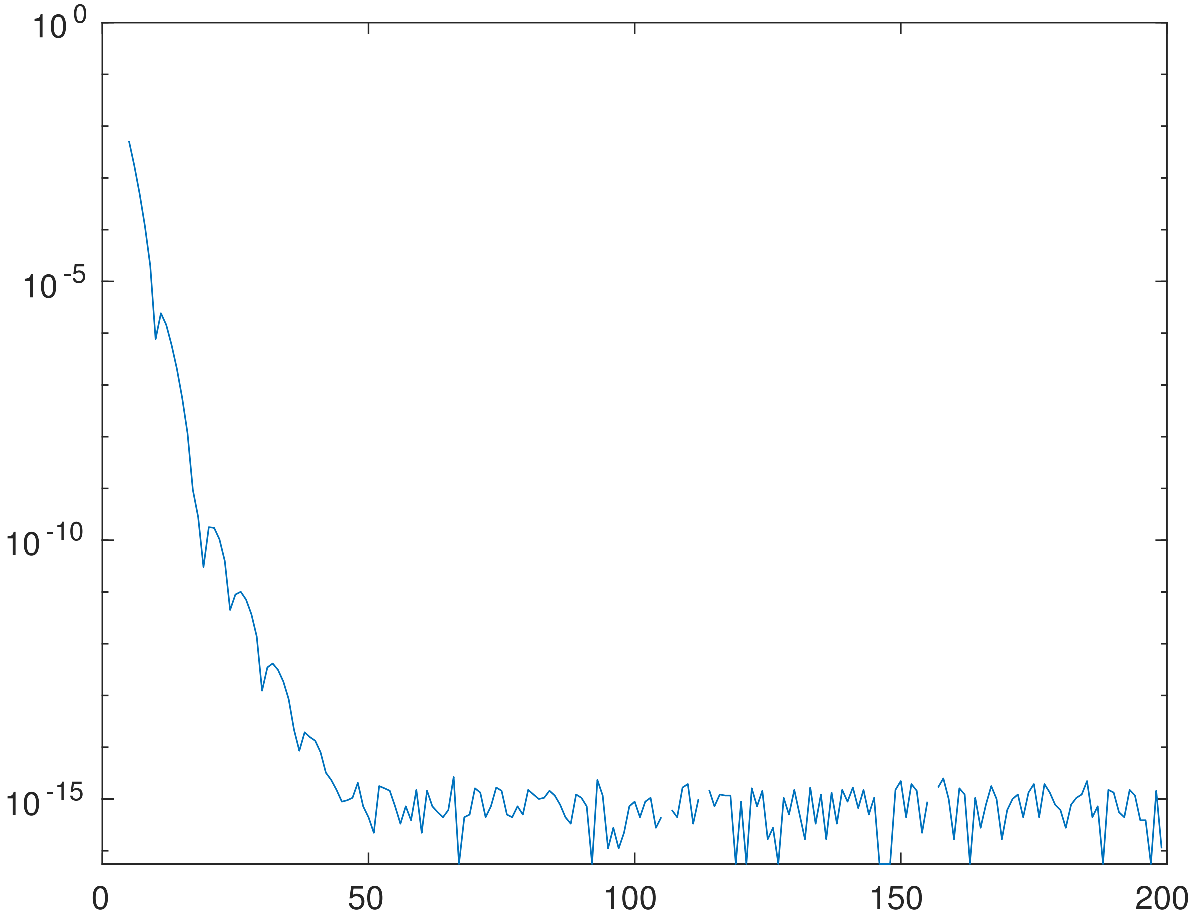


Figure 13: Here we see the convergence of the value of the spread option using the Fourier method as a function of how many weights were used. The convergence is fast and remains stable while increasing.

The value of the spread option using the Fourier method was tested against different strikes, $K = 2, 4, 6, 8, \dots, 40$, and the result is presented in the image below.

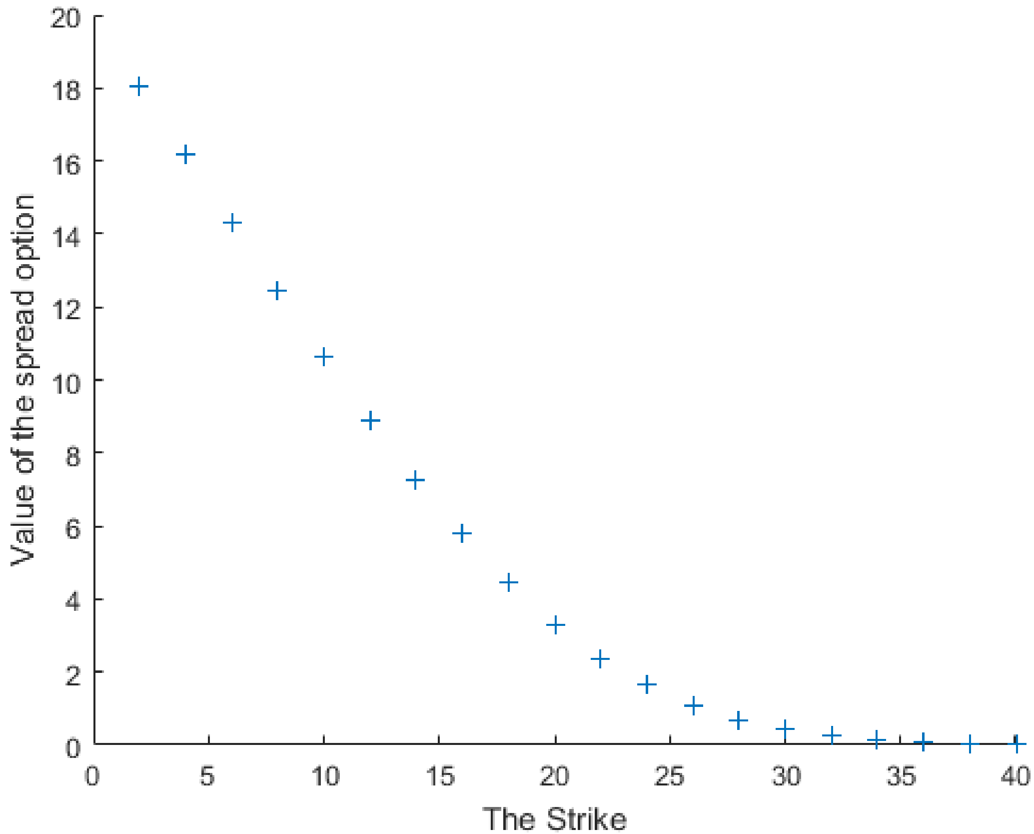


Figure 14: Here we see the value of the spread option using the Fourier method for different strikes.

## 2.9 Monte Carlo vs Fourier Method

Using the values presented in the simulation section all results are presented below.

Table 8: The result for the different methods

| Kind of simulation | Value | Standard deviation | Confidence interval |
|---|---|---|---|
| Monte Carlo | 3.302221 | 0.0147 | $[3.2728, 3.3317]$ |
| Monte Carlo with control variate | 3.305545 | 0.0076 | $[3.2903, 3.3208]$ |
| Fourier Method | 3.3175 | | |

Below follows images showing the value of the spread option according to the Fourier method and a confidence interval from the Monte Carlo simulation illustrating its performance and accuracy.

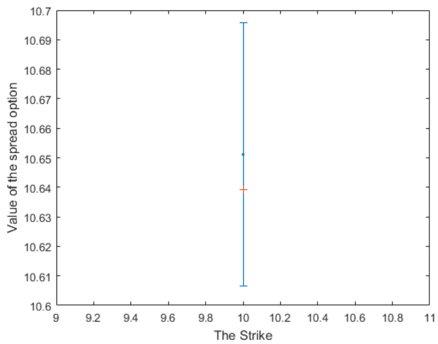


Figure 15: Here we see the confidence interval of the Monte Carlo simulation of the spread option for the strike $K = 10$. The blue dot represents the mean and the red '+' represent the value according to the Fourier method.

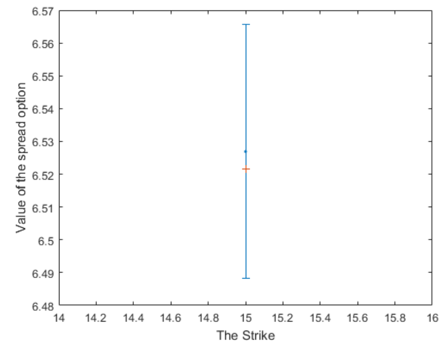


Figure 16: Here we see the confidence interval of the Monte Carlo simulation of the spread option for the strike $K = 15$. The blue dot represents the mean and the red '+' represent the value according to the Fourier method.

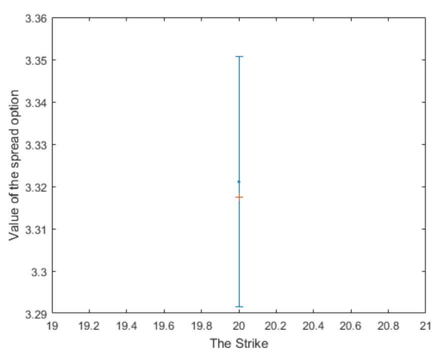


Figure 17: Here we see the confidence interval of the Monte Carlo simulation of the spread option for the strike $K = 20$. The blue dot represents the mean and the red '+' represent the value according to the Fourier method.

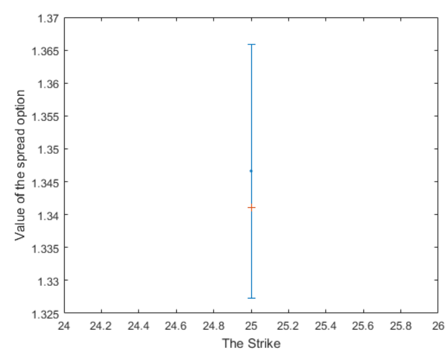


Figure 18: Here we see the confidence interval of the Monte Carlo simulation of the spread option for the strike $K = 25$. The blue dot represents the mean and the red '+' represent the value according to the Fourier method.

We see that the value of the Fourier method lies in the confidence interval which is what we aimed for. There fore we conclude that the calculations are correct and valuation of the spread option using the Fourier method is accomplished.

# References

[1] Sebastian Åberg. *Derivative Pricing*. Lund University, 2018.

[2] Tomas Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, 2009.

[3] Magnus Wiktorsson. Notes on the benchop implementations for the fourier gauss laguerre fgl method. 2017.

[4] Peter DEMARZO Jonathan Berk. *CORPORATE FINANCE*. PEARSON, 2013.

[5] Stegun Abramowitz. *Handbook of mathematical functions*. United States department of commerce, 1972.

[6] Magnus Wiktorsson. Valuation of derivative assets, lecture 11, 2017. Slides available at: http://www.maths.lth.se/matstat/kurser/fmsn25masm24/ht17/L11.pdf.