# New rate 1/2, 1/3, and 1/4 binary convolutional encoders with an optimum distance profile

Johannesson, Rolf; Ståhl, Per

[Link to publication](Link to publication)

Finally, some comments about the decoding complexity. Suppose that the decoder of woven convolutional codes with outer warp consists of one Viterbi decoder for the inner code and $l_o$ Viterbi decoders for the outer codes. The decoding complexity is proportional to

$$\Gamma = 2^{m_i} + l_o 2^{m_o}. \tag{30}$$

Let us choose $m_i = m_o = \sqrt{m}$. Then, we have

*Theorem 2:* Suppose that a decoder for a woven convolutional code with outer warp consists of $l_o$ Viterbi decoder for the outer convolutional codes and one Viterbi decoder for the inner convolutional code. If both the outer and inner convolutional codes have memory $\sqrt{m}$, then the complexity of the decoder is proportional to

$$\Gamma = (1 + l_o) 2^{\sqrt{m}}. \tag{31}$$

$\square$

From Theorems 1 and 2 it follows, somewhat surprisingly, that for woven convolutional codes with $m_o = m_i = \sqrt{m}$ the decoding error probability decreases exponentially with $m$ while the decoding complexity increases exponentially only with $\sqrt{m}$.

### REFERENCES

[1] S. Höst, R. Johannesson, and V. V. Zyablov, "A first encounter with binary woven convolutional codes," in *Proc. 4th Int. Symp. Communication Theory and Applications* (Lake District, U.K., July 13–18, 1997).

[2] V. V. Zyablov and S. A. Shavgulidze, "Error exponent of concatenated decoding of block-convolutional concatenated codes," in *Proc. Int. Workshop "Convolutional Codes; Multi-User Communication"* (Sochi, USSR, May 30–June 6, 1983).

[3] ——, "Generalized convolutional concatenated codes with unit memory," *Probl. Pered. Inform.*, vol. 22, no. 4, pp. 9–28, 1986.

[4] S. Höst, R. Johannesson, D. K. Zigangirov, K. Sh. Zigangirov, and V. V. Zyablov, "On the distribution of the output error burst lengths for Viterbi decoding of convolutional codes," in *Proc. 1997 IEEE Int. Symp. Information Theory* (Ulm, Germany, June 29–July 4, 1997).

[5] R. G. Gallager, *Information Theory and Reliable Communication*. New York: Wiley, 1968.

[6] R. Johannesson and K. Sh. Zigangirov, *Fundamentals of Convolutional Coding*. Piscataway, NJ: IEEE Press, 1999.

[7] G. D. Forney Jr., "Convolutional codes II: Maximum-likelihood decoding," *Inform. Contr.*, vol. 25, pp. 222–266, 1974.

[8] A. J. Viterbi and J. K. Omura, *Principles of Digital Communication and Coding*. New York: McGraw-Hill, 1979.

[9] D. J. Costello, "Free distance bounds for convolutional codes," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 356–365, May 1974.

# New Rate $1/2$, $1/3$, and $1/4$ Binary Convolutional Encoders with an Optimum Distance Profile

Rolf Johannesson, *Fellow, IEEE*, and Per Ståhl, *Student Member, IEEE*

*Abstract*— Tabulations of binary systematic and nonsystematic polynomial convolutional encoders with an optimum distance profile for rate $1/2$, $1/3$, and $1/4$ are given. The reported encoders are found by computer searches that optimize over the weight spectra. The free distances for rate $1/3$ and $1/4$ are compared with Heller's and Griesmer's upper bounds.

*Index Terms*—Convolutional encoders, free distance, optimum distance profile.

The *distance profile* [1] $\boldsymbol{d} = [d_0, d_1, \cdots, d_m]$, where $d_j$ is the $j$th-order *column distance* [2] and $m$ is the memory of the convolutional encoder, is an important distance parameter for convolutional encoders. It is an encoder property but if we limit our interest to consider only encoding matrices $G(D)$ with $G(0)$ having full rank we can regard the distance profile as a code property [3]. When comparing codes with the same rate and memory, we say that a distance profile $\boldsymbol{d}$ is superior to a distance profile $\boldsymbol{d}'$ if $d_i > d_i'$ for the smallest $i, 0 \leq i \leq m$, where $d_i \neq d_i'$. The code with the superior $\boldsymbol{d}$ will generally require less computation with sequential decoding than the other code [1], [4].

In [5], extensive tables of rate $1/2$ convolutional encoders were given. In Tables I and II we give rate $1/2$ polynomial systematic and nonsystematic convolutional encoders, respectively, with an *optimum distance profile* (ODP encoders), i.e., with a distance profile equal to or superior to that of any other encoder. The generators are written in an octal form according to the convention introduced in [1]. For each value of the memory, we give the encoder with the largest free distance $d_{\text{free}}$ among ODP encoders. (The free distance is the minimum Hamming distance between any two differing codewords.) Ties were resolved by comparing their weight spectra, i.e., by successively using the number of low-weight paths $n_{d_{\text{free}}+i}$ for $i = 0, 1, \cdots, 9$ as a further optimality criterion. The generators marked with "*" have better spectra than those given in [5].

In an earlier paper [6], systematic convolutional encoders of rate $1/3$ and $1/4$ were published together with a few short nonsystematic encoders of rate $1/3$. Only one spectral component, viz., the number of paths of weight $d_{\text{free}}$, was given. Here we give ten spectral components as well as extensive lists of nonsystematic encoders. We list rate $1/3$ and $1/4$ systematic as well as nonsystematic polynomial convolutional ODP encoders. The free distances are compared with Heller's and Griesmer's upper bounds on the free distances for nonlinear trellis and linear convolutional codes, respectively.

The free distance for any binary, rate $R = b/c$ convolutional code encoded by a polynomial, nonsystematic encoding matrix of memory

TABLE I
$n_{d_{\text{free}}+i}, i = 0, \cdots, 9$ for Systematic Rate $R = 1/2$ ODP Encoding Matrices $G = (4 \ g_{12})$. For Memories $m$ Marked with "*" These Encoders Have Better Spectra Than Those Given in [5]

| | | | $i$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $g_{12}$ | $d_{\text{free}}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 6 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 7 | 4 | 2 | 0 | 5 | 0 | 13 | 0 | 34 | 0 | 89 | 0 |
| 3 | 64 | 4 | 1 | 0 | 6 | 0 | 16 | 0 | 69 | 0 | 232 | 0 |
| 4* | 66 | 5 | 2 | 2 | 1 | 10 | 21 | 29 | 77 | 180 | 332 | 711 |
| 5 | 73 | 6 | 3 | 0 | 13 | 0 | 55 | 0 | 298 | 0 | 1401 | 0 |
| 6* | 674 | 6 | 1 | 3 | 4 | 11 | 25 | 53 | 118 | 274 | 654 | 1430 |
| 7* | 714 | 6 | 2 | 0 | 9 | 0 | 46 | 0 | 248 | 0 | 1289 | 0 |
| 8* | 671 | 7 | 1 | 5 | 5 | 17 | 35 | 70 | 173 | 452 | 993 | 2415 |
| 9* | 7154 | 8 | 4 | 0 | 19 | 0 | 94 | 0 | 542 | 0 | 3159 | 0 |
| 10 | 7152 | 8 | 3 | 0 | 16 | 0 | 79 | 0 | 457 | 0 | 2618 | 0 |
| 11* | 7153 | 9 | 3 | 5 | 11 | 26 | 52 | 124 | 317 | 821 | 1870 | 4364 |
| 12 | 67114 | 9 | 1 | 4 | 10 | 15 | 46 | 104 | 224 | 576 | 1368 | 3322 |
| 13 | 67116 | 10 | 5 | 0 | 27 | 0 | 124 | 0 | 777 | 0 | 4529 | 0 |
| 14* | 71447 | 10 | 4 | 0 | 12 | 0 | 105 | 0 | 517 | 0 | 3138 | 0 |
| 15* | 671174 | 10 | 1 | 0 | 16 | 0 | 78 | 0 | 437 | 0 | 2391 | 0 |
| 16 | 671166 | 12 | 13 | 0 | 46 | 0 | 263 | 0 | 1486 | 0 | 9019 | 0 |
| 17* | 671166 | 12 | 13 | 0 | 46 | 0 | 263 | 0 | 1486 | 0 | 9019 | 0 |
| 18* | 6711454 | 12 | 4 | 0 | 23 | 0 | 154 | 0 | 817 | 0 | 4896 | 0 |
| 19 | 7144616 | 12 | 3 | 0 | 23 | 0 | 92 | 0 | 556 | 0 | 3472 | 0 |
| 20* | 7144761 | 12 | 1 | 3 | 10 | 25 | 53 | 110 | 263 | 676 | 1593 | 3838 |
| 21* | 71447614 | 12 | 1 | 0 | 7 | 0 | 66 | 0 | 314 | 0 | 1842 | 0 |
| 22* | 71446166 | 14 | 6 | 0 | 44 | 0 | 189 | 0 | 1132 | 0 | 6570 | 0 |
| 23* | 67115143 | 14 | 2 | 0 | 38 | 0 | 168 | 0 | 947 | 0 | 5726 | 0 |
| 24* | 714461654 | 15 | 5 | 7 | 23 | 62 | 115 | 256 | 669 | 1648 | 3999 | 9703 |
| 25* | 671145536 | 15 | 3 | 7 | 16 | 44 | 112 | 244 | 578 | 1312 | 3267 | 8097 |
| 26* | 714476053 | 16 | 8 | 0 | 54 | 0 | 289 | 0 | 1691 | 0 | 9609 | 0 |
| 27* | 7144760524 | 16 | 7 | 0 | 73 | 0 | 350 | 0 | 1971 | 0 | 11624 | 0 |
| 28* | 7144616566 | 16 | 3 | 0 | 38 | 0 | 134 | 0 | 834 | 0 | 5052 | 0 |
| 29 | 7144760535 | 18 | 22 | 0 | 118 | 0 | 695 | 0 | 3926 | 0 | 22788 | 0 |
| 30* | 67114543064 | 16 | 1 | 1 | 10 | 15 | 36 | 101 | 225 | 596 | 1342 | 3298 |
| 31 | 67114543066 | 18 | 11 | 0 | 53 | 0 | 307 | 0 | 1742 | 0 | 10218 | 0 |

TABLE II
$n_{d_{\text{free}}+i}, i = 0, \cdots, 9$ for Nonsystematic Rate $R = 1/2$ ODP Encoders $G = (g_{11} \ g_{12})$. For Memories $m$ Marked with "*" These Encoders Have Better Spectra Than Those Given in [5]

| | | | | $i$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $g_{11}$ | $g_{12}$ | $d_{\text{free}}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 7 | 5 | 5 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| 3 | 74 | 54 | 6 | 1 | 3 | 5 | 11 | 25 | 55 | 121 | 267 | 589 | 1299 |
| 4 | 62 | 56 | 7 | 2 | 3 | 4 | 16 | 37 | 68 | 176 | 432 | 925 | 2156 |
| 5* | 77 | 45 | 8 | 2 | 3 | 8 | 15 | 41 | 90 | 224 | 515 | 1239 | 2896 |
| 6 | 634 | 564 | 10 | 12 | 0 | 53 | 0 | 234 | 0 | 1517 | 0 | 8862 | 0 |
| 7 | 626 | 572 | 10 | 1 | 6 | 13 | 20 | 64 | 123 | 321 | 764 | 1858 | 4442 |
| 8 | 751 | 557 | 12 | 10 | 9 | 30 | 51 | 156 | 340 | 875 | 1951 | 5127 | 11589 |
| 9 | 7664 | 5714 | 12 | 1 | 8 | 8 | 31 | 73 | 150 | 441 | 940 | 2214 | 5531 |
| 10 | 7512 | 5562 | 14 | 19 | 0 | 80 | 0 | 450 | 0 | 2615 | 0 | 15276 | 0 |
| 11 | 6643 | 5175 | 14 | 1 | 10 | 25 | 46 | 105 | 258 | 616 | 1531 | 3611 | 8675 |
| 12 | 63374 | 47244 | 15 | 2 | 10 | 29 | 55 | 138 | 301 | 692 | 1720 | 4199 | 10245 |
| 13 | 45332 | 77136 | 16 | 5 | 15 | 21 | 56 | 161 | 381 | 879 | 2095 | 5085 | 12207 |
| 14 | 65231 | 43677 | 17 | 3 | 16 | 44 | 62 | 172 | 455 | 1025 | 2395 | 5853 | 14487 |
| 15* | 727144 | 424374 | 18 | 5 | 15 | 21 | 56 | 161 | 381 | 879 | 2095 | 5085 | 12207 |
| 16 | 717066 | 522702 | 19 | 9 | 16 | 48 | 112 | 259 | 596 | 1457 | 3460 | 8257 | 20562 |
| 17* | 745705 | 546153 | 20 | 6 | 31 | 58 | 125 | 314 | 711 | 1819 | 4222 | 10502 | 25222 |
| 18* | 6302164 | 5634554 | 21 | 13 | 34 | 72 | 161 | 369 | 914 | 2167 | 5318 | 12937 | 31241 |
| 19 | 5122642 | 7315626 | 22 | 26 | 0 | 160 | 0 | 916 | 0 | 5154 | 0 | 29386 | 0 |
| 20* | 7375407 | 4313045 | 22 | 1 | 17 | 49 | 108 | 234 | 521 | 1310 | 3099 | 7433 | 18264 |
| 21 | 67520654 | 50371444 | 24 | 40 | 0 | 251 | 0 | 1379 | 0 | 7812 | 0 | 45858 | 0 |
| 22* | 64553062 | 42533736 | 24 | 4 | 27 | 75 | 147 | 331 | 817 | 1956 | 4578 | 11053 | 27282 |
| 23 | 55076157 | 75501351 | 26 | 65 | 0 | 331 | 0 | 2014 | 0 | 11359 | 0 | 65585 | 0 |
| 24* | 744537344 | 472606614 | 26 | 10 | 45 | 91 | 235 | 465 | 1186 | 2882 | 6790 | 16618 | 39794 |
| 25 | 665041116 | 516260772 | 27 | 24 | 54 | 125 | 278 | 637 | 1599 | 3779 | 9073 | 21831 | 52929 |

TABLE III
$n_{d_\text{free}+i}, i = 0, \cdots, 9$ FOR SYSTEMATIC RATE $R = 1/3$ ODP ENCODERS $G = (4\ g_{12}\ g_{13})$

| | | | | | | | | $i$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $g_{12}$ | $g_{13}$ | $d_\text{free}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 6 | 6 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 5 | 7 | 6 | 1 | 0 | 2 | 0 | 4 | 0 | 8 | 0 | 16 | 0 |
| 3 | 64 | 74 | 8 | 2 | 0 | 3 | 0 | 8 | 0 | 19 | 0 | 46 | 0 |
| 4 | 56 | 72 | 9 | 1 | 2 | 3 | 3 | 1 | 2 | 9 | 24 | 48 | 62 |
| 5 | 57 | 73 | 10 | 1 | 3 | 1 | 0 | 4 | 10 | 13 | 21 | 35 | 48 |
| 6 | 564 | 754 | 12 | 4 | 0 | 8 | 0 | 12 | 0 | 39 | 0 | 140 | 0 |
| 7 | 516 | 676 | 12 | 1 | 2 | 1 | 7 | 8 | 7 | 19 | 20 | 43 | 93 |
| 8 | 531 | 676 | 13 | 1 | 3 | 3 | 6 | 8 | 9 | 27 | 25 | 46 | 110 |
| 9 | 5314 | 6764 | 15 | 3 | 5 | 2 | 4 | 11 | 11 | 29 | 49 | 80 | 144 |
| 10 | 5312 | 6766 | 16 | 4 | 0 | 8 | 0 | 20 | 0 | 64 | 0 | 168 | 0 |
| 11 | 5317 | 6767 | 16 | 1 | 2 | 4 | 7 | 7 | 15 | 18 | 31 | 66 | 95 |
| 12 | 65304 | 71274 | 17 | 1 | 2 | 6 | 8 | 4 | 12 | 24 | 38 | 65 | 118 |
| 13 | 65306 | 71276 | 18 | 1 | 3 | 4 | 7 | 5 | 9 | 31 | 35 | 71 | 123 |
| 14 | 65305 | 71273 | 19 | 2 | 2 | 3 | 5 | 9 | 23 | 25 | 35 | 65 | 98 |
| 15 | 653764 | 712614 | 20 | 2 | 0 | 8 | 0 | 19 | 0 | 41 | 0 | 138 | 0 |
| 16 | 531206 | 676672 | 20 | 1 | 0 | 5 | 0 | 11 | 0 | 25 | 0 | 106 | 0 |
| 17 | 653055 | 712737 | 22 | 2 | 0 | 7 | 0 | 20 | 0 | 65 | 0 | 176 | 0 |
| 18 | 5144574 | 7325154 | 24 | 6 | 0 | 19 | 0 | 40 | 0 | 92 | 0 | 294 | 0 |
| 19 | 6530576 | 7127306 | 24 | 2 | 0 | 11 | 0 | 27 | 0 | 71 | 0 | 197 | 0 |
| 20 | 6530547 | 7127375 | 26 | 4 | 0 | 21 | 0 | 43 | 0 | 131 | 0 | 307 | 0 |
| 21 | 65376114 | 71261054 | 26 | 3 | 0 | 11 | 0 | 23 | 0 | 88 | 0 | 261 | 0 |
| 22 | 51445036 | 73251266 | 26 | 1 | 0 | 4 | 0 | 18 | 0 | 49 | 0 | 132 | 0 |
| 23 | 65305477 | 71273753 | 28 | 3 | 4 | 3 | 9 | 17 | 25 | 46 | 75 | 142 | 201 |
| 24 | 514453214 | 732513134 | 28 | 1 | 0 | 8 | 0 | 29 | 0 | 50 | 0 | 169 | 0 |
| 25 | 653761172 | 712610566 | 30 | 2 | 0 | 14 | 0 | 46 | 0 | 100 | 0 | 276 | 0 |
| 26 | 514450363 | 732512675 | 31 | 2 | 5 | 9 | 14 | 17 | 25 | 58 | 94 | 136 | 229 |
| 27 | 6537616604 | 7126106264 | 32 | 10 | 0 | 19 | 0 | 43 | 0 | 140 | 0 | 449 | 0 |
| 28 | 6537616606 | 7126106264 | 33 | 5 | 13 | 12 | 13 | 23 | 54 | 75 | 145 | 261 | 412 |
| 29 | 5312071307 | 6766735721 | 33 | 3 | 2 | 5 | 12 | 16 | 31 | 54 | 80 | 119 | 192 |
| 30 | 51445320354 | 73251313564 | 34 | 1 | 6 | 6 | 5 | 14 | 31 | 43 | 75 | 136 | 200 |

TABLE IV
$n_{d_\text{free}+i}, i = 0, \cdots, 9$ FOR NONSYSTEMATIC RATE $R = 1/3$ ODP ENCODERS $G = (g_{11}\ g_{12}\ g_{13})$

| | | | | | | | | | $i$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $g_{11}$ | $g_{12}$ | $g_{13}$ | $d_\text{free}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 4 | 6 | 6 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 5 | 7 | 7 | 8 | 2 | 0 | 5 | 0 | 13 | 0 | 34 | 0 | 89 | 0 |
| 3 | 54 | 64 | 74 | 10 | 3 | 0 | 2 | 0 | 15 | 0 | 24 | 0 | 87 | 0 |
| 4 | 52 | 66 | 76 | 12 | 5 | 0 | 3 | 0 | 13 | 0 | 62 | 0 | 108 | 0 |
| 5 | 47 | 53 | 75 | 13 | 1 | 3 | 6 | 4 | 5 | 12 | 14 | 33 | 66 | 106 |
| 6 | 574 | 664 | 744 | 14 | 1 | 0 | 8 | 0 | 11 | 0 | 35 | 0 | 97 | 0 |
| 7 | 536 | 656 | 722 | 16 | 1 | 5 | 2 | 6 | 14 | 18 | 34 | 44 | 65 | 125 |
| 8 | 435 | 526 | 717 | 17 | 1 | 2 | 6 | 7 | 6 | 13 | 30 | 38 | 73 | 140 |
| 9 | 5674 | 6304 | 7524 | 20 | 7 | 0 | 19 | 0 | 40 | 0 | 99 | 0 | 321 | 0 |
| 10 | 5136 | 6642 | 7166 | 21 | 4 | 1 | 4 | 14 | 18 | 28 | 39 | 60 | 114 | 225 |
| 11 | 4653 | 5435 | 6257 | 22 | 3 | 0 | 9 | 0 | 32 | 0 | 70 | 0 | 190 | 0 |
| 12 | 47164 | 57254 | 76304 | 24 | 2 | 8 | 10 | 15 | 18 | 29 | 74 | 101 | 155 | 267 |
| 13 | 47326 | 61372 | 74322 | 26 | 7 | 0 | 23 | 0 | 64 | 0 | 166 | 0 | 426 | 0 |
| 14 | 47671 | 55245 | 63217 | 27 | 6 | 4 | 6 | 21 | 24 | 37 | 69 | 112 | 166 | 328 |
| 15 | 447454 | 632734 | 766164 | 28 | 1 | 6 | 5 | 17 | 24 | 34 | 67 | 90 | 155 | 266 |
| 16 | 552334 | 614426 | 772722 | 30 | 3 | 9 | 20 | 21 | 29 | 49 | 83 | 145 | 241 | 409 |
| 17 | 552137 | 614671 | 772233 | 32 | 7 | 15 | 11 | 21 | 58 | 82 | 94 | 177 | 288 | 523 |
| 18 | 4550704 | 6246334 | 7731724 | 34 | 28 | 0 | 53 | 0 | 112 | 0 | 357 | 0 | 994 | 0 |
| 19 | 5531236 | 6151572 | 7731724 | 35 | 8 | 18 | 29 | 32 | 54 | 78 | 130 | 267 | 431 | 693 |
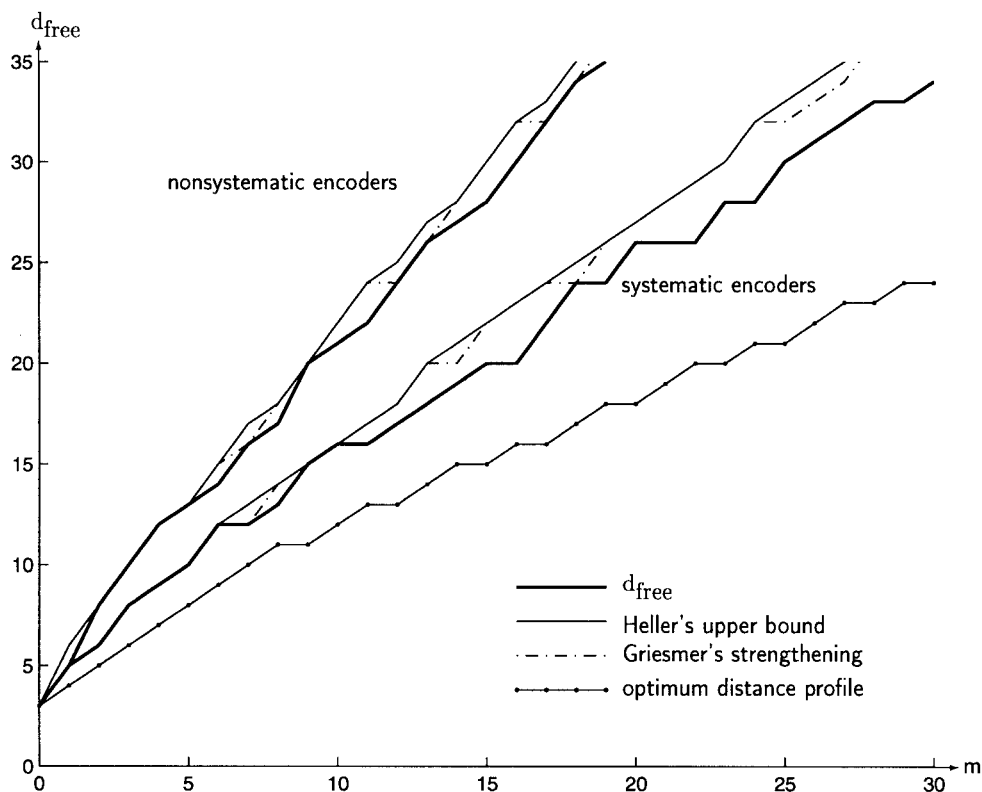
Fig. 1. The free distances for rate $R = 1/3$ systematic and nonsystematic ODP convolutional encoders and comparisons with Heller's and Griesmer's upper bounds and with the optimum distance profile.

TABLE V
$n_{d_{\text{free}}+i}, i = 0, \cdots, 9$ FOR SYSTEMATIC RATE $R = 1/4$ ODP ENCODERS $G = (4\ g_{12}\ g_{13}\ g_{14})$

| | | | | | $i$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $g_{12}$ | $g_{13}$ | $g_{14}$ | $d_{\text{free}}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 4 | 6 | 6 | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 5 | 6 | 7 | 8 | 1 | 0 | 1 | 0 | 2 | 0 | 3 | 0 | 5 | 0 |
| 3 | 54 | 64 | 74 | 11 | 1 | 0 | 1 | 0 | 5 | 0 | 4 | 0 | 8 | 0 |
| 4 | 56 | 62 | 72 | 12 | 1 | 0 | 1 | 0 | 5 | 0 | 4 | 0 | 8 | 0 |
| 5 | 51 | 67 | 73 | 14 | 1 | 0 | 2 | 0 | 5 | 0 | 6 | 0 | 10 | 0 |
| 6 | 534 | 634 | 754 | 16 | 1 | 3 | 0 | 0 | 3 | 5 | 5 | 9 | 9 | 9 |
| 7 | 516 | 676 | 732 | 18 | 1 | 3 | 1 | 2 | 3 | 3 | 5 | 10 | 11 | 13 |
| 8 | 535 | 637 | 755 | 20 | 2 | 4 | 0 | 3 | 2 | 4 | 8 | 9 | 19 | 21 |
| 9 | 5350 | 6370 | 7554 | 20 | 2 | 0 | 2 | 0 | 9 | 0 | 8 | 0 | 29 | 0 |
| 10 | 5156 | 6272 | 7404 | 20 | 1 | 0 | 1 | 0 | 4 | 0 | 5 | 0 | 17 | 0 |
| 11 | 5351 | 6371 | 7557 | 24 | 1 | 0 | 7 | 0 | 7 | 0 | 16 | 0 | 35 | 0 |
| 12 | 53514 | 63714 | 75574 | 24 | 1 | 0 | 1 | 2 | 1 | 8 | 8 | 3 | 9 | 10 |
| 13 | 51056 | 63116 | 76472 | 26 | 2 | 0 | 2 | 0 | 5 | 0 | 12 | 0 | 23 | 0 |
| 14 | 51055 | 63117 | 76473 | 28 | 1 | 0 | 5 | 0 | 4 | 0 | 13 | 0 | 30 | 0 |
| 15 | 515630 | 627350 | 740424 | 27 | 1 | 0 | 0 | 2 | 3 | 2 | 3 | 5 | 9 | 18 |
| 16 | 530036 | 611516 | 747332 | 30 | 3 | 0 | 2 | 0 | 6 | 0 | 10 | 0 | 21 | 0 |
| 17 | 535154 | 637141 | 755775 | 30 | 1 | 0 | 1 | 0 | 5 | 0 | 12 | 0 | 20 | 0 |
| 18 | 5105444 | 6311614 | 7647074 | 32 | 1 | 0 | 1 | 0 | 3 | 0 | 12 | 0 | 17 | 0 |
| 19 | 5105446 | 6311616 | 7647072 | 34 | 2 | 0 | 3 | 0 | 4 | 0 | 12 | 0 | 21 | 0 |
| 20 | 5105447 | 6311617 | 7647073 | 36 | 2 | 2 | 3 | 5 | 1 | 4 | 7 | 9 | 13 | 26 |
| 21 | 51054474 | 63116164 | 76470730 | 36 | 1 | 0 | 3 | 0 | 4 | 0 | 12 | 0 | 21 | 0 |
| 22 | 51563362 | 62735066 | 74040356 | 38 | 2 | 0 | 3 | 0 | 5 | 0 | 7 | 0 | 21 | 0 |
| 23 | 51054477 | 63116167 | 76470731 | 40 | 1 | 1 | 3 | 4 | 1 | 3 | 10 | 10 | 11 | 32 |
| 24 | 510544764 | 631161674 | 764707304 | 42 | 2 | 0 | 7 | 0 | 5 | 0 | 30 | 0 | 29 | 0 |
| 25 | 510544770 | 631161666 | 764707302 | 42 | 2 | 0 | 3 | 0 | 10 | 0 | 17 | 0 | 37 | 0 |
| 26 | 510544771 | 631161667 | 764707303 | 44 | 1 | 0 | 1 | 3 | 2 | 12 | 5 | 10 | 14 | 18 |
| 27 | 5105447710 | 6311616664 | 7647073024 | 47 | 3 | 3 | 6 | 9 | 13 | 9 | 23 | 25 | 35 | 60 |
| 28 | 5105447714 | 6311616664 | 7647073032 | 48 | 3 | 4 | 2 | 7 | 8 | 17 | 20 | 33 | 38 | 48 |
| 29 | 5105447715 | 6311616671 | 7647073025 | 51 | 6 | 3 | 4 | 8 | 15 | 25 | 28 | 39 | 44 | 58 |
| 30 | 51054477154 | 63116166734 | 76470730324 | 52 | 3 | 2 | 7 | 6 | 9 | 13 | 14 | 35 | 37 | 60 |
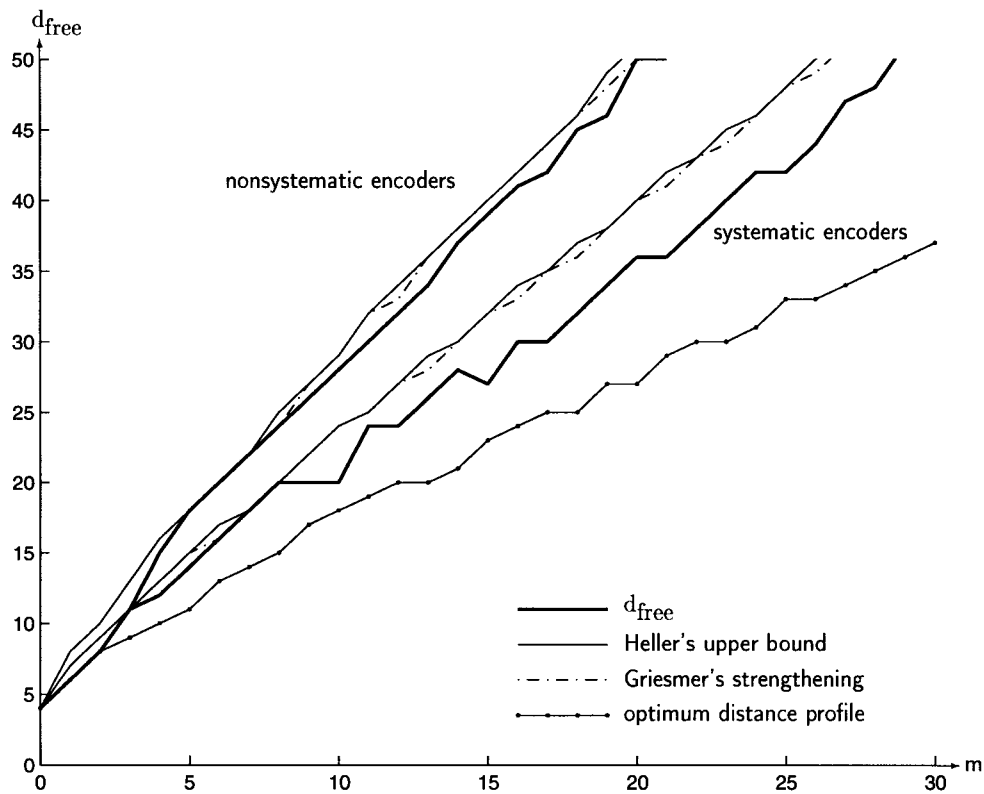
Fig. 2. The free distances for rate $R = 1/4$ systematic and nonsystematic ODP convolutional encoders and comparisons with Heller's and Griesmer's upper bounds and with the optimum distance profile.

TABLE VI
$n_{d_{\mathrm{free}}+i}, i = 0, \cdots, 9$ FOR NONSYSTEMATIC RATE $R = 1/4$ ODP ENCODERS $G = (g_{11} \ g_{12} \ g_{13} \ g_{14})$

| | | | | | $i$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $g_{11}$ | $g_{12}$ | $g_{13}$ | $g_{14}$ | $d_{\mathrm{free}}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 4 | 4 | 6 | 6 | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 4 | 5 | 6 | 7 | 8 | 1 | 0 | 1 | 0 | 2 | 0 | 3 | 0 | 5 | 0 |
| 3 | 44 | 54 | 64 | 70 | 11 | 1 | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 4 |
| 4 | 46 | 52 | 66 | 76 | 15 | 1 | 2 | 1 | 1 | 1 | 3 | 7 | 4 | 7 | 18 |
| 5 | 47 | 53 | 67 | 75 | 18 | 3 | 0 | 5 | 0 | 8 | 0 | 10 | 0 | 25 | 0 |
| 6 | 454 | 574 | 664 | 724 | 20 | 3 | 0 | 4 | 0 | 7 | 0 | 15 | 0 | 27 | 0 |
| 7 | 476 | 556 | 672 | 712 | 22 | 1 | 5 | 2 | 2 | 4 | 4 | 5 | 15 | 21 | 14 |
| 8 | 457 | 575 | 663 | 723 | 24 | 1 | 3 | 4 | 7 | 2 | 1 | 6 | 13 | 31 | 29 |
| 9 | 4730 | 5574 | 6564 | 7104 | 26 | 3 | 0 | 4 | 0 | 12 | 0 | 20 | 0 | 41 | 0 |
| 10 | 4266 | 5362 | 6136 | 7722 | 28 | 4 | 0 | 5 | 0 | 10 | 0 | 17 | 0 | 40 | 0 |
| 11 | 4227 | 5177 | 6225 | 7723 | 30 | 4 | 0 | 4 | 0 | 11 | 0 | 23 | 0 | 39 | 0 |
| 12 | 46554 | 56174 | 66450 | 72374 | 32 | 1 | 3 | 6 | 9 | 6 | 13 | 10 | 15 | 31 | 37 |
| 13 | 45562 | 57052 | 64732 | 73176 | 34 | 1 | 0 | 11 | 0 | 11 | 0 | 33 | 0 | 39 | 0 |
| 14 | 47633 | 57505 | 66535 | 71145 | 37 | 3 | 5 | 6 | 10 | 11 | 11 | 25 | 32 | 45 | 56 |
| 15 | 454374 | 574624 | 662564 | 723354 | 39 | 5 | 7 | 10 | 4 | 5 | 10 | 15 | 33 | 40 | 62 |
| 16 | 463712 | 566132 | 661562 | 727446 | 41 | 3 | 7 | 7 | 10 | 19 | 11 | 21 | 35 | 52 | 75 |
| 17 | 415727 | 523133 | 624577 | 744355 | 42 | 1 | 0 | 14 | 0 | 17 | 0 | 24 | 0 | 72 | 0 |
| 18 | 4653444 | 5426714 | 6477354 | 7036504 | 45 | 3 | 5 | 8 | 13 | 16 | 14 | 25 | 42 | 64 | 87 |
| 19 | 4654522 | 5617436 | 6645066 | 7237532 | 46 | 1 | 0 | 13 | 0 | 20 | 0 | 28 | 0 | 89 | 0 |
| 20 | 4712241 | 5763615 | 6765523 | 7330467 | 50 | 13 | 0 | 18 | 0 | 39 | 0 | 91 | 0 | 168 | 0 |
| 21 | 45724414 | 55057474 | 65556514 | 72624710 | 50 | 1 | 7 | 6 | 13 | 15 | 13 | 36 | 36 | 55 | 79 |

$m$ satisfies [7], [8]

$$\text{Heller:} \quad d_{\text{free}} \leq \min_{i \geq 1} \left\lfloor \frac{(m+i)c}{2(1 - 2^{-bi})} \right\rfloor \quad (1)$$

$$\text{Griesmer:} \quad \sum_{j=0}^{bi-1} \left\lceil \frac{d_{\text{free}}}{2^j} \right\rceil \leq (m+i)c, \qquad i = 1, 2, \cdots. \quad (2)$$

For systematic encoding matrices we have the corresponding bounds [3]

$$\text{Heller:} \quad d_{\text{free}} \leq \min_{i \geq 1} \left\lfloor \frac{(m(1-R)+i)c}{2(1-2^{-bi})} \right\rfloor \quad (3)$$

$$\text{Griesmer:} \quad \sum_{j=0}^{bi-1} \left\lceil \frac{d_{\text{free}}}{2^j} \right\rceil \leq (m(1-R)+i)c, \qquad i = 1, 2, \cdots. \quad (4)$$

In Table III we list rate $1/3$ systematic polynomial ODP encoders for memories $1 \leq m \leq 30$. The corresponding nonsystematic encoders for memories $1 \leq m \leq 19$ are listed in Table IV. In Fig. 1 the free distances are compared with Heller's and Griesmer's upper bounds. For comparison we also show the optimum distance profile. (The distance profile is always the same for systematic and nonsystematic encoders [1], [3].)

Rate $1/4$ systematic polynomial ODP convolutional encoders for memories $1 \leq m \leq 30$ are listed in Table V and rate $1/4$ nonsystematic polynomial ODP convolutional encoders for memories $1 \leq m \leq 21$ are listed in Table VI. Finally, in Fig. 2 the free distances are related to Heller's and Griesmer's bounds.

The new convolutional codes combine a large free distance with an optimum distance profile and, thus might be attractive for use in various communication systems.

### References

[1] R. Johannesson, "Robustly-optimal rate one-half binary convolutional codes," *IEEE Trans. Inform. Theory*, vol. IT-21, pp. 464–468, 1975.

[2] D. J. Costello, Jr., "A construction technique for random-error-correcting convolutional codes," *IEEE Trans. Inform. Theory*, vol. IT-15, pp. 631–636, 1969.

[3] R. Johannesson and K. Sh. Zigangirov, *Fundamentals of Convolutional Coding*. Piscataway, NJ: IEEE Press, 1999.

[4] P. R. Chevillat and D. J. Costello, Jr., "Distance and computation in sequential decoding," *IEEE Trans. Commun. Technol.*, vol. COM-24, pp. 440–447, 1976.

[5] M. Cedervall and R. Johannesson, "A fast algorithm for computing distance spectrum of convolutional codes," *IEEE Trans. Inform. Theory*, vol. 35, pp. 1146–1159, 1989.

[6] R. Johannesson, "Some rate 1/3 and 1/4 binary convolutional codes with an optimum distance profile," *IEEE Trans. Inform. Theory*, vol. IT-23, pp. 281–283, 1977.

[7] J. A. Heller, "Sequential decoding: Short constraint length convolutional codes," Jet Propulsion Lab., California Inst. Technol., Pasadena, Space Program Summary 37–54, vol. 3, pp. 171–174, Dec. 1968.

[8] J. H. Griesmer, "A bound for error-correcting codes," *IBM J. Res. Develop.*, vol. 4, no. 5, 1960.

# The Weighted Coordinates Bound and Trellis Complexity of Block Codes and Periodic Packings

Ilan Reuven, *Student Member, IEEE*,
and Yair Be'ery, *Senior Member, IEEE*

*Abstract*— Weighted entropy profiles and a new bound, the weighted coordinates bound, on the state complexity profile of block codes are presented. These profiles and bound generalize the notion of dimension/length profile (DLP) and entropy/length profile (ELP) to block codes whose symbols are not drawn from a common alphabet set, and in particular, group codes. Likewise, the new bound may improve upon the DLP and ELP bounds for linear and nonlinear block codes over fields. However, it seems that the major contribution of the proposed bound is to the study of trellis complexity of block codes whose different coordinates are drawn from different alphabet sets. The label code of lattice and nonlattice periodic packings usually has this property. The construction of a trellis diagram for a lattice and some related bounds are generalized to periodic packings by introducing the fundamental module of the packing, and using the new bound on the state complexity profile. This generalization is limited to a given coordinate system. We show that any bounds on the trellis structure of block codes, and in particular, the bound presented in this work, are applicable to periodic packings.

*Index Terms*— Entropy/dimension profiles, entropy/length profiles, lattices, periodic packings, trellis complexity.

## I. Introduction

Trellis diagrams suggest an efficient framework for soft-decision decoding algorithms for codes and lattices, such as the maximum-likelihood or the maximum *a posteriori* algorithms. Trellis complexity is a fundamental descriptive characteristic of both codes and lattices since it reflects the decoding complexity of these algorithms. The investigation of trellis diagrams of linear block codes has been an active research area during the last decade. Less attention has been directed to group codes and lattices in recent literature hitherto.

Under a given symbol permutation, any group code has a unique minimal biproper trellis [14]. An algorithm for computing the minimal trellis for a group code over a finite Abelian group has been presented by Vazirani *et al.* [27]. This algorithm extends the work of Kschischang and Sorokine [15] which treats linear codes over fields. The generalization of Vazirani *et al.* introduces the notions of $p$-*linear combinations* and $p$-*generator sequences*. The trellis product of the codewords of a $p$-generator sequence is minimal if and only if this sequence is *two-way proper*. A two-way proper $p$-generator sequence is a generalization of the trellis-oriented generator matrix [6], [15], for linear block codes over fields.

Measures of trellis complexity of block codes over a fixed alphabet set are bounded by the *entropy/length profile* (ELP) [18] which extends the *dimension/length profile* (DLP) of linear codes [7] to nonlinear codes. Several studies have addressed the problem of finding efficient permutations that meet the DLP bound, and hence minimize measures of trellis complexity (e.g., [3], [12], [13]). There is no measure equivalent to the DLP and ELP for block codes whose symbols are taken from alphabets of different sizes, such as