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Johannesson, Rolf; Wittenmark, Emma

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Two 16-State, Rate $R = 2/4$ Trellis Codes Whose Free Distances Meet the Heller Bound

Rolf Johannesson, *Fellow, IEEE*, and
Emma Wittenmark, *Student Member, IEEE*

***Abstract*—For rate $R = 1/2$ convolutional codes with 16 states there exists a gap between Heller's upper bound on the free distance and its optimal value. This correspondence reports on the construction of 16-state, binary, rate $R = 2/4$ nonlinear trellis and convolutional codes having $d_{\rm free} = 8$; a free distance that meets the Heller upper bound. The nonlinear trellis code is constructed from a 16-state, rate $R = 1/2$ convolutional code over $\mathbb{Z}_4$ using the Gray map to obtain a binary code. Both convolutional codes are obtained by computer search. Systematic feedback encoders for both codes are potential candidates for use in combination with iterative decoding. Regarded as modulation codes for 4-PSK, these codes have free squared Euclidean distance $d^2_{\rm E,\,free} = 16$.**

***Index Terms*—Convolutional codes, free distance, Heller bound, trellis codes.**

I. Introduction

It is well known that the free distance $d_{\rm free}$ is the principal determiner for the error correcting capability of a trellis code when we communicate over a channel with high signal-to-noise ratio and use maximum-likelihood (or nearly so) decoding. In Fig. 1, we show the free distances for rate $R = 1/2$, binary, optimum free-distance (OFD) convolutional codes together with Heller's and Griesmer's upper bounds for rate $R = 1/2$ [1], [2].

For any binary, rate $R = b/c$ trellis code with memory m, Heller's upper bound on the free distance is given by

$$d_{\rm free} \le \min_{i \ge 1} \left\{ \left\lfloor \frac{(m+i)c}{2(1-2^{-bi})} \right\rfloor \right\}. \tag{1}$$

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The authors are with the Department of Information Technology, Information Theory Group, Lund University, S-221 00 Lund, Sweden (e-mail: rolf@it.lth.se, emma@it.lth.se).

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Heller's bound is not only valid for convolutional codes, i.e., linear, time-invariant trellis codes, but also for a larger class of codes, viz., the class of nonlinear, time-varying trellis codes.

For any binary, rate $R = b/c$ convolutional code with memory m, Griesmer's upper bound on the free distance says that the inequality

$$\sum_{j=0}^{bi-1} \left\lceil \frac{d_{\rm free}}{2^j} \right\rceil \le (m+i)c \tag{2}$$

is satisfied for $i = 1, 2, \cdots$. Griesmer's bound is valid only for (linear) convolutional codes.

It is interesting to notice that for codes with as few as 16 states there exists a gap between Heller's bound (and Griesmer's bound), $d_{\rm free} \le 8$, and the optimal value of the free distance, $d_{\rm free} = 7$, for rate $R = 1/2$ (linear) convolutional codes.

Inspired by the recent advances on block codes over $\mathbb{Z}_4$, see for example [3] and [4], we have searched for 16-state, rate $R = 1/2$ convolutional codes over the ring $\mathbb{Z}_4$ such that they, when combined with the Gray map, can be regarded as 16-state, binary, rate $R = 2/4$, nonlinear trellis codes with a free distance as large as $d_{\rm free} = 8$. We also searched for 16-state, rate $R = 2/4$, binary convolutional codes whose free distance closes the gap. Such low-complexity codes with large free distance are strong candidates for use in practical systems.

In Section II we describe the codes and compare their spectra with that of the best rate $R = 1/2$ convolutional code. Since these codes might be excellent choices for iterative decoding we give systematic, feedback encoders in Section III. Comments on related previous works are given in Section IV.

II. 16-State Codes with $d_{\rm free} = 8$

The computer search was concentrated on encoders with 16 states, since this corresponds to the smallest memory for which a gap exists. The optimal free distance for rate $R = 1/2$ convolutional codes over $\mathbb{Z}_4$ and Gray mapped to a binary trellis code was found to be $d_{\rm free} = 8$, which meets the Heller bound. We also found a binary, rate $R = 2/4$ convolutional code over $\mathbb{F}_2$ with free distance $d_{\rm free} = 8$, which meets the Griesmer bound.

Several codes within each class were found to have the same free distance. The codes reported are those with the best spectra, i.e., as few codewords as possible successively at $d_{\rm free}$, $d_{\rm free} + 1$, and so forth.

The optimum free distance, rate $R = 1/2$ convolutional code over $\mathbb{Z}_4$ has generator matrix (Fig. 2)

$$G(D) = (3 + 3D + D^2 \quad 2 + D + 2D^2) \tag{3}$$

and the optimum free distance, binary, rate $R = 2/4$ convolutional code has generator matrix

$$G(D) = \begin{pmatrix} D + D^2 & 1 + D & D^2 & 1 + D + D^2 \\ 1 & D + D^2 & 1 + D & 1 + D + D^2 \end{pmatrix}. \tag{4}$$

In Table I we compare the spectra, $n(d_{\rm free} + i)$, $i = 0, 1, \cdots, 6$, for these codes with that of the optimum free distance, binary, rate $R = 1/2$ convolutional code with generator matrix

$$G(D) = (1 + D + D^4 \quad 1 + D^2 + D^3 + D^4). \tag{5}$$

For 32-state encoders, as can be seen in Fig. 1, there exist convolutional codes which reach Griesmer's upper bound on the free distance for linear codes, $d_{\rm free} = 8$, but Heller's upper bound indicates that there might exist codes with this complexity and a larger free distance,

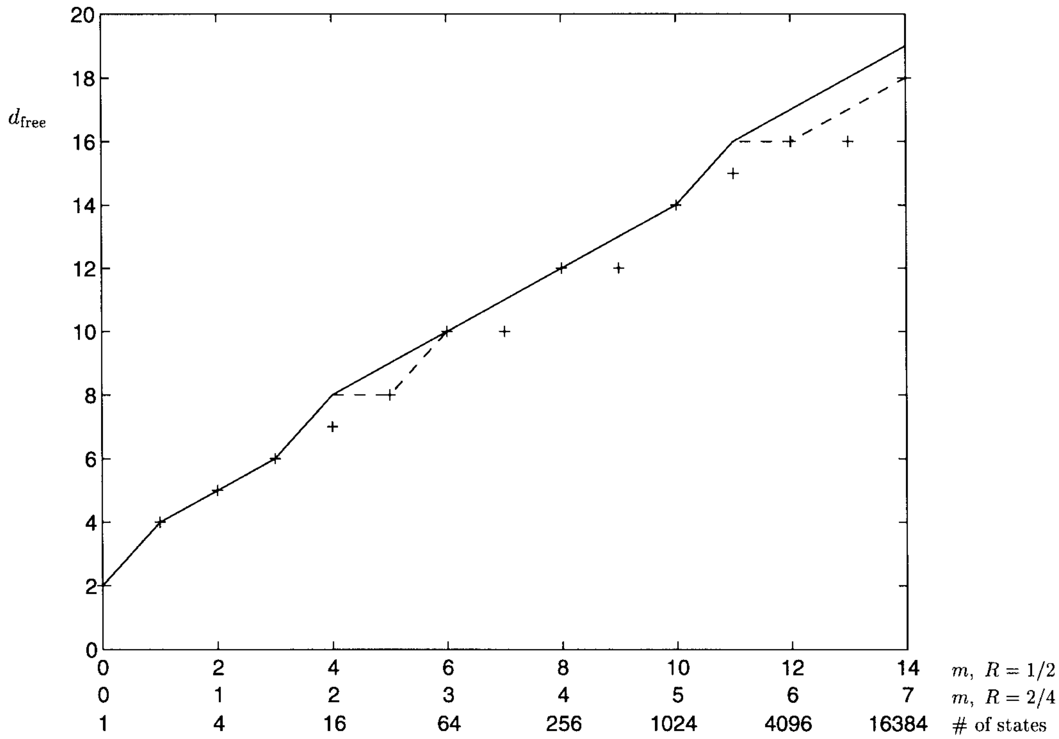

Fig. 1. Heller's (solid line) and Griesmer's (dashed line) bounds together with the optimal free distance for rate $R = 1/2$ convolutional codes ($+$).

TABLE I
SPECTRA OF THE THREE CODES

$n(d_{\text{free}} + i)$	d_{free}	i: 0	1	2	3	4	5	6
$R = 1/2$, $\mathbb{Z}_4$	8	14	0	46	0	281	0	1528
$R = 2/4$, binary	8	12	0	52	0	260	0	1483
$R = 1/2$, binary	7	2	3	4	16	37	68	176

viz., $d_{\text{free}} = 9$. However, a computer search shows that this gap cannot be closed using a trellis code generated as a rate $R = 1/2$ convolutional code over $\mathbb{Z}_4$. Whether it can be closed by any trellis code remains an open question.

III. Systematic Feedback Encoders

It was recently shown that systematic, feedback convolutional encoders can be efficiently decoded by iterative decoding also for low signal-to-noise ratios [5].

A rate $R = b/c$ convolutional code over a ring is systematic if and only if it has a generator matrix that has a $b \times b$ subdeterminant which is a unit in the ring of realizable functions [6]. Since $3 + 3D + D^2$ is such a unit we have the following systematic, feedback encoder for our convolutional code over $\mathbb{Z}_4$:

$$G(D) = \begin{pmatrix} 1 & \dfrac{2 + D + 2D^2}{3 + 3D + D^2} \end{pmatrix}. \tag{6}$$

It is well known that every convolutional code over a field has a systematic generator matrix [7]. Our binary, rate $R = 2/4$ convolutional code can be encoded by the following systematic, rational generator matrix:

$$G(D) = \begin{pmatrix} 1 & 0 & \dfrac{1 + D + D^3}{1 + D^2 + D^3} & \dfrac{1 + D^3}{1 + D^2 + D^3} \\ 0 & 1 & \dfrac{D + D^2 + D^3}{1 + D + D^2 + D^4} & \dfrac{1 + D^2 + D^4}{1 + D + D^2 + D^4} \end{pmatrix}. \tag{7}$$

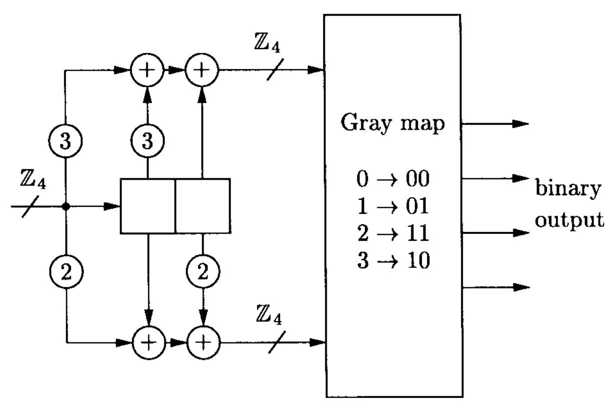

Fig. 2. A rate $R = 1/2$ convolutional encoder over $\mathbb{Z}_4$ together with a Gray map.

A realization in controller canonical form requires 128 states. Since the generator matrix is systematic it is also minimal [7] and, hence, it can be realized with 16 states. Such a minimal realization is shown in Fig. 3.

IV. Comments

In [8], Lee reported a 16-state, rate $R = 4/8$, unit-memory convolutional code with $d_{\text{free}} = 8$.

Regarded as modulation codes for 4-PSK, the codes given in (3) and (4) have free squared Euclidean distance $d^2_{\text{E, free}} = 16$. In his comments on our manuscript, Garello [9] has drawn our attention to the following earlier results:

In [10], Benedetto *et al.* published a systematic 16-state $R = 2/4$ binary convolutional encoder with feedback that over a 4-PSK constellation with binary Gray mapping achieves $d^2_{\text{E, free}} = 16$ and has the same spectrum as (4), thus meeting an upper bound on the free distance for group codes given in [11]. Clearly, their constituent

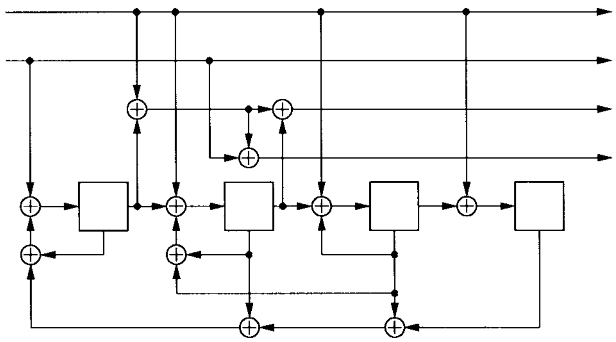

Fig. 3. A minimal realization of the systematic rate $R = 2/4$ binary encoder.

convolutional encoder has $d_{\rm free} = 8$ which meets the Griesmer bound.

Furthermore, in [12] Benedetto *et al.* reported a 16-state $R = 1/2$ convolutional code over $\mathbb{Z}_4$ with generator

$$G(D) = (1 + D + D^2 \quad 2 + 3D + 2D^2) \tag{8}$$

that achieves $d^2_{\rm E,\,free} = 16$. This code has $n(12) = 289$ and, hence, is slightly inferior to (3).

Recently, Calderbank *et al.* [13] used "unwrapping" of their tail-biting representation of the $(24, 12, 8)$ extended Golay code to construct a most interesting 16-state convolutional code with $d_{\rm free} = 8$. Their GCC (Golay convolutional code) can be encoded by a rate $R = 4/8$ time-invariant convolutional encoder or with a rate $R = 1/2$ time-varying, period 4 convolutional encoder; see also [14].

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Convolutional Encoder State Estimation

A. J. Han Vinck, *Senior Member, IEEE*,
Petr Dolezal, and Young-Gil Kim

***Abstract*— To estimate the convolutional encoder state from received data, one may use the inverse to the encoder G. However, channel errors make this method unreliable. We propose a method that uses the received data in the following way. We calculate the syndrome, and after a specific number of received syndrome values equal to zero, we expect that the corresponding received data is also error-free. The received data is then used to build the inverse and give an estimate for the encoder state. The method can be used in situations where knowledge of the encoder state helps the decoding process or for synchronization purposes. We analyze the performance of the described method with respect to state estimation error probability and the average time it takes before we can estimate the encoder state with a certain desired reliability.**

***Index Terms*—Convolutional codes, state recovery.**

I. INTRODUCTION

A general convolutional encoder is specified by its $k \times n$ generator matrix G. For minimal encoders, we can derive the delay-free right inverse G^{-1} and the syndrome former H^T, see [1]. We concentrate on rate $R = 1/2$, or $k = 1$ and $n = 2$ convolutional encoders, with a standard constraint length 6 encoder as a working example. Nonsystematic, noncatastrophic encoders are described by the pair of binary polynomials (g_1, g_2). Generally, one assumes that both polynomials have zero delay and maximum degree or constraint length m. The inverse consists of two polynomials, d_1 and d_2 such that $g_1 d_1 + g_2 d_2 = 1$, where all operations are modulo 2. The realization of the inverse is called invertor. The syndrome former is the pair $(g_2, g_1)^T$. Using the delay operator D notation, the encoder input sequence, the encoder output sequence pair and the received channel output sequence pair are given by $X(D)$, $(C_1(D), C_2(D)) = X(D) * G$, and $(R_1(D), R_2(D))$, respectively. The received sequence

$$(R_1(D), R_2(D)) = (C_1(D) + n_1(D), C_2(D) + n_2(D))$$

where $n_i(D)$, $i = 1, 2$ is the channel error sequence that results from a hard-decision detection and a $+$ denotes modulo two addition. Note

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A. J. Han Vinck and P. Dolezal are with the Institute for Experimental Mathematics, University of Essen, 45326, Essen, Germany.
Y.-G. Kim is with KAIST, Taejon, South Korea.
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