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On Decoding Bit Error Probability for Binary Convolutional Codes

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Abstract — **An explanation is given for the paradoxical fact that, at low signal-to-noise ratios, the systematic feedback encoder results in fewer decoding bit errors than does a nonsystematic feedforward encoder for the same code. The analysis identifies a new code property, the *d-distance weight density* of the code. For a given *d*-distance weight density, the decoding bit error probability depends on the number of taps in the realization of the encoder inverse. Among all encoders for a given convolutional code, the systematic one has the simplest encoder inverse and, hence, gives the smallest bit error probability.**

I. Introduction

It is well-known that the free distance of a convolutional code is the principle determiner of the burst error probability (first-event error probability) for large signal-to-noise ratios when maximum-likelihood decoding is used [1]. Since the free distance is a code parameter, the burst error probability is the same whether the convolutional code was encoded by a nonsystematic feedforward encoder or by a systematic feedback encoder. The decoding bit error probability, however, depends on the encoder used. Typically at high signal-to-noise ratios where most of the decoding burst errors are made to codewords at the free distance from the transmitted codeword, the systematic feedback encoder results in more bit errors than a nonsystematic encoder. We now explain the paradoxical fact, often observed in practice, that, at low signal-to-noise ratios, the systematic feedback encoder results in fewer bit errors than does a nonsystematic feedforward encoder.

II. d-distance weight density

Our analysis is based on consideration of what we call the *d-distance weight density* of the code, p_d, and define as the fraction of 1's in the "detours" of weight d in the binary convolutional code. We use this parameter in a model of the internal codeword structure, together with the structure of the encoder inverse, to estimate the number of information bit errors that result from each 1 in a burst error that forms a codeword of weight d. The weights of codewords are code parameters, not encoder parameters, and hence these estimates reveal which convolutional encoders give the best bit error probability performance. At low signal-to-noise ratios, i.e., at code rates close to channel capacity, error bursts are typically very long so that the codewords with weights substantially larger than the free distance of the code primarily determine the decoding bit-error probability.

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Consider all codewords of weight d in a rate $R = b/c$ fixed binary convolutional code. For small d, the number of codewords of weight d, n_d, is also small and the value of p_d fluctuates widely. For larger d, however, the number of codewords of weight d increases rapidly and the value of p_d stabilizes. One finds that p_d tends towards an asymptotic value as d increases. This asymptote is slightly memory dependent. For small memory, m, the asymptotic value is larger than for large m. As m grows, however, p_d quickly decreases to its asymptotic value, which we denote by p_∞. The d-distance weight density, p_d, also depends on the code rate, the lower the rate, the higher the value of p_d. To determine the asymptote p_∞, we analyzed randomly chosen rate $R = b/c$, time-varying binary convolutional codes. We calculated the following values of p_∞ for some interesting rates: $p_\infty = 0.29$ for $R = 1/2$, $p_\infty = 0.37$ for $R = 1/3$ and $p_\infty = 0.40$ for $R = 1/4$.

III. Bit Error Probabilities via Encoder Inverses

We now compare the decoding bit error probability for systematic and nonsystematic encoders. For a particular encoder, let q_d denote the arithmetic average of the number of decoding bit errors per codeword 1 taken over all codewords of weight d. Somewhat surprisingly, q_d turns out to be an affine function of d with a slope that depends on the encoder type. The different slopes can be explained using an argument involving encoder inverses.

We model the appearance of 1's within a codeword of weight d by a binary memoryless source which outputs a 1 with probability p_d. For brevity, we consider here only rate $R = 1/2$ codes. For systematic encoders, whose encoder inverse has only one tap, the average number of bit errors per codeword 1 is then $q_d = 1/2$. This is reasonable since one would expect that half of all codeword 1's occur in the systematic bit-stream and thus create information bit errors. For quick-look-in encoders whose encoder inverses have two taps [2], we obtain $q_d = 1-p_d$. Inserting the asymptote $p_\infty = 0.293$ for $R = 1/2$, we get $q_d = 0.71$. As the number of taps in the encoder inverse increases, q_d increases monotonically to its asymptotic value of 0.85. This explains why the average number of bit errors per codeword at distance d is larger for nonsystematic encoders than for the systematic ones.

References

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