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# IDEALS AND MAXIMAL COMMUTATIVE SUBRINGS OF GRADED RINGS

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Abstract This thesis is mainly concerned with the intersection between ideals and (maximal commutative) subrings of graded rings. The motivation for this investigation originates in the theory of $C^*$ -crossed product algebras associated to topological dynamical systems, where connections between intersection properties of ideals and maximal commutativity of certain subalgebras are well-known. In the last few years, algebraic analogues of these $C^*$ -algebra theorems have been proven by C. Svensson, S. Silvestrov and M. de Jeu for different kinds of skew group algebras arising from actions of the group $Z$ . This raised the question whether or not this could be further generalized to other types of (strongly) graded rings. In this thesis we show that it can indeed be done for many other types of graded rings and actions!  Given any (category) graded ring, there is a canonical subring which is referred to as the <i>neutral component</i> or the <i>coefficient subring</i> . Through this thesis we successively show that for algebraic crossed products, crystalline graded rings, general strongly graded rings and (under some conditions) groupoid crossed products, each nonzero ideal of the ring has a nonzero intersection with the commutant of the center of the neutral component subring. In particular, if the neutral component subring is maximal commutative in the ring this yields that each nonzero ideal of the ring has a nonzero intersection with the neutral component subring.  Not only are ideal intersection properties interesting in their own right, they also play a key role when investigating simplicity of the ring itself. For strongly group graded rings, there is a canonical action such that the grading group acts as automorphisms of certain subrings of the graded ring. By using the previously mentioned ideal intersection properties we are able to relate $G$ -simplicity of these subrings to simplicity of the ring itself. It turns out that maximal commutativity of the subrings plays a key role here! Necessary and sufficient conditions for simplicity of a general skew group ring are not known. In this thesis we resolve this problem for skew group rings with commutative coefficient rings.			
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# IDEALS AND MAXIMAL COMMUTATIVE SUBRINGS OF GRADED RINGS

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*“My conclusion is that there is no reason to believe any of the dogmas of traditional theology and, further, that there is no reason to wish that they were true. Man, in so far as he is not subject to natural forces, is free to work out his own destiny. The responsibility is his, and so is the opportunity.”*

— **Bertrand Russell**, 1952



# Preface

This thesis is based on six papers (A, B, C, D, E and F). In Part I of the thesis, an introduction to the subject and a summary of the results obtained in papers A to F, shall be given. The papers, each of which shall be presented in Part II of the thesis, are the following:

- A. J. Öinert, S. D. Silvestrov, *Commutativity and Ideals in Algebraic Crossed Products*, J. Gen. Lie. T. Appl. **2** (2008), no. 4, 287–302.
- B. J. Öinert, S. D. Silvestrov, *Commutativity and Ideals in Pre-Crystalline Graded Rings*. To appear in Acta Appl. Math.
- C. J. Öinert, S. D. Silvestrov, *Crossed Product-Like and Pre-Crystalline Graded Rings*, Chapter 24 in S. Silvestrov, E. Paal, V. Abramov, A. Stolin (Eds.), *Generalized Lie theory in Mathematics, Physics and Beyond*, pp. 281–296, Springer-Verlag, Berlin, Heidelberg, 2009.
- D. J. Öinert, S. Silvestrov, T. Theohari-Apostolidi, H. Vavatsoulas, *Commutativity and Ideals in Strongly Graded Rings*. To appear in Acta Appl. Math.
- E. J. Öinert, *Simple Group Graded Rings and Maximal Commutativity*. To appear in the Contemp. Math. series of the AMS.
- F. J. Öinert, P. Lundström, *Commutativity and Ideals in Category Crossed Products*. To appear in the Proceedings of the Estonian Academy of Sciences.

In addition to the above, there are two other papers by the author. These, however, will not be included in the thesis:

- J. Öinert, S. D. Silvestrov, *On a Correspondence Between Ideals and Commutativity in Algebraic Crossed Products*, J. Gen. Lie. T. Appl. **2** (2008), no. 3, 216–220.
- J. Öinert, P. Lundström, *Noncrossed Product Matrix Subrings and Ideals of Graded Rings*, Preprints in Mathematical Sciences 2009:10, ISSN 1403-9338, LUTFMA-5112-2009, Centre for Mathematical Sciences, Lund University, 2009.





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Malmö, Midsummer's Eve 2009

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## **Part I**

# **Introduction and summary**





# Chapter 1

## Introduction

In this chapter we shall give a review of the basics of graded ring theory and also describe the background to the problems that we are considering. More specifically, in Section 1.1 we lay out the general theory of (group) graded rings and describe some special cases; skew group rings, twisted group rings, crossed products, strongly graded rings, pre-crystalline graded rings, crystalline graded rings and crossed product-like rings. In Section 1.2 we establish the notation and terminology that we use for the more general situation of category graded rings. In Section 1.3 we describe the background to the research, stemming from  $C^*$ -crossed product algebras associated to topological dynamical systems. Finally, in Chapter 2 we summarize the results which are obtained in papers A, B, C, D, E and F.

### 1.1 Graded rings

Throughout this thesis all rings are assumed to be associative and any ring  $\mathcal{R}$  is unital, with a multiplicative identity  $1_{\mathcal{R}} \in \mathcal{R}$ . We assume that ring homomorphisms are unital, i.e. they respect the multiplicative identities. If  $X$  and  $Y$  are nonempty subsets of a ring  $\mathcal{R}$ , then  $XY$  denotes the set of all finite sums of elements of the form  $xy$  where  $x \in X$  and  $y \in Y$ . The group of multiplication invertible elements of a ring  $\mathcal{R}$  will be denoted by  $U(\mathcal{R})$ .

**Definition 1.1.1** (Graded ring). Let  $M$  be a monoid with neutral element  $e$ . A ring  $\mathcal{R}$  is said to be *graded by  $M$* , or  *$M$ -graded*, if there is a family  $\{\mathcal{R}_s\}_{s \in M}$  of additive subgroups  $\mathcal{R}_s$  of  $\mathcal{R}$  such that

$$\mathcal{R} = \bigoplus_{s \in M} \mathcal{R}_s$$

and

$$\mathcal{R}_s \mathcal{R}_t \subseteq \mathcal{R}_{st} \tag{1.1}$$

for all  $s, t \in M$ . The additive subgroup  $\mathcal{R}_s$  is called the *homogeneous component* of  $\mathcal{R}$  of degree  $s \in M$ . If  $\mathcal{R}$  is an  $M$ -graded ring such that  $\mathcal{R}_s \mathcal{R}_t = \mathcal{R}_{st}$  holds for all  $s, t \in M$ , then we say that  $\mathcal{R}$  is *strongly graded by  $M$*  or *strongly  $M$ -graded*.

The set  $h(\mathcal{R}) = \bigcup_{s \in M} \mathcal{R}_s$  is the set of *homogeneous elements* of  $\mathcal{R}$ . A nonzero element  $x \in \mathcal{R}_s$  is said to be *homogeneous of degree  $s$*  and we write  $\deg(x) = s$ . Each element  $r \in \mathcal{R}$  has a unique decomposition  $r = \sum_{s \in M} r_s$  with  $r_s \in \mathcal{R}_s$  for all  $s \in M$ , and the sum is finite, i.e. almost all  $r_s$  are zero. The *support of  $r$  in  $M$*  is denoted by  $\text{supp}(r) = \{s \in M \mid r_s \neq 0\}$ .

It may happen that  $M$  is in fact a group and when we want to emphasize that a ring  $\mathcal{R}$  is graded by a monoid respectively a group we shall say that it is *monoid graded* respectively *group graded*. Whenever we say that a ring is  $G$ -graded it should be understood that  $G$  is a group with neutral element  $e$ , unless otherwise specified. Similarly, when we say that a ring is  $M$ -graded it should be understood that  $M$  is a monoid with neutral element  $e$ .

**Remark 1.1.2.** Note that any ring  $\mathcal{R}$  is (strongly) group graded by choosing the trivial group  $G = \{e\}$  as grading group and putting  $\mathcal{R}_e = \mathcal{R}$ . This grading is called the *trivial grading* (or *trivial gradation*). We shall mainly be interested in nontrivial grading groups.

Strongly group graded rings have different names in the literature. They are sometimes referred to as *generalized crossed products* (see e.g. [3]) or as *fully graded rings* (see e.g. [4]).

**Remark 1.1.3.** If  $\mathcal{R} = \bigoplus_{s \in M} \mathcal{R}_s$  is an  $M$ -graded ring, then it follows from (1.1) that  $\mathcal{R}_e$  is a subring of  $\mathcal{R}$ . We shall refer to  $\mathcal{R}_e$  as the *neutral component subring* or simply the *neutral component*. Furthermore, for each  $s \in M$ , the homogeneous component  $\mathcal{R}_s$  is an  $\mathcal{R}_e$ -bimodule.

**Proposition 1.1.4.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a  $G$ -graded ring. The following assertions hold:

- (i)  $\mathcal{R}_e$  is a subring of  $\mathcal{R}$  and  $1_{\mathcal{R}} \in \mathcal{R}_e$ .
- (ii) If  $r \in U(\mathcal{R})$  is a homogeneous element of degree  $h \in G$ , then its inverse  $r^{-1}$  is a homogeneous element of degree  $h^{-1}$ .
- (iii)  $\mathcal{R}$  is a strongly  $G$ -graded ring if and only if  $1_{\mathcal{R}} \in \mathcal{R}_g \mathcal{R}_{g^{-1}}$  for each  $g \in G$ .

*Proof.* (i) It is clear that  $\mathcal{R}_e$  is a subring of  $\mathcal{R}$ . To prove that  $1_{\mathcal{R}} \in \mathcal{R}_e$ , let  $1_{\mathcal{R}} = \sum_{g \in G} r_g$  be the decomposition of  $1_{\mathcal{R}}$  with  $r_g \in \mathcal{R}_g$  for each  $g \in G$ . For any  $s_h \in \mathcal{R}_h$ ,  $h \in G$ , we have

$$s_h = s_h 1_{\mathcal{R}} = \sum_{g \in G} s_h r_g$$

and  $s_h r_g \in \mathcal{R}_{hg}$ . Consequently, for each  $g \neq e$  we have  $s_h r_g = 0$  and hence  $s r_g = 0$  for any  $s \in \mathcal{R}$ . In particular, for  $s = 1$  we obtain  $r_g = 0$  for any  $g \neq e$ . Thus,  $1_{\mathcal{R}} = r_e \in \mathcal{R}_e$ .

(ii) Assume that  $r \in U(\mathcal{R}) \cap \mathcal{R}_h$  for some  $h \in G$ . If  $r^{-1} = \sum_{g \in G} (r^{-1})_g$  with  $(r^{-1})_g \in \mathcal{R}_g$ , then  $1_{\mathcal{R}} = r r^{-1} = \sum_{g \in G} r (r^{-1})_g$ . Since  $1_{\mathcal{R}} \in \mathcal{R}_e$  and  $r (r^{-1})_g \in \mathcal{R}_{hg}$  we have that  $r (r^{-1})_g = 0$  for any  $g \neq h^{-1}$ . Since  $r \in U(\mathcal{R})$  we get that  $(r^{-1})_g = 0$  for  $g \neq h^{-1}$ , and therefore  $r^{-1} = (r^{-1})_{h^{-1}} \in \mathcal{R}_{h^{-1}}$ .

(iii) Suppose that  $1_{\mathcal{R}} \in \mathcal{R}_g \mathcal{R}_{g^{-1}}$  for each  $g \in G$ . For each  $g, h \in G$  we have

$$\mathcal{R}_{gh} = 1_{\mathcal{R}} \mathcal{R}_{gh} \subseteq \mathcal{R}_g \mathcal{R}_{g^{-1}} \mathcal{R}_{gh} \subseteq \mathcal{R}_g \mathcal{R}_{g^{-1}gh} = \mathcal{R}_g \mathcal{R}_h \subseteq \mathcal{R}_{gh}$$

which shows that  $\mathcal{R}_{gh} = \mathcal{R}_g \mathcal{R}_h$ , and hence  $\mathcal{R}$  is strongly  $G$ -graded. Conversely, if  $\mathcal{R}$  is strongly  $G$ -graded, then it follows from (i) that  $1_{\mathcal{R}} \in \mathcal{R}_e = \mathcal{R}_g \mathcal{R}_{g^{-1}}$  for each  $g \in G$ .  $\square$

### 1.1.1 The commutant of $\mathcal{R}_e$ in a group graded ring $\mathcal{R}$

Let  $S$  be a subset of a ring  $\mathcal{R}$ . The commutant of  $S$  in  $\mathcal{R}$  is defined to be the set

$$C_{\mathcal{R}}(S) = \{a \in \mathcal{R} \mid ab = ba, \quad \forall b \in S\}$$

and one easily verifies that  $C_{\mathcal{R}}(S)$  is a subring of  $\mathcal{R}$ . The *center* of  $\mathcal{R}$ , i.e.  $C_{\mathcal{R}}(\mathcal{R})$ , is denoted by  $Z(\mathcal{R})$ . If  $S$  is a subring of  $\mathcal{R}$ , then  $Z(S) \subseteq C_{\mathcal{R}}(S)$  always holds and in particular if  $S$  is commutative then  $S \subseteq C_{\mathcal{R}}(S)$ .

**Definition 1.1.5.** A commutative subring  $S$  of a ring  $\mathcal{R}$  is said to be *maximal commutative* in  $\mathcal{R}$  if and only if

$$S = C_{\mathcal{R}}(S).$$

**Remark 1.1.6.** For any  $G$ -graded ring  $\mathcal{R}$ , the commutant of  $\mathcal{R}_e$  in  $\mathcal{R}$  is always a  $G$ -graded ring. Indeed, write  $D_g = C_{\mathcal{R}}(\mathcal{R}_e) \cap \mathcal{R}_g$  for each  $g \in G$ . It is clear that  $C_{\mathcal{R}}(\mathcal{R}_e) = \bigoplus_{g \in G} D_g$  and it is readily verified that  $D_g D_h \subseteq D_{gh}$  for all  $g, h \in G$ .

### 1.1.2 Skew group rings

Let  $\mathcal{R}_0$  be a ring and  $G$  a group. In this construction, the group  $G$  acts as automorphisms of the ring  $\mathcal{R}_0$ , i.e. there is a group homomorphism  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_0)$ . Let  $\{u_g\}_{g \in G}$  be a copy (as a set) of  $G$ . The *skew group ring*  $\mathcal{R}_0 \rtimes_{\sigma} G$  is a free left  $\mathcal{R}_0$ -module with the basis  $\{u_g\}_{g \in G}$ , that is

$$\mathcal{R}_0 \rtimes_{\sigma} G = \left\{ \sum_{g \in G} r_g u_g \mid r_g \in \mathcal{R}_0 \text{ and } r_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

The multiplication on  $\mathcal{R}_0 \rtimes_{\sigma} G$  is defined as the bilinear extension of the rule

$$(a u_g)(b u_h) = a \sigma_g(b) u_{gh}$$

for  $a, b \in \mathcal{R}_0$  and  $g, h \in G$  and one easily verifies that this makes  $\mathcal{R}_0 \rtimes_{\sigma} G$  into a unital and associative ring. The multiplicative identity is given by  $1_{\mathcal{R}_0} u_e$ .

### 1.1.3 Twisted group rings

Let  $\mathcal{R}_0$  be a ring,  $G$  a group and  $\alpha : G \times G \rightarrow U(\mathcal{R}_0)$  a map such that for any triple  $g, h, s \in G$  the following equalities hold:

$$\begin{aligned} \alpha(g, h) \alpha(gh, s) &= \alpha(h, s) \alpha(g, hs) \\ \alpha(g, e) &= \alpha(e, g) = 1_{\mathcal{R}_0} \end{aligned}$$

Let  $\{u_g\}_{g \in G}$  be a copy (as a set) of  $G$ . The *twisted group ring*  $\mathcal{R}_0 \rtimes^\alpha G$  is a free left  $\mathcal{R}_0$ -module with the basis  $\{u_g\}_{g \in G}$ , that is

$$\mathcal{R}_0 \rtimes^\alpha G = \left\{ \sum_{g \in G} r_g u_g \mid r_g \in \mathcal{R}_0 \text{ and } r_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

The multiplication on  $\mathcal{R}_0 \rtimes^\alpha G$  is defined as the bilinear extension of the rule

$$(a u_g)(b u_h) = a b \alpha(g, h) u_{gh}$$

for  $a, b \in \mathcal{R}_0$  and  $g, h \in G$ . One may verify that this multiplication makes  $\mathcal{R}_0 \rtimes^\alpha G$  into a unital and associative ring.

### 1.1.4 Crossed products

A *G-crossed system* is a quadruple  $\{\mathcal{R}_0, G, \sigma, \alpha\}$  consisting of a ring  $\mathcal{R}_0$ , a group  $G$  and two maps  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_0)$  and  $\alpha : G \times G \rightarrow U(\mathcal{R})$ , satisfying the following three conditions for any triple  $g, h, s \in G$  and  $a \in \mathcal{R}_0$ :

- (i)  $\sigma_g(\sigma_h(a)) = \alpha(g, h) \sigma_{gh}(a) \alpha(g, h)^{-1}$
- (ii)  $\alpha(g, h) \alpha(gh, s) = \sigma_g(\alpha(h, s)) \alpha(g, hs)$
- (iii)  $\alpha(g, e) = \alpha(e, g) = 1_{\mathcal{R}_0}$

Let  $\{u_g\}_{g \in G}$  be a copy (as a set) of  $G$ . The *crossed product*  $\mathcal{R}_0 \rtimes_\sigma^\alpha G$  is a free left  $\mathcal{R}_0$ -module with the basis  $\{u_g\}_{g \in G}$ , that is

$$\mathcal{R}_0 \rtimes_\sigma^\alpha G = \left\{ \sum_{g \in G} r_g u_g \mid r_g \in \mathcal{R}_0 \text{ and } r_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

The multiplication on  $\mathcal{R}_0 \rtimes_\sigma^\alpha G$  is defined as the bilinear extension of the rule

$$(a u_g)(b u_h) = a \sigma_g(b) \alpha(g, h) u_{gh}$$

for  $a, b \in \mathcal{R}_0$  and  $g, h \in G$ . This multiplication makes  $\mathcal{R}_0 \rtimes_\sigma^\alpha G$  into a unital and associative ring with multiplicative identity  $1_{\mathcal{R}_0} u_e$ .

We shall now see how the two previous constructions can be obtained as special cases of the crossed product construction.

**Example 1.1.7** (Skew group ring). Let  $\{\mathcal{R}_0, G, \sigma, \alpha\}$  be a  $G$ -crossed system where  $\alpha$  is the trivial map, i.e.  $\alpha(g, h) = 1_{\mathcal{R}_0}$  for all  $g, h \in G$ . First of all, it is easy to see that  $\alpha$  must satisfy conditions (ii) and (iii) of the above. Furthermore, since  $\alpha$  is trivial we see by (i) that  $G$  acts as automorphisms of  $\mathcal{R}_0$ . Hence, the crossed product that is given by this  $G$ -crossed system is a skew group ring.

**Example 1.1.8** (Twisted group ring). Let  $\{\mathcal{R}_0, G, \sigma, \alpha\}$  be a  $G$ -crossed system where  $\sigma$  is the trivial map, i.e.  $\sigma_g = \text{id}_{\mathcal{R}_0}$  for each  $g \in G$ . It is easy to see that the crossed product that is given by this  $G$ -crossed system is a twisted group ring.

We shall now describe another, more abstract, way to look at crossed products. After stating the definition we shall see that it gives rise to exactly the same type of rings as in the concrete construction of crossed products given above.

**Definition 1.1.9** ( $G$ -crossed product). Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a  $G$ -graded ring. If

$$U(\mathcal{R}) \cap \mathcal{R}_g \neq \emptyset$$

for each  $g \in G$ , then  $\mathcal{R}$  is called a  $G$ -crossed product.

Let  $U^{\text{gr}}(\mathcal{R})$  denote the graded units of  $\mathcal{R}$ , i.e.  $U^{\text{gr}}(\mathcal{R}) = \bigcup_{g \in G} (\mathcal{R}_g \cap U(\mathcal{R}))$ . The degree map  $\text{deg} : U^{\text{gr}}(\mathcal{R}) \rightarrow G$  is defined as  $\text{deg}(a) = g$  if and only if  $a \in \mathcal{R}_g$ , for  $g \in G$ . From Definition 1.1.9 it is clear that a  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a  $G$ -crossed product if and only if the following sequence is an exact sequence of groups.

$$1_{\mathcal{R}} \longrightarrow U(\mathcal{R}_e) \xrightarrow{\iota} U^{\text{gr}}(\mathcal{R}) \xrightarrow{\text{deg}} G \longrightarrow e$$

The map  $\iota$  denotes the inclusion. In general, the map  $\text{deg}$  need not be surjective, but for  $G$ -crossed products it is.

**Proposition 1.1.10.** *The ring  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$  (coming from the  $G$ -crossed system  $\{\mathcal{R}_0, G, \sigma, \alpha\}$ ) is  $G$ -graded by  $(\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G)_g = \mathcal{R}_0 u_g$ , for  $g \in G$ , and it is a  $G$ -crossed product.*

*Proof.* From the definition of  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$  we have

$$\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G = \bigoplus_{g \in G} \mathcal{R}_0 u_g$$

and

$$(\mathcal{R}_0 u_g)(\mathcal{R}_0 u_h) = \mathcal{R}_0 u_{gh}$$

for each  $g, h \in G$  and therefore  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$  is a (strongly)  $G$ -graded ring. For each  $g \in G$  we have  $u_g u_{g^{-1}} = \alpha(g, g^{-1}) u_e$  and  $u_{g^{-1}} u_g = \alpha(g^{-1}, g) u_e$ . Since  $\alpha(g, g^{-1})$  and  $\alpha(g^{-1}, g)$  are invertible elements of  $\mathcal{R}_0$ , we conclude that  $u_g$  and  $u_{g^{-1}}$  are invertible elements of  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$ , and this shows that  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$  is a  $G$ -crossed product.  $\square$

**Proposition 1.1.11.** *Every  $G$ -crossed product  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is of the form  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$  for some  $G$ -crossed system  $\{\mathcal{R}_0, G, \sigma, \alpha\}$ .*

*Proof.* Start by putting  $\mathcal{R}_0 = \mathcal{R}_e$ . Since  $\mathcal{R}_g \cap U(\mathcal{R}) \neq \emptyset$  for each  $g \in G$ , we may choose some  $u_g \in \mathcal{R}_g \cap U(\mathcal{R})$  for each  $g \in G$ . We take  $u_e = 1_{\mathcal{R}}$ . Then it is clear that  $\mathcal{R}_g = \mathcal{R}_e u_g = u_g \mathcal{R}_e$ , and that the set  $\{u_g\}_{g \in G}$  is a basis for  $\mathcal{R}$  as a left (and right)  $\mathcal{R}_e$ -module. Let us define the map  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$  by  $\sigma_g(a) = u_g a u_g^{-1}$  for  $g \in G, a \in \mathcal{R}_e$  and  $\alpha : G \times G \rightarrow U(\mathcal{R}_e)$  by  $\alpha(g, h) = u_g u_h u_{gh}^{-1}$  for  $g, h \in G$ . We shall now show that  $\sigma$  and  $\alpha$  satisfy conditions (i), (ii) and (iii) in the definition of a  $G$ -crossed system. Indeed, for any  $g, h \in G$  and  $a \in \mathcal{R}_e$  we get

$$\begin{aligned} \sigma_g(\sigma_h(a)) &= u_g (u_h a u_h^{-1}) u_g^{-1} = u_g u_h u_{gh}^{-1} (u_{gh} a u_{gh}^{-1}) u_{gh} u_h^{-1} u_g^{-1} \\ &= \alpha(g, h) \sigma_{gh}(a) \alpha(g, h)^{-1} \end{aligned}$$

and therefore (i) holds. For any triple  $g, h, s \in G$  we get

$$\begin{aligned} \alpha(g, h) \alpha(gh, s) &= u_g u_h u_{gh}^{-1} u_{gh} u_s u_{ghs}^{-1} \\ &= u_g u_h u_s u_{ghs}^{-1} = u_g u_h u_s u_{hs}^{-1} u_{hs} u_{ghs}^{-1} \\ &= u_g \alpha(h, s) u_g^{-1} u_g u_{hs} u_{ghs}^{-1} = \sigma_g(\alpha(h, s)) \alpha(g, hs) \end{aligned}$$

so (ii) holds too. Since  $u_e = 1_{\mathcal{R}}$  we have

$$\alpha(g, e) = u_g u_e u_g^{-1} = 1_{\mathcal{R}}$$

and similarly we get  $\alpha(e, g) = 1_{\mathcal{R}}$  and therefore (iii) holds. Let  $a \in \mathcal{R}_g$  and  $b \in \mathcal{R}_h$  be homogeneous elements of  $\mathcal{R}$  for  $g, h \in G$ . We compute the product  $ab$  via the maps  $\sigma$  and  $\alpha$ . We have that  $a$  and  $b$  can be uniquely expressed as  $a = a_g u_g$  and  $b = b_h u_h$  for some  $a_g, b_h \in \mathcal{R}_e$ . Then

$$\begin{aligned} ab &= (a_g u_g)(b_h u_h) = a_g (u_g b_h u_g^{-1}) u_g u_h \\ &= a_g (u_g b_h u_g^{-1}) (u_g u_h u_{gh}^{-1}) u_{gh} \\ &= a_g \sigma_g(b_h) \alpha(g, h) u_{gh}. \end{aligned}$$

This entails that the ring  $\mathcal{R}$  is isomorphic to  $\mathcal{R}_0 \rtimes_{\sigma}^{\alpha} G$ . □

### 1.1.5 Strongly graded rings

Each  $G$ -crossed product is a strongly  $G$ -graded ring. Indeed, if  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a  $G$ -crossed product, then for each  $g \in G$  we can find some  $u_g \in U(\mathcal{R}) \cap \mathcal{R}_g$  and by Proposition 1.1.4 (ii) it follows that  $u_g^{-1} \in \mathcal{R}_{g^{-1}}$ . Hence,  $1_{\mathcal{R}} \in \mathcal{R}_g \mathcal{R}_{g^{-1}}$  for each  $g \in G$  and by Proposition 1.1.4 (iii) we conclude that  $\mathcal{R}$  is a strongly  $G$ -graded ring.

The converse is not true in general, i.e. strongly graded rings need not be crossed products, as the following example will show. This explains why strongly group graded rings are sometimes referred to as *generalized crossed products*.

**Example 1.1.12.** Let  $A$  be a ring and  $\mathcal{R} = M_3(A)$  the matrix ring over  $A$ . By putting

$$\mathcal{R}_0 = \begin{pmatrix} A & A & 0 \\ A & A & 0 \\ 0 & 0 & A \end{pmatrix} \quad \text{and} \quad \mathcal{R}_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & A \\ A & A & 0 \end{pmatrix}$$

one may verify that this defines a strong  $\mathbb{Z}_2$ -gradation on  $\mathcal{R}$ . Note that  $\mathcal{R}$  is not a  $\mathbb{Z}_2$ -crossed product with this grading since  $\mathcal{R}_1$  contains only singular matrices.

**Lemma 1.1.13.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $a \in \mathcal{R}$  is such that

$$a \mathcal{R}_g = \{0\} \quad \text{or} \quad \mathcal{R}_g a = \{0\}$$

for some  $g \in G$ , then  $a = 0$ .

*Proof.* Suppose that  $a \mathcal{R}_g = \{0\}$  for some  $g \in G$ ,  $a \in \mathcal{R}$ . We then have  $a \mathcal{R}_g \mathcal{R}_{g^{-1}} = \{0\}$  or equivalently  $a \mathcal{R}_e = \{0\}$ . From the fact that  $1_{\mathcal{R}} \in \mathcal{R}_e$ , we conclude that  $a = 0$ . The other case is treated analogously.  $\square$

From the preceding lemma we see that for a strongly  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  we must always have  $\mathcal{R}_g \neq \{0\}$ , for each  $g \in G$ .

**Remark 1.1.14.** If  $\mathcal{R}$  is a strongly group graded ring which is commutative, then one can easily show that the grading group is necessarily abelian.

**Lemma 1.1.15.** If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a  $G$ -graded ring and  $N$  is a normal subgroup of  $G$ , then  $\mathcal{R}$  can be regarded as a  $G/N$ -graded ring, where the homogeneous components are given by

$$\mathcal{R}_{gN} = \bigoplus_{x \in gN} \mathcal{R}_x$$

for  $gN \in G/N$ . Moreover, if  $\mathcal{R}$  is a crossed product (or strongly graded ring) of  $G$  over  $\mathcal{R}_e$ , then  $\mathcal{R}$  can also be regarded as a crossed product (or strongly  $G/N$ -graded ring) of  $G/N$  over

$$\mathcal{R}_N = \bigoplus_{x \in N} \mathcal{R}_x.$$

*Proof.* Let  $T$  be a transversal for  $N$  in  $G$ . It is obvious that

$$\mathcal{R} = \bigoplus_{t \in T} \mathcal{R}_{tN}.$$

For any  $t_1, t_2 \in T$ , we have

$$\begin{aligned} \mathcal{R}_{t_1N} \mathcal{R}_{t_2N} &= \left( \bigoplus_{x \in t_1N} \mathcal{R}_x \right) \left( \bigoplus_{y \in t_2N} \mathcal{R}_y \right) \subseteq \bigoplus_{\substack{x \in t_1N \\ y \in t_2N}} \mathcal{R}_{xy} \\ &= \bigoplus_{z \in t_1 t_2 N} \mathcal{R}_z = \mathcal{R}_{t_1 t_2 N} \end{aligned}$$



which shows that  $\mathcal{R}$  is  $G/N$ -graded. Assume that  $\mathcal{R}$  is a  $G$ -crossed product. There exists a unit  $u_g \in U(\mathcal{R}) \cap \mathcal{R}_g$  for each  $g \in G$ , and in particular  $u_g \in \mathcal{R}_g \subseteq \mathcal{R}_{gN}$  so  $\mathcal{R}$  is a  $G/N$ -crossed product with the new grading. Assume that  $\mathcal{R}$  is a strongly  $G$ -graded ring, then for each  $t \in T$  we get

$$\begin{aligned} \mathcal{R}_{tN} \mathcal{R}_{t^{-1}N} &= \left( \bigoplus_{x \in tN} \mathcal{R}_x \right) \left( \bigoplus_{y \in t^{-1}N} \mathcal{R}_y \right) = \bigoplus_{\substack{x \in tN \\ y \in t^{-1}N}} \mathcal{R}_{xy} \\ &= \bigoplus_{z \in tt^{-1}N} \mathcal{R}_z = \mathcal{R}_{eN} \supseteq \mathcal{R}_e \ni 1_{\mathcal{R}} \end{aligned}$$

and by Proposition 1.1.4 (iii) we conclude that  $\mathcal{R}$  is a strongly  $G/N$ -graded ring.  $\square$

If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring, then it follows that  $1_{\mathcal{R}} \in \mathcal{R}_e = \mathcal{R}_g \mathcal{R}_{g^{-1}}$  for each  $g \in G$ . Thus, for each  $g \in G$  there exists a positive integer  $n_g$  and elements  $a_g^{(i)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$ , such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}.$$

For every  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ , and in particular for every  $\lambda \in Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$ , and  $g \in G$  we define

$$\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}.$$

**Lemma 1.1.16.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring,  $g \in G$  and write  $\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}$  for some  $n_g > 0$  and  $a_g^{(i)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$ . For each  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  define  $\sigma_g(\lambda)$  by  $\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}$ . The following properties hold:*

(i)  $\sigma_g(\lambda)$  is the unique element of  $\mathcal{R}$  satisfying

$$r_g \lambda = \sigma_g(\lambda) r_g, \quad \forall r_g \in \mathcal{R}_g.$$

Furthermore,  $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$  and if  $\lambda \in Z(\mathcal{R}_e)$ , then  $\sigma_g(\lambda) \in Z(\mathcal{R}_e)$ .

(ii) The group  $G$  acts as automorphisms of the rings  $C_{\mathcal{R}}(\mathcal{R}_e)$  and  $Z(\mathcal{R}_e)$ , with each  $g \in G$  sending any  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  and  $\lambda \in Z(\mathcal{R}_e)$ , respectively, to  $\sigma_g(\lambda)$ .

(iii)  $Z(\mathcal{R}) = \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_g(\lambda) = \lambda, \forall g \in G\}$ , i.e.  $Z(\mathcal{R})$  is the fixed subring  $C_{\mathcal{R}}(\mathcal{R}_e)^G$  of  $C_{\mathcal{R}}(\mathcal{R}_e)$  with respect to the action of  $G$ .

*Proof.* See the proof of Lemma D.4.3 in Paper D.  $\square$

### 1.1.6 Crossed product-like rings

This class of rings were introduced in Paper C in order to generalize pre-crystalline graded rings and allow broader classes of examples to fit in.

**Definition 1.1.17** (Crossed product-like ring). An associative and unital ring  $\mathcal{A}$  is said to be *crossed product-like* if

- There is a monoid  $M$  (with neutral element  $e$ ) and a map  $u : M \rightarrow \mathcal{A}$ ,  $s \mapsto u_s$  such that  $u_e = 1_{\mathcal{A}}$  and  $u_s \neq 0_{\mathcal{A}}$  for every  $s \in M$ .
- There is a subring  $\mathcal{A}_0 \subseteq \mathcal{A}$  containing  $1_{\mathcal{A}}$ .

such that the following conditions are satisfied:

(P1)  $\mathcal{A} = \bigoplus_{s \in M} \mathcal{A}_0 u_s$ ;

(P2) For each  $s \in M$ ,  $u_s \mathcal{A}_0 \subseteq \mathcal{A}_0 u_s$  and  $\mathcal{A}_0 u_s$  is a free left  $\mathcal{A}_0$ -module of rank one;

(P3) The decomposition in P1 makes  $\mathcal{A}$  into an  $M$ -graded ring with  $\mathcal{A}_0 = \mathcal{A}_e$ .

**Lemma 1.1.18.** *With notation and definitions as above:*

- (i) For every  $s \in M$ , there is a set map  $\sigma_s : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  defined by  $u_s a = \sigma_s(a) u_s$  for  $a \in \mathcal{A}_0$ . The map  $\sigma_s$  is additive and multiplicative. Moreover,  $\sigma_e = \text{id}_{\mathcal{A}_0}$ .
- (ii) There is a set map  $\alpha : M \times M \rightarrow \mathcal{A}_0$  defined by  $u_s u_t = \alpha(s, t) u_{st}$  for  $s, t \in M$ . For any triple  $s, t, w \in M$  and  $a \in \mathcal{A}_0$  the following equalities hold:

$$\begin{aligned} \alpha(s, t) \alpha(st, w) &= \sigma_s(\alpha(t, w)) \alpha(s, tw) \\ \sigma_s(\sigma_t(a)) \alpha(s, t) &= \alpha(s, t) \sigma_{st}(a) \end{aligned}$$

- (iii) For every  $s \in M$  we have  $\alpha(s, e) = \alpha(e, s) = 1_{\mathcal{A}_0}$ .

*Proof.* (i) It follows from (P2) that  $\sigma_s$ ,  $s \in M$ , is well-defined. For  $a, b \in \mathcal{A}_0$  we get:

$$\begin{aligned} u_s(ab) &= (u_s a) b \implies \sigma_s(ab) = \sigma_s(a) \sigma_s(b) \\ u_s(a+b) &= u_s a + u_s b \implies \sigma_s(a+b) = \sigma_s(a) + \sigma_s(b) \end{aligned}$$

using that  $\mathcal{A}_0 u_s$  is a free left  $\mathcal{A}_0$ -module with basis  $u_s$  and that  $\mathcal{A}$  is associative. For any  $a \in \mathcal{A}_0$  we have

$$a = u_e a = \sigma_e(a)$$

and hence  $\sigma_e = \text{id}_{\mathcal{A}_0}$ .

- (ii) It follows from (P3) that  $\alpha$  is well-defined. For any triple  $s, t, w \in M$  we have  $(u_s u_t) u_w = u_s (u_t u_w)$ . Hence,

$$\alpha(s, t) \alpha(st, w) u_{stw} = \sigma_s(\alpha(t, w)) \alpha(s, tw) u_{stw}$$

and the claim follows from the fact that  $\mathcal{A}_0 u_{stw}$  is a free left  $\mathcal{A}_0$ -module with basis  $u_{stw}$ . Secondly, for any  $a \in \mathcal{A}_0$  we have  $u_s (u_t a) = (u_s u_t) a$  and this yields

$$\sigma_s(\sigma_t(a)) \alpha(s, t) u_{st} = \alpha(s, t) \sigma_{st}(a) u_{st}.$$

Thus proving the second claim.

- (iii) For any  $s \in M$  we have  $u_s = u_s u_e = \alpha(s, e) u_s$  which yields  $\alpha(s, e) = 1_{\mathcal{A}_0}$ . Analogously we obtain  $\alpha(e, s) = 1_{\mathcal{A}_0}$ . □

By the foregoing lemma we see that, for arbitrary  $a, b \in \mathcal{A}_0$  and  $s, t \in M$ , the product of  $a u_s$  and  $b u_t$  in the crossed product-like ring  $\mathcal{A}$  may be written as

$$(a u_s)(b u_t) = a \sigma_s(b) \alpha(s, t) u_{st}$$

and this is the motivation for the name *crossed product-like*. A crossed product-like ring  $\mathcal{A}$  with the above properties will be denoted by  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ , indicating the maps  $\sigma$  and  $\alpha$ .

**Remark 1.1.19.** Note that for  $s \in M \setminus \{e\}$  we need not necessarily have  $\sigma_s(1_{\mathcal{A}_0}) = 1_{\mathcal{A}_0}$  and hence  $\sigma_s$  need not be a ring morphism.

### 1.1.7 Pre-crystalline and crystalline graded rings

Crystalline graded rings were introduced by E. Nauwelaerts and F. Van Oystaeyen in [28] and have been further studied in [29–32, 34, 39–41].

**Definition 1.1.20** (Pre-crystalline graded ring). A crossed product-like ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  where for each  $s \in M$ ,  $\mathcal{A}_0 u_s = u_s \mathcal{A}_0$ , is said to be a *pre-crystalline graded ring*.

**Lemma 1.1.21.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is a pre-crystalline graded ring, then the following holds:*

- (i) *For every  $s \in M$ , the map  $\sigma_s : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  is a surjective ring morphism.*

- (ii) *If  $M$  is a group, then*

$$\alpha(g, g^{-1}) = \sigma_g(\alpha(g^{-1}, g))$$

*for each  $g \in M$ .*

*Proof:* (i) It follows from Lemma 1.1.18 that  $\sigma_s, s \in M$ , is additive and multiplicative. The surjectivity of  $\sigma_s$  follows from  $\mathcal{A}_0 u_s \subseteq u_s \mathcal{A}_0$ , i.e. for an arbitrary  $b \in \mathcal{A}_0$  we have that  $b u_s = u_s a$  for some  $a \in \mathcal{A}_0$ , thus  $b = \sigma_s(a)$ . In particular the surjectivity implies  $\sigma_s(1_{\mathcal{A}_0}) = 1_{\mathcal{A}_0}$  for each  $s \in M$ .

(ii) By putting  $s = g, t = g^{-1}$  and  $w = g$  in Lemma 1.1.18, we obtain

$$\alpha(g, g^{-1}) \alpha(e, g) = \sigma_g(\alpha(g^{-1}, g)) \alpha(g, e)$$

and by using  $\alpha(e, g) = \alpha(g, e) = 1_{\mathcal{A}_0}$  we conclude that

$$\alpha(g, g^{-1}) = \sigma_g(\alpha(g^{-1}, g))$$

for any  $g \in G$ . □

In a pre-crystalline graded ring, one may show that for  $s, t \in M$ , the  $\alpha(s, t)$  are normalizing elements of  $\mathcal{A}_0$  in the sense that  $\mathcal{A}_0 \alpha(s, t) = \alpha(s, t) \mathcal{A}_0$  (see [28, Proposition 2.3]). If we in addition assume that  $\mathcal{A}_0$  is commutative, then we see by Lemma 1.1.18 that the map  $\sigma : M \rightarrow \text{End}(\mathcal{A}_0)$  is a monoid morphism.

For a pre-crystalline group graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we let  $S(G)$  denote the multiplicative set in  $\mathcal{A}_0$  generated by  $\{\alpha(g, g^{-1}) \mid g \in G\}$  and let  $S(G \times G)$  be the multiplicative set generated by  $\{\alpha(g, h) \mid g, h \in G\}$ . Recall that  $\mathcal{A}_0$  is said to be  $S(G)$ -torsion free if

$$t_{S(G)}(\mathcal{A}_0) = \{a \in \mathcal{A}_0 \mid sa = 0 \text{ for some } s \in S(G)\} = \{0\}.$$

**Lemma 1.1.22** (see [28]). *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline group graded ring, then the following are equivalent:*

- $\mathcal{A}_0$  is  $S(G)$ -torsion free.
- $\mathcal{A}$  is  $S(G)$ -torsion free.
- $\alpha(g, g^{-1})a_0 = 0$  for some  $g \in G$  implies  $a_0 = 0$ .
- $\alpha(g, h)a_0 = 0$  for some  $g, h \in G$  implies  $a_0 = 0$ .
- $\mathcal{A}_0 u_g = u_g \mathcal{A}_0$  is also free as a right  $\mathcal{A}_0$ -module, with basis  $u_g$ , for every  $g \in G$ .
- For every  $g \in G$ ,  $\sigma_g$  is bijective and hence a ring automorphism of  $\mathcal{A}_0$ .

From Lemma 1.1.22 we see that when  $\mathcal{A}_0$  is  $S(G)$ -torsion free in a pre-crystalline group graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we have  $\text{im}(\sigma) \subseteq \text{Aut}(\mathcal{A}_0)$ . We shall now state the definition of a crystalline graded ring.

**Definition 1.1.23** (Crystalline graded ring). A pre-crystalline group graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  which is  $S(G)$ -torsion free is said to be a *crystalline graded ring*.

### 1.1.8 Examples of graded rings

Recall that every ring  $\mathcal{R}$  is (strongly) group graded by the trivial group  $G = \{e\}$  by putting  $\mathcal{R}_e = \mathcal{R}$ . We shall now give examples of some nontrivial monoid gradings and group gradings.

**Example 1.1.24** (Polynomial ring). Let  $A$  be a ring and consider the polynomial ring  $\mathcal{R} = A[X]$  in the indeterminate  $X$ . By putting

- $\mathcal{R}_n = AX^n$ , for  $n \in \mathbb{Z}_{\geq 0}$
- $\mathcal{R}_m = \{0\}$ , for  $m \in \mathbb{Z}_{<0}$

we have defined a  $\mathbb{Z}$ -gradation on  $\mathcal{R}$ . Note, however, that this is not a strong gradation.

In the preceding example, the ring  $\mathcal{R} = A[X]$  may also be regarded as strongly graded by the monoid  $(\mathbb{Z}_{\geq 0}, +)$ .

**Example 1.1.25** (Laurent polynomial ring). Let  $A$  be a ring and consider the Laurent polynomial ring  $\mathcal{R} = A[X, X^{-1}]$  in the indeterminate  $X$ . By putting

- $\mathcal{R}_n = AX^n$ , for  $n \in \mathbb{Z}$

we have defined a  $\mathbb{Z}$ -gradation on  $\mathcal{R}$ , which clearly makes  $\mathcal{R}$  into a  $\mathbb{Z}$ -crossed product.

**Example 1.1.26** (The first Weyl algebra). Let  $\mathcal{R} = \frac{\mathbb{C}\langle x, y \rangle}{(yx - xy - 1)}$ , the so called *first Weyl algebra*. If we put

- $\mathcal{R}_0 = \mathbb{C}[xy]$
- $\mathcal{R}_n = \mathbb{C}[xy]x^n$  for  $n \in \mathbb{Z}_{>0}$
- $\mathcal{R}_m = \mathbb{C}[xy]y^{-m}$  for  $m \in \mathbb{Z}_{<0}$

then one may verify that this defines a  $\mathbb{Z}$ -gradation on  $\mathcal{R}$ .

**Example 1.1.27** (Field extensions). Let  $K \subseteq E$  be a field extension and suppose that  $E = K(\alpha)$ , where  $\alpha$  is algebraic over  $K$ , and has a minimal polynomial of the form  $p(X) = X^n - a$  for some  $a \in K$  and  $n \in \mathbb{Z}_{>0}$ . Then the elements  $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$  form a basis for  $E$  over  $K$ . Thus,

$$E = \bigoplus_{i=0}^{n-1} K \alpha^i$$

and this defines a  $\mathbb{Z}_n$ -gradation on  $E$  with  $E_0 = K$ . Moreover,  $E$  is a crossed product with this gradation.

A special case of Example 1.1.27 is given by the following familiar example.

**Example 1.1.28** (Complex numbers). The field of complex numbers is an extension of the field of real numbers, where the imaginary unit  $i$  is algebraic over  $\mathbb{R}$  with minimal polynomial  $p(X) = X^2 + 1$ . Thus,  $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$  is clearly a  $\mathbb{Z}_2$ -graded ring.

**Example 1.1.29** (The real quaternion algebra). Let  $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$  with multiplication defined by  $i^2 = -1$ ,  $j^2 = -1$  and  $ij = -ji = k$ . This is a 4-dimensional  $\mathbb{R}$ -algebra with center  $\mathbb{R}$ . By putting

$$\deg(1) = (0, 0), \quad \deg(i) = (0, 1), \quad \deg(j) = (1, 0) \quad \text{and} \quad \deg(k) = (1, 1)$$

one may verify that  $\mathbb{H}$  is a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra, which is in fact a crossed product. This means that  $\mathbb{H} = \mathbb{R} \rtimes_{\sigma}^{\alpha} (\mathbb{Z}_2 \times \mathbb{Z}_2)$  and we shall now determine the maps  $\sigma$  and  $\alpha$  explicitly. The coefficient ring  $\mathbb{R}$  is the center of  $\mathbb{H}$  and hence we must have  $\sigma_1 = \sigma_i = \sigma_j = \sigma_k = \text{id}_{\mathbb{R}}$  which means that  $\mathbb{H}$  is a twisted group algebra. The map  $\alpha : (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rightarrow U(\mathbb{R})$  is defined by

$$\begin{array}{cccc} \alpha(1, 1) = 1 & \alpha(1, i) = 1 & \alpha(1, j) = 1 & \alpha(1, k) = 1 \\ \alpha(i, 1) = 1 & \alpha(i, i) = -1 & \alpha(i, j) = 1 & \alpha(i, k) = -1 \\ \alpha(j, 1) = 1 & \alpha(j, i) = -1 & \alpha(j, j) = -1 & \alpha(j, k) = 1 \\ \alpha(k, 1) = 1 & \alpha(k, i) = 1 & \alpha(k, j) = -1 & \alpha(k, k) = -1. \end{array}$$

In the above description, the elements  $1, i, j, k$  are to be identified with their respective group elements in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  corresponding to their degree.

**Remark 1.1.30.** The real quaternion algebra is an example of a so called *division ring* (sometimes also referred to as a *skew field*), i.e. a ring with  $0 \neq 1$  for which every nonzero element has a multiplicative inverse. It was a common belief that all division rings were crossed products, until S. Amitsur [1] constructed the first example of a *noncrossed product division ring* in 1972.

## 1.2 Category and groupoid graded rings

In papers A to E we mainly consider group gradings (and occasionally monoid gradings). In Paper F we consider category gradings which is the most general type of grading. For this reason we include this short section where we shall recall some basics and particularly highlight the difference to the situation of Definition 1.1.1.

Let  $\mathcal{C}$  be a small category. We shall denote by  $\text{ob}(\mathcal{C})$  the set of objects of  $\mathcal{C}$  and by  $\text{mor}(\mathcal{C})$  the set of morphisms of  $\mathcal{C}$ . If  $\alpha$  is a morphism of  $\mathcal{C}$  then we shall indicate this by writing  $\alpha \in \text{mor}(\mathcal{C})$  or simply  $\alpha \in \mathcal{C}$ . The domain and codomain of a morphism  $\alpha$  is denoted by  $d(\alpha)$  and  $c(\alpha)$  respectively. An object  $a$  of  $\mathcal{C}$  will often be identified with its identity morphism  $\text{id}_a$ . The set of identity morphisms is denoted by  $\mathcal{C}_0$ .

Given two objects  $a, b \in \text{ob}(\mathcal{C})$ , the set of morphisms from  $a$  to  $b$  will be denoted by  $\text{hom}_{\mathcal{C}}(a, b)$  or  $\text{hom}(a, b)$ . By  $\mathcal{C}^{(2)}$  we denote the set of all *composable pairs* of morphisms of  $\mathcal{C}$ . This means that  $(\alpha, \beta) \in \mathcal{C}^{(2)}$  if and only if  $d(\alpha) = c(\beta)$ . If  $(\alpha, \beta) \in \mathcal{C}^{(2)}$ , then the composition morphism

$$d(\beta) \xrightarrow{\beta} c(\beta) \xrightarrow{\alpha} c(\alpha)$$

is denoted by  $\alpha\beta$ . We say that a category is *cancellable* if each morphism of the category is both an epimorphism and a monomorphism.

**Definition 1.2.1.** A category is called a *groupoid* if each morphism is an isomorphism.

Obviously, a groupoid is cancellable.

**Definition 1.2.2** (Category graded ring). Let  $\mathcal{C}$  be a category. A ring  $\mathcal{R}$  is said to be graded by  $\mathcal{C}$  if there is a family  $\{\mathcal{R}_s\}_{s \in \mathcal{C}}$  of additive subgroups of  $\mathcal{R}$ , such that

$$\mathcal{R} = \bigoplus_{s \in \mathcal{C}} \mathcal{R}_s \quad \text{and} \quad \mathcal{R}_s \mathcal{R}_t \subseteq \begin{cases} \mathcal{R}_{st} & \text{if } (s, t) \in \mathcal{C}^{(2)} \\ \{0\} & \text{otherwise.} \end{cases}$$

If  $\mathcal{R}_s \mathcal{R}_t = \mathcal{R}_{st}$  holds for all  $(s, t) \in \mathcal{C}^{(2)}$ , then we say that  $\mathcal{R}$  is *strongly graded by  $\mathcal{C}$* . Definition 1.2.2 is a generalization of Definition 1.1.1 from rings graded by a one-object category to rings graded by a general category.

### The category of graded rings

The category of all rings is denoted by **RING**. If  $\mathcal{C}$  is a category, then the category of  $\mathcal{C}$ -graded rings, denoted by  **$\mathcal{C}$ -RING**, is obtained by taking all  $\mathcal{C}$ -graded rings as the objects and for the morphisms between  $\mathcal{C}$ -graded rings  $\mathcal{R} = \bigoplus_{s \in \mathcal{C}} \mathcal{R}_s$  and  $\mathcal{S} = \bigoplus_{t \in \mathcal{C}} \mathcal{S}_t$  we take the ring morphisms  $\varphi : \mathcal{R} \rightarrow \mathcal{S}$  such that  $\varphi(\mathcal{R}_s) \subseteq \mathcal{S}_s$  for every  $s \in \mathcal{C}$ .

### Different gradations on a given ring

Let  $G$  be a group and  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  a  $G$ -graded ring. For any nonempty subset  $X$  of  $G$  we denote

$$\mathcal{R}_X = \bigoplus_{x \in X} \mathcal{R}_x.$$

In particular, if  $H$  is a subgroup of  $G$ , then  $\mathcal{R}_H = \bigoplus_{h \in H} \mathcal{R}_h$  is a subring of  $\mathcal{R}$ , and it is in fact an  $H$ -graded ring. Clearly the correspondence  $\mathcal{R} \mapsto \mathcal{R}_H$  defines a functor  $(\ )_H : G\text{-RING} \rightarrow H\text{-RING}$ . If  $N$  is a normal subgroup of  $G$ , then  $\mathcal{R}$  can be regarded as a  $G/N$ -graded ring (as seen in Lemma 1.1.15) by writing

$$\mathcal{R} = \bigoplus_{gN \in G/N} \left( \bigoplus_{x \in gN} \mathcal{R}_x \right).$$

Therefore  $\mathcal{R}$  has a natural  $G/N$ -grading and we can define a functor  $U_{G/N} : G\text{-RING} \rightarrow G/N\text{-RING}$ , associating to the  $G$ -graded ring  $\mathcal{R}$  the same ring with the  $G/N$ -grading described above. If  $N = G$ , then  $G/G\text{-RING} = \text{RING}$ , and the functor  $U_{G/G}$  is exactly the forgetful functor  $U : G\text{-RING} \rightarrow \text{RING}$ , which associates to the  $G$ -graded ring  $\mathcal{R}$  the underlying ring  $\mathcal{R}$ .

### 1.3 Motivation coming from $C^*$ -algebras

In this section we shall briefly explain how one constructs a  $C^*$ -crossed product algebra from a  $C^*$ -dynamical system. Furthermore, we will show how to associate a transformation group  $C^*$ -algebra to any given topological dynamical system. Finally, we state Theorem 1.3.3 which establishes a connection between the dynamics of a topological dynamical system and algebraic properties of its associated  $C^*$ -crossed product algebra.

#### $C^*$ -dynamical systems and $C^*$ -crossed product algebras

Recall that a  $C^*$ -algebra  $\mathfrak{A}$  is a Banach  $*$ -algebra over  $\mathbb{C}$ , the field of complex numbers, satisfying the so called  $C^*$ -identity, i.e.

$$\|a^*a\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}^2, \quad \forall a \in \mathfrak{A}.$$

**Definition 1.3.1.** A  $C^*$ -dynamical system is a triple  $\{\mathfrak{A}, G, \alpha\}$ , consisting of a  $C^*$ -algebra  $\mathfrak{A}$ , a locally compact group  $G$  and a strongly continuous representation  $\alpha : G \rightarrow \text{Aut}(\mathfrak{A})$ , i.e.  $s_n \rightarrow s$  in  $G$  implies  $\|\alpha_{s_n}(a) - \alpha_s(a)\|_{\mathfrak{A}} \rightarrow 0$  for all  $a \in \mathfrak{A}$ .

To each  $C^*$ -dynamical system one may associate a  $C^*$ -crossed product algebra. We are now going to explain how this can be done for a  $C^*$ -dynamical system  $\{\mathfrak{A}, G, \alpha\}$  by restricting our attention to the case when  $G$  is a countable discrete group and  $\mathfrak{A}$  is a unital  $C^*$ -algebra.

A *covariant representation* of the  $C^*$ -dynamical system, is a pair  $(\pi, V)$  where  $\pi$  is a  $*$ -representation of  $\mathfrak{A}$  on a Hilbert space  $\mathcal{H}$  and  $s \rightarrow V_s$  is a unitary representation of  $G$  on the same space such that

$$V_s \pi(A) V_s^* = \pi(\alpha_s(A)) \quad \text{for all } A \in \mathfrak{A}, s \in G.$$

The space of continuous compactly supported  $\mathfrak{A}$ -valued functions on  $G$  is just the space of all finite formal sums  $f = \sum_{t \in G} A_t u_t$  with coefficients  $A_t \in \mathfrak{A}$ , for  $t \in G$ . Let  $\mathfrak{A} \rtimes_{\alpha} G$  denote the usual algebraic skew group algebra, i.e. the multiplication is defined by the rule  $u_t A u_{t^{-1}} = \alpha_t(A)$  for  $A \in \mathfrak{A}$  and  $t \in G$ . One may define an involution on  $\mathfrak{A} \rtimes_{\alpha} G$  by putting  $u_s^* = u_{s^{-1}}$  for  $s \in G$ . This yields

$$(A u_s)^* = u_s^* A^* = u_{s^{-1}} A^* u_s u_{s^{-1}} = \alpha_s^{-1}(A^*) u_{s^{-1}}$$



for  $A \in \mathfrak{A}$  and  $s \in G$ . Hence, for an arbitrary  $f = \sum_{t \in G} A_t u_t \in \mathfrak{A} \rtimes_{\alpha} G$ , we get

$$f^* = \sum_{t \in G} \alpha_t(A_{t^{-1}}) u_t.$$

The covariant representation  $(\pi, V)$  of  $\{\mathfrak{A}, G, \alpha\}$  yields a  $*$ -representation of  $\{\mathfrak{A}, G, \alpha\}$  by

$$\varphi(f) = \sum_{t \in G} \pi(A_t) V_t$$

for  $f = \sum_{t \in G} A_t u_t \in \mathfrak{A} \rtimes_{\alpha} G$ . Indeed,

$$\begin{aligned} \varphi(f)^* &= \sum_{t \in G} V_t^* \pi(A_t)^* = \sum_{t \in G} V_{t^{-1}} \pi(A_t^*) V_t V_{t^{-1}} \\ &= \sum_{s \in G} \pi(\alpha_s(A_{s^{-1}})) V_s = \varphi(f^*) \end{aligned}$$

and

$$\begin{aligned} \varphi(f)\varphi(g) &= \sum_{t \in G} \sum_{w \in G} \pi(A_t) V_t \pi(B_w) V_w \\ &= \sum_{t \in G} \sum_{w \in G} \pi(A_t) (V_t \pi(B_w) V_t^*) V_t V_w \\ &= \sum_{t \in G} \sum_{w \in G} \pi(A_t) \pi(\alpha_t(B_w)) V_{tw} \\ &= \sum_{s \in G} \left( \sum_{t \in G} \pi(A_t \alpha_t(B_{t^{-1}s})) \right) V_s = \varphi(fg) \end{aligned}$$

for  $f = \sum_{t \in G} A_t u_t$  and  $g = \sum_{w \in G} B_w u_w$  in  $\mathfrak{A} \rtimes_{\alpha} G$ . Conversely, any  $*$ -representation of  $\mathfrak{A} \rtimes_{\alpha} G$  yields a covariant representation of  $\{\mathfrak{A}, G, \alpha\}$  simply by the restrictions

$$\pi(A) = \varphi(Au_e) \text{ and } V_s = \varphi(u_s).$$

for  $A \in \mathfrak{A}$  and  $s \in G$ . Indeed,

$$V_s \pi(A) V_s^* = \varphi(u_s) \varphi(Au_e) \varphi(u_s)^* = \varphi(u_s A u_{s^{-1}}) = \varphi(\alpha_s(A) u_e) = \pi(\alpha_s(A)).$$

We can introduce a norm on  $\mathfrak{A} \rtimes_{\alpha} G$  by

$$\|f\|_1 = \sum_{t \in G} \|A_t\|_{\mathfrak{A}}$$

for  $f = \sum_{t \in G} A_t u_t$ . The completion of  $\mathfrak{A} \rtimes_{\alpha} G$  with respect to this norm is a Banach  $*$ -algebra which we denote by  $\ell^1(\mathfrak{A} \rtimes_{\alpha} G)$ . One can show that there is a natural one-to-one

correspondence between  $*$ -representations of  $\ell^1(\mathfrak{A} \rtimes_\alpha G)$  and covariant representations of  $\{\mathfrak{A}, G, \alpha\}$  (see [2]). The  $C^*$ -crossed product algebra, which we shall denote by  $\mathfrak{A} \rtimes_{\alpha}^{C^*} G$ , is defined as the *enveloping  $C^*$ -algebra* of  $\ell^1(\mathfrak{A} \rtimes_\alpha G)$ . That is, one defines a  $C^*$ -norm by

$$\|f\| = \sup_{\varphi} \|\varphi(f)\|_1$$

and  $\varphi$  runs over all  $*$ -representations of  $\ell^1(\mathfrak{A} \rtimes_\alpha G)$ , which can be shown to be a nonempty family of representations [2, 9].

### $C^*$ -crossed products associated to topological dynamical systems

A *topological dynamical system*  $\Sigma = (X, h)$  is a pair consisting of a compact Hausdorff space  $X$  and a homeomorphism

$$h : X \rightarrow X.$$

**Proposition** (see [33]). Let  $C(X)$  be the algebra of all complex-valued continuous functions on a compact space  $X$ , where the involution on  $C(X)$  is given by pointwise complex conjugation, i.e.  $f^*(x) = \overline{f(x)}$  for all  $x \in X$  and the norm is given by  $\|f\| = \sup_{x \in X} |f(x)|$ . Then  $C(X)$  is a unital  $C^*$ -algebra.

To each topological dynamical system  $\Sigma = (X, h)$ , one may associate a  $C^*$ -dynamical system. Indeed, choose  $\mathfrak{A} = C(X)$  and  $G = (\mathbb{Z}, +)$  (a countable discrete group), and let  $\tilde{h} : \mathbb{Z} \rightarrow \text{Aut}(C(X))$  be the action defined by

$$\tilde{h}_s(f)(x) = f(h^{\circ(-s)}(x)), \quad f \in C(X), x \in X$$

for  $s \in \mathbb{Z}$ . One may verify that  $\tilde{h}$  is a strongly continuous group representation of  $\mathbb{Z}$  in  $\text{Aut}(C(X))$ . This shows that  $(C(X), \mathbb{Z}, \tilde{h})$  is a  $C^*$ -dynamical system and hence we may define its associated  $C^*$ -crossed product.

**Definition 1.3.2.** Let  $\Sigma = (X, h)$  be a topological dynamical system. Then

$$C(X) \rtimes_{\tilde{h}}^{C^*} \mathbb{Z}$$

is called the *transformation group  $C^*$ -algebra* associated to  $\Sigma = (X, h)$ .

Given a topological dynamical system,  $\mathbb{Z}$  acts on  $X$  in an obvious way by taking iterations of  $h$  and  $h^{-1}$ . For this dynamical system we shall now define the following sets:

- (i)  $\text{Per}^n(X, h) := \{x \in X \mid x = h^{\circ(n)}(x)\}$
- (ii)  $\text{Per}_n(X, h) := \text{Per}^n(X, h) \setminus \{\bigcup_{m=1}^{n-1} \text{Per}^m(X, h)\}$

$$(iii) \text{Per}(X, h) := \bigcup_{m \in \mathbb{Z}_{>0}} \text{Per}^m(X, h)$$

$$(iv) \text{Per}^\infty(X, h) := X \setminus \text{Per}(X, h)$$

From this definition we see that  $\text{Per}^n(X, h)$  contains points which have period  $n$  (or fixed points if  $n = 1$ ) or a period of which  $n$  is a multiple. The set  $\text{Per}_n(X, h)$  contains points of exactly period  $n$ . The set  $\text{Per}(X, h)$  contains all points that are periodic or fixed points. Points which belong to  $\text{Per}^\infty(X, h)$  are called *aperiodic points*.

The following theorem can be found in the book [49] by J. Tomiyama.

**Theorem 1.3.3.** *Let  $\Sigma = (X, h)$  be a topological dynamical system. The following three assertions are equivalent:*

(i)  $\text{Per}^\infty(X, h)$  is dense in  $X$ ;

(ii)  $I \cap C(X) \neq \{0\}$ , for any closed nonzero ideal  $I$  of  $C(X) \rtimes_{\tilde{h}}^* \mathbb{Z}$ ;

(iii)  $C(X)$  is a maximal commutative  $C^*$ -subalgebra of  $C(X) \rtimes_{\tilde{h}}^* \mathbb{Z}$ .

In the work of C. Svensson, S. Silvestrov and M. de Jeu [46–48], an algebraic analogue of the above theorem is proven for the  $\mathbb{Z}$ -graded skew group algebra which embeds densely into this  $C^*$ -crossed product algebra. Their work also includes other algebraic generalizations of the above theorem.

## Chapter 2

# Summary of the thesis

For  $C^*$ -crossed product algebras associated to topological dynamical systems, there is a well-known connection between maximal commutativity of a certain commutative  $C^*$ -subalgebra of the  $C^*$ -crossed product algebra, and the way in which ideals intersect this  $C^*$ -subalgebra, as displayed by Theorem 1.3.3 in Section 1.3.

In [46–48], C. Svensson, S. Silvestrov and M. de Jeu prove various analogues of Theorem 1.3.3, for (algebraic) skew group algebras graded by  $(\mathbb{Z}, +)$ . This makes it natural to ask whether or not this can be further generalized to more general classes of graded rings.

Let  $\mathcal{A}$  be a (noncommutative) ring which contains a commutative subring  $\mathcal{A}_0$  such that  $1_{\mathcal{A}} \in \mathcal{A}_0$ . Consider the following two assertions:

**S1:**  $\mathcal{A}_0$  is a maximal commutative subring of  $\mathcal{A}$ .

**S2:**  $I \cap \mathcal{A}_0 \neq \{0\}$ , for each nonzero twosided ideal  $I$  of  $\mathcal{A}$ .

If **S2** is satisfied, then  $\mathcal{A}_0$  is said to have the *ideal intersection property*. The question that we are asking is: When are **S1** and **S2** equivalent? We consider this question for different types of graded rings, where  $\mathcal{A}_0$  is chosen to be the *neutral component* of a graded ring  $\mathcal{A}$ . As we shall see, for almost all the types of graded rings that we consider, the assertion **S1** implies the assertion **S2**. The converse, however, need not always hold.

Ideal intersection properties of this kind are not only interesting in their own right, they also play a key role when describing simplicity of the ring itself. When investigating simplicity of strongly group graded rings, this will be very useful.

Skew group rings have been studied in depth, see e.g. [25], but necessary and sufficient conditions for a skew group ring to be simple are not known. For skew group rings with commutative neutral component, we resolve this problem (see Paper E).

The results in this thesis generalize results from [6, 7, 15, 45–48]. In the following sections we shall give an overview of the results obtained in the papers of Part II and also make a short comment on some further results.

## 2.1 Overview of Paper A

In this paper the focus lies on *algebraic crossed products*. We give an explicit description of the center of a general crossed product ring  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  (Proposition A.3.1) and describe the commutant  $\mathcal{C}_{\mathcal{A} \rtimes_{\alpha}^{\sigma} G}(\mathcal{A})$  of the coefficient ring  $\mathcal{A}$  (Theorem A.4.1). From this we directly obtain necessary and sufficient conditions for maximal commutativity of  $\mathcal{A}$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$

(Corollary A.4.3). We generalize [46, Proposition 2.1] by giving sufficient conditions for  $C_{\mathcal{A} \rtimes_{\alpha}^{\sigma} G}(\mathcal{A})$  to be commutative (Proposition A.4.11). The main theorem of the paper is the following (Theorem A.5.1).

**Theorem.** If  $\mathcal{A}$  is commutative and  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is a  $G$ -crossed product, then

$$I \cap C_{\mathcal{A} \rtimes_{\alpha}^{\sigma} G}(\mathcal{A}) \neq \{0\}$$

for each nonzero twosided ideal  $I$  of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

From the above theorem, we immediately conclude that if the coefficient ring  $\mathcal{A}$  is maximal commutative in the algebraic crossed product  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , then  $\mathcal{A}$  has the ideal intersection property. A method to construct nonzero ideals which have zero intersection with the coefficient ring  $\mathcal{A}$  of a  $G$ -crossed product  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is given by Theorem A.5.4. As a corollary to this result, we obtain the following (Corollary A.5.8).

**Corollary.** Let  $\mathcal{A} \rtimes^{\sigma} G$  be a skew group ring where  $G$  is abelian. If  $\mathcal{A}$  has the ideal intersection property, then  $\ker(\sigma) = \{e\}$ .

This result indicates that for the neutral component of a skew group ring one can still hope to obtain an equivalence between having the ideal intersection property and being maximal commutative. This investigation is continued in Paper B and eventually completed in Paper E. If we, in addition to the above corollary, assume that  $\mathcal{A}$  is an integral domain, then we conclude that if  $\mathcal{A}$  has the ideal intersection property then  $\mathcal{A}$  is maximal commutative in the skew group ring  $\mathcal{A} \rtimes^{\sigma} G$  (Theorem A.5.11). Several results obtained in this paper generalize results in [45–48].

## 2.2 Overview of Paper B

In this paper we turn our focus to *pre-crystalline graded rings* and *skew group rings*. Recall that algebraic crossed products are examples of (pre-) crystalline graded rings, and hence we are now considering a broader class of rings than in Paper A. Generalizing the work in Paper A, we give an explicit description of both the center of a pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  (Proposition B.3.9) and the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  of the neutral component subring  $\mathcal{A}_0$  (Theorem B.3.1). We generalize some of the results obtained in Paper A, and in particular we obtain the following (Corollary B.3.13) which generalizes Theorem A.5.1 in Paper A.

**Corollary.** If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring where  $\mathcal{A}_0$  is commutative, then

$$I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$$

for each nonzero twosided ideal  $I$  of  $\mathcal{A}$ .

The conclusion is that even for a crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , maximal commutativity of the neutral component subring  $\mathcal{A}_0$  implies that it has the ideal intersection property.

Continuing the work on skew group rings from Paper A, we now consider skew group rings  $\mathcal{A}_0 \rtimes_{\sigma} G$  where  $\mathcal{A}_0$  is commutative and  $G$  is a torsion-free abelian group. We show that if  $\mathcal{A}_0$  has the ideal intersection property, then it is maximal commutative in the skew group ring. This is summed up in the following (Theorem B.3.16).

**Theorem.** Let  $\mathcal{A}_0 \rtimes_{\sigma} G$  be a skew group ring. If either of the following two conditions is satisfied:

- (i)  $\mathcal{A}_0$  is an integral domain and  $G$  is an abelian group;
- (ii)  $\mathcal{A}_0$  is commutative and  $G$  is a torsion-free abelian group.

Then  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \rtimes_{\sigma} G$  if and only if  $\mathcal{A}_0$  has the ideal intersection property.

By giving an example of a *twisted group ring* we show that in general the ideal intersection property is not enough to ensure maximal commutativity of the neutral component of a graded ring (Example B.4.2). We also provide sufficient conditions for maximal commutativity of  $\mathcal{A}_0$  to be equivalent to  $\mathcal{A}_0$  having the ideal intersection property in a crystalline graded ring (Theorem B.3.17).

## 2.3 Overview of Paper C

In this paper we introduce *crossed product-like rings* as a class of rings (see Definition C.2.2) containing the pre-crystalline graded rings, and therefore also the crossed product rings, as special examples. These rings share many properties with classical crossed product, but they need not be group graded and they allow more general examples to fit in. We provide explicit descriptions of the center (Proposition C.4.1) and the commutant of the neutral component (Theorem C.3.1) in these rings and give an example of a crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  in which there actually exists a nonzero ideal  $I$  for which  $I \cap C_{\mathcal{A}}(\mathcal{A}_0) = \{0\}$  (Proposition C.5.1). This displays a difference between the group graded and the monoid graded situation. The rest of this paper has a substantial overlap with Paper B.

## 2.4 Overview of Paper D

In this paper we turn our attention to general *strongly graded rings*, not necessarily crossed products. Given a  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  and a subgroup  $H$  of  $G$ , one may consider the restriction of  $\mathcal{R}$  to  $H$ , i.e.  $\mathcal{R}_H = \bigoplus_{h \in H} \mathcal{R}_h$ , which is an  $H$ -graded

subring of  $\mathcal{R}$ . We prove the following theorem which gives an explicit description of the commutant of  $\mathcal{R}_H$  in  $\mathcal{R}$  (Theorem D.4.7) and this generalizes [15, Proposition 1.8 iii].

**Theorem.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring,  $H$  a subgroup of  $G$  and denote  $\mathcal{R}_H = \bigoplus_{h \in H} \mathcal{R}_h$ . If  $\sigma : G \rightarrow \text{Aut}(C_{\mathcal{R}}(\mathcal{R}_e))$  is the action defined in (D.5), then it follows that

$$\begin{aligned} C_{\mathcal{R}}(\mathcal{R}_H) &= \left\{ \lambda = \sum_{g \in G} \lambda_g \in \mathcal{R} \mid \lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e) \cap \mathcal{R}_g, \right. \\ &\quad \left. \sigma_h(\lambda_g) = \lambda_{hg h^{-1}}, \forall g \in G, \forall h \in H \right\} \\ &= \{ \lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_h(\lambda) = \lambda, \forall h \in H \}. \end{aligned}$$

This theorem also generalizes the previous description given in Paper A, where we only considered  $H = \{e\}$  and only for algebraic crossed products, not general strongly graded rings. We also prove the following theorem (Theorem D.4.9).

**Theorem.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is commutative and  $\ker(\sigma)$  is the kernel of the previously defined action  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$ , i.e.  $\ker(\sigma) = \{g \in G \mid \sigma_g(\lambda_e) = \lambda_e, \forall \lambda_e \in \mathcal{R}_e\}$ . If  $H$  is a subgroup of  $G$  which is contained in  $\ker(\sigma) \cap Z(G)$ , then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_H) \neq \{0\}$$

for each nonzero twosided ideal  $I$  of  $\mathcal{R}$ .

As a corollary to this we get the following (Corollary D.4.11) which generalizes Theorem A.5.1 in Paper A, from a  $G$ -crossed product to a general strongly graded ring.

**Corollary.** If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is commutative, then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$$

for each nonzero twosided ideal  $I$  of  $\mathcal{R}$ .

From this we conclude that for strongly group graded rings where the neutral component is maximal commutative, it will also have the ideal intersection property. Finally, we consider crystalline graded rings. Given a subgroup  $H$  of  $G$  we give a description of the commutant of  $\mathcal{A}_H$  in the crystalline graded ring  $\mathcal{A}$  and give sufficient conditions for each nonzero twosided ideal  $I$  of  $\mathcal{A}$  to have a nonzero intersection with  $C_{\mathcal{A}}(\mathcal{A}_H)$  (Theorem D.5.7).

## 2.5 Overview of Paper E

In this paper, the connection between simplicity of group graded rings and  $G$ -simplicity of certain subrings is studied. For this purpose, results about intersections between ideals and certain subrings are very important. We begin by improving some of the results from the previous papers. First we prove the following result (Theorem E.3.1) which generalizes Corollary D.4.11.

**Theorem.** If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring, then

$$I \cap C_{\mathcal{R}}(Z(\mathcal{R}_e)) \neq \{0\}$$

for each nonzero ideal  $I$  of  $\mathcal{R}$ .

An analogue of this theorem is also proven for crystalline graded rings under a certain condition (Theorem E.3.3). We move on to prove the following theorem (Theorem E.3.5) and thereby generalizing Theorem A.5.11, Theorem B.3.16 and several results in [45–48].

**Theorem.** Let  $\mathcal{R} = \mathcal{R}_e \rtimes_{\sigma} G$  be a skew group ring with  $\mathcal{R}_e$  commutative. The following two assertions are equivalent:

- (i)  $\mathcal{R}_e$  is a maximal commutative subring of  $\mathcal{R}$ .
- (ii)  $I \cap \mathcal{R}_e \neq \{0\}$  for each nonzero ideal  $I$  of  $\mathcal{R}$ .

For each strongly group graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  there is a canonical action  $\sigma : G \rightarrow \text{Aut}(C_{\mathcal{R}}(\mathcal{R}_e))$  and the main goal of the second half of this paper is to relate  $G$ -simplicity of  $C_{\mathcal{R}}(\mathcal{R}_e)$  and  $Z(\mathcal{R}_e)$  to simplicity of the graded ring itself. The following theorem gives sufficient conditions for a strongly graded ring to be simple.

**Theorem** (F. Van Oystaeyen, 1984). Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring such that the morphism  $G \rightarrow \text{Pic}(\mathcal{R}_e)$ , defined by  $g \rightarrow [\mathcal{R}_g]$ , is injective. If  $\mathcal{R}_e$  is a simple ring, then  $\mathcal{R}$  is a simple ring.

For a general strongly graded ring with commutative neutral component, we prove the following (Theorem E.6.6) which does not require  $\mathcal{R}_e$  to be simple, and which relates simplicity of the graded ring to  $G$ -simplicity of the commutative neutral component.

**Theorem.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then the following two assertions are equivalent:

- (i)  $\mathcal{R}_e$  is a  $G$ -simple ring (with respect to the canonical action).
- (ii)  $\mathcal{R}$  is a simple ring.



Necessary and sufficient conditions for a general skew group ring to be simple are not known, but for skew group rings with a commutative neutral component the following result (Theorem E.6.13) resolves this problem.

**Theorem.** Let  $\mathcal{R} = \mathcal{R}_e \rtimes_{\sigma} G$  be a skew group ring with  $\mathcal{R}_e$  commutative. The following two assertions are equivalent:

- (i)  $\mathcal{R}_e$  is a maximal commutative subring of  $\mathcal{R}$  and  $\mathcal{R}_e$  is  $G$ -simple.
- (ii)  $\mathcal{R}$  is a simple ring.

The following theorem appears in [7, Theorem 2.2] and is one of the main results in the thesis [6].

**Theorem** (K. Crow, 2005). Suppose  $A$  is a commutative semiprime ring and  $\sigma$  is an action of a group  $G$  on  $A$ . Assume that  $e$  is the only element of  $G$  whose image under  $\sigma$  is the identity on some nonzero ideal of  $A$ . Then  $A \rtimes_{\sigma} G$  is simple if and only if  $A$  is  $G$ -simple.

In Paper E it is pointed out that Theorem E.6.13 is a generalization of Crow's result. To make this clear, we are now going to show explicitly how Crow's result can be retrieved directly from either Theorem E.6.6 or Theorem E.6.13. For a subset  $S$  of a commutative ring  $A$ , the annihilator ideal of  $S$  in  $A$  is defined to be the set  $\text{Ann}_A(S) = \{b \in A \mid sb = 0, \forall s \in S\}$ . The following useful proposition appears in [7, Proposition 2.2 (a)].

**Proposition 2.5.1.** *Let  $A$  be a semiprime ring and  $f$  an automorphism of  $A$ . If  $A$  is commutative, then the following are equivalent:*

- (i)  $f$  is  $X$ -inner.
- (ii)  $\text{Ann}_A((f - \text{id}_A)(A)) \neq \{0\}$ .
- (iii) *There is a nonzero ideal  $I$  in  $A$  so that  $f$  is the identity on  $I$ .*

The following proposition shows that the assumptions made in [7, Theorem 2.2] actually force the coefficient ring to be maximal commutative in the skew group ring and hence both Theorem E.6.6 and Theorem E.6.13 are applicable, thus yielding the desired conclusion.

**Proposition 2.5.2.** *If  $A$  is a commutative semiprime ring and  $\sigma$  is an action of a group  $G$  on  $A$ , such that  $e$  is the only element of  $G$  whose image under  $\sigma$  is the identity on some nonzero ideal of  $A$ , then  $A$  is maximal commutative in  $A \rtimes_{\sigma} G$ .*

*Proof.* Pick an arbitrary pair  $(s, r_s) \in (G \setminus \{e\}) \times (A \setminus \{0\})$ . By Proposition 2.5.1 we conclude that  $\text{Ann}_A(\sigma_s - \text{id}_A)(A) = \{0\}$ . Hence we can pick some  $a \in A$  such that  $\sigma_s(a) - a \notin \text{Ann}(r_s)$ . By Corollary A.4.3 we conclude that  $A$  is maximal commutative in  $A \rtimes_{\sigma} G$ .  $\square$

At the end of the paper we consider the skew group algebra  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  associated to a topological dynamical system  $(X, h)$  and prove the following (Theorem E.7.6).

**Theorem.** If  $(X, h)$  is a topological dynamical system with  $X$  infinite, then the following assertions are equivalent:

- (i)  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  is a simple algebra.
- (ii)  $C(X)$  is maximal commutative in  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  and  $C(X)$  is  $\mathbb{Z}$ -simple.
- (iii)  $(X, h)$  is a minimal dynamical system.

This result is an analogue of a well-known result in the theory of  $C^*$ -algebras and topological dynamics.

## 2.6 Overview of Paper F

In this paper we introduce *category crossed products* as a natural generalization of group graded algebraic crossed products and crystalline graded rings. This class of rings also generalize matrix rings. There is one big difference between the rings considered in this paper, compared to the rings considered in the previous papers. In papers A, B, C, D and E the rings that we study are all group graded (and occasionally monoid graded). However, in this paper the rings are usually graded by a general category and sometimes a groupoid. We are able to generalize most of the results in Paper A to groupoid crossed products. We give an explicit description of the center of a category crossed product (Proposition F.2.4) and also describe the commutant of the neutral component  $A = \bigoplus_{e \in \text{ob}(G)} A_e u_e$  in a category crossed product  $A \rtimes_{\alpha}^{\sigma} G$  (Proposition F.3.1). For a groupoid  $G$  with a finite number of objects, we let  $A$  denote the neutral component subring and show the following (Theorem F.4.1).

**Theorem.** If  $A \rtimes_{\alpha}^{\sigma} G$  is a groupoid crossed product such that for every  $s \in G$ ,  $\alpha(s, s^{-1})$  is not a zero divisor in  $A_{c(s)}$ , then every intersection of a nonzero twosided ideal of  $A \rtimes_{\alpha}^{\sigma} G$  with the commutant of  $Z(A)$  in  $A \rtimes_{\alpha}^{\sigma} G$  is nonzero.

The preceding theorem is a generalization of Theorem A.5.1 in a direction which is different compared to the previous generalizations. We conclude that if the neutral component subring  $A$  is maximal commutative in the groupoid crossed product, then it has the ideal intersection property. In the last part of the paper we provide different ways to construct nonzero ideals of the category crossed product, which have zero intersection with the neutral component subring (Proposition F.4.5, Proposition F.4.6 and Proposition F.4.7). For skew groupoid rings we give sufficient conditions to obtain an equivalence between maximal commutativity of the neutral component  $A$  and the neutral component having the ideal intersection property (Proposition F.4.8). In this paper we also give some examples of category crossed products.

## **2.7 A comment on further results**

In the recent preprint [36] it has actually been shown that Theorem E.3.1, Theorem E.3.3 and Theorem F.4.1 can be simultaneously generalized to general groupoid graded rings with a certain ideal property. Furthermore, Theorem E.3.5 can be generalized to more general types of groupoid graded rings.

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**Part II**

**Scientific papers**







A

## **Paper A**

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## Paper A

# Commutativity and ideals in algebraic crossed products

Johan Öinert and Sergei D. Silvestrov

**Abstract.** *We investigate properties of commutative subrings and ideals in non-commutative algebraic crossed products for actions by arbitrary groups. A description of the commutant of the coefficient subring in the crossed product ring is given. Conditions for commutativity and maximal commutativity of the commutant of the coefficient subring are provided in terms of the action as well as in terms of the intersection of ideals in the crossed product ring with the coefficient subring, specially taking into account both the case of coefficient rings without non-trivial zero-divisors and the case of coefficient rings with non-trivial zero-divisors.*

## A.1 Introduction

The description of commutative subrings and commutative subalgebras and of the ideals in non-commutative rings and algebras are important directions of investigation for any class of non-commutative algebras or rings, because it allows one to relate representation theory, non-commutative properties, graded structures, ideals and subalgebras, homological and other properties of non-commutative algebras to spectral theory, duality, algebraic geometry and topology naturally associated with the commutative subalgebras. In representation theory, for example, one of the keys to the construction and classification of representations is the method of induced representations. The underlying structures behind this method are the semi-direct products or crossed products of rings and algebras by various actions. When a non-commutative ring or algebra is given, one looks for a subring or a subalgebra such that its representations can be studied and classified more easily, and such that the whole ring or algebra can be decomposed as a crossed product of this subring or subalgebra by a suitable action. Then the representations for the subring or subalgebra are extended to representations of the whole ring or algebra using the action and its properties. A description of representations is most tractable for commutative subrings or subalgebras as being, via the spectral theory and duality, directly connected to algebraic geometry, topology or measure theory.

If one has found a way to present a non-commutative ring or algebra as a crossed product of a commutative subring or subalgebra by some action on it of the elements from

outside the subring or subalgebra, then it is important to know whether this subring or subalgebra is maximal abelian or, if not, to find a maximal abelian subring or subalgebra containing the given subalgebra, since if the selected subring or subalgebra is not maximal abelian, then the action will not be entirely responsible for the non-commutative part as one would hope, but will also have the commutative trivial part taking care of the elements commuting with everything in the selected commutative subring or subalgebra. This maximality of a commutative subring or subalgebra and associated properties of the action are intimately related to the description and classifications of representations of the non-commutative ring or algebra.

Little is known in general about connections between properties of the commutative subalgebras of crossed product rings and algebras and properties of the action. A remarkable result in this direction is known, however, in the context of crossed product  $C^*$ -algebras. In the case of the crossed product  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$  of the  $C^*$ -algebra of complex-valued continuous functions on a compact Hausdorff space  $X$  by an action of  $\mathbb{Z}$  via the composition automorphism associated with a homeomorphism  $\sigma : X \rightarrow X$ , it is known that  $C(X)$  sits inside the  $C^*$ -crossed product as a maximal abelian  $C^*$ -subalgebra if and only if for every positive integer  $n$ , the set of points in  $X$  having period  $n$  under iterations of  $\sigma$  has no interior points [26, Theorem 5.4], [25, Corollary 3.3.3], [27, Proposition 4.14], [10, Lemma 7.3.11]. This condition is equivalent to the action of  $\mathbb{Z}$  on  $X$  being topologically free in the sense that the non-periodic points of  $\sigma$  are dense in  $X$ . In [24], a purely algebraic variant of the crossed product allowing for more general classes of algebras than merely continuous functions on compact Hausdorff spaces serving as coefficient algebras in the crossed products was considered. In the general set theoretical framework of a crossed product algebra  $A \rtimes_{\alpha} \mathbb{Z}$  of an arbitrary subalgebra  $A$  of the algebra  $\mathbb{C}^X$  of complex-valued functions on a set  $X$  (under the usual pointwise operations) by  $\mathbb{Z}$  acting on  $A$  via a composition automorphism defined by a bijection of  $X$ , the essence of the matter is revealed. Topological notions are not available here and thus the condition of freeness of the dynamics as described above is not applicable, so that it has to be generalized in a proper way in order to be equivalent to the maximal commutativity of  $A$ . In [24] such a generalization was provided by involving separation properties of  $A$  with respect to the space  $X$  and the action for significantly more arbitrary classes of coefficient algebras and associated spaces and actions. The (unique) maximal abelian subalgebra containing  $A$  was described as well as general results and examples and counterexamples on equivalence of maximal commutativity of  $A$  in the crossed product and the generalization of topological freeness of the action.

In this article, we bring these results and interplay into a more general algebraic context of crossed product rings (or algebras) for crossed systems with arbitrary group actions and twisting cocycle maps [17]. We investigate the connections with the ideal structure of a general crossed product ring, describe the center of crossed product rings, describe the commutant of the coefficient subring in a crossed product ring of a general crossed system, and obtain conditions for maximal commutativity of the commutant of the co-

efficient subring in terms of the action as well as in terms of intersection of ideals in the crossed product ring with the coefficient subring, specially taking into account both the case of coefficient rings without non-trivial zero-divisors and the case of coefficient rings with non-trivial zero-divisors.

## A.2 Preliminaries

In this section we recall the notation from [17], which is necessary for the understanding of the rest of this article. Throughout this article all rings are assumed to be associative rings.

**Definition A.2.1.** Let  $G$  be a group with unit element  $e$ . The ring  $\mathcal{R}$  is  $G$ -graded if there is a family  $\{\mathcal{R}_\sigma\}_{\sigma \in G}$  of additive subgroups  $\mathcal{R}_\sigma$  of  $\mathcal{R}$  such that  $\mathcal{R} = \bigoplus_{\sigma \in G} \mathcal{R}_\sigma$  and  $\mathcal{R}_\sigma \mathcal{R}_\tau \subseteq \mathcal{R}_{\sigma\tau}$  (*strongly  $G$ -graded* if, in addition,  $\supseteq$  also holds) for every  $\sigma, \tau \in G$ .

**Definition A.2.2.** A unital and  $G$ -graded ring  $\mathcal{R}$  is called a  $G$ -crossed product if  $U(\mathcal{R}) \cap \mathcal{R}_\sigma \neq \emptyset$  for every  $\sigma \in G$ , where  $U(\mathcal{R})$  denotes the group of multiplication invertible elements of  $\mathcal{R}$ . Note that every  $G$ -crossed product is strongly  $G$ -graded, as explained in [17, p.2].

**Definition A.2.3.** A  $G$ -crossed system is a quadruple  $\{\mathcal{A}, G, \sigma, \alpha\}$ , consisting of a unital ring  $\mathcal{A}$ , a group  $G$  (with unit element  $e$ ), a map  $\sigma : G \rightarrow \text{Aut}(\mathcal{A})$  and a  $\sigma$ -cocycle map  $\alpha : G \times G \rightarrow U(\mathcal{A})$  such that for any  $x, y, z \in G$  and  $a \in \mathcal{A}$  the following conditions hold:

- (i)  $\sigma_x(\sigma_y(a)) = \alpha(x, y) \sigma_{xy}(a) \alpha(x, y)^{-1}$
- (ii)  $\alpha(x, y) \alpha(xy, z) = \sigma_x(\alpha(y, z)) \alpha(x, yz)$
- (iii)  $\alpha(x, e) = \alpha(e, x) = 1_{\mathcal{A}}$

**Remark A.2.4.** Note that, by combining conditions (i) and (iii), we get  $\sigma_e(\sigma_e(a)) = \sigma_e(a)$  for all  $a \in \mathcal{A}$ . Furthermore,  $\sigma_e : \mathcal{A} \rightarrow \mathcal{A}$  is an automorphism and hence  $\sigma_e = \text{id}_{\mathcal{A}}$ . Also note that, from the definition of  $\text{Aut}(\mathcal{A})$ , we have  $\sigma_g(0_{\mathcal{A}}) = 0_{\mathcal{A}}$  and  $\sigma_g(1_{\mathcal{A}}) = 1_{\mathcal{A}}$  for any  $g \in G$ . From condition (i) it immediately follows that  $\sigma$  is a group homomorphism if  $\mathcal{A}$  is commutative or if  $\alpha$  is trivial.

**Definition A.2.5.** Let  $\overline{G}$  be a copy (as a set) of  $G$ . Given a  $G$ -crossed system  $\{\mathcal{A}, G, \sigma, \alpha\}$ , we denote by  $\mathcal{A} \rtimes_{\alpha}^{\sigma} \overline{G}$  the free left  $\mathcal{A}$ -module having  $\overline{G}$  as its basis and we define a multiplication on this set by

$$(a_1 \overline{x})(a_2 \overline{y}) = a_1 \sigma_x(a_2) \alpha(x, y) \overline{xy} \tag{A.1}$$

for all  $a_1, a_2 \in \mathcal{A}$  and  $x, y \in G$ . Each element of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  may be expressed as a sum  $\sum_{g \in G} a_g \bar{g}$  where  $a_g \in \mathcal{A}$  and  $a_g = 0_{\mathcal{A}}$  for all but a finite number of  $g \in G$ . Explicitly, the addition and multiplication of two arbitrary elements  $\sum_{s \in G} a_s \bar{s}, \sum_{t \in G} b_t \bar{t} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is given by

$$\begin{aligned} \sum_{s \in G} a_s \bar{s} + \sum_{t \in G} b_t \bar{t} &= \sum_{g \in G} (a_g + b_g) \bar{g} \\ \left( \sum_{s \in G} a_s \bar{s} \right) \left( \sum_{t \in G} b_t \bar{t} \right) &= \sum_{(s,t) \in G \times G} (a_s \bar{s})(b_t \bar{t}) = \sum_{(s,t) \in G \times G} a_s \sigma_s(b_t) \alpha(s,t) \overline{st} \\ &= \sum_{g \in G} \left( \sum_{\{(s,t) \in G \times G \mid st=g\}} a_s \sigma_s(b_t) \alpha(s,t) \right) \bar{g} \quad (\text{A.2}) \end{aligned}$$

**Remark A.2.6.** The ring  $\mathcal{A}$  is unital, with unit element  $1_{\mathcal{A}}$ , and it is easy to see that  $(1_{\mathcal{A}} \bar{e})$  is the multiplicative identity in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

By abuse of notation, we shall sometimes let 0 denote the zero element in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  and sometimes the unit element in the abelian group  $(\mathbb{Z}, +)$ . The proofs of the two following propositions can be found in [17, Proposition 1.4.1, p.11] and [17, Proposition 1.4.2, pp.12-13] respectively (see also [18], [19]).

**Proposition A.2.7.** *Let  $\{\mathcal{A}, G, \sigma, \alpha\}$  be a  $G$ -crossed system. Then  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is an associative ring (with the multiplication defined in (A.1)). Moreover, this ring is  $G$ -graded,  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G = \bigoplus_{g \in G} \mathcal{A} \bar{g}$ , and it is a  $G$ -crossed product.*

**Proposition A.2.8.** *Every  $G$ -crossed product  $\mathcal{R}$  is of the form  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  for some ring  $\mathcal{A}$  and some maps  $\sigma, \alpha$ .*

**Remark A.2.9.** If  $k$  is a field and  $\mathcal{A}$  is a  $k$ -algebra, then so is  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

The coefficient ring  $\mathcal{A}$  is naturally embedded as a subring into  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  via the canonical isomorphism  $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  defined by  $a \mapsto a \bar{e}$ . We denote by  $\tilde{\mathcal{A}}$  the image of  $\mathcal{A}$  under  $\iota$  and by  $\mathcal{A}^G = \{a \in \mathcal{A} \mid \sigma_s(a) = a, \forall s \in G\}$  the *fixed ring* of  $\mathcal{A}$ . If  $\mathcal{A}$  is commutative we define  $\text{Ann}(r) = \{c \in \mathcal{A} \mid r \cdot c = 0_{\mathcal{A}}\}$  for  $r \in \mathcal{A}$ .

**Remark A.2.10.** Obviously,  $\mathcal{A}$  is commutative if and only if  $\tilde{\mathcal{A}}$  is commutative.

**Example A.2.11.** Let  $\mathcal{A}$  be commutative and  $\mathcal{B} = \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  a crossed product. For  $x \in G$  and  $c, d \in \mathcal{A}$  we may write

$$(c \bar{x})(d \bar{e}) = c \sigma_x(d) \bar{x} = (\sigma_x(d) \bar{e})(c \bar{x})$$

Let  $b = c \bar{x}$ ,  $a = d \bar{e}$  and  $f : \mathcal{B} \rightarrow \mathcal{B}$  be a map defined by  $f = \iota \circ \sigma_x \circ \iota^{-1}$ . Then the above relation may be written as  $ba = f(a)b$ , which is a re-ordering formula frequently appearing in physical applications.

### A.3 Commutativity in $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$

From the definition of the product in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , given by (A.2), we see that two elements  $\sum_{s \in G} a_s \bar{s}$  and  $\sum_{t \in G} b_t \bar{t}$  commute if and only if

$$\sum_{\{(s,t) \in G \times G | st=g\}} a_s \sigma_s(b_t) \alpha(s,t) = \sum_{\{(s,t) \in G \times G | st=g\}} b_s \sigma_s(a_t) \alpha(s,t) \quad (\text{A.3})$$

for each  $g \in G$ . The crossed product  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is in general non-commutative and in the following proposition we give a description of its center.

**Proposition A.3.1.** *The center of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G) = \left\{ \sum_{g \in G} r_g \bar{g} \mid \begin{aligned} & r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t), \\ & r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}, (s, t) \in G \times G \end{aligned} \right\}$$

*Proof.* Let  $\sum_{g \in G} r_g \bar{g} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be an element which commutes with every element of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ . Then, in particular  $\sum_{g \in G} r_g \bar{g}$  must commute with  $a \bar{s}$  for every  $a \in \mathcal{A}$ . From (A.3) we immediately see that this implies  $r_s \sigma_s(a) = a r_s$  for every  $a \in \mathcal{A}$  and  $s \in G$ . Furthermore,  $\sum_{g \in G} r_g \bar{g}$  must commute with  $1_{\mathcal{A}} \bar{s}$  for any  $s \in G$ . This yields

$$\begin{aligned} \sum_{t \in G} r_{ts^{-1}} \alpha(ts^{-1}, s) \bar{t} &= \sum_{g \in G} r_g \alpha(g, s) \bar{g} s = \sum_{g \in G} r_g \sigma_g(1_{\mathcal{A}}) \alpha(g, s) \bar{g} s \\ &= \left( \sum_{g \in G} r_g \bar{g} \right) (1_{\mathcal{A}} \bar{s}) = (1_{\mathcal{A}} \bar{s}) \left( \sum_{g \in G} r_g \bar{g} \right) \\ &= \sum_{g \in G} 1_{\mathcal{A}} \sigma_s(r_g) \alpha(s, g) \bar{s} g = \sum_{g \in G} \sigma_s(r_g) \alpha(s, g) \bar{s} g \\ &= \sum_{t \in G} \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t) \bar{t} \end{aligned}$$

and hence, for each  $(s, t) \in G \times G$ , we have  $r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t)$ .

Conversely, suppose that  $\sum_{g \in G} r_g \bar{g} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is an element satisfying  $r_s \sigma_s(a) = a r_s$  and  $r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t)$  for every  $a \in \mathcal{A}$  and  $(s, t) \in G \times G$ .



Let  $\sum_{s \in G} a_s \bar{s} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be arbitrary. Then

$$\begin{aligned}
 \left( \sum_{g \in G} r_g \bar{g} \right) \left( \sum_{s \in G} a_s \bar{s} \right) &= \sum_{(g,s) \in G \times G} r_g \sigma_g(a_s) \alpha(g, s) \bar{g} \bar{s} \\
 &= \sum_{(g,s) \in G \times G} a_s r_g \alpha(g, s) \bar{g} \bar{s} \\
 &= \sum_{(t,s) \in G \times G} a_s (r_{ts^{-1}} \alpha(ts^{-1}, s)) \bar{t} \\
 &= \sum_{(t,s) \in G \times G} a_s \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t) \bar{t} \\
 &= \sum_{(g,s) \in G \times G} a_s \sigma_s(r_g) \alpha(s, g) \bar{s} \bar{g} \\
 &= \left( \sum_{s \in G} a_s \bar{s} \right) \left( \sum_{g \in G} r_g \bar{g} \right)
 \end{aligned}$$

and hence  $\sum_{g \in G} r_g \bar{g}$  commutes with every element of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .  $\square$

A few corollaries follow from Proposition A.3.1, showing how a successive addition of restrictions on the corresponding  $G$ -crossed system, leads to a simplified description of  $Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G)$ .

**Corollary A.3.2** (Center of a twisted group ring). *If  $\sigma \equiv \text{id}_{\mathcal{A}}$ , then the center of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G) = \left\{ \sum_{g \in G} r_g \bar{g} \mid r_s \in Z(\mathcal{A}), \quad r_{ts^{-1}} \alpha(ts^{-1}, s) = r_{s^{-1}t} \alpha(s, s^{-1}t), \right. \\
 \left. \forall a \in \mathcal{A}, (s, t) \in G \times G \right\}$$

**Corollary A.3.3.** *If  $G$  is abelian and  $\alpha$  is symmetric<sup>1</sup>, then the center of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G) = \left\{ \sum_{g \in G} r_g \bar{g} \mid r_s \sigma_s(a) = a r_s, \quad r_s \in \mathcal{A}^G, \quad \forall a \in \mathcal{A}, s \in G \right\}$$

**Corollary A.3.4.** *If  $\mathcal{A}$  is commutative,  $G$  is abelian and  $\alpha \equiv 1_{\mathcal{A}}$ , then the center of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G) = \left\{ \sum_{g \in G} r_g \bar{g} \mid r_s \in \mathcal{A}^G, \quad \sigma_s(a) - a \in \text{Ann}(r_s), \quad \forall a \in \mathcal{A}, s \in G \right\}$$

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<sup>1</sup>Symmetric in the sense that  $\alpha(x, y) = \alpha(y, x)$  for every  $(x, y) \in G \times G$ .

**Remark A.3.5.** Note that in the proof of Theorem A.3.1, the property that the image of  $\alpha$  is contained in  $U(\mathcal{A})$  is not used and therefore the theorem is true in greater generality. Consider the case when  $\mathcal{A}$  is an integral domain and let  $\alpha$  take its values in  $\mathcal{A} \setminus \{0_{\mathcal{A}}\}$ . In this case it is clear that  $r_s \sigma_s(a) = a r_s$  for all  $a \in \mathcal{A} \iff r_s(\sigma_s(a) - a) = 0$  for all  $a \in \mathcal{A} \iff r_s = 0$  for  $s \notin \sigma^{-1}(\text{id}_{\mathcal{A}}) = \{g \in G \mid \sigma_g = \text{id}_{\mathcal{A}}\}$ . After a change of variable via  $x = s^{-1}t$  the first condition in the description of the center may be written as  $\sigma_s(r_x) \alpha(s, x) = r_{sxs^{-1}} \alpha(sxs^{-1}, s)$  for all  $(s, x) \in G \times G$ . From this relation we conclude that  $r_x = 0$  if and only if  $r_{sxs^{-1}} = 0$ , and hence it is trivially satisfied if we put  $r_x = 0$  whenever  $x \notin \sigma^{-1}(\text{id}_{\mathcal{A}})$ . This case has been presented in [19, Proposition 2.2] with a more elaborate proof.

The final corollary describes the exceptional situation when  $Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G)$  coincides with  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , that is when  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is commutative.

**Corollary A.3.6.**  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is commutative if and only if all of the following hold:

- (i)  $\mathcal{A}$  is commutative
- (ii)  $\sigma_s = \text{id}_{\mathcal{A}}$  for each  $s \in G$
- (iii)  $G$  is abelian
- (iv)  $\alpha$  is symmetric

*Proof.* Suppose that  $Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G) = \mathcal{A} \rtimes_{\alpha}^{\sigma} G$ . Then,  $\tilde{\mathcal{A}} \subseteq \mathcal{A} \rtimes_{\alpha}^{\sigma} G = Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G)$  and hence (i) follows by Remark A.2.10. By assumption,  $1_{\mathcal{A}} \bar{s} \in Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G)$  for any  $s \in G$  and by Proposition A.3.1 we see that  $\sigma_s = \text{id}_{\mathcal{A}}$  for every  $s \in G$ , and hence (ii). For any  $(x, y) \in G \times G$  we have  $\alpha(x, y) \overline{xy} = (1_{\mathcal{A}} \bar{x})(1_{\mathcal{A}} \bar{y}) = (1_{\mathcal{A}} \bar{y})(1_{\mathcal{A}} \bar{x}) = \alpha(y, x) \overline{yx}$ , but  $\alpha(x, y) \neq 0_{\mathcal{A}}$  which implies  $xy = yx$  and also  $\alpha(x, y) = \alpha(y, x)$ , which shows (iii) and (iv). The converse implication is easily verified.  $\square$

## A.4 The commutant of $\tilde{\mathcal{A}}$ in $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$

From now on we shall assume that  $G \neq \{e\}$ . As we have seen,  $\tilde{\mathcal{A}}$  is a subring of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  and we define its commutant by  $\text{Comm}(\tilde{\mathcal{A}}) = \{b \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G \mid ab = ba, \forall a \in \tilde{\mathcal{A}}\}$ . Theorem A.4.1 tells us exactly when an element of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  lies in  $\text{Comm}(\tilde{\mathcal{A}})$ .

**Theorem A.4.1.** *The commutant of  $\tilde{\mathcal{A}}$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$\text{Comm}(\tilde{\mathcal{A}}) = \left\{ \sum_{s \in G} r_s \bar{s} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G \mid r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}, s \in G \right\}$$

*Proof.* The proof is established through the following sequence of equivalences:

$$\begin{aligned}
 \sum_{s \in G} r_s \bar{s} \in \text{Comm}(\tilde{\mathcal{A}}) &\iff \left( \sum_{s \in G} r_s \bar{s} \right) (a \bar{e}) = (a \bar{e}) \left( \sum_{s \in G} r_s \bar{s} \right), \quad \forall a \in \mathcal{A} \\
 &\iff \sum_{s \in G} r_s \sigma_s(a) \alpha(s, e) \bar{s} \bar{e} \\
 &= \sum_{s \in G} a \sigma_e(r_s) \alpha(e, s) \bar{e} \bar{s}, \quad \forall a \in \mathcal{A} \\
 &\iff \sum_{s \in G} r_s \sigma_s(a) \bar{s} = \sum_{s \in G} a r_s \bar{s}, \quad \forall a \in \mathcal{A} \\
 &\iff \text{For each } s \in G : r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}
 \end{aligned}$$

Here we have used the fact that  $\alpha(s, e) = \alpha(e, s) = 1_{\mathcal{A}}$  for all  $s \in G$ . The above equivalence can also be deduced directly from (A.3).  $\square$

When  $\mathcal{A}$  is commutative we get the following description of the commutant by Theorem A.4.1.

**Corollary A.4.2.** *If  $\mathcal{A}$  is commutative, then the commutant of  $\tilde{\mathcal{A}}$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$\text{Comm}(\tilde{\mathcal{A}}) = \left\{ \sum_{s \in G} r_s \bar{s} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G \mid \sigma_s(a) - a \in \text{Ann}(r_s), \quad \forall a \in \mathcal{A}, s \in G \right\}$$

When  $\mathcal{A}$  is commutative it is clear that  $\tilde{\mathcal{A}} \subseteq \text{Comm}(\tilde{\mathcal{A}})$ . Using the explicit description of  $\text{Comm}(\tilde{\mathcal{A}})$  in Corollary A.4.2, we are now able to state exactly when  $\tilde{\mathcal{A}}$  is maximal commutative, i.e.  $\text{Comm}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$ .

**Corollary A.4.3.** *Let  $\mathcal{A}$  be commutative.  $\tilde{\mathcal{A}}$  is maximal commutative in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  if and only if, for each pair  $(s, r_s) \in (G \setminus \{e\}) \times (\mathcal{A} \setminus \{0_{\mathcal{A}}\})$ , there exists  $a \in \mathcal{A}$  such that  $\sigma_s(a) - a \notin \text{Ann}(r_s)$ .*

**Example A.4.4** (The crossed product associated to a dynamical system). In this example we follow the notation of [24]. Let  $\sigma : X \rightarrow X$  be a bijection on a non-empty set  $X$ , and  $A \subseteq \mathbb{C}^X$  an algebra of functions, such that if  $h \in A$  then  $h \circ \sigma \in A$  and  $h \circ \sigma^{-1} \in A$ . Let  $\tilde{\sigma} : \mathbb{Z} \rightarrow \text{Aut}(A)$  be defined by  $\tilde{\sigma}_n : f \mapsto f \circ \sigma^{\circ(-n)}$  for  $f \in A$ . We now have a  $\mathbb{Z}$ -crossed system (with trivial  $\tilde{\sigma}$ -cocycle) and we may form the crossed product  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . Recall the definition of the set  $\text{Sep}_A^n(X) = \{x \in X \mid \exists h \in A, \text{ s.t. } h(x) \neq (\tilde{\sigma}_n(h))(x)\}$ . Corollary A.4.3 is a generalization of [24, Theorem 3.5] and the easiest way to see this is by negating the statements. Suppose that  $A$  is not maximal commutative in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . Then, by Corollary A.4.3, there exists a pair  $(n, f_n) \in (\mathbb{Z} \setminus \{0\}) \times (A \setminus \{0\})$  such that  $\tilde{\sigma}_n(g) - g \in \text{Ann}(f_n)$  for every  $g \in A$ , i.e.  $\text{supp}(\tilde{\sigma}_n(g) - g) \cap \text{supp}(f_n) = \emptyset$

for every  $g \in A$ . In particular, this means that  $f_n$  is identically zero on  $\text{Sep}_{\mathcal{A}}^n(X)$ . However,  $f_n \in A \setminus \{0\}$  is not identically zero on  $X$  and hence  $\text{Sep}_{\mathcal{A}}^n(X)$  is not a *domain of uniqueness* (as defined in [24, Definition 3.2]). The converse can be proved similarly.

**Corollary A.4.5.** *Let  $\mathcal{A}$  be commutative. If for each  $s \in G \setminus \{e\}$  it is always possible to find some  $a \in \mathcal{A}$  such that  $\sigma_s(a) - a$  is not a zero-divisor in  $\mathcal{A}$ , then  $\tilde{\mathcal{A}}$  is maximal commutative in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .*

The next corollary is a consequence of Corollary A.4.3 and shows how maximal commutativity of the coefficient ring in the crossed product has an impact on the non-triviality of the action  $\sigma$ .

**Corollary A.4.6.** *If  $\tilde{\mathcal{A}}$  is maximal commutative in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , then  $\sigma_g \neq \text{id}_{\mathcal{A}}$  for every  $g \in G \setminus \{e\}$ .*

The description of the commutant  $\text{Comm}(\tilde{\mathcal{A}})$  from Corollary A.4.2 can be further refined in the case when  $\mathcal{A}$  is an integral domain.

**Corollary A.4.7.** *If  $\mathcal{A}$  is an integral domain<sup>2</sup>, then the commutant of  $\tilde{\mathcal{A}}$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is*

$$\text{Comm}(\tilde{\mathcal{A}}) = \left\{ \sum_{s \in \sigma^{-1}(\text{id}_{\mathcal{A}})} r_s \bar{s} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G \mid r_s \in \mathcal{A} \right\}$$

where  $\sigma^{-1}(\text{id}_{\mathcal{A}}) = \{g \in G \mid \sigma_g = \text{id}_{\mathcal{A}}\}$ .

**Corollary A.4.8.** *Let  $\mathcal{A}$  be an integral domain.  $\tilde{\mathcal{A}}$  is maximal commutative in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  if and only if  $\sigma_g \neq \text{id}_{\mathcal{A}}$  for every  $g \in G \setminus \{e\}$ .*

Corollary A.4.8 can be derived directly from Corollary A.4.6 together with either Corollary A.4.5 or A.4.7.

**Remark A.4.9.** Recall that when  $\mathcal{A}$  is commutative,  $\sigma$  is a group homomorphism. Thus, to say that  $\sigma_g \neq \text{id}_{\mathcal{A}}$  for all  $g \in G \setminus \{e\}$  is another way of saying that  $\ker(\sigma) = \{e\}$ , i.e.  $\sigma$  is injective.

**Example A.4.10.** Let  $\mathcal{A} = \mathbb{C}[x_1, \dots, x_n]$  be the polynomial ring in  $n$  commuting variables  $x_1, \dots, x_n$  and  $G = S_n$  the symmetric group on  $n$  elements. An element  $\tau \in S_n$  is a permutation which maps the sequence  $(1, \dots, n)$  into  $(\tau(1), \dots, \tau(n))$ . The group  $S_n$  acts on  $\mathbb{C}[x_1, \dots, x_n]$  in a natural way. To each  $\tau \in S_n$  we may associate a map  $\mathcal{A} \rightarrow \mathcal{A}$ , which sends any polynomial  $f(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$  into a new polynomial  $g$ , defined by  $g(x_1, \dots, x_n) = f(x_{\tau(1)}, \dots, x_{\tau(n)})$ . It is clear that each such mapping is a ring automorphism on  $\mathcal{A}$ . Let  $\sigma$  be the embedding  $S_n \hookrightarrow \text{Aut}(\mathcal{A})$  and  $\alpha \equiv 1_{\mathcal{A}}$ . Note that  $\mathbb{C}[x_1, \dots, x_n]$  is an integral domain and that  $\sigma$  is injective. Hence, by Corollary A.4.8 and Remark A.4.9 it is clear that the embedding of  $\mathbb{C}[x_1, \dots, x_n]$  is maximal commutative in  $\mathbb{C}[x_1, \dots, x_n] \rtimes_{\alpha}^{\sigma} S_n$ .

<sup>2</sup>By an *integral domain* we shall mean a commutative ring with an additive identity  $0_{\mathcal{A}}$  and a multiplicative identity  $1_{\mathcal{A}}$  such that  $0_{\mathcal{A}} \neq 1_{\mathcal{A}}$ , in which the product of any two non-zero elements is always non-zero.

One might want to describe properties of the  $\sigma$ -cocycle in the case when  $\tilde{\mathcal{A}}$  is maximal commutative, but unfortunately this will lead to a dead end. The explanation for this is revealed by condition (iii) in the definition of a  $G$ -crossed system, where we see that  $\alpha(e, g) = \alpha(g, e) = 1_{\mathcal{A}}$  for all  $g \in G$  and hence we are not able to extract any interesting information about  $\alpha$  by assuming that  $\tilde{\mathcal{A}}$  is maximal commutative. Also note that in a *twisted group ring*  $\mathcal{A} \rtimes_{\alpha} G$ , i.e. with  $\sigma \equiv \text{id}_{\mathcal{A}}$ ,  $\tilde{\mathcal{A}}$  can never be maximal commutative (when  $G \neq \{e\}$ ), since for each  $g \in G$ ,  $\bar{g}$  centralizes  $\tilde{\mathcal{A}}$ . If  $\mathcal{A}$  is commutative, then this follows immediately from Corollary A.4.6. We shall now give a sufficient condition for  $\text{Comm}(\tilde{\mathcal{A}})$  to be commutative.

**Proposition A.4.11.** *If  $\mathcal{A}$  is a commutative ring,  $G$  is an abelian group and  $\alpha$  is symmetric, then  $\text{Comm}(\tilde{\mathcal{A}})$  is commutative.*

*Proof.* Let  $\sum_{s \in G} r_s \bar{s}$  and  $\sum_{t \in G} p_t \bar{t}$  be arbitrary elements of  $\text{Comm}(\tilde{\mathcal{A}})$ . By our assumptions and Corollary A.4.2 we get

$$\begin{aligned} \left( \sum_{s \in G} r_s \bar{s} \right) \left( \sum_{t \in G} p_t \bar{t} \right) &= \sum_{(s,t) \in G \times G} r_s \sigma_s(p_t) \alpha(s, t) \bar{s} \bar{t} \\ &= \sum_{(s,t) \in G \times G} r_s p_t \alpha(s, t) \bar{s} \bar{t} \\ &= \sum_{(s,t) \in G \times G} p_t \sigma_t(r_s) \alpha(t, s) \bar{t} \bar{s} \\ &= \left( \sum_{t \in G} p_t \bar{t} \right) \left( \sum_{s \in G} r_s \bar{s} \right) \end{aligned}$$

This shows that  $\text{Comm}(\tilde{\mathcal{A}})$  is commutative. □

This proposition is a generalization of [24, Proposition 2.1] from a function algebra to an arbitrary unital associative commutative ring  $\mathcal{A}$ , from  $\mathbb{Z}$  to an arbitrary abelian group  $G$  and from a trivial to a possibly non-trivial symmetric  $\sigma$ -cocycle  $\alpha$ .

**Remark A.4.12.** By using Proposition A.4.11 and the arguments made in the previous example on *the crossed product associated to a dynamical system* it is clear that Corollary A.4.2 is a generalization of [24, Theorem 3.3]. Furthermore, we see that Corollary A.3.4 is a generalization of [24, Theorem 3.6].

## A.5 Ideals in $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$

In this section we describe properties of the ideals in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  in connection with maximal commutativity and properties of the action  $\sigma$ .

**Theorem A.5.1.** *If  $\mathcal{A}$  is commutative, then*

$$I \cap \text{Comm}(\tilde{\mathcal{A}}) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

*Proof:* Let  $\mathcal{A}$  be commutative. Then  $\tilde{\mathcal{A}}$  is also commutative. Let  $I \subseteq \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be an arbitrary non-zero two-sided ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

*Part 1:*

For each  $g \in G$  we define a map  $T_g : \mathcal{A} \rtimes_{\alpha}^{\sigma} G \rightarrow \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  by  $\sum_{s \in G} a_s \bar{s} \mapsto (\sum_{s \in G} a_s \bar{s}) \bar{g}$ . Note that, for any  $g \in G$ ,  $I$  is invariant<sup>3</sup> under  $T_g$ . We have

$$T_g \left( \sum_{s \in G} a_s \bar{s} \right) = \left( \sum_{s \in G} a_s \bar{s} \right) \bar{g} = \sum_{s \in G} a_s \alpha(s, g) \overline{sg}$$

for every  $g \in G$ . It is important to note that if  $a_s \neq 0_{\mathcal{A}}$ , then  $a_s \alpha(s, g) \neq 0_{\mathcal{A}}$  and hence this operation does not kill coefficients, it only translates and deforms them. If we have a non-zero element  $\sum_{s \in G} a_s \bar{s}$  for which  $a_e = 0_{\mathcal{A}}$ , then we may pick some non-zero coefficient, say  $a_p$  and apply the map  $T_{p^{-1}}$  to end up with

$$T_{p^{-1}} \left( \sum_{s \in G} a_s \bar{s} \right) = \sum_{s \in G} a_s \alpha(s, p^{-1}) \overline{sp^{-1}} = \sum_{t \in G} d_t \bar{t}$$

This resulting element will then have the following properties:

- $d_e = a_p \alpha(p, p^{-1}) \neq 0_{\mathcal{A}}$
- $\#\{s \in G \mid a_s \neq 0_{\mathcal{A}}\} = \#\{t \in G \mid d_t \neq 0_{\mathcal{A}}\}$

*Part 2:*

For each  $a \in \mathcal{A}$  we define a map  $D_a : \mathcal{A} \rtimes_{\alpha}^{\sigma} G \rightarrow \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  by

$$\sum_{s \in G} a_s \bar{s} \mapsto (a \bar{e}) \left( \sum_{s \in G} a_s \bar{s} \right) - \left( \sum_{s \in G} a_s \bar{s} \right) (a \bar{e})$$

Note that, for each  $a \in \mathcal{A}$ ,  $I$  is invariant under  $D_a$ . By assumption  $\mathcal{A}$  is commutative

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<sup>3</sup>By *invariant* we mean that the set is *closed* under this operation.

and hence the above expression can be simplified.

$$\begin{aligned}
 D_a \left( \sum_{s \in G} a_s \bar{s} \right) &= (a \bar{e}) \left( \sum_{s \in G} a_s \bar{s} \right) - \left( \sum_{s \in G} a_s \bar{s} \right) (a \bar{e}) \\
 &= \left( \sum_{s \in G} a \sigma_e(a_s) \alpha(e, s) \bar{e} \bar{s} \right) - \left( \sum_{s \in G} a_s \sigma_s(a) \alpha(s, e) \bar{s} \bar{e} \right) \\
 &= \sum_{s \in G} \underbrace{a a_s}_{=a_s a} \bar{s} - \sum_{s \in G} a_s \sigma_s(a) \bar{s} = \sum_{s \in G} a_s (a - \sigma_s(a)) \bar{s} \\
 &= \sum_{s \neq e} a_s (a - \sigma_s(a)) \bar{s} = \sum_{s \neq e} d_s \bar{s}
 \end{aligned}$$

The maps  $\{D_a\}_{a \in \mathcal{A}}$  all share the property that they kill the coefficient in front  $\bar{e}$ . Hence, if  $a_e \neq 0_{\mathcal{A}}$ , then the number of non-zero coefficients of the resulting element will always be reduced by at least one. Note that  $\text{Comm}(\tilde{\mathcal{A}}) = \bigcap_{a \in \mathcal{A}} \ker(D_a)$ . This means that for each non-zero  $\sum_{s \in G} a_s \bar{s}$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G \setminus \text{Comm}(\tilde{\mathcal{A}})$  we may always choose some  $a \in \mathcal{A}$  such that  $\sum_{s \in G} a_s \bar{s} \notin \ker(D_a)$ . By choosing such an  $a$  we note that, using the same notation as above, we get

$$\#\{s \in G \mid a_s \neq 0_{\mathcal{A}}\} \geq \#\{s \in G \mid d_s \neq 0_{\mathcal{A}}\} \geq 1$$

for each non-zero  $\sum_{s \in G} a_s \bar{s} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G \setminus \text{Comm}(\tilde{\mathcal{A}})$ .

*Part 3:*

The ideal  $I$  is assumed to be non-zero, which means that we can pick some non-zero element  $\sum_{s \in G} r_s \bar{s} \in I$ . If  $\sum_{s \in G} r_s \bar{s} \in \text{Comm}(\tilde{\mathcal{A}})$ , then we are finished, so assume that this is not the case. Note that  $r_s \neq 0_{\mathcal{A}}$  for finitely many  $s \in G$ . Recall that the ideal  $I$  is invariant under  $T_g$  and  $D_a$  for all  $g \in G$  and  $a \in \mathcal{A}$ . We may now use the maps  $\{T_g\}_{g \in G}$  and  $\{D_a\}_{a \in \mathcal{A}}$  to generate new elements of  $I$ . More specifically, we may use the  $T_g$ 's to translate our element  $\sum_{s \in G} r_s \bar{s}$  into a new element which has a non-zero coefficient in front of  $\bar{e}$  (if needed) after which we use the map  $D_a$  to kill this coefficient and end up with yet another new element of  $I$  which is non-zero but has a smaller number of non-zero coefficients. We may repeat this procedure and in a finite number of iterations arrive at an element of  $I$  which lies in  $\text{Comm}(\tilde{\mathcal{A}}) \setminus \tilde{\mathcal{A}}$  and if not we continue the above procedure until we reach an element which is of the form  $b \bar{e}$  with some non-zero  $b \in \mathcal{A}$ . In particular  $\tilde{\mathcal{A}} \subseteq \text{Comm}(\tilde{\mathcal{A}})$  and hence  $I \cap \text{Comm}(\tilde{\mathcal{A}}) \neq \{0\}$ .  $\square$

The embedded coefficient ring  $\tilde{\mathcal{A}}$  is maximal commutative if and only if  $\tilde{\mathcal{A}} = \text{Comm}(\tilde{\mathcal{A}})$  and hence we have the following corollary.

**Corollary A.5.2.** *If the subring  $\tilde{\mathcal{A}}$  is maximal commutative in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , then*

$$I \cap \tilde{\mathcal{A}} \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

**Proposition A.5.3.** *Let  $I$  be a subset of  $\mathcal{A}$  and define*

$$J = \left\{ \sum_{s \in G} a_s \bar{s} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G \mid a_s \in I \right\}$$

The following assertions hold:

- (i) *If  $I$  is a right ideal in  $\mathcal{A}$ , then  $J$  is a right ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$*
- (ii) *If  $I$  is a two-sided ideal in  $\mathcal{A}$  such that  $I \subseteq \mathcal{A}^G$ , then  $J$  is a two-sided ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$*

*Proof.* If  $I$  is a (possibly one-sided) ideal in  $\mathcal{A}$ , then  $J$  is an additive subgroup of  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ .

(i). Let  $I$  be a right ideal in  $\mathcal{A}$ . Then

$$\left( \sum_{s \in G} a_s \bar{s} \right) \left( \sum_{t \in G} b_t \bar{t} \right) = \sum_{(s,t) \in G \times G} \underbrace{a_s \sigma_s(b_t) \alpha(s,t)}_{\in I} \bar{st} \in J$$

for arbitrary  $\sum_{s \in G} a_s \bar{s} \in J$  and  $\sum_{t \in G} b_t \bar{t} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  and hence  $J$  is a right ideal.

(ii). Let  $I$  be a two-sided ideal in  $\mathcal{A}$  such that  $I \subseteq \mathcal{A}^G$ . By (i) it is clear that  $J$  is a right ideal. Let  $\sum_{s \in G} a_s \bar{s} \in J$  and  $\sum_{t \in G} b_t \bar{t} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be arbitrary. Then

$$\begin{aligned} \left( \sum_{t \in G} b_t \bar{t} \right) \left( \sum_{s \in G} a_s \bar{s} \right) &= \sum_{(t,s) \in G \times G} b_t \sigma_t(a_s) \alpha(t,s) \bar{ts} \\ &= \sum_{(t,s) \in G \times G} \underbrace{b_t a_s \alpha(t,s)}_{\in I} \bar{ts} \in J \end{aligned}$$

which shows that  $J$  is also a left ideal.  $\square$

**Theorem A.5.4.** *Let  $\sigma : G \rightarrow \text{Aut}(\mathcal{A})$  be a group homomorphism and  $N$  be a normal subgroup of  $G$ , contained in  $\sigma^{-1}(\text{id}_{\mathcal{A}}) = \{g \in G \mid \sigma_g = \text{id}_{\mathcal{A}}\}$ . Let  $\varphi : G \rightarrow G/N$  be the quotient group homomorphism and suppose that  $\alpha$  is such that  $\alpha(s,t) = 1_{\mathcal{A}}$  whenever  $s \in N$  or  $t \in N$ . Furthermore, suppose that there exists a map  $\beta : G/N \times G/N \rightarrow U(\mathcal{A})$  such that  $\beta(\varphi(s), \varphi(t)) = \alpha(s,t)$  for each  $(s,t) \in G \times G$ . If  $I$  is an ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  generated by an element  $\sum_{s \in N} a_s \bar{s}$  for which the coefficients (of which all but finitely many are zero) satisfy  $\sum_{s \in N} a_s = 0_{\mathcal{A}}$ , then*

$$I \cap \tilde{\mathcal{A}} = \{0\}$$



*Proof.* Let  $I \subseteq \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be the ideal generated by an element  $\sum_{s \in N} a_s \bar{s}$ , which satisfies  $\sum_{s \in N} a_s = 0_{\mathcal{A}}$ . The quotient homomorphism  $\varphi : G \rightarrow G/N$ ,  $s \mapsto sN$  satisfies  $\ker(\varphi) = N$ . By assumption, the map  $\sigma$  is a group homomorphism and  $\sigma(N) = \text{id}_{\mathcal{A}}$ . Hence by the universal property, see for example [7, p.16], there exists a unique group homomorphism  $\rho$  making the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & \text{Aut}(\mathcal{A}) \\ \varphi \downarrow & \nearrow \rho & \\ G/N & & \end{array}$$

By assumption there exists  $\beta$  such that  $\beta(\varphi(s), \varphi(t)) = \alpha(s, t)$  for each  $(s, t) \in G \times G$ . One may verify that  $\beta$  is a  $\rho$ -cocycle and hence we can define a new crossed product  $\mathcal{A} \rtimes_{\beta}^{\rho} G/N$ . Let  $T$  be a transversal to  $N$  in  $G$  and define  $\Gamma$  to be the map

$$\Gamma : \mathcal{A} \rtimes_{\alpha}^{\sigma} G \rightarrow \mathcal{A} \rtimes_{\beta}^{\rho} G/N, \quad \sum_{s \in G} a_s \bar{s} \mapsto \sum_{t \in T} \left( \sum_{s \in tN} a_s \right) \overline{tN}$$

which is a ring homomorphism. Indeed,  $\Gamma$  is clearly additive and due to the assumptions, for any two elements  $\sum_{g \in G} a_g \bar{g}$  and  $\sum_{h \in G} b_h \bar{h}$  in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , the multiplicativity of  $\Gamma$  follows by

$$\begin{aligned} & \Gamma \left( \sum_{g \in G} a_g \bar{g} \right) \Gamma \left( \sum_{h \in G} b_h \bar{h} \right) \\ &= \left( \sum_{s \in T} \left( \sum_{g \in sN} a_g \right) \overline{sN} \right) \left( \sum_{t \in T} \left( \sum_{h \in tN} b_h \right) \overline{tN} \right) \\ &= \sum_{q \in T} \left( \sum_{\substack{\{(s,t) \in T \times T\} \\ sNtN=qN}} \left( \left( \sum_{g \in sN} a_g \right) \rho_{sN} \left( \sum_{h \in tN} b_h \right) \beta(sN, tN) \right) \right) \overline{qN} \\ &= \sum_{q \in T} \left( \sum_{\substack{\{(s,t) \in T \times T\} \\ sNtN=qN}} \left( \sum_{(g,h) \in sN \times tN} a_g \rho_{sN}(b_h) \beta(sN, tN) \right) \right) \overline{qN} \end{aligned}$$

$$\begin{aligned}
&= \sum_{q \in T} \left( \sum_{\substack{\{(s,t) \in T \times T\} \\ sNtN=qN}} \left( \sum_{(g,h) \in sN \times tN} a_g \rho_{sN}(b_h) \beta(gN, hN) \right) \right) \overline{qN} \\
&= \sum_{q \in T} \left( \sum_{\substack{\{(g,h) \in G \times G\} \\ gNhN=qN}} a_g \rho_{gN}(b_h) \beta(gN, hN) \right) \overline{qN} \\
&= \sum_{q \in T} \left( \sum_{\substack{\{(g,h) \in G \times G\} \\ ghN=qN}} a_g \sigma_g(b_h) \alpha(g, h) \right) \overline{qN} \\
&= \sum_{q \in T} \left( \sum_{p \in qN} \left( \sum_{\substack{\{(g,h) \in G \times G\} \\ gh=p}} a_g \sigma_g(b_h) \alpha(g, h) \right) \right) \overline{qN} \\
&= \Gamma \left( \sum_{p \in G} \left( \sum_{\substack{\{(g,h) \in G \times G\} \\ gh=p}} a_g \sigma_g(b_h) \alpha(g, h) \right) \overline{p} \right) \\
&= \Gamma \left( \left( \sum_{g \in G} a_g \overline{g} \right) \left( \sum_{h \in G} b_h \overline{h} \right) \right)
\end{aligned}$$

and hence  $\Gamma$  defines a ring homomorphism. We shall note that the generator of  $I$  is mapped onto zero, i.e.

$$\Gamma \left( \sum_{s \in N} a_s \overline{s} \right) = \left( \sum_{s \in N} a_s \right) \overline{N} = 0_{\mathcal{A}} \overline{N} = 0$$

and hence  $\Gamma|_I \equiv 0$ . Furthermore, we see that

$$\Gamma(b\overline{e}) = 0 \implies b\overline{N} = 0 \iff b = 0_{\mathcal{A}} \iff b\overline{e} = 0$$

and hence  $\Gamma|_{\tilde{\mathcal{A}}}$  is injective. We may now conclude that if  $c \in I \cap \tilde{\mathcal{A}}$ , then  $\Gamma(c) = 0$  and so necessarily  $c = 0$ . This shows that  $I \cap \tilde{\mathcal{A}} = \{0\}$ .  $\square$

If  $\mathcal{A}$  is commutative, then  $\sigma$  is automatically a group homomorphism and we get the following.

**Corollary A.5.5.** *Let  $\mathcal{A}$  be commutative and  $N \subseteq \sigma^{-1}(\text{id}_{\mathcal{A}}) = \{g \in G \mid \sigma_g = \text{id}_{\mathcal{A}}\}$  a normal subgroup of  $G$ . Let  $\varphi : G \rightarrow G/N$  be the quotient group homomorphism and suppose that  $\alpha$  is such that  $\alpha(s, t) = 1_{\mathcal{A}}$  whenever  $s \in N$  or  $t \in N$ . Furthermore, suppose that there exists a map  $\beta : G/N \times G/N \rightarrow U(\mathcal{A})$  such that  $\beta(\varphi(s), \varphi(t)) = \alpha(s, t)$  for each  $(s, t) \in G \times G$ . If  $I$  is an ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  generated by an element  $\sum_{s \in N} a_s \bar{s}$  for which the coefficients (of which all but finitely many are zero) satisfy  $\sum_{s \in N} a_s = 0_{\mathcal{A}}$ , then  $I \cap \tilde{\mathcal{A}} = \{0\}$ .*

When  $\alpha \equiv 1_{\mathcal{A}}$  we need not assume that  $\mathcal{A}$  is commutative, in order to make  $\sigma$  a group homomorphism. In this case we may choose  $\beta \equiv 1_{\mathcal{A}}$  and by Theorem A.5.4 we have the following corollaries.

**Corollary A.5.6.** *Let  $\alpha \equiv 1_{\mathcal{A}}$  and  $N \subseteq \sigma^{-1}(\text{id}_{\mathcal{A}}) = \{g \in G \mid \sigma_g = \text{id}_{\mathcal{A}}\}$  be a normal subgroup of  $G$ . If  $I$  is an ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  generated by an element  $\sum_{s \in N} a_s \bar{s}$  for which the coefficients (of which all but finitely many are zero) satisfy  $\sum_{s \in N} a_s = 0_{\mathcal{A}}$ , then  $I \cap \tilde{\mathcal{A}} = \{0\}$ .*

**Corollary A.5.7.** *If  $\alpha \equiv 1_{\mathcal{A}}$ , then the following implication holds:*

- $$\begin{aligned} & \text{(i) } Z(G) \cap \sigma^{-1}(\text{id}_{\mathcal{A}}) \neq \{e\} \\ & \quad \downarrow \\ & \text{(ii) For each } g \in Z(G) \cap \sigma^{-1}(\text{id}_{\mathcal{A}}), \text{ the ideal } I_g \text{ generated by the element } \sum_{n \in \mathbb{Z}} a_n \bar{g}^n \\ & \quad \text{for which } \sum_{n \in \mathbb{Z}} a_n = 0_{\mathcal{A}} \text{ has the property } I_g \cap \tilde{\mathcal{A}} = \{0\} \end{aligned}$$

*Proof.* Suppose that there exists a  $g \in (Z(G) \cap \sigma^{-1}(\text{id}_{\mathcal{A}})) \setminus \{e\}$ . Let  $I_g \subseteq \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be the ideal generated by  $\sum_{n \in \mathbb{Z}} a_n \bar{g}^n$ , where  $\sum_{n \in \mathbb{Z}} a_n = 0_{\mathcal{A}}$ . The element  $g$  commutes with each element of  $G$  and hence the cyclic subgroup  $N = \langle g \rangle$  generated by  $g$  is normal in  $G$  and since  $\sigma$  is a group homomorphism  $N \subseteq \sigma^{-1}(\text{id}_{\mathcal{A}})$ . Hence  $I_g \cap \tilde{\mathcal{A}} = \{0\}$  by Corollary A.5.6.  $\square$

**Corollary A.5.8.** *If  $\alpha \equiv 1_{\mathcal{A}}$  and  $G$  is abelian, then the following implication holds:*

- $$\begin{aligned} & \text{(i): } I \cap \tilde{\mathcal{A}} \neq \{0\}, \text{ for every non-zero two-sided ideal } I \text{ in } \mathcal{A} \rtimes_{\alpha}^{\sigma} G \\ & \quad \downarrow \\ & \text{(ii): } \sigma_g \neq \text{id}_{\mathcal{A}} \text{ for all } g \in G \setminus \{e\} \end{aligned}$$

*Proof by contraposition.* Since  $G$  is abelian,  $G = Z(G)$ . Suppose that (ii) is false, i.e. there exists  $g \in G \setminus \{e\}$  such that  $\sigma_g = \text{id}_{\mathcal{A}}$ . Pick such a  $g$  and let  $I_g \subseteq \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  be the ideal generated by  $1_{\mathcal{A}} \bar{e} - 1_{\mathcal{A}} \bar{g}$ . Then obviously  $I_g \neq \{0\}$  and by Corollary A.5.7 we get  $I_g \cap \tilde{\mathcal{A}} = \{0\}$  and hence (i) is false. This concludes the proof.  $\square$

**Example A.5.9.** We should note that in the proof of Corollary A.5.8 one could have chosen the ideal in many different ways. The ideal generated by  $1_{\mathcal{A}} \bar{e} - 1_{\mathcal{A}} \bar{g} + 1_{\mathcal{A}} \bar{g}^2 - 1_{\mathcal{A}} \bar{g}^3 + \dots + 1_{\mathcal{A}} \bar{g}^{2n} - 1_{\mathcal{A}} \bar{g}^{2n+1} = (1_{\mathcal{A}} \bar{e} - 1_{\mathcal{A}} \bar{g}) \sum_{k=0}^n 1_{\mathcal{A}} \bar{g}^{2k}$  is contained in

the ideal  $I_g$ , generated by  $1_{\mathcal{A}}\bar{e} - 1_{\mathcal{A}}\bar{g}$ , and therefore it has zero intersection with  $\tilde{\mathcal{A}}$  if  $I_g \cap \tilde{\mathcal{A}} = \{0\}$ . Also note that for  $\alpha \equiv 1_{\mathcal{A}}$  we may always write

$$1_{\mathcal{A}}\bar{e} - 1_{\mathcal{A}}\bar{g}^n = (1_{\mathcal{A}}\bar{e} - 1_{\mathcal{A}}\bar{g}) \left( \sum_{k=0}^{n-1} 1_{\mathcal{A}}\bar{g}^k \right)$$

and hence  $1_{\mathcal{A}}\bar{e} - 1_{\mathcal{A}}\bar{g}$  is a zero-divisor in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  whenever  $g$  is a torsion element.

**Example A.5.10.** We now give an example of how one may choose  $\beta$  as in Theorem A.5.4. Let  $N \subseteq \sigma^{-1}(\text{id}_{\mathcal{A}})$  be a normal subgroup of  $G$  such that for  $g \in N$ ,  $\alpha(s, g) = 1_{\mathcal{A}}$  for all  $s \in G$  and let  $\alpha$  be symmetric. Since  $\alpha$  is the  $\sigma$ -cocycle map of a  $G$ -system, we get

$$\begin{aligned} \alpha(g, s) \alpha(gs, t) = \sigma_g(\alpha(s, t)) \alpha(g, st) &\iff \alpha(g, s) \alpha(gs, t) = \alpha(s, t) \alpha(g, st) \\ &\iff \alpha(gs, t) = \alpha(s, t) \end{aligned}$$

for all  $(s, t) \in G \times G$ . Using the last equality and the symmetry of  $\alpha$  we immediately see that

$$\alpha(gs, ht) = \alpha(s, t) \quad \forall s, t \in G$$

for all  $g, h \in N$ . The last equality means that  $\alpha$  is constant on the pairs of right cosets which coincide with the left cosets by normality of  $N$ . It is therefore clear that we can define

$$\beta : G/N \times G/N \rightarrow \text{Aut}(\mathcal{A}) \text{ by } \beta(\varphi(s), \varphi(t)) = \alpha(s, t) \text{ for } s, t \in G.$$

**Theorem A.5.11.** *If  $\mathcal{A}$  is an integral domain,  $G$  is an abelian group and  $\alpha \equiv 1_{\mathcal{A}}$ , then the following implication holds:*

$$\begin{aligned} \text{(i): } I \cap \tilde{\mathcal{A}} \neq \{0\}, \text{ for every non-zero two-sided ideal } I \text{ in } \mathcal{A} \rtimes_{\alpha}^{\sigma} G \\ \downarrow \\ \text{(ii): } \tilde{\mathcal{A}} \text{ is a maximal commutative subring in } \mathcal{A} \rtimes_{\alpha}^{\sigma} G \end{aligned}$$

*Proof.* This follows from Corollary A.4.8 and Corollary A.5.8. □

**Example A.5.12** (The quantum torus). Let  $q \in \mathbb{C} \setminus \{0, 1\}$  and denote by  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  the *twisted Laurent polynomial ring* in two non-commuting variables under the twisting

$$yx = qxy \tag{A.4}$$

The ring  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  is known as the *quantum torus*. Now let  $\mathcal{A} = \mathbb{C}[x, x^{-1}]$ ,  $G = (\mathbb{Z}, +)$ ,  $\sigma_n : P(x) \mapsto P(q^n x)$  for  $n \in G$  and  $P(x) \in \mathcal{A}$ , and let  $\alpha(s, t) = 1_{\mathcal{A}}$  for all  $s, t \in G$ . It is easily verified that  $\sigma$  and  $\alpha$  together satisfy conditions (i)-(iii) of a  $G$ -system and it is not hard to see that  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G \cong \mathbb{C}_q[x, x^{-1}, y, y^{-1}]$ . In the current

example,  $\mathcal{A}$  is an integral domain,  $G$  is abelian,  $\alpha \equiv 1_{\mathcal{A}}$  and hence all the conditions of Theorem A.5.11 are satisfied. Note that the commutation relation (A.4) implies

$$y^n x^m = q^{mn} x^m y^n, \quad \forall n, m \in \mathbb{Z} \quad (\text{A.5})$$

It is important to distinguish between two different cases:

**Case 1** ( $q$  is a root of unity). Suppose that  $q^n = 1$  for some  $n \neq 0$ . From equality (A.5) we note that  $y^n \in Z(\mathbb{C}_q[x, x^{-1}, y, y^{-1}])$  and hence  $\mathbb{C}[x, x^{-1}]$  is not maximal commutative in  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$ . Thus, according to Theorem A.5.11, there must exist some non-zero ideal  $I$  which has zero intersection with  $\mathbb{C}[x, x^{-1}]$ .

**Case 2** ( $q$  is not a root of unity). Suppose that  $q^n \neq 1$  for all  $n \in \mathbb{Z} \setminus \{0\}$ . One can show that this implies that  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  is *simple*. This means that the only non-zero ideal is  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  itself and this ideal obviously intersects  $\mathbb{C}[x, x^{-1}]$  non-trivially. Hence, by Theorem A.5.11, we conclude that  $\mathbb{C}[x, x^{-1}]$  is maximal commutative in  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$ .

## A.6 Ideals, intersections and zero-divisors

Let  $D$  denote the subset of zero-divisors in  $\mathcal{A}$  and note that  $D$  is always non-empty since  $0_{\mathcal{A}} \in D$ . By  $\tilde{D}$  we denote the image of  $D$  under the embedding  $\iota$ .

**Theorem A.6.1.** *If  $\mathcal{A}$  is commutative, then the following implication holds:*

$$(i): I \cap (\tilde{\mathcal{A}} \setminus \tilde{D}) \neq \emptyset, \text{ for every non-zero two-sided ideal } I \text{ in } \mathcal{A} \rtimes_{\alpha}^{\sigma} G$$

$\Downarrow$

$$(ii): D \cap \mathcal{A}^G = \{0_{\mathcal{A}}\}, \text{ i.e. the only zero-divisor that is fixed under all automorphisms is } 0_{\mathcal{A}}$$

*Proof by contraposition.* Let  $\mathcal{A}$  be commutative. Suppose that  $D \cap \mathcal{A}^G \neq \{0_{\mathcal{A}}\}$ . Then there exist some  $c \in D \setminus \{0_{\mathcal{A}}\}$  such that  $\sigma_s(c) = c$  for all  $s \in G$ . There is also some  $d \in D \setminus \{0_{\mathcal{A}}\}$ , such that  $c \cdot d = 0_{\mathcal{A}}$ . Consider the ideal  $\text{Ann}(c) = \{a \in \mathcal{A} \mid a \cdot c = 0_{\mathcal{A}}\}$  in  $\mathcal{A}$ . It is clearly non-empty since we always have  $0_{\mathcal{A}} \in \text{Ann}(c)$  and  $d \in \text{Ann}(c)$ . Let  $\theta : \mathcal{A} \rightarrow \mathcal{A} / \text{Ann}(c)$  be the quotient homomorphism defined by  $a \mapsto a + \text{Ann}(c)$ . Let us define a map  $\rho : G \rightarrow \text{Aut}(\mathcal{A} / \text{Ann}(c))$  by  $\rho_s(a + \text{Ann}(c)) = \sigma_s(a) + \text{Ann}(c)$  for  $a + \text{Ann}(c) \in \mathcal{A} / \text{Ann}(c)$  and  $s \in G$ . Note that  $\text{Ann}(c)$  is invariant under  $\sigma_s$  for every  $s \in G$  and thus it is easily verified that  $\rho_s$  is a well-defined automorphism on  $\mathcal{A} / \text{Ann}(c)$  for each  $s \in G$ . Define a map  $\beta : G \times G \rightarrow U(\mathcal{A} / \text{Ann}(c))$  by  $(s, t) \mapsto (\theta \circ \alpha)(s, t)$ . It is not hard to see that  $\{\mathcal{A} / \text{Ann}(c), G, \rho, \beta\}$  is in fact a  $G$ -crossed system. Consider the map  $\Gamma : \mathcal{A} \rtimes_{\alpha}^{\sigma} G \rightarrow \mathcal{A} / \text{Ann}(c) \rtimes_{\beta}^{\rho} G$  defined by  $\sum_{s \in G} a_s \bar{s} \mapsto \sum_{s \in G} \theta(a_s) \bar{s}$ .

For any two elements  $\sum_{s \in G} a_s \bar{s}, \sum_{t \in G} b_t \bar{t} \in \mathcal{A} \rtimes_{\alpha}^{\sigma} G$  the additivity of  $\Gamma$  follows by

$$\begin{aligned} \Gamma \left( \sum_{s \in G} a_s \bar{s} + \sum_{t \in G} b_t \bar{t} \right) &= \Gamma \left( \sum_{s \in G} (a_s + b_s) \bar{s} \right) = \sum_{s \in G} \theta(a_s + b_s) \bar{s} \\ &= \sum_{s \in G} \theta(a_s) \bar{s} + \sum_{t \in G} \theta(b_t) \bar{t} \\ &= \Gamma \left( \sum_{s \in G} a_s \bar{s} \right) + \Gamma \left( \sum_{t \in G} b_t \bar{t} \right) \end{aligned}$$

and due to the assumptions, the multiplicativity follows by

$$\begin{aligned} \Gamma \left( \sum_{s \in G} a_s \bar{s} \sum_{t \in G} b_t \bar{t} \right) &= \Gamma \left( \sum_{(s,t) \in G \times G} a_s \sigma_s(b_t) \alpha(s,t) \bar{s} \bar{t} \right) \\ &= \sum_{(s,t) \in G \times G} \theta(a_s \sigma_s(b_t) \alpha(s,t)) \bar{s} \bar{t} \\ &= \sum_{(s,t) \in G \times G} \theta(a_s) \theta(\sigma_s(b_t)) \theta(\alpha(s,t)) \bar{s} \bar{t} \\ &= \sum_{(s,t) \in G \times G} \theta(a_s) \rho_s(\theta(b_t)) \beta(s,t) \bar{s} \bar{t} \\ &= \left( \sum_{s \in G} \theta(a_s) \bar{s} \right) \left( \sum_{t \in G} \theta(b_t) \bar{t} \right) \\ &= \Gamma \left( \sum_{s \in G} a_s \bar{s} \right) \Gamma \left( \sum_{t \in G} b_t \bar{t} \right) \end{aligned}$$

where we have used that  $\beta = \theta \circ \alpha$  and  $\theta(\sigma_s(b_t)) = \rho_s(\theta(b_t))$  for all  $b_t \in \mathcal{A}$  and  $s \in G$ . This shows that  $\Gamma$  is a ring homomorphism. Now, pick some  $g \neq e$  and let  $I$  be the ideal generated by  $d\bar{g}$ . Clearly  $I \neq \{0\}$  and  $\Gamma|_I \equiv 0$ . Note that  $\ker(\theta) = \text{Ann}(c)$  and in particular  $\Gamma(a\bar{e}) = 0$  implies  $a \in \text{Ann}(c)$ . Take  $m\bar{e} \in I \cap (\tilde{\mathcal{A}} \setminus \tilde{D})$ . Then  $\Gamma(m\bar{e}) = 0$  and hence  $m \in \text{Ann}(c) \subseteq D$ , which is a contradiction. Thus,  $I \cap (\tilde{\mathcal{A}} \setminus \tilde{D}) = \emptyset$  and by contrapositivity this concludes the proof.  $\square$

**Example A.6.2** (The truncated quantum torus). Let  $q \in \mathbb{C} \setminus \{0, 1\}$ ,  $m \in \mathbb{N}$  and consider the ring  $\frac{\mathbb{C}[x, y, y^{-1}]}{(y x - q x y, x^m)}$  which is commonly referred to as the *truncated quantum torus*. It is easily verified that this ring is isomorphic to  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  with  $\mathcal{A} = \mathbb{C}[x]/(x^m)$ ,  $G = (\mathbb{Z}, +)$ ,  $\sigma_n : P(x) \mapsto P(q^n x)$  for  $n \in G$  and  $P(x) \in \mathcal{A}$ , and  $\alpha(s, t) = 1_{\mathcal{A}}$  for

all  $s, t \in G$ . One should note that in this case  $\mathcal{A}$  is commutative, but not an integral domain. In fact, the zero-divisors in  $\mathbb{C}[x]/(x^m)$  are precisely those polynomials where the constant term is zero, i.e.  $p(x) = \sum_{i=0}^{m-1} a_i x^i$ , with  $a_i \in \mathbb{C}$ , such that  $a_0 = 0$ . It is also important to remark that, unlike the *quantum torus*,  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is never simple (for  $m > 1$ ). In fact we always have a chain of two-sided ideals

$$\frac{\mathbb{C}[x, y, y^{-1}]}{(y x - q x y, x^m)} \supset \langle x \rangle \supset \langle x^2 \rangle \supset \dots \supset \langle x^{m-1} \rangle \supset \{0\}$$

independent of the value of  $q$ . Moreover, the two-sided ideal  $J = \langle x^{m-1} \rangle$  is contained in  $\text{Comm}(\mathbb{C}[x]/(x^m))$  and contains elements outside of  $\mathbb{C}[x]/(x^m)$ . Hence we conclude that  $\mathbb{C}[x]/(x^m)$  is not maximal commutative in  $\frac{\mathbb{C}[x, y, y^{-1}]}{(y x - q x y, x^m)}$ . When  $q$  is a root of unity, with  $q^n = 1$  for some  $n < m$ , we are able to say more. Consider the polynomial  $p(x) = x^n$ , which is a non-trivial zero-divisor in  $\mathbb{C}[x]/(x^m)$ . For every  $s \in \mathbb{Z}$  we see that  $p(x) = x^n$  is fixed under the automorphism  $\sigma_s$  and therefore, by Theorem A.6.1, we conclude that there exists a non-zero two-sided ideal in  $\frac{\mathbb{C}[x, y, y^{-1}]}{(y x - q x y, x^m)}$  such that its intersection with  $\tilde{\mathcal{A}} \setminus \tilde{D}$  is empty.

## A.7 Comments to the literature

The literature contains several different types of intersection theorems for group rings, Ore extensions and crossed products. Typically these theorems rely on heavy restrictions on the coefficient rings and the groups involved. We shall now give references to some interesting results in the literature.

It was proven in [23, Theorem 1, Theorem 2] that the center of a semiprimitive (semisimple in the sense of Jacobson [6]) P.I. ring respectively semiprime P.I. ring has a non-zero intersection with every non-zero ideal in such a ring. For crossed products satisfying the conditions in [23, Theorem 2], it offers a more precise result than Theorem A.5.1 since  $Z(\mathcal{A} \rtimes_{\alpha}^{\sigma} G) \subseteq \text{Comm}(\tilde{\mathcal{A}})$ . However, every crossed product need not be semiprime nor a P.I. ring and this justifies the need for Theorem A.5.1.

In [12, Lemma 2.6] it was proven that if the coefficient ring  $\mathcal{A}$  of a crossed product  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is prime,  $P$  is a prime ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  such that  $P \cap \tilde{\mathcal{A}} = 0$  and  $I$  is an ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  properly containing  $P$ , then  $I \cap \tilde{\mathcal{A}} \neq 0$ . Furthermore, in [12, Proposition 5.4] it was proven that the crossed product  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  with  $G$  abelian and  $\mathcal{A}$  a  $G$ -prime ring has the property that, if  $G_{\text{inn}} = \{e\}$ , then every non-zero ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  has a non-zero intersection with  $\tilde{\mathcal{A}}$ . It was shown in [2, Corollary 3] that if  $\mathcal{A}$  is semiprime and  $G_{\text{inn}} = \{e\}$ , then every non-zero ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  has a non-zero intersection with  $\tilde{\mathcal{A}}$ . In [13, Lemma 3.8] it was shown that if  $\mathcal{A}$  is a  $G$ -prime ring,  $P$  a prime ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  with  $P \cap \tilde{\mathcal{A}} = 0$  and if  $I$  is an ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  properly containing  $P$ , then  $I \cap \tilde{\mathcal{A}} \neq 0$ . In [16, Proposition 2.6] it was shown that if  $\mathcal{A}$  is a prime ring and  $I$  is a non-zero ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , then  $I \cap (\mathcal{A} \rtimes_{\alpha}^{\sigma} G_{\text{inn}}) \neq 0$ . In [16, Proposition 2.11] it was shown that

for a crossed product  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  with prime ring  $\mathcal{A}$ , every non-zero ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  has a non-zero intersection with  $\tilde{\mathcal{A}}$  if and only if  $C^t[G_{\text{inn}}]$  is  $G$ -simple and in particular if  $|G_{\text{inn}}| < \infty$ , then every non-zero ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  has a non-zero intersection with  $\tilde{\mathcal{A}}$  if and only if  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  is prime.

Corollary A.5.2 shows that if  $\tilde{\mathcal{A}}$  is maximal commutative in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$ , without any further conditions on the coefficient ring or the group, we are able to conclude that every non-zero ideal in  $\mathcal{A} \rtimes_{\alpha}^{\sigma} G$  has a non-zero intersection with  $\tilde{\mathcal{A}}$ .

In the theory of group rings (crossed products with no action or twisting) the intersection properties of ideals with certain subrings have played an important role and are studied in depth in for example [3], [11] and [22]. Some further properties of intersections of ideals and homogeneous components in graded rings have been studied in for example [1], [14].

For ideals in Ore extensions there are interesting results in [4, Theorem 4.1] and [8, Lemma 2.2, Theorem 2.3, Corollary 2.4], explaining a correspondence between certain ideals in the Ore extension and certain ideals in its coefficient ring. Given a domain  $\mathcal{A}$  of characteristic 0 and a non-zero derivation  $\delta$  it is shown in [5, Proposition 2.6] that every non-zero ideal in the Ore extension  $R = \mathcal{A}[x; \delta]$  intersects  $\mathcal{A}$  in a non-zero  $\delta$ -invariant ideal. Similar types of intersection results for ideals in Ore extension rings can be found in for example [9] and [15].

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**B**

## **Paper B**

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## Paper B

# Commutativity and ideals in pre-crystalline graded rings

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**Abstract.** *Pre-crystalline graded rings constitute a class of rings which share many properties with classical crossed products. Given a pre-crystalline graded ring  $\mathcal{A}$ , we describe its center, the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  of the degree zero grading part, and investigate the connection between maximal commutativity of  $\mathcal{A}_0$  in  $\mathcal{A}$  and the way in which two-sided ideals intersect  $\mathcal{A}_0$ .*

## B.1 Introduction

Given a ring  $\mathcal{A}$  containing a commutative subring  $\mathcal{A}_0$ , one may consider the following two assertions:

- S1:** The ring  $\mathcal{A}_0$  is a maximal commutative subring in  $\mathcal{A}$ .
- S2:** For every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ ,  $I \cap \mathcal{A}_0 \neq \{0\}$ .

Different types of intersection properties, closely related to **S2**, have been studied for rings with specific restrictions like primeness, semi-primeness, semisimplicity, P.I. property and semiprimitivity in [2–6, 11–18, 25, 26].

It has been shown in [22, 23, 27–33], that for some types of algebraic crossed products as well as  $C^*$ -crossed products, there is a connection between these two assertions. Under some conditions on the crossed products the two statements are in fact equivalent, but not in general. In the recent paper [20], by E. Nauwelaerts and F. Van Oystaeyen, so called *crystalline graded rings*, which generalize algebraic crossed products, were defined. In the paper [21], by T. Neijens, F. Van Oystaeyen and W.W. Yu, the structure of the center of special classes of crystalline graded rings and generalized Clifford algebras was studied. In this paper we describe the center  $Z(\mathcal{A})$  and the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  of the degree zero grading part in general pre-crystalline graded rings. Furthermore, we show that for some types of pre-crystalline and crystalline graded rings there is a close connection between the two assertions **S1** and **S2**. In particular, for crystalline graded rings and skew group rings, we provide sufficient conditions on the degree zero grading component  $\mathcal{A}_0$ , the grading group  $G$ , the cocycle and the action for equivalence between **S1** and

**S2** (Theorems B.3.16 and B.3.17). We also provide an example of a situation in which the equivalence does not hold (Example B.4.2). For pre-crystalline graded rings with commutative  $\mathcal{A}_0$ , we show that under a certain condition each non-zero two-sided ideal has a non-zero intersection with the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  (Theorem B.3.11). In particular this yields sufficient conditions for **S1** to imply **S2** for general pre-crystalline graded rings (Corollary B.3.12). For crystalline graded rings we show that if  $\mathcal{A}_0$  is commutative, then each non-zero two-sided ideal always has a non-zero intersection with the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  and this immediately implies that when  $\mathcal{A}_0$  is maximal commutative, **S1** implies **S2** (Corollary B.3.13 and B.3.14).

## B.2 Definitions and background

We shall begin by recalling some basic definitions and properties following [20]. For a thorough exposition of the theory of graded rings we refer to [1, 19].

**Definition B.2.1** (Pre-crystalline graded ring). An associative and unital ring  $\mathcal{A}$  is said to be *pre-crystalline graded* if

- (i) there is a group  $G$  (with neutral element  $e$ ),
- (ii) there is a map  $u : G \rightarrow \mathcal{A}$ ,  $g \mapsto u_g$  such that  $u_e = 1_{\mathcal{A}}$  and  $u_g \neq 0$  for every  $g \in G$ ,
- (iii) there is a subring  $\mathcal{A}_0 \subseteq \mathcal{A}$  containing  $1_{\mathcal{A}} = 1_{\mathcal{A}_0}$ ,

such that the following conditions are satisfied:

- (P1)  $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_0 u_g$ ;
- (P2) For every  $g \in G$ ,  $u_g \mathcal{A}_0 = \mathcal{A}_0 u_g$  and this is a free left  $\mathcal{A}_0$ -module of rank one;
- (P3) The decomposition in P1 makes  $\mathcal{A}$  into a  $G$ -graded ring with  $\mathcal{A}_0 = \mathcal{A}_e$ .

**Lemma B.2.2** (see [20]). *With notation and definitions as above:*

- (i) For every  $g \in G$ , there is a set map  $\sigma_g : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  defined by  $u_g a = \sigma_g(a) u_g$  for  $a \in \mathcal{A}_0$ . The map  $\sigma_g$  is a surjective ring morphism. Moreover,  $\sigma_e = \text{id}_{\mathcal{A}_0}$ .
- (ii) There is a set map  $\alpha : G \times G \rightarrow \mathcal{A}_0$  defined by  $u_s u_t = \alpha(s, t) u_{st}$  for  $s, t \in G$ . For any triple  $s, t, w \in G$  and  $a \in \mathcal{A}_0$ , the following equalities hold:

$$\alpha(s, t) \alpha(st, w) = \sigma_s(\alpha(t, w)) \alpha(s, tw) \tag{B.1}$$

$$\sigma_s(\sigma_t(a)) \alpha(s, t) = \alpha(s, t) \sigma_{st}(a) \tag{B.2}$$

- (iii) For every  $g \in G$  we have  $\alpha(g, e) = \alpha(e, g) = 1_{\mathcal{A}_0}$  and  $\alpha(g, g^{-1}) = \sigma_g(\alpha(g^{-1}, g))$ .

A pre-crystalline graded ring  $\mathcal{A}$  with the above properties will be denoted by  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ . In [20] it was shown that for pre-crystalline graded rings, the elements  $\alpha(s, t)$  are normalizing elements of  $\mathcal{A}_0$ , i.e.  $\mathcal{A}_0 \alpha(s, t) = \alpha(s, t) \mathcal{A}_0$  for each  $s, t \in G$ . For a pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we let  $S(G)$  denote the multiplicative set in  $\mathcal{A}_0$  generated by  $\{\alpha(g, g^{-1}) \mid g \in G\}$  and let  $S(G \times G)$  denote the multiplicative set generated by  $\{\alpha(g, h) \mid g, h \in G\}$ .

**Lemma B.2.3** (see [20]). *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring, then the following are equivalent:*

- (i)  $\mathcal{A}_0$  is  $S(G)$ -torsion free.
- (ii)  $\mathcal{A}$  is  $S(G)$ -torsion free.
- (iii)  $\alpha(g, g^{-1}) a_0 = 0$  for some  $g \in G$  implies  $a_0 = 0$ .
- (iv)  $\alpha(g, h) a_0 = 0$  for some  $g, h \in G$  implies  $a_0 = 0$ .
- (v)  $\mathcal{A}_0 u_g = u_g \mathcal{A}_0$  is also free as a right  $\mathcal{A}_0$ -module, with basis  $u_g$ , for every  $g \in G$ .
- (vi) For every  $g \in G$ ,  $\sigma_g$  is bijective and hence a ring automorphism of  $\mathcal{A}_0$ .

**Definition B.2.4** (Crystalline graded ring). A pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , which is  $S(G)$ -torsion free, is said to be a *crystalline graded ring*.

### B.3 Commutant, center and ideals

The commutant of the subring  $\mathcal{A}_0$  in the pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  will be denoted by

$$C_{\mathcal{A}}(\mathcal{A}_0) = \{b \in \mathcal{A} \mid ab = ba, \quad \forall a \in \mathcal{A}_0\}.$$

In this section we give a description of the commutant of  $\mathcal{A}_0$  in various pre-crystalline graded rings. Theorem B.3.1 tells us exactly when an element of a pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  lies in  $C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Theorem B.3.1.** *In a pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we have*

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{s \in G} r_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G \mid r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0, s \in G \right\}.$$

*Proof.* The proof is established through the following sequence of equivalences:

$$\begin{aligned} \sum_{s \in G} r_s u_s \in C_{\mathcal{A}}(\mathcal{A}_0) &\iff \left( \sum_{s \in G} r_s u_s \right) a = a \left( \sum_{s \in G} r_s u_s \right), \quad \forall a \in \mathcal{A}_0 \\ &\iff \sum_{s \in G} r_s \sigma_s(a) u_s = \sum_{s \in G} a r_s u_s, \quad \forall a \in \mathcal{A}_0 \\ &\iff \text{For each } s \in G : r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0. \end{aligned}$$



□

If  $\mathcal{A}_0$  is commutative, then for  $r \in \mathcal{A}_0$  we denote its annihilator ideal in  $\mathcal{A}_0$  by  $\text{Ann}(r) = \{a \in \mathcal{A}_0 \mid ar = 0\}$  and get a simplified description of  $C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Corollary B.3.2.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring and  $\mathcal{A}_0$  is commutative, then*

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{s \in G} r_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G \mid \sigma_s(a) - a \in \text{Ann}(r_s), \quad \forall a \in \mathcal{A}_0, s \in G \right\}.$$

When  $\mathcal{A}_0$  is commutative it is clear that  $\mathcal{A}_0 \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$ . Using the explicit description of  $C_{\mathcal{A}}(\mathcal{A}_0)$  in Corollary B.3.2, we immediately get necessary and sufficient conditions for  $\mathcal{A}_0$  to be maximal commutative, i.e.  $\mathcal{A}_0 = C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Corollary B.3.3.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring where  $\mathcal{A}_0$  is commutative, then  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  if and only if, for each pair  $(s, r_s) \in (G \setminus \{e\}) \times (\mathcal{A}_0 \setminus \{0_{\mathcal{A}_0}\})$ , there exists  $a \in \mathcal{A}_0$  such that  $\sigma_s(a) - a \notin \text{Ann}(r_s)$ .*

**Corollary B.3.4.** *Let  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  be a pre-crystalline graded ring where  $\mathcal{A}_0$  is commutative. If for each  $s \in G \setminus \{e\}$  it is always possible to find some  $a \in \mathcal{A}_0$  such that  $\sigma_s(a) - a$  is not a zero-divisor in  $\mathcal{A}_0$ , then  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ .*

The next corollary is a consequence of Corollary B.3.3.

**Corollary B.3.5.** *If the subring  $\mathcal{A}_0$  of a pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is maximal commutative, then  $\sigma_s \neq \text{id}_{\mathcal{A}_0}$  for each  $s \in G \setminus \{e\}$ .*

The description of the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  from Corollary B.3.2 can be further refined in the case when  $\mathcal{A}_0$  is an integral domain.

**Corollary B.3.6.** *If  $\mathcal{A}_0$  is an integral domain, then the commutant of  $\mathcal{A}_0$  in the pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is*

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{s \in \sigma^{-1}(\text{id}_{\mathcal{A}_0})} r_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G \mid r_s \in \mathcal{A}_0 \right\}$$

where  $\sigma^{-1}(\text{id}_{\mathcal{A}_0}) = \{s \in G \mid \sigma_s = \text{id}_{\mathcal{A}_0}\}$ .

The following corollary can be derived directly from Corollary B.3.5 together with either Corollary B.3.4 or Corollary B.3.6.

**Corollary B.3.7.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring where  $\mathcal{A}_0$  is an integral domain, then  $\sigma_s \neq \text{id}_{\mathcal{A}_0}$  for each  $s \in G \setminus \{e\}$  if and only if  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ .*

We will now give a sufficient condition for  $C_{\mathcal{A}}(\mathcal{A}_0)$  to be commutative.

**Proposition B.3.8.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring where  $\mathcal{A}_0$  is commutative,  $G$  is abelian and  $\alpha(s, t) = \alpha(t, s)$  for all  $s, t \in G$ , then  $C_{\mathcal{A}}(\mathcal{A}_0)$  is commutative.*

*Proof.* Let  $\sum_{s \in G} r_s u_s$  and  $\sum_{t \in G} p_t u_t$  be arbitrary elements of  $C_{\mathcal{A}}(\mathcal{A}_0)$ , then by our assumptions and Corollary B.3.2 we get

$$\begin{aligned} \left( \sum_{s \in G} r_s u_s \right) \left( \sum_{t \in G} p_t u_t \right) &= \sum_{(s,t) \in G \times G} r_s \sigma_s(p_t) \alpha(s, t) u_{st} \\ &= \sum_{(s,t) \in G \times G} r_s p_t \alpha(s, t) u_{st} \\ &= \sum_{(s,t) \in G \times G} p_t \sigma_t(r_s) \alpha(t, s) u_{ts} \\ &= \left( \sum_{t \in G} p_t u_t \right) \left( \sum_{s \in G} r_s u_s \right). \end{aligned}$$

□

### B.3.1 The center in $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$

In this section we will describe the center  $Z(\mathcal{A})$  of a pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ . Note that  $Z(\mathcal{A}) \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Proposition B.3.9.** *The center of a pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is*

$$Z(\mathcal{A}) = \left\{ \sum_{g \in G} r_g u_g \mid r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t), \right. \\ \left. r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0, (s, t) \in G \times G \right\}.$$

*Proof.* Let  $\sum_{g \in G} r_g u_g \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  be an element which commutes with every element in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ . In particular  $\sum_{g \in G} r_g u_g$  must commute with every  $a \in \mathcal{A}_0$ . From Theorem B.3.1 we immediately see that this implies  $r_s \sigma_s(a) = a r_s$  for every  $a \in \mathcal{A}_0$  and  $s \in G$ . Furthermore,  $\sum_{g \in G} r_g u_g$  must commute with  $u_s$  for every  $s \in G$ . This yields

$$\begin{aligned} \sum_{t \in G} r_{ts^{-1}} \alpha(ts^{-1}, s) u_t &= \sum_{g \in G} r_g \alpha(g, s) u_{gs} = \left( \sum_{g \in G} r_g u_g \right) u_s \\ &= u_s \left( \sum_{g \in G} r_g u_g \right) = \sum_{g \in G} \sigma_s(r_g) \alpha(s, g) u_{sg} = \sum_{t \in G} \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t) u_t \end{aligned}$$

and hence, for each  $(s, t) \in G \times G$ , we have  $r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t)$ . Conversely, suppose that  $\sum_{g \in G} r_g u_g \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is an element satisfying  $r_s \sigma_s(a) = a r_s$  and  $r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t)$  for every  $a \in \mathcal{A}$  and  $(s, t) \in G \times G$ . Let  $\sum_{s \in G} a_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  be arbitrary. Then

$$\begin{aligned}
 \left( \sum_{g \in G} r_g u_g \right) \left( \sum_{s \in G} a_s u_s \right) &= \sum_{(g,s) \in G \times G} r_g \sigma_g(a_s) \alpha(g, s) u_{gs} \\
 &= \sum_{(g,s) \in G \times G} a_s r_g \alpha(g, s) u_{gs} \\
 &= \sum_{(t,s) \in G \times G} a_s (r_{ts^{-1}} \alpha(ts^{-1}, s)) u_t \\
 &= \sum_{(t,s) \in G \times G} a_s \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t) u_t \\
 &= \sum_{(g,s) \in G \times G} a_s \sigma_s(r_g) \alpha(s, g) u_{sg} \\
 &= \left( \sum_{s \in G} a_s u_s \right) \left( \sum_{g \in G} r_g u_g \right)
 \end{aligned}$$

and hence  $\sum_{g \in G} r_g u_g$  commutes with every element in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ .  $\square$

**Remark B.3.10.** Consider the case when  $\mathcal{A}_0$  is an integral domain and let  $\alpha$  take its values in  $\mathcal{A}_0 \setminus \{0\}$ . In this case it is clear that the following three assertions are equivalent:

1.  $r_s \sigma_s(a) = a r_s$  for all  $a \in \mathcal{A}_0$
2.  $r_s(\sigma_s(a) - a) = 0$  for all  $a \in \mathcal{A}_0$
3.  $r_s = 0$  if  $s \notin \sigma^{-1}(\text{id}_{\mathcal{A}_0}) = \{g \in G \mid \sigma_g = \text{id}_{\mathcal{A}_0}\}$

After changing the variable via  $x = s^{-1}t$ , the first condition in Proposition B.3.9 may be written as  $\sigma_s(r_x) \alpha(s, x) = r_{sxs^{-1}} \alpha(sxs^{-1}, s)$  for all  $(s, x) \in G \times G$ . From this relation we conclude that  $r_x = 0$  if and only if  $r_{sxs^{-1}} = 0$ , and hence it is trivially satisfied if we put  $r_x = 0$  whenever  $x \notin \sigma^{-1}(\text{id}_{\mathcal{A}_0})$ . This case has been presented in [21, Proposition 2.2] with a more elaborate proof.

### B.3.2 Ideals

Given a pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , for each  $b \in \mathcal{A}_0$  we define the commutator to be

$$D_b : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{s \in G} a_s u_s \mapsto b \left( \sum_{s \in G} a_s u_s \right) - \left( \sum_{s \in G} a_s u_s \right) b.$$

From the definition of the multiplication we have

$$\begin{aligned} D_b \left( \sum_{s \in G} a_s u_s \right) &= b \left( \sum_{s \in G} a_s u_s \right) - \left( \sum_{s \in G} a_s u_s \right) b \\ &= \left( \sum_{s \in G} b a_s u_s \right) - \left( \sum_{s \in G} a_s \sigma_s(b) u_s \right) \\ &= \sum_{s \in G} (b a_s - a_s \sigma_s(b)) u_s \end{aligned}$$

for each  $b \in \mathcal{A}_0$ .

**Theorem B.3.11** (see [24]). *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring where  $\mathcal{A}_0$  is commutative and for each  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  there exists  $s \in G$  such that  $a_s \notin \ker(\sigma_s \circ \sigma_{s^{-1}})$ , then*

$$I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

*Proof.* Let  $I$  be an arbitrary non-zero two-sided ideal in  $\mathcal{A}$  and assume that  $\mathcal{A}_0$  is commutative and that for each  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  there exists  $s \in G$  such that  $a_s \notin \ker(\sigma_s \circ \sigma_{s^{-1}})$ . For each  $g \in G$  we may define a *translation operator*

$$T_g : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{s \in G} a_s u_s \mapsto \left( \sum_{s \in G} a_s u_s \right) u_g.$$

Note that, for each  $g \in G$ ,  $I$  is invariant under  $T_g$ . We have

$$T_g \left( \sum_{s \in G} a_s u_s \right) = \left( \sum_{s \in G} a_s u_s \right) u_g = \sum_{s \in G} a_s \alpha(s, g) u_{sg}$$

for every  $g \in G$ . By the assumptions and together with [20, Corollary 2.4] it is clear that for each element  $c \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  it is always possible to choose some  $g \in G$  and let  $T_g$  operate on  $c$  to end up with an element where the coefficient in front of  $u_e$  is non-zero.

Note that, for each  $b \in \mathcal{A}_0$ ,  $I$  is invariant under  $D_b$ . Furthermore, we have

$$\begin{aligned} D_b \left( \sum_{s \in G} a_s u_s \right) &= \sum_{s \in G} (b a_s - a_s \sigma_s(b)) u_s = \\ &= \sum_{s \neq e} (b a_s - a_s \sigma_s(b)) u_s = \sum_{s \neq e} d_s u_s \end{aligned}$$

since  $(b a_e - a_e \sigma_e(b)) = b a_e - a_e b = 0$ . Note that  $C_{\mathcal{A}}(\mathcal{A}_0) = \bigcap_{b \in \mathcal{A}_0} \ker(D_b)$  and hence for any  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  we are always able to choose  $b \in \mathcal{A}_0$  and the corresponding  $D_b$  and have  $\sum_{s \in G} a_s u_s \notin \ker(D_b)$ . Therefore we can always pick an operator  $D_b$  which kills the coefficient in front of  $u_e$  without killing everything. Hence, if  $a_e \neq 0_{\mathcal{A}_0}$ , the number of non-zero coefficients of the resulting element will always be reduced by at least one.

The ideal  $I$  is assumed to be non-zero, which means that we can pick some non-zero element  $\sum_{s \in G} r_s u_s \in I$ . If  $\sum_{s \in G} r_s u_s \in C_{\mathcal{A}}(\mathcal{A}_0)$ , then we are finished, so assume that this is not the case. Note that  $r_s \neq 0_{\mathcal{A}_0}$  for finitely many  $s \in G$ . Recall that the ideal  $I$  is invariant under  $T_g$  and  $D_a$  for all  $g \in G$  and  $a \in \mathcal{A}_0$ . We may now use the operators  $\{T_g\}_{g \in G}$  and  $\{D_a\}_{a \in \mathcal{A}_0}$  to generate new elements of  $I$ . More specifically, we may use the  $T_g$ 's to translate our element  $\sum_{s \in G} r_s u_s$  into a new element which has a non-zero coefficient in front of  $u_e$  (if needed) after which we use the  $D_a$  operator to kill this coefficient and end up with yet another new element of  $I$  which is non-zero but has a smaller number of non-zero coefficients. We may repeat this procedure and in a finite number of iterations arrive at an element of  $I$  which lies in  $C_{\mathcal{A}}(\mathcal{A}_0) \setminus \mathcal{A}_0$ , and if not we continue the above procedure until we reach an element in  $\mathcal{A}_0 \setminus \{0_{\mathcal{A}_0}\}$ . In particular  $\mathcal{A}_0 \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$  since  $\mathcal{A}_0$  is commutative and hence  $I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$ .  $\square$

**Corollary B.3.12.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring where  $\mathcal{A}_0$  is maximal commutative and for each  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus \mathcal{A}_0$  there exists  $s \in G$  such that  $a_s \notin \ker(\sigma_s \circ \sigma_{s^{-1}})$ , then*

$$I \cap \mathcal{A}_0 \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

A crystalline graded ring has no  $S(G)$ -torsion and hence  $\ker(\sigma_s \circ \sigma_{s^{-1}}) = \{0_{\mathcal{A}_0}\}$  by [20, Corollary 2.4]. Therefore we get the following corollary which is a generalization of a result for algebraic crossed products in [22].

**Corollary B.3.13.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring where  $\mathcal{A}_0$  is commutative, then*

$$I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

When  $\mathcal{A}_0$  is maximal commutative we get the following corollary.

**Corollary B.3.14.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring where  $\mathcal{A}_0$  is maximal commutative, then*

$$I \cap \mathcal{A}_0 \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ .

This shows that in a crystalline graded ring where  $\mathcal{A}_0$  is commutative, the assertion **S1** always implies **S2**. The following lemma can be found in [22].

**Lemma B.3.15.** *If  $\mathcal{A}_0 \diamond_{\sigma} G$  is a skew group ring where  $G$  is abelian, then the assertion **S2** implies  $\sigma_g \neq \text{id}_{\mathcal{A}_0}$  for every  $g \in G \setminus \{e\}$ .*

**Theorem B.3.16.** *Let  $\mathcal{A}_0 \diamond_{\sigma} G$  be a skew group ring. If either of the following two conditions is satisfied:*

- (i)  $\mathcal{A}_0$  is an integral domain and  $G$  is an abelian group;
- (ii)  $\mathcal{A}_0$  is commutative and  $G$  is a torsion-free abelian group;

then the two assertions **S1** and **S2** are equivalent.

*Proof.* Let  $\mathcal{A}_0 \diamond_{\sigma} G$  be a skew group ring. In both of the two cases (i) and (ii),  $\mathcal{A}_0$  is commutative and hence it follows from Corollary B.3.14 that **S1** implies **S2**. Assume that (i) is satisfied. By Lemma B.3.15 and Corollary B.3.7 it follows that **S2** implies **S1**. Now assume that (ii) is satisfied. Suppose that  $\mathcal{A}_0$  is not maximal commutative. By Corollary B.3.3 there exists some  $s \in G \setminus \{e\}$  and  $r_s \in \mathcal{A}_0 \setminus \{0\}$  such that  $r_s \sigma_s(a) = r_s a$  for all  $a \in \mathcal{A}_0$ . Let us choose such an  $r_s$  and let  $I$  be the two-sided ideal in  $\mathcal{A}_0 \diamond_{\sigma} G$  generated by  $r_s + r_s u_s$ . The ideal  $I$  is obviously non-zero, and furthermore it is spanned by elements of the form  $a_g u_g (r_s + r_s u_s) a_h u_h$  where  $g, h \in G$  and  $a_g, a_h \in \mathcal{A}_0$ . We may now rewrite this expression.

$$\begin{aligned} a_g u_g (r_s + r_s u_s) a_h u_h &= a_g u_g (r_s a_h + \underbrace{r_s \sigma_s(a_h)}_{=r_s a_h} u_s) u_h \\ &= a_g \sigma_g(r_s a_h) u_g (1_{\mathcal{A}} + u_s) u_h \\ &= a_g \sigma_g(r_s a_h) (u_{gh} + u_{gsh}) \\ &= \underbrace{a_g \sigma_g(r_s a_h)}_{:=b} u_{gh} + a_g \sigma_g(r_s a_h) u_{gsh} \\ &= b u_{gh} + b u_{gsh} \end{aligned}$$

Since  $G$  is abelian, it is clear that any element of  $I$  may be written in the form

$$\sum_{t \in G} (c_t u_t + c_t u_{ts}) \tag{B.3}$$

for some  $c_t \in \mathcal{A}_0$ , where  $t$  only runs over a finite subset of  $G$ . By assumption  $s \neq e$  and hence  $t \neq ts$  for every  $t \in G$ . In particular this means that every contribution from  $c_e$  to the  $e$ -graded part of the element in (B.3) comes with an equal contribution to the  $s$ -graded part. Similarly  $c_s$ 's contribution to the  $s$ -graded part equals its contribution to the  $c_{s^2}$ -graded part. Recall that  $G$  is assumed to be torsion-free, i.e.  $s^n \neq e$  for every  $n \in \mathbb{Z} \setminus \{0\}$ , and hence the element in (B.3) can never be a non-zero element of degree  $e$ , which means  $I \cap \mathcal{A}_0 = \{0\}$ . By contra positivity we conclude that **S2**  $\implies$  **S1** and this finishes the proof.  $\square$

**Theorem B.3.17.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring where  $\mathcal{A}_0$  is an integral domain,  $G$  is an abelian torsion-free group and  $\alpha$  is such that  $\alpha(s, t) = 1_{\mathcal{A}_0}$  whenever  $\sigma_s = \text{id}_{\mathcal{A}_0}$  or  $\sigma_t = \text{id}_{\mathcal{A}_0}$ , then the two assertions **S1** and **S2** are equivalent.*

*Proof.* It is clear from Corollary B.3.14 that **S1**  $\implies$  **S2**. Suppose that  $\mathcal{A}_0$  is not maximal commutative. Since  $\mathcal{A}_0$  is an integral domain, by Corollary B.3.7 there exists some  $s \in G \setminus \{e\}$  such that  $\sigma_s = \text{id}_{\mathcal{A}_0}$ . For arbitrary  $g, h \in G$  we may use condition (2) in Lemma B.2.2 and the assumptions we made on  $\alpha$  to arrive at

$$\underbrace{\alpha(s, g)}_{=1_{\mathcal{A}_0}} \alpha(sg, h) = \underbrace{\sigma_s(\alpha(g, h))}_{=\alpha(g, h)} \underbrace{\alpha(s, gh)}_{=1_{\mathcal{A}_0}}$$

and since  $G$  is abelian we get  $\alpha(gs, h) = \alpha(sg, h) = \alpha(g, h)$ . Now, similarly to the proof of Theorem B.3.16, let  $I$  be the two-sided ideal in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  generated by  $1_{\mathcal{A}} + u_s$  and note that

$$\begin{aligned} a_g u_g (1_{\mathcal{A}} + u_s) a_h u_h &= a_g u_g a_h (1_{\mathcal{A}} + u_s) u_h \\ &= a_g u_g a_h u_h + a_g u_g a_h u_s u_h \\ &= a_g \sigma_g(a_h) u_g u_h + a_g \sigma_g(a_h) u_g u_s u_h \\ &= a_g \sigma_g(a_h) \alpha(g, h) u_{gh} + a_g \sigma_g(a_h) \underbrace{\alpha(g, s)}_{=1_{\mathcal{A}_0}} u_{gs} u_h \\ &= \underbrace{a_g \sigma_g(a_h) \alpha(g, h) u_{gh}}_{:=b} + a_g \sigma_g(a_h) \underbrace{\alpha(gs, h)}_{=\alpha(g, h)} u_{gsh} \\ &= b u_{gh} + b u_{gsh} \end{aligned}$$

for any  $g, h \in G$ ,  $a_g, b_h \in \mathcal{A}_0$ . The rest is analogous to the proof of Theorem B.3.16.  $\square$

**Remark B.3.18.** Note that, a twisted group ring can never fit into the conditions of Theorem B.3.17, because if  $\sigma$  is trivial then the conditions force  $\alpha$  to be trivial as well.

## B.4 Examples

**Example B.4.1** (Group rings). Let  $\mathcal{A}_0$  be a commutative (non-zero) ring and  $G$  any (non-trivial) group and denote the group ring by  $\mathcal{A}_0 \rtimes G$ . Note that this corresponds to the crossed product with trivial  $\sigma$  and  $\alpha$  maps. We may define the so called augmentation map

$$\epsilon : \mathcal{A}_0 \rtimes G \rightarrow \mathcal{A}_0, \quad \sum_{s \in G} a_s u_s \mapsto \sum_{s \in G} a_s$$

and it is straightforward to check that it is in fact a ring morphism. The kernel of this map,  $\ker(\epsilon)$  is a two-sided ideal in  $\mathcal{A}_0 \rtimes G$  and it is not hard to see that  $\ker(\epsilon) \cap \mathcal{A}_0 = \{0\}$ . This gives us an example of a non-zero two-sided ideal which has zero intersection with the coefficient ring  $\mathcal{A}_0$ , i.e. **S2** is false. However, for each  $s \in G$ ,  $u_s$  commutes with every element in  $\mathcal{A}_0$  and hence **S1** is never true for a group ring. In other words, in a group ring the two assertions **S1** and **S2** are always equivalent.

In a twisted group ring  $\mathcal{A}_0 \rtimes_{\alpha} G$ , just like for group rings mentioned above, the action  $\sigma$  is trivial and hence for each  $s \in G$  the element  $u_s$  commutes with every element in  $\mathcal{A}_0$ . In other words,  $\mathcal{A}_0$  is never maximal commutative in a twisted group ring.

**Example B.4.2** (The field of complex numbers). Let  $\mathcal{A}_0 = \mathbb{R}$ ,  $G = (\mathbb{Z}_2, +)$  and define the cocycle  $\alpha : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{R} \setminus \{0\}$  by  $\alpha(\bar{0}, \bar{0}) = 1$ ,  $\alpha(\bar{0}, \bar{1}) = 1$ ,  $\alpha(\bar{1}, \bar{0}) = 1$  and  $\alpha(\bar{1}, \bar{1}) = -1$ . It is easy to see that  $\mathbb{R} \rtimes_{\alpha} \mathbb{Z}_2 \cong \mathbb{C}$ . This twisted group ring is simple, hence  $\mathbb{C}$  is the only non-zero ideal and clearly  $\mathbb{C} \cap \mathbb{R} \neq \{0\}$ . However, as mentioned earlier, the coefficient ring  $\mathbb{R}$  is not maximal commutative in  $\mathbb{C}$ .

Example B.4.2 shows that in a twisted group ring, **S1** may be false even though **S2** is true.

**Example B.4.3** (The first Weyl algebra). Following [20], let  $A_1(\mathbb{C}) = \mathbb{C}\langle x, y \rangle / (yx - xy - 1)$  be the first Weyl algebra. If we put  $\deg(x) = 1$  and  $\deg(y) = -1$  and

$$\begin{aligned} A_1(\mathbb{C})_0 &= \mathbb{C}[xy] \\ A_1(\mathbb{C})_n &= \mathbb{C}[xy]x^n, \text{ for } n \geq 0 \\ A_1(\mathbb{C})_m &= \mathbb{C}[xy]y^{-m}, \text{ for } m \leq 0 \end{aligned}$$

then this defines a  $\mathbb{Z}$ -gradation on  $A_1(\mathbb{C})$ . We set  $u_n = x^n$  if  $n \geq 0$  and  $u_m = y^{-m}$  if  $m \leq 0$ . We put  $\sigma_n = \sigma_{x^n}$  for  $n \geq 0$  and  $\sigma_m = \sigma_{y^{-m}}$  for  $m \leq 0$ . It is clear that  $\sigma_x(xy) = xy - 1$  because  $x(xy) = (xy - 1)x$  and  $\sigma_y(xy) = xy + 1$  because  $y(xy) = (1 + xy)y$ . Let us put  $t = xy$ , then

$$\sigma : \mathbb{Z} \rightarrow \text{Aut}_{\mathbb{C}}(\mathbb{C}[t]), \quad n \mapsto (t \mapsto t - n).$$



We can also calculate, for example

$$\begin{aligned}\alpha(n, -n) &= x^n y^n = x^{n-1} t y^{n-1} = (t - (n-1)) x^{n-1} y^{n-1} \\ &= (t - n + 1) \cdot (t - n + 2) \cdot \dots \cdot (t - 2) \cdot (t - 1) \cdot t.\end{aligned}$$

Furthermore  $\alpha(n, -m)$  with  $n > m$  ( $n, m \in \mathbb{N}$ ) can be calculated from

$$x^n y^m = x^{n-m} x^m y^m = x^{n-m} \alpha(m, -m) = \sigma_x^{n-m}(\alpha(m, -m)) x^{n-m}$$

and so

$$\alpha(n, -m) = \sigma_x^{n-m}(\alpha(m, -m)).$$

Note that  $\sigma^{-1}(\text{id}_{\mathcal{A}_0}) = \{0\}$  and clearly  $\alpha(0, -m) = 1$  and  $\alpha(n, 0) = 1$ . By applying Theorem B.3.17 it is clear that the assertions **S1** and **S2** are equivalent for the first Weyl algebra. Of course, in this specific case this also follows easily since  $A_1(\mathbb{C})$  is known to be a simple algebra and hence the only non-zero two-sided ideal is  $A_1(\mathbb{C})$  itself, which clearly has a non-zero intersection with  $\mathbb{C}[xy]$ . Furthermore,  $\mathbb{C}[xy]$  is an integral domain and  $\sigma_n \neq \text{id}_{\mathbb{C}[xy]}$  for each  $n \neq 0$  and by Corollary B.3.7 the base ring  $\mathbb{C}[xy]$  is maximal commutative in  $A_1(\mathbb{C})$ . This shows that in this particular example of a crystalline graded ring, the two assertions **S1** and **S2** are always true.

**Example B.4.4** (The quantum Weyl algebra). Consider the quantum Weyl algebra  $\mathcal{A} = \mathbb{C}\langle x, y \rangle / (yx - qxy - 1)$  where  $q \in \mathbb{C} \setminus \{0, 1\}$  (see e.g. [8]). Put  $\mathcal{A}_0 = \mathbb{C}[xy]$  just like for the first Weyl algebra. If we set  $u_n = x^n$  for  $n \geq 0$  and  $u_m = y^{-m}$  for  $m \leq 0$ , then we obtain a  $\mathbb{Z}$ -grading on  $\mathcal{A}$ . Furthermore we get

$$\sigma_x^n(xy) = q^{-n}xy - q^{-n} - \dots - q^{-1} \tag{B.4}$$

and

$$\sigma_y^n(xy) = q^nxy + q^{n-1} + \dots + q + 1. \tag{B.5}$$

One may define  $\alpha(n, -n) = x^n y^n$  for  $n \in \mathbb{Z}_{\geq 0}$  and derive formulas for the values of  $\alpha$  in a similar fashion as for the first Weyl algebra. If  $q$  is not a root of unity, then  $\sigma^{-1}(\text{id}_{\mathcal{A}_0}) = \{0\}$  and we may apply Theorem B.3.17 to conclude that **S1** and **S2** are equivalent. Furthermore by Corollary B.3.7,  $\mathcal{A}_0$  is maximal commutative and hence we conclude that every non-zero two-sided ideal has a non-zero intersection with  $\mathcal{A}_0$ . Note that  $\mathcal{A}$  is not simple, in fact there is a whole collection of proper ideals as explained in [7].

If  $q \neq 1$  is a root of unity and  $n$  is the smallest positive integers such that  $q^n = 1$ , then from (B.4) and (B.5) and Corollary B.3.6 we see that

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{k \in n\mathbb{N}} p_k(xy) x^k + \sum_{l \in n\mathbb{N}} p_l(xy) y^l \mid p_k(xy), p_l(xy) \in \mathbb{C}[xy] \right\}$$

and hence  $\mathcal{A}_0$  is not maximal commutative.

Theorems B.3.16 and B.3.17 partly rely on the grading group  $G$  being torsion-free, which obviously excludes all finite groups. In the following we give an example of an algebra which is graded by a finite group and in which the two assertions **S1** and **S2** are in fact true and hence equivalent.

**Example B.4.5** (Central simple algebras). Let  $A$  be a finite-dimensional central simple algebra over a field  $F$ . By Wedderburn's theorem  $A \cong M_i(D)$  where  $D$  is a division algebra over  $F$  and  $i$  is some integer. If  $K$  is a maximal separable subfield of  $D$  then  $[K : F] = n$  where  $[D : F] = n^2$ . We shall assume that  $K$  is normal over  $F$  and that  $[A : F] = [K : F]^2$  (see [9] for motivation). Let  $\text{Gal}(K/F)$  be the Galois group of  $K$  over  $F$ . For  $k \in K$  and  $\sigma_s \in \text{Gal}(K/F)$  we shall write  $\sigma_s(k)$  for the image of  $k$  under  $\sigma_s$ . By the Noether-Skolem theorem there is an invertible element  $u_s \in A$  such that  $\sigma_s(k) = u_s k u_s^{-1}$  for every  $k \in K$ . One can show that the  $u_s$ 's are linearly independent over  $K$ . However, the linear span over  $K$  of the  $u_s$ 's has dimension  $n^2$  over  $F$ , hence must be all of  $A$ . In short  $A = \{\sum_{s \in G} k_s u_s \mid k_s \in K\}$ . If  $\sigma_s, \sigma_t \in \text{Gal}(K/F)$  and  $k \in K$ , then

$$u_s u_t k u_t^{-1} u_s^{-1} = u_s \sigma_t(k) u_s^{-1} = \sigma_{st}(k) = u_{st} k u_{st}^{-1}$$

This says that  $u_{st}^{-1}(u_s u_t) \in C_A(K) = K$ , in other words  $u_s u_t = f(s, t) u_{st}$  where  $f(s, t) \neq 0$  is in  $K$ . Since  $A$  is an associative algebra one may verify that  $f : \text{Gal}(K/F) \times \text{Gal}(K/F) \rightarrow K \setminus \{0\}$  is in fact a cocycle. By Theorem 4.4.1 in [9], if  $K$  is a normal extension of  $F$  with Galois group  $\text{Gal}(K/F)$  and  $f$  is a cocycle (factor set), then the crossed product  $K \rtimes_f^\sigma \text{Gal}(K/F)$  is a central simple algebra over  $F$  and hence in this situation both **S1** and **S2** are in fact true.

## B.5 Comments to the literature

As we have mentioned in the introduction, the literature contains several different types of intersection theorems for group rings, Ore extensions and crossed products. Typically these theorems rely on heavy restrictions on the coefficient rings and the groups involved.

It was proven in [26, Theorem 1, Theorem 2] that the center of a semiprimitive (semisimple in the sense of Jacobson [10]) P.I. ring respectively semiprime P.I. ring has a non-zero intersection with every non-zero ideal in such a ring. For crystalline graded rings satisfying the conditions in [26, Theorem 2], it offers a more precise result than Corollary B.3.13 since  $Z(\mathcal{A}) \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$ . However, every crystalline graded ring need not be semiprime nor a P.I. ring and this justifies the need for Corollary B.3.13.

In [14, Lemma 2.6] it was proven that if the coefficient ring  $\mathcal{A}_0$  of a crossed product  $\mathcal{A}_0 * G$  is prime,  $P$  is a prime ideal in  $\mathcal{A}_0 * G$  such that  $P \cap \mathcal{A}_0 = 0$  and  $I$  is an ideal in  $\mathcal{A}_0 * G$  properly containing  $P$ , then  $I \cap \mathcal{A}_0 \neq 0$ . Furthermore, in [14, Proposition 5.4] it was proven that the crossed product  $\mathcal{A}_0 * G$  with  $G$  abelian and  $\mathcal{A}_0$  a  $G$ -prime ring has the property that, if  $G_{\text{inn}} = \{e\}$ , then every non-zero ideal in  $\mathcal{A}_0 * G$  has a

non-zero intersection with  $\mathcal{A}_0$ . It was shown in [3, Corollary 3] that if  $\mathcal{A}_0$  is semiprime and  $G_{\text{inn}} = \{e\}$ , then every non-zero ideal in  $\mathcal{A}_0 * G$  has a non-zero intersection with  $\mathcal{A}_0$ . In [15, Lemma 3.8] it was shown that if  $\mathcal{A}_0$  is a  $G$ -prime ring,  $P$  a prime ideal in  $\mathcal{A}_0 * G$  with  $P \cap \mathcal{A}_0 = 0$  and if  $I$  is an ideal in  $\mathcal{A}_0 * G$  properly containing  $P$ , then  $I \cap \mathcal{A}_0 \neq 0$ . In [18, Proposition 2.6] it was shown that if  $\mathcal{A}_0$  is a prime ring and  $I$  is a non-zero ideal in  $\mathcal{A}_0 * G$ , then  $I \cap (\mathcal{A}_0 * G_{\text{inn}}) \neq 0$ . In [18, Proposition 2.11] it was shown that for a crossed product  $\mathcal{A}_0 * G$  with prime ring  $\mathcal{A}_0$ , every non-zero ideal in  $\mathcal{A}_0 * G$  has a non-zero intersection with  $\mathcal{A}_0$  if and only if  $C^t[G_{\text{inn}}]$  is  $G$ -simple and in particular if  $|G_{\text{inn}}| < \infty$ , then every non-zero ideal in  $\mathcal{A}_0 * G$  has a non-zero intersection with  $\mathcal{A}_0$  if and only if  $\mathcal{A}_0 * G$  is prime.

Corollary B.3.14 shows that if  $\mathcal{A}_0$  is maximal commutative in the crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , without any further conditions on the ring  $\mathcal{A}_0$  or the group  $G$ , we are able to conclude that every non-zero ideal in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  has a non-zero intersection with  $\mathcal{A}_0$ .

In the theory of group rings (crossed products with no action or twisting) the intersection properties of ideals with certain subrings have played an important role and are studied in depth in for example [4], [13] and [25]. Some further properties of intersections of ideals and homogeneous components in graded rings have been studied in for example [2], [16].

For ideals in Ore extensions there are interesting results in [5, Theorem 4.1] and [11, Lemma 2.2, Theorem 2.3, Corollary 2.4], explaining a correspondence between certain ideals in the Ore extension and certain ideals in its coefficient ring. Given a domain  $\mathcal{A}$  of characteristic 0 and a non-zero derivation  $\delta$  it is shown in [6, Proposition 2.6] that every non-zero ideal in the Ore extension  $R = \mathcal{A}[x; \delta]$  intersects  $\mathcal{A}$  in a non-zero  $\delta$ -invariant ideal. Similar types of intersection results for ideals in Ore extension rings can be found in for example [12] and [17].

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## Paper C

# Crossed product-like and pre-crystalline graded rings

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**Abstract.** *We introduce crossed product-like rings, as a natural generalization of crystalline graded rings, and describe their basic properties. Furthermore, we prove that for certain pre-crystalline graded rings and every crystalline graded ring  $\mathcal{A}$ , for which the base subring  $\mathcal{A}_0$  is commutative, each non-zero two-sided ideal has a non-zero intersection with  $C_{\mathcal{A}}(\mathcal{A}_0)$ , i.e. the commutant of  $\mathcal{A}_0$  in  $\mathcal{A}$ . We also show that in general this property need not hold for crossed product-like rings.*

## C.1 Introduction

In the recent paper [3], by E. Nauwelaerts and F. Van Oystaeyen, so called crystalline graded rings were introduced, as a general class of group graded rings containing as special examples the algebraic crossed products, the Weyl algebras, the generalized Weyl algebras and generalizations of Clifford algebras. In the paper [4], by T. Neijens, F. Van Oystaeyen and W.W. Yu, the structure of the center of special classes of crystalline graded rings and generalized Clifford algebras was studied.

In this paper we prove that if  $\mathcal{A}$  is a crystalline graded ring, where the base subring  $\mathcal{A}_0$  is commutative, then each non-zero two-sided ideal has a non-zero intersection with  $C_{\mathcal{A}}(\mathcal{A}_0) = \{b \in \mathcal{A} \mid ab = ba, \quad \forall a \in \mathcal{A}_0\}$ , i.e. the commutant of  $\mathcal{A}_0$  in  $\mathcal{A}$  (Corollary C.5.5). Furthermore, we define *pre-crystalline graded rings* as the obvious generalization of crystalline graded rings, and show that under certain conditions the previously mentioned property also holds for pre-crystalline group graded rings (Theorem C.5.3).

We also introduce *crossed product-like rings* which is a broad class of rings, containing as special cases the pre-crystalline graded rings and hence also the crystalline graded rings. Crossed product-like rings are in general only monoid graded in contrast to crystalline graded rings and algebraic crossed products, which are group graded. A crossed product-like ring graded by the monoid  $M$  and with base ring  $\mathcal{A}_0$  will be denoted by  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ , where  $\sigma_s : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  is an additive and multiplicative set map for each  $s \in M$  and  $\alpha : M \times M \rightarrow \mathcal{A}_0$  is another set map (see Lemma C.2.3 for details). The product in

$\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is given by the bilinear extension of the rule

$$(a u_s)(b u_t) = a \sigma_s(b) \alpha(s, t) u_{st}$$

for  $a, b \in \mathcal{A}_0$  and  $s, t \in M$ . In contrary to the case of algebraic crossed products or more generally crystalline graded rings, for crossed product-like rings the maps  $\sigma_s$ ,  $s \in M$ , may be non-surjective, thus allowing non-standard examples of rings to fit in (Example C.6.2).

The center  $Z(\mathcal{A})$  (Proposition C.4.1) and the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  of the base subring in a general crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  (Theorem C.3.1) will be described and we will give an example of a (possibly) non-invertible dynamical system, to which we associate a monoid graded crossed product-like ring and show that it contains a non-zero two-sided ideal which has zero intersection with the commutant of the base subring (Proposition C.5.1). This displays a difference between monoid graded and group graded rings with regards to the intersection property that we are investigating.

## C.2 Preliminaries and definitions

We shall begin by giving the definitions of the rings that we are investigating and also describe some of their basic properties.

**Definition C.2.1** (Graded ring). Let  $M$  be a monoid with neutral element  $e$ . A ring  $\mathcal{R}$  is said to be *graded by  $M$*  if  $\mathcal{R} = \bigoplus_{s \in M} \mathcal{R}_s$  for additive subgroups  $\mathcal{R}_s$ ,  $s \in M$ , satisfying  $\mathcal{R}_s \mathcal{R}_t \subseteq \mathcal{R}_{st}$  for all  $s, t \in M$ . Moreover, if  $\mathcal{R}_s \mathcal{R}_t = \mathcal{R}_{st}$  holds for all  $s, t \in M$ , then  $\mathcal{R}$  is said to be *strongly graded by  $M$* .

It may happen that  $M$  is in fact a group and when we want to emphasize that a ring  $\mathcal{R}$  is graded by a monoid respectively a group we shall say that it is *monoid graded* respectively *group graded*. It is worth noting that in an associative and graded ring  $\mathcal{R} = \bigoplus_{s \in M} \mathcal{R}_s$ , by the gradation property,  $\mathcal{R}_e$  will always be a subring and furthermore for every  $s \in M$ ,  $\mathcal{R}_s$  is an  $\mathcal{R}_e$ -bimodule. For a thorough exposition of the theory of graded rings we refer to [1, 2].

So called *crystalline graded rings* were defined in [3] and further investigated in [4]. Right now we wish to weaken some of the conditions in the definition of those rings, in order to allow more general rings to fit in.

**Definition C.2.2** (Crossed product-like ring). An associative and unital ring  $\mathcal{A}$  is said to be *crossed product-like* if

- There is a monoid  $M$  (with neutral element  $e$ ).
- There is a map  $u : M \rightarrow \mathcal{A}$ ,  $s \mapsto u_s$  such that  $u_e = 1_{\mathcal{A}}$  and  $u_s \neq 0_{\mathcal{A}}$  for every  $s \in M$ .

- There is a subring  $\mathcal{A}_0 \subseteq \mathcal{A}$  containing  $1_{\mathcal{A}}$ .

such that the following conditions are satisfied:

- (P1)  $\mathcal{A} = \bigoplus_{s \in M} \mathcal{A}_0 u_s$ .
- (P2) For every  $s \in M$ ,  $u_s \mathcal{A}_0 \subseteq \mathcal{A}_0 u_s$  and  $\mathcal{A}_0 u_s$  is a free left  $\mathcal{A}_0$ -module of rank one.
- (P3) The decomposition in P1 makes  $\mathcal{A}$  into an  $M$ -graded ring with  $\mathcal{A}_0 = \mathcal{A}_e$ .

If  $M$  is a group and we want to emphasize that, then we shall say that  $\mathcal{A}$  is a *crossed product-like group graded ring*. If we want to emphasize that  $M$  is only a monoid, we shall say that  $\mathcal{A}$  is a *crossed product-like monoid graded ring*.

Similarly to the case of algebraic crossed products, one is able to find maps that can be used to describe the formation of arbitrary products in the ring. Some key properties of these maps are highlighted in the following lemma.

**Lemma C.2.3.** *With notation and definitions as above:*

- (i) For every  $s \in M$ , there is a set map  $\sigma_s : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  defined by  $u_s a = \sigma_s(a) u_s$  for  $a \in \mathcal{A}_0$ . The map  $\sigma_s$  is additive and multiplicative. Moreover,  $\sigma_e = \text{id}_{\mathcal{A}_0}$ .
- (ii) There is a set map  $\alpha : M \times M \rightarrow \mathcal{A}_0$  defined by  $u_s u_t = \alpha(s, t) u_{st}$  for  $s, t \in M$ . For any triple  $s, t, w \in M$  and  $a \in \mathcal{A}_0$  the following equalities hold:

$$\alpha(s, t) \alpha(st, w) = \sigma_s(\alpha(t, w)) \alpha(s, tw) \quad (\text{C.1})$$

$$\sigma_s(\sigma_t(a)) \alpha(s, t) = \alpha(s, t) \sigma_{st}(a) \quad (\text{C.2})$$

- (iii) For every  $s \in M$  we have  $\alpha(s, e) = \alpha(e, s) = 1_{\mathcal{A}}$ .

*Proof.* The proof is analogous to the proof of Lemma 2.1 in [3]. □

By the foregoing lemma we see that, for arbitrary  $a, b \in \mathcal{A}_0$  and  $s, t \in M$ , the product of  $a u_s$  and  $b u_t$  in the crossed product-like ring  $\mathcal{A}$  may be written as

$$(a u_s)(b u_t) = a \sigma_s(b) \alpha(s, t) u_{st}$$

and this is the motivation for the name *crossed product-like*. A crossed product-like ring  $\mathcal{A}$  with the above properties will be denoted by  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ , indicating the maps  $\sigma$  and  $\alpha$ .

**Remark C.2.4.** Note that for  $s \in M \setminus \{e\}$  we need not necessarily have  $\sigma_s(1_{\mathcal{A}_0}) = 1_{\mathcal{A}_0}$  and hence  $\sigma_s$  need not be a ring morphism.

**Definition C.2.5** (Pre-crystalline graded ring). A crossed product-like ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  where for each  $s \in M$ ,  $\mathcal{A}_0 u_s = u_s \mathcal{A}_0$ , is said to be a *pre-crystalline graded ring*.

For a pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ , the following lemma gives us additional information about the maps  $\sigma$  and  $\alpha$  defined in Lemma C.2.3.

**Lemma C.2.6.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is a pre-crystalline graded ring, then the following holds:*

(i) *For every  $s \in M$ , the map  $\sigma_s : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  is a surjective ring morphism.*

(ii) *If  $M$  is a group, then*

$$\alpha(g, g^{-1}) = \sigma_g(\alpha(g^{-1}, g))$$

*for each  $g \in M$ .*

*Proof.* The proof is the same as for Lemma 2.1 in [3]. □

In a pre-crystalline graded ring, one may show that for  $s, t \in M$ , the  $\alpha(s, t)$  are normalizing elements of  $\mathcal{A}_0$  in the sense that  $\mathcal{A}_0 \alpha(s, t) = \alpha(s, t) \mathcal{A}_0$  (see [3, Proposition 2.3]). If we in addition assume that  $\mathcal{A}_0$  is commutative, then we see by Lemma C.2.3 that the map  $\sigma : M \rightarrow \text{End}(\mathcal{A}_0)$  is a monoid morphism.

For a pre-crystalline group graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we let  $S(G)$  denote the multiplicative set in  $\mathcal{A}_0$  generated by  $\{\alpha(g, g^{-1}) \mid g \in G\}$  and let  $S(G \times G)$  be the multiplicative set generated by  $\{\alpha(g, h) \mid g, h \in G\}$ .

**Lemma C.2.7** (Corollary 2.7 in [3]). *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline group graded ring, then the following are equivalent:*

- $\mathcal{A}_0$  is  $S(G)$ -torsion free.
- $\mathcal{A}$  is  $S(G)$ -torsion free.
- $\alpha(g, g^{-1})a_0 = 0$  for some  $g \in G$  implies  $a_0 = 0$ .
- $\alpha(g, h)a_0 = 0$  for some  $g, h \in G$  implies  $a_0 = 0$ .
- $\mathcal{A}_0 u_g = u_g \mathcal{A}_0$  is also free as a right  $\mathcal{A}_0$ -module, with basis  $u_g$ , for every  $g \in G$ .
- For every  $g \in G$ ,  $\sigma_g$  is bijective and hence a ring automorphism of  $\mathcal{A}_0$ .

From Lemma C.2.7 we see that when  $\mathcal{A}_0$  is  $S(G)$ -torsion free in a pre-crystalline group graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we have  $\text{im}(\sigma) \subseteq \text{Aut}(\mathcal{A}_0)$ . We shall now state the definition of a crystalline graded ring.

**Definition C.2.8** (Crystalline graded ring). A pre-crystalline group graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  which is  $S(G)$ -torsion free is said to be a *crystalline graded ring*.

### C.3 The commutant of $\mathcal{A}_0$ in a crossed product-like ring

The commutant of the subring  $\mathcal{A}_0$  in the crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is defined by

$$C_{\mathcal{A}}(\mathcal{A}_0) = \{b \in \mathcal{A} \mid ab = ba, \quad \forall a \in \mathcal{A}_0\}.$$

In this section we will describe  $C_{\mathcal{A}}(\mathcal{A}_0)$  in various crossed product-like rings. Theorem C.3.1 tells us exactly when an element of a crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  lies in  $C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Theorem C.3.1.** *In a crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ , we have*

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{s \in M} r_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M \mid r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0, s \in M \right\}.$$

*Proof.* The proof is established through the following sequence of equivalences:

$$\begin{aligned} \sum_{s \in M} r_s u_s \in C_{\mathcal{A}}(\mathcal{A}_0) &\iff \left( \sum_{s \in M} r_s u_s \right) a = a \left( \sum_{s \in M} r_s u_s \right), \quad \forall a \in \mathcal{A}_0 \\ &\iff \sum_{s \in M} r_s \sigma_s(a) u_s = \sum_{s \in M} a r_s u_s, \quad \forall a \in \mathcal{A}_0 \\ &\iff \text{For each } s \in M : r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0. \end{aligned}$$

□

If  $\mathcal{A}_0$  is commutative, then for each  $r \in \mathcal{A}_0$  we denote its annihilator ideal in  $\mathcal{A}_0$  by  $\text{Ann}(r) = \{a \in \mathcal{A}_0 \mid ar = 0\}$  and get a simplified description of  $C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Corollary C.3.2.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is a crossed product-like ring and  $\mathcal{A}_0$  is commutative, then*

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{s \in M} r_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M \mid \sigma_s(a) - a \in \text{Ann}(r_s), \quad \forall a \in \mathcal{A}_0, s \in M \right\}.$$

When  $\mathcal{A}_0$  is commutative it is clear that  $\mathcal{A}_0 \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$ . Using the explicit description of  $C_{\mathcal{A}}(\mathcal{A}_0)$  in Corollary C.3.2, we immediately get necessary and sufficient conditions for  $\mathcal{A}_0$  to be maximal commutative in  $\mathcal{A}$ , i.e.  $\mathcal{A}_0 = C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Corollary C.3.3.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is a crossed product-like ring where  $\mathcal{A}_0$  is commutative, then  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  if and only if, for each pair  $(s, r_s) \in (M \setminus \{e\}) \times (\mathcal{A}_0 \setminus \{0_{\mathcal{A}_0}\})$ , there exists  $a \in \mathcal{A}_0$  such that  $\sigma_s(a) - a \notin \text{Ann}(r_s)$ .*

**Corollary C.3.4.** *Let  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  be a crossed product-like ring where  $\mathcal{A}_0$  is commutative. If for each  $s \in M \setminus \{e\}$  it is always possible to find some  $a \in \mathcal{A}_0$  such that  $\sigma_s(a) - a$  is not a zero-divisor in  $\mathcal{A}_0$ , then  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ .*

The next corollary is a consequence of Corollary C.3.3.

**Corollary C.3.5.** *If the subring  $\mathcal{A}_0$  in the crossed product-like ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is maximal commutative, then  $\sigma_s \neq \text{id}_{\mathcal{A}_0}$  for each  $s \in M \setminus \{e\}$ .*

The description of the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  from Corollary C.3.2 can be further refined in the case when  $\mathcal{A}_0$  is an integral domain.

**Corollary C.3.6.** *If  $\mathcal{A}_0$  is an integral domain, then the commutant of  $\mathcal{A}_0$  in the crossed product-like ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is*

$$C_{\mathcal{A}}(\mathcal{A}_0) = \left\{ \sum_{s \in \sigma^{-1}(\text{id}_{\mathcal{A}_0})} r_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M \mid r_s \in \mathcal{A}_0 \right\}$$

where  $\sigma^{-1}(\text{id}_{\mathcal{A}_0}) = \{s \in M \mid \sigma_s = \text{id}_{\mathcal{A}_0}\}$ .

The following corollary can be derived directly from Corollary C.3.5 together with either Corollary C.3.4 or Corollary C.3.6.

**Corollary C.3.7.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is a crossed product-like ring where  $\mathcal{A}_0$  is an integral domain, then  $\sigma_s \neq \text{id}_{\mathcal{A}_0}$  for all  $s \in M \setminus \{e\}$  if and only if  $\mathcal{A}_0$  is maximal commutative in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ .*

We will now give a sufficient condition for  $C_{\mathcal{A}}(\mathcal{A}_0)$  to be commutative.

**Proposition C.3.8.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is a crossed product-like ring where  $\mathcal{A}_0$  is commutative,  $M$  is abelian and  $\alpha(s, t) = \alpha(t, s)$  for all  $s, t \in M$ , then  $C_{\mathcal{A}}(\mathcal{A}_0)$  is commutative.*

*Proof.* Let  $\sum_{s \in M} r_s u_s$  and  $\sum_{t \in M} p_t u_t$  be arbitrary elements in  $C_{\mathcal{A}}(\mathcal{A}_0)$ . By our assumptions and Corollary C.3.2 we get

$$\begin{aligned} \left( \sum_{s \in M} r_s u_s \right) \left( \sum_{t \in M} p_t u_t \right) &= \sum_{(s,t) \in M \times M} r_s \sigma_s(p_t) \alpha(s, t) u_{st} \\ &= \sum_{(s,t) \in M \times M} r_s p_t \alpha(s, t) u_{st} \\ &= \sum_{(s,t) \in M \times M} p_t \sigma_t(r_s) \alpha(t, s) u_{ts} \\ &= \left( \sum_{t \in M} p_t u_t \right) \left( \sum_{s \in M} r_s u_s \right). \end{aligned}$$

□

### C.4 The center of a crossed product-like ring $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$

In this section we will describe the center  $Z(\mathcal{A}) = \{b \in \mathcal{A} \mid ab = ba, \forall a \in \mathcal{A}\}$  of a crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ . Note that  $Z(\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M) \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$ .

**Proposition C.4.1.** *The center of a crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is*

$$Z(\mathcal{A}) = \left\{ \sum_{g \in M} r_g u_g \mid \sum_{g \in M} r_g \alpha(g, s) u_{gs} = \sum_{g \in M} \sigma_s(r_g) \alpha(s, g) u_{sg} \right. \\ \left. r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0, s \in M \right\}.$$

*Proof.* Let  $\sum_{g \in M} r_g u_g \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  be an element which commutes with every element in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ . Then, in particular  $\sum_{g \in M} r_g u_g$  must commute with every  $a \in \mathcal{A}_0$ . From Theorem C.3.1 we immediately see that this implies  $r_s \sigma_s(a) = a r_s$  for every  $a \in \mathcal{A}_0$  and  $s \in M$ . Furthermore,  $\sum_{g \in M} r_g u_g$  must commute with  $u_s$  for every  $s \in M$ . This yields

$$\begin{aligned} \sum_{g \in M} r_g \alpha(g, s) u_{gs} &= \left( \sum_{g \in M} r_g u_g \right) u_s = u_s \left( \sum_{g \in M} r_g u_g \right) \\ &= \sum_{g \in M} \sigma_s(r_g) \alpha(s, g) u_{sg} \end{aligned}$$

for each  $s \in M$ .

Conversely, suppose that  $\sum_{g \in M} r_g u_g \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  is an element satisfying  $r_s \sigma_s(a) = a r_s$  and  $\sum_{g \in M} r_g \alpha(g, s) u_{gs} = \sum_{g \in M} \sigma_s(r_g) \alpha(s, g) u_{sg}$  for every  $a \in \mathcal{A}_0$  and  $s \in M$ . Let  $\sum_{s \in M} a_s u_s \in \mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$  be arbitrary. Then

$$\begin{aligned} \left( \sum_{g \in M} r_g u_g \right) \left( \sum_{s \in M} a_s u_s \right) &= \sum_{(g,s) \in M \times M} r_g \sigma_g(a_s) \alpha(g, s) u_{gs} \\ &= \sum_{(g,s) \in M \times M} a_s r_g \alpha(g, s) u_{gs} \\ &= \sum_{s \in M} a_s \left( \sum_{g \in M} r_g \alpha(g, s) u_{gs} \right) \end{aligned}$$



$$\begin{aligned}
 &= \sum_{s \in M} a_s \left( \sum_{g \in M} \sigma_s(r_g) \alpha(s, g) u_{sg} \right) \\
 &= \sum_{(s, g) \in M \times M} a_s \sigma_s(r_g) \alpha(s, g) u_{sg} \\
 &= \left( \sum_{s \in M} a_s u_s \right) \left( \sum_{g \in M} r_g u_g \right)
 \end{aligned}$$

and hence  $\sum_{g \in M} r_g u_g$  commutes with every element in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} M$ . □

A crossed product-like group graded ring offers a more simple description of its center.

**Corollary C.4.2.** *The center of a crossed product-like group graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is*

$$Z(\mathcal{A}) = \left\{ \sum_{g \in G} r_g u_g \mid r_{ts^{-1}} \alpha(ts^{-1}, s) = \sigma_s(r_{s^{-1}t}) \alpha(s, s^{-1}t), \right. \\
 \left. r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0, (s, t) \in G \times G \right\}.$$

**Corollary C.4.3.** *The center of a crossed product-like ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  graded by an abelian group  $G$  is*

$$Z(\mathcal{A}) = \left\{ \sum_{g \in G} r_g u_g \mid r_g \alpha(g, s) = \sigma_s(r_g) \alpha(s, g), \right. \\
 \left. r_s \sigma_s(a) = a r_s, \quad \forall a \in \mathcal{A}_0, s \in G \right\}.$$

## C.5 Intersection theorems

For any group graded algebraic crossed product  $\mathcal{A}$ , where the base ring  $\mathcal{A}_0$  is commutative, it was shown in [5] that

$$I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$$

for each non-zero two-sided ideal  $I$  in  $\mathcal{A}$  (see also [6]). In this section we will investigate if this property holds for more general classes of rings. It turns out that it need not hold in a crossed product-like monoid graded ring. However, we shall see that under certain conditions it will hold for pre-crystalline graded rings and that it always holds for crystalline graded rings if  $\mathcal{A}_0$  is assumed to be commutative.

### C.5.1 Crossed product-like monoid graded rings

Let  $X$  be a non-empty set and  $\gamma : X \rightarrow X$  a map (not necessarily injective nor surjective). We define the following sets:

$$\begin{aligned} \text{Per}^n(\gamma) &= \{x \in X \mid \gamma^{\circ(n)}(x) = x\}, \quad n \in \mathbb{Z}_{\geq 0} \\ \text{Per}(\gamma) &= \bigcup_{n \in \mathbb{Z}_{\geq 0}} \text{Per}^n(\gamma) \\ \text{Aper}(\gamma) &= X \setminus \text{Per}(\gamma) \end{aligned}$$

We let  $\mathbb{C}^X$  denote the algebra of complex-valued functions on  $X$ , with addition and multiplication being the usual pointwise operations. We define a map

$$\sigma : \mathbb{Z}_{\geq 0} \rightarrow \text{End}(\mathbb{C}^X)$$

by

$$\sigma_n(f) = f \circ \gamma^{\circ(n)}, \quad f \in \mathbb{C}^X$$

for  $n \in \mathbb{Z}_{\geq 0}$ . Let  $\mathcal{A}$  be the ring consisting of finite sums of the form  $\sum_{s \in \mathbb{Z}_{\geq 0}} f_s u_s$  where  $f_s \in \mathbb{C}^X$  for  $s \in \mathbb{Z}_{\geq 0}$ , and where  $u_i$  are non-zero elements satisfying  $u_0 = 1$ ,  $u_s u_t = u_{s+t}$  for all  $s, t \in \mathbb{Z}_{\geq 0}$  and

$$u_s f = \sigma_s(f) u_s, \quad \text{for } f \in \mathbb{C}^X.$$

It is easy to see that  $\mathcal{A} = \mathbb{C}^X \diamond_{\sigma} \mathbb{Z}_{\geq 0}$  is a crossed product-like ring. We omit to indicate the map  $\alpha : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}^X$  since it is mapped onto the identity in  $\mathbb{C}^X$  everywhere.

**Proposition C.5.1.** *With conventions and notation as above, if  $\text{Aper}(\gamma) \neq \emptyset$ , then there exists a non-zero two-sided ideal  $I$  in  $\mathcal{A} = \mathbb{C}^X \diamond_{\sigma} \mathbb{Z}_{\geq 0}$  such that*

$$I \cap C_{\mathcal{A}}(\mathbb{C}^X) = \{0\}.$$

*Proof.* Suppose that  $\text{Aper}(\gamma) \neq \emptyset$ . Clearly,  $\gamma(\text{Per}(\gamma)) \subseteq \text{Per}(\gamma)$  and  $\gamma(\text{Aper}(\gamma)) \subseteq \text{Aper}(\gamma)$ . For the pre-images we have  $\gamma^{-1}(\text{Per}(\gamma)) \subseteq \text{Per}(\gamma)$  and  $\gamma^{-1}(\text{Aper}(\gamma)) \subseteq \text{Aper}(\gamma)$ . Choose some non-zero  $f \in \mathbb{C}^X$  such that  $\text{supp}(f) \subseteq \text{Aper}(\gamma)$  and let  $I$  be the ideal in  $\mathcal{A}$  generated by  $f u_1$ . An arbitrary element  $c \in I$  may be written as

$$c = \sum_{k \in \mathbb{Z}_{\geq 0}} \left( \sum_{r_1, r_2 \in \mathbb{Z}_{\geq 0}} g_{r_1, k} u_{r_1} f u_1 h_{r_2, k} u_{r_2} \right)$$

where  $g_{i, j}, h_{i, j} \in \mathbb{C}^X$  for  $i, j \in \mathbb{Z}_{\geq 0}$ . This may be rewritten as

$$c = \sum_{t \in \mathbb{Z}_{> 0}} \underbrace{\left( \sum_{\substack{r_1, r_2 \in \mathbb{Z}_{\geq 0} \\ r_1 + r_2 + 1 = t}} \sum_{k \in \mathbb{Z}_{\geq 0}} g_{r_1, k} \sigma_{r_1}(f) \sigma_{r_1+1}(h_{r_2, k}) \right)}_{=a_t} u_t.$$

By elementary properties of the support of a function, we get

$$\begin{aligned} \text{supp}(a_t) \subseteq \bigcup_{\substack{r_1, r_2 \in \mathbb{Z}_{\geq 0} \\ r_1 + r_2 + 1 = t}} \bigcup_{k \in \mathbb{Z}_{\geq 0}} \left( \text{supp}(g_{r_1, k}) \cap (\gamma^{\circ(r_1)})^{-1}(\text{supp}(f)) \right. \\ \left. \cap (\gamma^{\circ(r_1+1)})^{-1}(\text{supp}(h_{r_2, k})) \right) \end{aligned}$$

from which we conclude that

$$\text{supp}(a_t) \subseteq \bigcup_{\substack{r_1 \in \mathbb{Z}_{\geq 0} \\ r_1 + 1 = t}} (\gamma^{\circ(r_1)})^{-1}(\text{supp}(f)). \quad (\text{C.3})$$

It follows from Corollary C.3.2 that  $\sum_{t \in \mathbb{Z}_{\geq 0}} a_t u_t$  lies in  $C_{\mathcal{A}}(\mathbb{C}^X)$  if and only if  $\text{supp}(a_t) \subseteq \text{Per}^t(\gamma)$  for each  $t > 0$ . If we can show that  $\text{Per}^t(\gamma) \cap \text{supp}(a_t) = \emptyset$  for every  $t$ , then we are finished. Suppose that there exists some  $t$  for which  $\text{Per}^t(\gamma) \cap \text{supp}(a_t) \neq \emptyset$ . It now follows by (C.3) that

$$\text{Per}^t(\gamma) \cap \bigcup_{\substack{r_1 \in \mathbb{Z}_{\geq 0} \\ r_1 + 1 = t}} (\gamma^{\circ(r_1)})^{-1}(\text{supp}(f)) \neq \emptyset$$

which is absurd, after noting that  $\bigcup_{r_1 \in \mathbb{Z}_{\geq 0}, r_1 + 1 = t} (\gamma^{\circ(r_1)})^{-1}(\text{supp}(f)) \subseteq \text{Aper}(\gamma)$ ,  $\text{Per}^t(\gamma) \subseteq \text{Per}(\gamma)$  for each  $t \in \mathbb{Z}_{>0}$  and that  $\text{Per}(\gamma) \cap \text{Aper}(\gamma) = \emptyset$ . This concludes the proof.  $\square$

**Remark C.5.2.** Let  $X$  be a non-empty set and  $\gamma : X \rightarrow X$  a bijection. The bijection gives rise to a map  $\tilde{\gamma} : \mathbb{Z} \rightarrow \text{Aut}(\mathbb{C}^X)$  given by  $\tilde{\gamma}_n(f) = f \circ \gamma^{\circ(n)}$ , for  $f \in \mathbb{C}^X$ , and we may define the  $\mathbb{Z}$ -graded crossed product  $\mathcal{A} = \mathbb{C}^X \rtimes_{\tilde{\gamma}} \mathbb{Z}$  (see for example [7–9]). It follows from [9, Theorem 3.1] or more generally from [5, Theorem 2], that

$$I \cap C_{\mathcal{A}}(\mathbb{C}^X) \neq \{0\}$$

for each non-zero two-sided ideal  $I$  in  $\mathbb{C}^X \rtimes_{\tilde{\gamma}} \mathbb{Z}$ .

Proposition C.5.1 and Remark C.5.2 display a difference between monoid graded crossed product-like rings and group graded crossed products.

### C.5.2 Group graded pre-crystalline and crystalline graded rings

Given a pre-crystalline graded ring  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , for each  $b \in \mathcal{A}_0$  we define the commutator to be

$$D_b : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{s \in G} a_s u_s \mapsto b \left( \sum_{s \in G} a_s u_s \right) - \left( \sum_{s \in G} a_s u_s \right) b.$$

From the definition of the multiplication we have

$$\begin{aligned} D_b \left( \sum_{s \in G} a_s u_s \right) &= b \left( \sum_{s \in G} a_s u_s \right) - \left( \sum_{s \in G} a_s u_s \right) b \\ &= \left( \sum_{s \in G} b a_s u_s \right) - \left( \sum_{s \in G} a_s \sigma_s(b) u_s \right) \\ &= \sum_{s \in G} (b a_s - a_s \sigma_s(b)) u_s \end{aligned}$$

for each  $b \in \mathcal{A}_0$ .

**Theorem C.5.3.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline group graded ring where  $\mathcal{A}_0$  is commutative and for each  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  there exists  $s \in G$  such that  $a_s \notin \ker(\sigma_s \circ \sigma_{s^{-1}})$ , then*

$$I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

*Proof.* Let  $I$  be an arbitrary non-zero two-sided ideal in  $\mathcal{A}$  and assume that  $\mathcal{A}_0$  is commutative and that for each  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  there exists  $s \in G$  such that  $a_s \notin \ker(\sigma_s \circ \sigma_{s^{-1}})$ . For each  $g \in G$  we may define a *translation operator*

$$T_g : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{s \in G} a_s u_s \mapsto \left( \sum_{s \in G} a_s u_s \right) u_g.$$

Note that, for each  $g \in G$ ,  $I$  is invariant under  $T_g$ . We have

$$T_g \left( \sum_{s \in G} a_s u_s \right) = \left( \sum_{s \in G} a_s u_s \right) u_g = \sum_{s \in G} a_s \alpha(s, g) u_{sg}$$

for every  $g \in G$ . By the assumptions and together with [3, Corollary 2.4] it is clear that for each element  $c \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  it is always possible to choose some  $g \in G$  and let  $T_g$  operate on  $c$  to end up with an element where the coefficient in front of  $u_e$  is non-zero.

Note that, for each  $b \in \mathcal{A}_0$ ,  $I$  is invariant under  $D_b$ . Furthermore, we have

$$\begin{aligned} D_b \left( \sum_{s \in G} a_s u_s \right) &= \sum_{s \in G} (b a_s - a_s \sigma_s(b)) u_s = \\ &= \sum_{s \neq e} (b a_s - a_s \sigma_s(b)) u_s = \sum_{s \neq e} d_s u_s \end{aligned}$$

since  $(b a_e - a_e \sigma_e(b)) = b a_e - a_e b = 0$ . Note that  $C_{\mathcal{A}}(\mathcal{A}_0) = \bigcap_{b \in \mathcal{A}_0} \ker(D_b)$  and hence for any  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_0)$  we are always able to choose  $b \in \mathcal{A}_0$  and the

corresponding  $D_b$  and have  $\sum_{s \in G} a_s u_s \notin \ker(D_b)$ . Therefore we can always pick an operator  $D_b$  which kills the coefficient in front of  $u_e$  without killing everything. Hence, if  $a_e \neq 0_{\mathcal{A}_0}$ , the number of non-zero coefficients of the resulting element will always be reduced by at least one.

The ideal  $I$  is assumed to be non-zero, which means that we can pick some non-zero element  $\sum_{s \in G} r_s u_s \in I$ . If  $\sum_{s \in G} r_s u_s \in C_{\mathcal{A}}(\mathcal{A}_0)$ , then we are finished, so assume that this is not the case. Note that  $r_s \neq 0_{\mathcal{A}_0}$  for finitely many  $s \in G$ . Recall that the ideal  $I$  is invariant under  $T_g$  and  $D_a$  for all  $g \in G$  and  $a \in \mathcal{A}_0$ . We may now use the operators  $\{T_g\}_{g \in G}$  and  $\{D_a\}_{a \in \mathcal{A}_0}$  to generate new elements of  $I$ . More specifically, we may use the  $T_g$ 's to translate our element  $\sum_{s \in G} r_s u_s$  into a new element which has a non-zero coefficient in front of  $u_e$  (if needed) after which we use the  $D_a$  operator to kill this coefficient and end up with yet another new element of  $I$  which is non-zero but has a smaller number of non-zero coefficients. We may repeat this procedure and in a finite number of iterations arrive at an element of  $I$  which lies in  $C_{\mathcal{A}}(\mathcal{A}_0) \setminus \mathcal{A}_0$ , and if not we continue the above procedure until we reach an element in  $\mathcal{A}_0 \setminus \{0_{\mathcal{A}_0}\}$ . In particular  $\mathcal{A}_0 \subseteq C_{\mathcal{A}}(\mathcal{A}_0)$  since  $\mathcal{A}_0$  is commutative and hence  $I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$ .  $\square$

**Corollary C.5.4.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline group graded ring where  $\mathcal{A}_0$  is maximal commutative and for each  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus \mathcal{A}_0$  there exists  $s \in G$  such that  $a_s \notin \ker(\sigma_s \circ \sigma_{s^{-1}})$ , then*

$$I \cap \mathcal{A}_0 \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

A crystalline graded ring has no  $S(G)$ -torsion and hence  $\ker(\sigma_s \circ \sigma_{s^{-1}}) = \{0_{\mathcal{A}_0}\}$  by [3, Corollary 2.4]. Therefore we get the following corollary which is a generalization of a result for algebraic crossed products in [5, Theorem 2].

**Corollary C.5.5.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring where  $\mathcal{A}_0$  is commutative, then*

$$I \cap C_{\mathcal{A}}(\mathcal{A}_0) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

When  $\mathcal{A}_0$  is maximal commutative we get the following corollary.

**Corollary C.5.6.** *If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring where  $\mathcal{A}_0$  is maximal commutative, then*

$$I \cap \mathcal{A}_0 \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ .

## C.6 Examples of crossed product-like and crystalline graded rings

**Example C.6.1** (The quantum torus). Let  $q \in \mathbb{C} \setminus \{0, 1\}$ . Denote by  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  the unital ring of polynomials over  $\mathbb{C}$  with four generators  $\{x, x^{-1}, y, y^{-1}\}$  and the defining commutation relations  $xx^{-1} = x^{-1}x = 1$ ,  $yy^{-1} = y^{-1}y = 1$  and

$$yx = qxy. \quad (\text{C.4})$$

This means that  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}] = \frac{\mathbb{C}\langle x, x^{-1}, y, y^{-1} \rangle}{\{xx^{-1}=x^{-1}x=1, yy^{-1}=y^{-1}y=1, yx=qxy\}}$ . This ring is sometimes called *the twisted Laurent polynomial ring* or *the quantum torus*. Following the notation of Definition C.2.2, let  $\mathcal{A}_0 = \mathbb{C}[x, x^{-1}]$ ,  $M = (\mathbb{Z}, +)$  and  $u_n = y^n$  for  $n \in M$ . It is not difficult to show that

$$\mathcal{A} = \mathbb{C}_q[x, x^{-1}, y, y^{-1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}[x, x^{-1}]y^n = \bigoplus_{n \in M} \mathcal{A}_0 u_n$$

and that the other conditions in Definition C.2.2 are satisfied as well. Therefore,  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  can be viewed as a crossed product-like group graded ring. From the defining commutation relations and the choice of  $u_n$ , it follows that  $\sigma_n : P(x) \mapsto P(q^n x)$  for  $n \in M$  and  $P(x) \in \mathcal{A}_0$ , and that  $\alpha(s, t) = 1_{\mathcal{A}}$  for all  $s, t \in M$ , following the notation of Lemma C.2.3.

In the current example,  $\mathcal{A}_0$  is an integral domain. Thus, by Corollary C.3.7, the subring  $\mathcal{A}_0 = \mathbb{C}[x, x^{-1}]$  is maximal commutative in  $\mathbb{C}_q[x, x^{-1}, y, y^{-1}]$  if and only if  $q^n x \neq x$  for every  $n \neq 0$  or equivalently if and only if  $q$  is not a root of unity.

**Example C.6.2** (Twisted functions in the complex plane). Let  $\mathbb{C}^{\mathbb{C}}$  be the algebra of functions  $\mathbb{C} \rightarrow \mathbb{C}$ , with addition and multiplication being the usual pointwise operations. Fix a pair of numbers  $q \in \mathbb{C} \setminus \{0\}$  and  $d \in \mathbb{Z}_{>0}$  and consider the map

$$\gamma : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto qz^d.$$

We define

$$\sigma_n : \mathbb{C}^{\mathbb{C}} \rightarrow \mathbb{C}^{\mathbb{C}}, \quad f \mapsto f \circ \gamma^{\circ(n)}$$

for  $n \in \mathbb{Z}_{>0}$  and set  $\sigma_0 = \text{id}_{\mathbb{C}^{\mathbb{C}}}$ . If  $n \geq 2$ , then this yields

$$(\sigma_n(f))(z) = f(q^{1+d+\dots+d^{n-1}} z^{d^n})$$

for  $f \in \mathbb{C}^{\mathbb{C}}$  and  $z \in \mathbb{C}$ . Let  $\mathcal{A}$  be the algebra consisting of finite sums of the form  $\sum_{i \in \mathbb{Z}_{\geq 0}} f_i u_i$ , where  $f_i \in \mathbb{C}^{\mathbb{C}}$  for each  $i \in \mathbb{Z}_{\geq 0}$ , and  $u_i$  are non-zero elements satisfying

1.  $u_0 = 1$

$$2. \quad u_n u_m = u_{n+m}, \quad \text{for all } n, m \in \mathbb{Z}_{\geq 0}$$

and also such that  $u_i$  does not in general commute with  $\mathbb{C}^{\mathbb{C}}$ , but satisfy

$$u_n f = \sigma_n(f) u_n, \quad \text{for } f \in \mathbb{C}^{\mathbb{C}} \text{ and } n \in \mathbb{Z}_{\geq 0}.$$

One may easily verify that this corresponds to the crossed product-like monoid graded ring  $\mathbb{C}^{\mathbb{C}} \diamond_{\sigma} \mathbb{Z}_{\geq 0}$ . Consider the set

$$\text{Per}^n(\gamma) = \{z \in \mathbb{C} \mid \sigma_n(f)(z) = f(z), \quad \forall f \in \mathbb{C}^{\mathbb{C}}\}. \quad (\text{C.5})$$

Note that  $\text{Per}^0(\gamma) = \mathbb{C}$  and that  $\text{Per}^1(\gamma)$  contains the solutions to the equation  $qz^d = z$ . For  $n \in \mathbb{Z}_{\geq 2}$  we have

$$\text{Per}^n(\gamma) = \{z \in \mathbb{C} \mid q^{1+d+\dots+d^{n-1}} z^{d^n} = z\}. \quad (\text{C.6})$$

By using the formula for a geometric series, it is easy to see that, for each  $n \in \mathbb{Z}_{\geq 1}$ ,  $\text{Per}^n$  contains the point  $z = 0$  and  $d^n - 1$  points equally distributed along a circle with radius  $r = |q|^{\frac{1}{1-d}}$  and with its center in the origin in the complex plane.

The following proposition follows from Corollary C.3.2.

**Proposition C.6.3.** *With notation and definitions as above, for  $\mathcal{A} = \mathbb{C}^{\mathbb{C}} \diamond_{\sigma} \mathbb{Z}_{\geq 0}$  we have*

$$C_{\mathcal{A}}(\mathbb{C}^{\mathbb{C}}) = \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} f_i u_i \mid \text{supp}(f_i) \subseteq \text{Per}^i(\gamma), \quad i \in \mathbb{Z}_{\geq 0} \right\}.$$

It now becomes clear that  $\mathbb{C}^{\mathbb{C}}$  is never maximal commutative in  $\mathbb{C}^{\mathbb{C}} \diamond_{\sigma} \mathbb{Z}_{\geq 0}$ . In fact Proposition C.6.3 makes it possible to explicitly provide the elements in the commutant  $C_{\mathcal{A}}(\mathbb{C}^{\mathbb{C}})$  that are not in  $\mathbb{C}^{\mathbb{C}}$ . If  $f_0, f_1, f_2 \in \mathbb{C}^{\mathbb{C}}$ , then by using (C.5) and (C.6) and Proposition C.6.3, we conclude that  $a = f_0 + f_1 u_1 + f_2 u_2$  lies in  $C_{\mathcal{A}}(\mathbb{C}^{\mathbb{C}})$  if and only if,  $\text{supp}(f_1)$  respectively  $\text{supp}(f_2)$  is contained in  $\text{Per}^1(\gamma)$  respectively  $\text{Per}^2(\gamma)$ .

**Example C.6.4** (The first Weyl algebra). Following [3], let

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y \rangle / (yx - xy - 1)$$

be the first Weyl algebra. If we put  $\deg(x) = 1$  and  $\deg(y) = -1$  and

$$\begin{aligned} A_1(\mathbb{C})_0 &= \mathbb{C}[xy] \\ A_1(\mathbb{C})_n &= \mathbb{C}[xy]x^n, \quad \text{for } n \geq 0 \\ A_1(\mathbb{C})_m &= \mathbb{C}[xy]y^{-m}, \quad \text{for } m \leq 0 \end{aligned}$$

then this defines a  $\mathbb{Z}$ -gradation on  $A_1(\mathbb{C})$ . We set  $u_n = x^n$  if  $n \geq 0$  and  $u_m = y^{-m}$  if  $m \leq 0$ . It is clear that  $\sigma_1(xy) = xy - 1$  because  $x(xy) = (xy - 1)x$  and  $\sigma_{-1}(xy) = xy + 1$  because  $y(xy) = (1 + xy)y$ . Let us put  $t = xy$ , then

$$\sigma : \mathbb{Z} \rightarrow \text{Aut}_{\mathbb{C}}(\mathbb{C}[t]), \quad n \mapsto (t \mapsto t - n).$$

We can also calculate, for example

$$\begin{aligned} \alpha(n, -n) &= x^n y^n = x^{n-1} t y^{n-1} = (t - (n-1)) x^{n-1} y^{n-1} \\ &= (t - n + 1) \cdot (t - n + 2) \cdot \dots \cdot (t - 2) \cdot (t - 1) \cdot t. \end{aligned}$$

Furthermore  $\alpha(n, -m)$  with  $n > m$  ( $n, m \in \mathbb{Z}_{\geq 0}$ ) can be calculated from

$$x^n y^m = x^{n-m} x^m y^m = x^{n-m} \alpha(m, -m) = \sigma_{n-m}(\alpha(m, -m)) x^{n-m}$$

and so

$$\alpha(n, -m) = \sigma_{n-m}(\alpha(m, -m)).$$

It is well-known that  $A_1(\mathbb{C})$  is a simple algebra and hence the only non-zero two-sided ideal is  $A_1(\mathbb{C})$  itself, which clearly has a non-zero intersection with  $\mathbb{C}[xy]$ . Furthermore,  $\mathbb{C}[xy]$  is an integral domain and  $\sigma_n \neq \text{id}_{\mathbb{C}[xy]}$  for each  $n \neq 0$  and by Corollary C.3.7, the base ring  $\mathbb{C}[xy]$  is maximal commutative in  $A_1(\mathbb{C})$ .

**Example C.6.5** (Generalized Weyl algebras). Let  $\mathcal{A}_0$  be an associative and unital ring and fix a positive integer  $n$  and set  $\mathbf{n} = \{1, 2, \dots, n\}$  and let  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$  be a set of commuting automorphisms of  $\mathcal{A}_0$ . Let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  be an  $n$ -tuple with nonzero entries in  $Z(\mathcal{A}_0)$  such that  $\sigma_i(a_j) = a_j$  for  $i \neq j$ . The *generalized Weyl algebra*  $\mathcal{A} = \mathcal{A}_0(\sigma, \mathbf{a})$  is defined as the ring generated by  $\mathcal{A}_0$  and  $2n$  symbols  $X_1, \dots, X_n, Y_1, \dots, Y_n$  satisfying the following rules:

1. For each  $i \in \{1, \dots, n\}$  :

$$Y_i X_i = a_i, \quad \text{and} \quad X_i Y_i = \sigma_i(a_i)$$

2. For all  $d \in \mathcal{A}_0$  and each  $i \in \{1, \dots, n\}$ ,

$$X_i d = \sigma_i(d) X_i, \quad \text{and} \quad Y_i d = \sigma_i^{-1}(d) Y_i$$

3. For  $i \neq j$ ,

$$\begin{aligned} [Y_i, Y_j] &= 0 \\ [X_i, X_j] &= 0 \\ [X_i, Y_j] &= 0 \end{aligned}$$

where  $[x, y] = xy - yx$ .



Now for  $m \in \mathbb{Z}$  we write  $u_m(i) = (X_i)^m$  if  $m > 0$  and  $u_m(i) = (Y_i)^{-m}$  if  $m < 0$ . For  $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$  we set  $u_{\mathbf{k}} = u_{k_1}(1) \cdot \dots \cdot u_{k_n}(n)$ . By putting  $\mathcal{A} = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} \mathcal{A}_{\mathbf{k}}$  where  $\mathcal{A}_{\mathbf{k}} = \mathcal{A}_0 u_{\mathbf{k}}$ , we see that  $\mathcal{A}$  is a  $\mathbb{Z}^n$ -graded ring, which is crystalline graded (see [3]). If the base ring  $\mathcal{A}_0$  is commutative, just like in many of the examples of this class of rings, we get by Corollary C.5.5 that each nonzero two-sided ideal in the generalized Weyl algebra  $\mathcal{A}$  contains a non-zero element which commutes with all of  $\mathcal{A}_0$ .

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## **Paper D**

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**D**



## Paper D

# Commutativity and ideals in strongly graded rings

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**Abstract.** *In some recent papers by the first two authors it was shown that for any algebraic crossed product  $\mathcal{A}$ , where  $\mathcal{A}_0$ , the subring in the degree zero component of the grading, is a commutative ring, each non-zero two-sided ideal in  $\mathcal{A}$  has a non-zero intersection with the commutant  $C_{\mathcal{A}}(\mathcal{A}_0)$  of  $\mathcal{A}_0$  in  $\mathcal{A}$ . This result has also been generalized to crystalline graded rings; a more general class of graded rings to which algebraic crossed products belong. In this paper we generalize this result in another direction, namely to strongly graded rings (in some literature referred to as generalized crossed products) where the subring  $\mathcal{A}_0$ , the degree zero component of the grading, is a commutative ring. We also give a description of the intersection between arbitrary ideals and commutants to bigger subrings than  $\mathcal{A}_0$ , and this is done both for strongly graded rings and crystalline graded rings.*

## D.1 Introduction

Dynamical systems, generated by the iteration of homeomorphisms of compact Hausdorff spaces, lead to crossed product algebras of continuous functions on the space, by the action of the additive group of integers via composition of continuous functions with the iterations of the homeomorphisms. In the context of  $C^*$ -algebras, the interplay between topological properties of the dynamical system such as minimality, transitivity, freeness on one hand, and properties of ideals, subalgebras and representations of the corresponding crossed product on the other hand, has been a subject of intensive investigation at least since the 1960's, both for single map dynamics and for more general actions of groups and semigroups, that is in particular iterations of several transformations called iterated function systems in the literature on fractals and dynamical systems. Such constant and growing interest to this interplay between dynamics, actions and non-commutative algebras can be explained by the fundamental importance of this interplay and its implications for operator representations of the corresponding crossed product algebras, spectral and harmonic analysis and non-commutative analysis and non-commutative geometry fundamental for the mathematical foundations of quantum mechanics, quantum field

theory, string theory, integrable systems, lattice models, quantization, symmetry analysis, renormalization, and recently in analysis and geometry of fractals and in wavelet analysis and its applications in signal and image processing (see [1–5, 7, 9–14, 19–21, 28–30, 33, 39, 42, 44, 55, 56, 58] and references therein).

There has been a substantial progress on the interplay between  $C^*$ -algebras and dynamics of iterations of continuous transformations and more general actions of groups on compact Hausdorff spaces [2, 10, 44, 55–57]. However, the investigation of actions of not necessarily continuous transformations on more general and more irregular spaces than Hausdorff spaces requires an extension of this interplay beyond  $C^*$ -algebras to a purely algebraic framework of general algebras and rings. Only partial progress in this important direction has been made. In [45–48], extensions and modifications of this result and the interplay between dynamics and maximal commutativity properties of the canonical coefficient subalgebra, the degree zero component of the grading, and its intersection with ideals was investigated for dynamical systems that are not topologically free on more general spaces than Hausdorff spaces both in the context of algebraic crossed products by  $\mathbb{Z}$  and for the corresponding Banach algebra and  $C^*$ -algebra crossed products in the case of single homeomorphism dynamical systems or more general dynamical systems generated by an invertible map. Also in these works, this interplay has been considered from the point of view of representations as well as with respect to duality in the crossed product algebras. Some results, that could be considered as related to this direction of interplay, have been scattered within the purely algebraic literature on graded rings and algebras [6, 8, 15–18, 22–27, 31–35, 40, 41, 43, 49–54]. In many of these related results, very special properties are assumed for the coefficient subring or for the whole crossed product or graded ring or algebra, such as being a ring without zero-divisors, semi-simple or simple ring, etc. This has been motivated in most cases by the desire to use the algebraic constructions, tools and techniques that were available at the time. However, it turns out that these restrictions often exclude for example many important classes of algebras arising in physics and associated to actions on algebras and rings of functions on infinite spaces or other algebras and rings with zero-divisors, and many other situations. Thus, it is desirable to investigate the above mentioned interplay between actions and properties of ideals and subalgebras for general graded rings and crossed product rings and algebras and their generalizations, without any restrictive artificially imposed conditions. It turns out that many interesting properties and results hold in such much greater generality and also as a consequence, new previously not noticed results and constructions come to light.

In this paper we focus on the connections between the structure of ideals and the commutant of subrings in generalizations of crossed product rings and in general classes of graded rings. This, in particular, provides a general understanding of the conditions for maximal commutativity of the degree zero component subalgebra of the grading and properties of more general subalgebras important for representation theory. In this paper we substantially extend the approach and some of the key results in [36–38] to more general subrings than the degree zero component of the grading in crossed products, or

more general graded rings.

In Section 2 we briefly recall the basics of graded rings and crossed products. Given a ring  $\mathcal{R}$  and a subset  $\mathcal{S} \subseteq \mathcal{R}$ , we denote by

$$C_{\mathcal{R}}(\mathcal{S}) = \{r \in \mathcal{R} \mid rs = sr, \forall s \in \mathcal{S}\}$$

the commutant of  $\mathcal{S}$  in  $\mathcal{R}$ . The following theorem was shown in [36].

**Theorem D.1.1.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a  $G$ -crossed product where  $\mathcal{R}_e$  is commutative, then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$$

*for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .*

Given a normal subgroup  $N$  of  $G$  one can consider the subring  $\mathcal{R}_N = \bigoplus_{n \in N} \mathcal{R}_n$  in  $\mathcal{R}$  and obtain a generalization of Theorem D.1.1 by considering the intersection between arbitrary non-zero ideals and  $C_{\mathcal{R}}(\mathcal{R}_N)$ . This is done in Section 3 (Theorem D.3.3).

In Section 4 we consider general strongly graded rings  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ , which are not necessarily crossed products. Given any subgroup  $H$  of  $G$  we give a description of the commutant of  $\mathcal{R}_H$  in  $\mathcal{R}$  (Theorem D.4.7) and prove the main theorem (Theorem D.4.9). We obtain some interesting corollaries (Corollary D.4.10, Corollary D.4.12 and Corollary D.4.13) which generalize the results obtained in Section 2 and generalize Theorem D.1.1 to general strongly graded rings.

In Section 5 we recall the definition and basic properties of *crystalline graded rings*, a class of graded rings which are not necessarily strongly graded (for more details see [34, 35, 38]). Given a subgroup  $H$  of  $G$  we give a description of the commutant of  $\mathcal{A}_H$  in the crystalline graded ring  $\mathcal{A}$  and give sufficient conditions for each non-zero two-sided ideal  $I$  in  $\mathcal{A}$  to have a non-zero intersection with  $C_{\mathcal{A}}(\mathcal{A}_H)$  (Theorem D.5.7).

## D.2 Preliminaries

Throughout this paper all rings are assumed to be unital and associative and we let  $G$  be an arbitrary group with neutral element  $e$ .

A ring  $\mathcal{R}$  is said to be  *$G$ -graded* if

$$\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g \quad \text{and} \quad \mathcal{R}_g \mathcal{R}_h \subseteq \mathcal{R}_{gh}$$

for all  $g, h \in G$ , where  $\{\mathcal{R}_g\}_{g \in G}$  is a family of additive subgroups in  $\mathcal{R}$ . The additive subgroup  $\mathcal{R}_g$  is called the homogeneous component of  $\mathcal{R}$  of degree  $g \in G$ . Moreover, if  $\mathcal{R}_g \mathcal{R}_h = \mathcal{R}_{gh}$  holds for all  $g, h \in G$ , then  $\mathcal{R}$  is said to be *strongly graded by  $G$*  and if we in addition have

$$U(\mathcal{R}) \cap \mathcal{R}_g \neq \emptyset$$



for each  $g \in G$ , where  $U(\mathcal{R})$  denotes the group of multiplication invertible elements in  $\mathcal{R}$ , then  $\mathcal{R}$  is said to be a  $G$ -crossed product.

Suppose that we are given a group  $G$ , a ring  $\mathcal{R}_e$  and two maps  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$  and  $\alpha : G \times G \rightarrow U(\mathcal{R}_e)$  satisfying the following conditions

$$\sigma_g(\sigma_h(a)) \alpha(g, h) = \alpha(g, h) \sigma_{gh}(a) \quad (\text{D.1})$$

$$\alpha(g, h) \alpha(gh, s) = \sigma_g(\alpha(h, s)) \alpha(g, hs) \quad (\text{D.2})$$

$$\alpha(g, e) = \alpha(e, g) = 1_{\mathcal{R}_e} \quad (\text{D.3})$$

for all  $g, h, s \in G$  and  $a \in \mathcal{R}_e$ . We may then choose a family of symbols  $\{v_g\}_{g \in G}$  and define  $\mathcal{R}'$  to be the free left  $\mathcal{R}_e$ -module with basis  $\{v_g\}_{g \in G}$  and define a multiplication on the set  $\mathcal{R}'$  by

$$(a_1 v_g)(a_2 v_h) = a_1 \sigma_g(a_2) \alpha(g, h) v_{gh}$$

for  $a_1, a_2 \in \mathcal{R}_e$  and  $g, h \in G$ . It turns out that  $\mathcal{R}'$  is an associative and unital ring with this multiplication and that it is in fact a  $G$ -crossed product, where the homogeneous component of degree  $g \in G$  is given by  $\mathcal{R}_e v_g$ .

Conversely, given a  $G$ -crossed product  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ , one can choose a family of elements  $\{u_g\}_{g \in G}$  in  $\mathcal{R}$  such that  $u_g \in U(\mathcal{R}) \cap \mathcal{R}_g$  for each  $g \in G$  and put  $u_e = 1_{\mathcal{R}}$ . It is clear that  $\mathcal{R}_g = \mathcal{R}_e u_g = u_g \mathcal{R}_e$  and that the set  $\{u_g\}_{g \in G}$  is a basis for  $\mathcal{R}$  as a left (and right)  $\mathcal{R}_e$ -module. We may now define a map

$$\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$$

by  $u_g a = \sigma_g(a) u_g$  for all  $a \in \mathcal{R}$  and  $g \in G$ . Furthermore, we define a map

$$\alpha : G \times G \rightarrow U(\mathcal{R}_e)$$

by  $\alpha(g, h) = u_g u_h u_{gh}^{-1}$  and it is straight forward to check that these maps satisfy conditions (1)-(3) above. Furthermore, one can use these maps together with  $G$  and  $\mathcal{R}_e$  and make the previous construction and obtain a  $G$ -crossed product  $\mathcal{R}'$  which actually turns out to be isomorphic to the  $G$ -crossed product  $\mathcal{R}$  that we started with. For more details on this we refer to [33, Proposition 1.4.1 and Proposition 1.4.2].

**Remark D.2.1.** The above crossed product will be denoted by  $\mathcal{R}_e \rtimes_{\alpha}^{\sigma} G$ , to indicate the maps  $\sigma$  and  $\alpha$ .

### D.3 Subrings graded by subgroups

Given a  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  and a non-empty subset  $X$  of  $G$ , we denote

$$\mathcal{R}_X = \bigoplus_{x \in X} \mathcal{R}_x.$$

In particular, if  $H$  is a subgroup of  $G$ , then  $\mathcal{R}_H = \bigoplus_{h \in H} \mathcal{R}_h$  is a subring in  $\mathcal{R}$ , and it is in fact an  $H$ -graded ring. The following lemma can be found in [22, Proposition 1.7].

**Lemma D.3.1.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a  $G$ -graded ring and  $N$  is a normal subgroup of  $G$ , then  $\mathcal{R}$  can be regarded as a  $G/N$ -graded ring, where the homogeneous components are given by*

$$\mathcal{R}_{gN} = \bigoplus_{x \in gN} \mathcal{R}_x$$

for  $gN \in G/N$ . Moreover, if  $\mathcal{R}$  is a crossed product of  $G$  over  $\mathcal{R}_e$ , then  $\mathcal{R}$  can also be regarded as a crossed product of  $G/N$  over

$$\mathcal{R}_N = \bigoplus_{x \in N} \mathcal{R}_x.$$

**Proposition D.3.2.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g = \mathcal{R}_e \rtimes_{\alpha}^{\sigma} G$  be a  $G$ -crossed product and suppose that  $N$  is a subgroup of  $G$ . If the following conditions are satisfied*

- (i)  $\mathcal{R}_e$  is commutative
- (ii)  $N \subseteq Z(G) \cap \ker(\sigma)$
- (iii)  $\alpha(x, y) = \alpha(y, x)$  for all  $(x, y) \in N \times N$

then  $\mathcal{R}_N$  is commutative.

*Proof.* Let the family of elements  $\{u_g\}_{g \in G}$  be chosen as in Section 2. To prove that  $\mathcal{R}_N$  is commutative, it suffices to show that for any  $g, h \in N$  and  $a_g, b_h \in \mathcal{R}_e$  the two elements  $a_g u_g$  and  $b_h u_h$  commute. By our assumptions, we have

$$\begin{aligned} (a_g u_g) (b_h u_h) &= a_g \sigma_g(b_h) \alpha(g, h) u_{gh} = a_g b_h \alpha(g, h) u_{gh} \\ &= b_h a_g \alpha(h, g) u_{hg} = b_h \sigma_h(a_g) \alpha(h, g) u_{hg} = (b_h u_h) (a_g u_g) \end{aligned}$$

and hence  $\mathcal{R}_N$  is commutative.  $\square$

**Theorem D.3.3.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g = \mathcal{R}_e \rtimes_{\alpha}^{\sigma} G$  is a  $G$ -crossed product and the following conditions are satisfied*

- (i)  $\mathcal{R}_e$  is commutative
- (ii)  $N$  is a subgroup of  $G$ , such that  $N \subseteq Z(G) \cap \ker(\sigma)$
- (iii)  $\alpha(x, y) = \alpha(y, x)$  for all  $(x, y) \in N \times N$

then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_N) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .

*Proof.* It is clear that  $N$  is normal in  $G$  and it follows from Lemma D.3.1 that  $\mathcal{R}_e \rtimes_{\alpha}^{\sigma} G = \mathcal{R}_N \rtimes_{\alpha'}^{\sigma'} G/N$  for some maps  $\sigma'$  and  $\alpha'$ . By our assumptions and Proposition D.3.2 we see that  $\mathcal{R}_N$  is commutative, and hence by Theorem D.1.1 it follows that each non-zero two-sided ideal in  $\mathcal{R}$  has a non-zero intersection with  $C_{\mathcal{R}}(\mathcal{R}_N)$ .  $\square$

**Corollary D.3.4.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g = \mathcal{R}_e \rtimes^{\sigma} G$  is a  $G$ -graded skew group ring where  $\mathcal{R}_e$  is commutative and  $N \subseteq Z(G) \cap \ker(\sigma)$  is a subgroup of  $G$ , then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_N) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .

**Remark D.3.5.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a  $G$ -graded ring. If

$$\{e\} \subseteq \dots \subseteq G_k \subseteq Z(G) \subseteq G_n \subseteq \dots \subseteq G$$

is an increasing chain of subgroups of  $G$ , then we get

$$\mathcal{R}_e \subseteq \dots \subseteq \mathcal{R}_{G_k} \subseteq \mathcal{R}_{Z(G)} \subseteq \mathcal{R}_{G_n} \subseteq \dots \subseteq \mathcal{R}$$

as an increasing chain of subrings in  $\mathcal{R}$  and the corresponding

$$C_{\mathcal{R}}(\mathcal{R}_e) \supseteq \dots \supseteq C_{\mathcal{R}}(\mathcal{R}_{G_k}) \supseteq C_{\mathcal{R}}(\mathcal{R}_{Z(G)}) \supseteq C_{\mathcal{R}}(\mathcal{R}_{G_n}) \supseteq \dots \supseteq C_{\mathcal{R}}(\mathcal{R}) = Z(\mathcal{R})$$

as a decreasing chain of subrings in  $\mathcal{R}$ . The existence of non-trivial subgroups  $N$  of  $G$  satisfying the conditions of Theorem D.3.3 therefore provides more precise information about the ideals in the crossed product than the previous Theorem D.1.1. By the arguments made above it is clear that  $N = Z(G) \cap \ker(\sigma)$ , the biggest normal subgroup to fit into our theorems, is the most interesting case to consider since it makes  $C_{\mathcal{R}}(\mathcal{R}_N)$  as small as possible.

## D.4 Strongly graded rings

In this section we let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring, not necessarily a crossed product. It follows that  $1_{\mathcal{R}} \in \mathcal{R}_e$  since  $\mathcal{R}$  is  $G$ -graded (see [33, Proposition 1.1.1]), and that  $\mathcal{R}_g \mathcal{R}_{g^{-1}} = \mathcal{R}_e$  for each  $g \in G$  since  $\mathcal{R}$  is strongly  $G$ -graded. Thus, for each  $g \in G$  there exists a positive integer  $n_g$  and elements  $a_g^{(i)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$ , such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}. \quad (\text{D.4})$$

For every  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ , and in particular for every  $\lambda \in Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$ , and  $g \in G$  we define

$$\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}. \quad (\text{D.5})$$

**Remark D.4.1.** The definition of  $\sigma_g$  is independent of the choice of the  $a_g^{(i)}$ 's and  $b_{g^{-1}}^{(i)}$ 's. Indeed, given positive integers  $n_g, n'_g$  and elements  $a_g^{(i)}, c_g^{(j)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)}, d_{g^{-1}}^{(j)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$  and  $j \in \{1, \dots, n'_g\}$  such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}} \quad \text{and} \quad \sum_{j=1}^{n'_g} c_g^{(j)} d_{g^{-1}}^{(j)} = 1_{\mathcal{R}},$$

for  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  we get

$$\begin{aligned} & \left( \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)} \right) - \left( \sum_{j=1}^{n'_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)} \right) \\ &= 1_{\mathcal{R}} \left( \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)} \right) - \left( \sum_{j=1}^{n'_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)} \right) 1_{\mathcal{R}} \\ &= \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \underbrace{d_{g^{-1}}^{(j)} a_g^{(i)}}_{\in \mathcal{R}_e} \lambda b_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)} a_g^{(i)} b_{g^{-1}}^{(i)} \\ &= \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)} a_g^{(i)} b_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)} a_g^{(i)} b_{g^{-1}}^{(i)} = 0. \end{aligned}$$

**Lemma D.4.2.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $a \in \mathcal{R}$  is such that

$$a \mathcal{R}_g = \{0\}$$

for some  $g \in G$ , then  $a = 0$ .

*Proof.* Suppose that  $a \mathcal{R}_g = \{0\}$  for some  $g \in G$ ,  $a \in \mathcal{R}$ . We then have  $a \mathcal{R}_g \mathcal{R}_{g^{-1}} = \{0\}$  or equivalently  $a \mathcal{R}_e = \{0\}$ . From the fact that  $1_{\mathcal{R}} \in \mathcal{R}_e$ , we conclude that  $a = 0$ .  $\square$

For the convenience of the reader we include the following lemma from [22, Proposition 1.8].

**Lemma D.4.3.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring,  $g \in G$  and write

$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}$  for some  $n_g > 0$  and  $a_g^{(i)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$ .

For each  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  define  $\sigma_g(\lambda)$  by  $\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}$ . The following properties hold:

- (i)  $\sigma_g(\lambda)$  is a unique element of  $\mathcal{R}$  satisfying

$$r_g \lambda = \sigma_g(\lambda) r_g, \quad \forall r_g \in \mathcal{R}_g. \quad (\text{D.6})$$

Furthermore,  $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$  and if  $\lambda \in Z(\mathcal{R}_e)$ , then  $\sigma_g(\lambda) \in Z(\mathcal{R}_e)$ .

- (ii) The group  $G$  acts as automorphisms of the rings  $C_{\mathcal{R}}(\mathcal{R}_e)$  and  $Z(\mathcal{R}_e)$ , with each  $g \in G$  sending any  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  and  $\lambda \in Z(\mathcal{R}_e)$ , respectively, into  $\sigma_g(\lambda)$ .
- (iii)  $Z(\mathcal{R}) = \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_g(\lambda) = \lambda, \forall g \in G\}$ , i.e.  $Z(\mathcal{R})$  is the fixed subring  $C_{\mathcal{R}}(\mathcal{R}_e)^G$  of  $C_{\mathcal{R}}(\mathcal{R}_e)$  with respect to the action of  $G$ .

*Proof.* (i) Let  $g \in G$ . If  $r_g \in \mathcal{R}_g$ , then  $b_{g^{-1}}^{(i)} r_g \in \mathcal{R}_{g^{-1}} \mathcal{R}_g = \mathcal{R}_e$  and so  $b_{g^{-1}}^{(i)} r_g$  commutes with  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  for each  $i \in \{1, \dots, n_g\}$ . It follows that

$$\sigma_g(\lambda) r_g = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)} r_g = \sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} r_g \lambda = r_g \lambda.$$

Take an arbitrary  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  and let  $x \in \mathcal{R}$  be an element satisfying  $a_g^{(i)} \lambda = x a_g^{(i)}$  for all  $i \in \{1, \dots, n_g\}$ . This yields

$$\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)} = \sum_{i=1}^{n_g} x a_g^{(i)} b_{g^{-1}}^{(i)} = x$$

which shows that  $\sigma_g(\lambda)$  is a unique element satisfying (D.6). By the strong gradation it follows that if  $\lambda \in \mathcal{R}_e$ , then  $\sigma_g(\lambda) \in \mathcal{R}_e$ . In particular if  $\lambda \in Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$ , then  $\sigma_g(\lambda) \in Z(\mathcal{R}_e)$ . Indeed, for  $\lambda \in Z(\mathcal{R}_e)$  and  $c \in \mathcal{R}_e$  we have

$$\begin{aligned} c \sigma_g(\lambda) &= 1_{\mathcal{R}} c \sigma_g(\lambda) = \sum_{i=1}^{n_g} \sum_{j=1}^{n'_g} a_g^{(j)} b_{g^{-1}}^{(j)} \underbrace{c a_g^{(i)}}_{\in \mathcal{R}_e} \lambda b_{g^{-1}}^{(i)} \\ &= \sum_{i=1}^{n_g} \sum_{j=1}^{n'_g} a_g^{(j)} \lambda b_{g^{-1}}^{(j)} c a_g^{(i)} b_{g^{-1}}^{(i)} = \sigma_g(\lambda) c 1_{\mathcal{R}} = \sigma_g(\lambda) c \end{aligned}$$

where  $\sum_{j=1}^{n'_g} a_g^{(j)} b_{g^{-1}}^{(j)} = 1_{\mathcal{R}}$ , and hence it only remains to verify that  $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$  for an arbitrary  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ . If  $r_g \in \mathcal{R}_g$  and  $z \in \mathcal{R}_e$ , then  $z r_g \in \mathcal{R}_e \mathcal{R}_g = \mathcal{R}_g$ , so we have

$$\begin{aligned} (\sigma_g(\lambda) z) r_g &= \sigma_g(\lambda) (z r_g) = (z r_g) \lambda \\ &= z (r_g \lambda) = z (\sigma_g(\lambda) r_g) = (z \sigma_g(\lambda)) r_g \end{aligned}$$

which means that  $(\sigma_g(\lambda) z - z \sigma_g(\lambda)) \mathcal{R}_g = \{0\}$ . By Lemma D.4.2 we conclude that  $\sigma_g(\lambda) z = z \sigma_g(\lambda)$  and hence  $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$ .

(ii) Since  $1_{\mathcal{R}} \in \mathcal{R}_e$ , we have for each  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  that

$$\lambda = 1_{\mathcal{R}} \lambda = \sigma_e(\lambda) 1_{\mathcal{R}} = \sigma_e(\lambda).$$

If  $g, h \in G$ ,  $r_g \in \mathcal{R}_g$  and  $r_h \in \mathcal{R}_h$ , then  $r_g r_h \in \mathcal{R}_{gh}$  and for  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  we have

$$\begin{aligned} \sigma_{gh}(\lambda) (r_g r_h) &= (r_g r_h) \lambda = r_g (r_h \lambda) = r_g (\sigma_h(\lambda) r_h) \\ &= (r_g \sigma_h(\lambda)) r_h = (\sigma_g(\sigma_h(\lambda)) r_g) r_h = \sigma_g(\sigma_h(\lambda)) (r_g r_h). \end{aligned}$$

The products of the form  $r_g r_h$  generate the submodule  $\mathcal{R}_{gh}$  and by Lemma D.4.2 we conclude that

$$\sigma_g(\sigma_h(\lambda)) = \sigma_{gh}(\lambda)$$

proving that  $(g, \lambda) \mapsto \sigma_g(\lambda)$  is an action of  $G$  on the set  $C_{\mathcal{R}}(\mathcal{R}_e)$ . Now take an arbitrary  $g \in G$  and fix it. By the definition of  $\sigma_g(\lambda)$ , the map  $\lambda \mapsto \sigma_g(\lambda)$  is clearly additive. For some positive integer  $n_{g-1}$  we may choose  $c_{g-1}^{(j)} \in \mathcal{R}_{g-1}$  and  $d_g^{(j)} \in \mathcal{R}$  for  $j \in \{1, \dots, n_{g-1}\}$ , such that  $1_{\mathcal{R}} = \sum_{j=1}^{n_{g-1}} c_{g-1}^{(j)} d_g^{(j)}$  and define  $\sigma_{g-1}$  following (D.5). Then, for each  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ , we get

$$\begin{aligned} \sigma_{g-1}(\sigma_g(\lambda)) &= \sum_{j=1}^{n_{g-1}} c_{g-1}^{(j)} \left( \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g-1}^{(i)} \right) d_g^{(j)} \\ &= \sum_{j=1}^{n_{g-1}} c_{g-1}^{(j)} \left( \sum_{i=1}^{n_g} a_g^{(i)} \lambda \underbrace{b_{g-1}^{(i)} d_g^{(j)}}_{\in \mathcal{R}_e} \right) \\ &= \sum_{j=1}^{n_{g-1}} c_{g-1}^{(j)} \left( \sum_{i=1}^{n_g} a_g^{(i)} b_{g-1}^{(i)} d_g^{(j)} \lambda \right) \\ &= \sum_{j=1}^{n_{g-1}} c_{g-1}^{(j)} \underbrace{\left( \sum_{i=1}^{n_g} a_g^{(i)} b_{g-1}^{(i)} \right)}_{=1_{\mathcal{R}}} d_g^{(j)} \lambda = \underbrace{\sum_{j=1}^{n_{g-1}} c_{g-1}^{(j)} d_g^{(j)}}_{=1_{\mathcal{R}}} \lambda = \lambda \end{aligned}$$

and hence  $\sigma_{g-1}$  is the inverse of  $\sigma_g$ . For any  $\lambda, t \in C_{\mathcal{R}}(\mathcal{R}_e)$  and  $r_g \in \mathcal{R}_g$ , we get

$$\begin{aligned} \sigma_g(\lambda t) r_g &= r_g (\lambda t) = (r_g \lambda) t = (\sigma_g(\lambda) r_g) t \\ &= \sigma_g(\lambda) (r_g t) = \sigma_g(\lambda) (\sigma_g(t) r_g) = (\sigma_g(\lambda) \sigma_g(t)) r_g. \end{aligned}$$

By Lemma D.4.2 this implies  $\sigma_g(\lambda t) = \sigma_g(\lambda) \sigma_g(t)$ . Therefore, for each  $g \in G$ , the map  $\lambda \mapsto \sigma_g(\lambda)$  is an automorphism of the ring  $C_{\mathcal{R}}(\mathcal{R}_e)$ .

(iii) Since  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is strongly  $G$ -graded, we have

$$Z(\mathcal{R}) = \bigcap_{g \in G} C_{\mathcal{R}}(\mathcal{R}_g) = \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \lambda \in C_{\mathcal{R}}(\mathcal{R}_g), \forall g \in G\}$$

and the result now follows from the fact that an element  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  centralizes  $\mathcal{R}_g$ ,  $g \in G$ , if and only if  $\sigma_g(\lambda) = \lambda$ . Indeed, if  $\sigma_g(\lambda) = \lambda$  for some  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ , then clearly  $\lambda$  centralizes  $\mathcal{R}_g$ . Conversely, if we suppose that  $\mathcal{R}_g$  is centralized by some  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ , then we have  $(\sigma_g(\lambda) - \lambda)\mathcal{R}_g = \{0\}$  and hence by Lemma D.4.2 we have  $\sigma_g(\lambda) = \lambda$ . □

**Remark D.4.4.** We have shown that  $G$  acts as automorphisms of  $C_{\mathcal{R}}(\mathcal{R}_e)$ . However, note that since  $\mathcal{R}_e$  is not assumed to be commutative, we may have  $\mathcal{R}_e \not\subseteq C_{\mathcal{R}}(\mathcal{R}_e)$  and hence  $G$  does not necessarily act as automorphisms of  $\mathcal{R}_e$ . This should be compared to the case of an algebraic crossed product as described in the previous section. For crossed products, if  $\mathcal{R}_e$  is commutative, then we see that  $G$  acts as automorphisms of  $\mathcal{R}_e$ , but in general this is not true.

**Lemma D.4.5.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring and  $\sigma : G \rightarrow \text{Aut}(C_{\mathcal{R}}(\mathcal{R}_e))$  defined as in (D.5). If  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then  $\ker(\sigma) = \{e\}$ .*

*Proof.* By our assumption  $\mathcal{R}_e = C_{\mathcal{R}}(\mathcal{R}_e)$  is maximal commutative in  $\mathcal{R}$  and hence for each  $g \neq e$  and all  $r_g \in \mathcal{R}_g$ , there must exist some  $\lambda \in \mathcal{R}_e$  such that  $\lambda r_g \neq r_g \lambda = \sigma_g(\lambda) r_g$ , using the definition of  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$ . Hence  $\sigma_g \neq \text{id}_{\mathcal{R}_e}$  for each  $g \neq e$ . □

We shall now state an obvious, but useful lemma.

**Lemma D.4.6.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring, then*

$$\begin{aligned} C_{\mathcal{R}}(\mathcal{R}_e) &= \left\{ \lambda = \sum_{g \in G} \lambda_g \in \mathcal{R} \mid \lambda_g \in \mathcal{R}_g, r_e \lambda_g = \lambda_g r_e, \forall g \in G, \forall r_e \in \mathcal{R}_e \right\} \\ &= \left\{ \lambda = \sum_{g \in G} \lambda_g \in \mathcal{R} \mid \lambda_g \in \mathcal{R}_g \cap C_{\mathcal{R}}(\mathcal{R}_e), \forall g \in G \right\} \\ &= \bigoplus_{g \in G} (\mathcal{R}_g \cap C_{\mathcal{R}}(\mathcal{R}_e)) \end{aligned}$$

The following theorem is a generalization of (iii) of Lemma D.4.3.

**Theorem D.4.7.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring,  $H$  a subgroup of  $G$  and denote  $\mathcal{R}_H = \bigoplus_{h \in H} \mathcal{R}_h$ . If  $\sigma : G \rightarrow \text{Aut}(C_{\mathcal{R}}(\mathcal{R}_e))$  is the action defined in (D.5), then it follows that*

$$\begin{aligned} C_{\mathcal{R}}(\mathcal{R}_H) &= \left\{ \lambda = \sum_{g \in G} \lambda_g \in \mathcal{R} \mid \lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e) \cap \mathcal{R}_g, \right. \\ &\quad \left. \sigma_h(\lambda_g) = \lambda_{hgh^{-1}}, \forall g \in G, \forall h \in H \right\} \\ &= \{ \lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_h(\lambda) = \lambda, \forall h \in H \}. \end{aligned}$$

*Proof.* Let  $\lambda = \sum_{g \in G} \lambda_g \in C_{\mathcal{R}}(\mathcal{R}_H)$ , with  $\lambda_g \in \mathcal{R}_g$ , be arbitrary. Since  $\mathcal{R}_e \subseteq \mathcal{R}_H$ , we have  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  and from Lemma D.4.6 we see that  $\lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e)$  for each  $g \in G$ . For every  $r_h \in \mathcal{R}_h$ ,  $h \in H$ , we have

$$r_h \sum_{g \in G} \lambda_g = \sum_{g \in G} \lambda_g r_h$$

since  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_H)$ , but  $\lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e)$  so from (D.6) we get

$$\sum_{g \in G} \sigma_h(\lambda_g) r_h = \sum_{g \in G} \lambda_g r_h$$

which is an equality in  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ . If we look in the  $\mathcal{R}_h \mathcal{R}_g \mathcal{R}_{h^{-1}} \mathcal{R}_h = \mathcal{R}_{hg}$  component, for all  $g \in G$ ,  $h \in H$ , we deduce that

$$\sigma_h(\lambda_g) r_h = \lambda_{hgh^{-1}} r_h, \quad \forall r_h \in \mathcal{R}_h$$

since the sum  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is direct. Applying the above equality to the elements  $a_h^{(i)}$  of  $\mathcal{R}_h$  in (D.4), we get

$$\sigma_h(\lambda_g) a_h^{(i)} = \lambda_{hgh^{-1}} a_h^{(i)}$$

for each  $i \in \{1, \dots, n_h\}$ , which implies

$$\begin{aligned} \sigma_h(\lambda_g) &= \sigma_h(\lambda_g) 1_{\mathcal{R}} = \sigma_h(\lambda_g) \sum_{i=1}^{n_h} a_h^{(i)} b_{h^{-1}}^{(i)} = \sum_{i=1}^{n_h} \sigma_h(\lambda_g) a_h^{(i)} b_{h^{-1}}^{(i)} \\ &= \sum_{i=1}^{n_h} \lambda_{hgh^{-1}} a_h^{(i)} b_{h^{-1}}^{(i)} = \lambda_{hgh^{-1}} \sum_{i=1}^{n_h} a_h^{(i)} b_{h^{-1}}^{(i)} = \lambda_{hgh^{-1}}. \end{aligned}$$

Conversely, let  $\lambda = \sum_{g \in G} \lambda_g \in \mathcal{R}$ , where  $\lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e) \cap \mathcal{R}_g$  and  $\sigma_h(\lambda_g) = \lambda_{hgh^{-1}}$ , for all  $g \in G$ ,  $h \in H$ . Then, for every  $r_h \in \mathcal{R}_h$ ,

$$r_h \lambda = \sum_{g \in G} r_h \lambda_g = \sum_{g \in G} \sigma_h(\lambda_g) r_h = \sum_{g \in G} \lambda_{hgh^{-1}} r_h = \sum_{k \in G} \lambda_k r_h = \lambda r_h$$



and hence  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_H)$ . This concludes the proof.  $\square$

**Remark D.4.8.** If  $\mathcal{R}_e$  is commutative, then  $\mathcal{R}_e = Z(\mathcal{R}_e)$ . Thus, if  $\mathcal{R}_e$  is commutative, then  $G$  acts as automorphisms of the ring  $\mathcal{R}_e$ .

**Theorem D.4.9.** Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is commutative and  $\ker(\sigma)$  is the kernel of the previously defined action  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$ , i.e.  $\ker(\sigma) = \{g \in G \mid \sigma_g(\lambda_e) = \lambda_e, \forall \lambda_e \in \mathcal{R}_e\}$ . If  $H$  is a subgroup of  $G$  which is contained in  $\ker(\sigma) \cap Z(G)$ , then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_H) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .

*Proof.* Let  $I$  be an arbitrary non-zero two-sided ideal in  $\mathcal{R}$ . For every  $h \in H$  and  $r_h \in \mathcal{R}_h$  we define a kill operator

$$D_{r_h} : \mathcal{R} \rightarrow \mathcal{R}, \quad D_{r_h} \left( \sum_{g \in G} \lambda_g \right) = r_h \sum_{g \in G} \lambda_g - \sum_{g \in G} \lambda_g r_h = \sum_{k \in G} d_k.$$

Note that for every non-zero summand  $\lambda_g \in \mathcal{R}_g$  of  $\lambda = \sum_{g \in G} \lambda_g$ , we take a summand  $d_{hg} = r_h \lambda_g - \lambda_g r_h \in \mathcal{R}_{gh} = \mathcal{R}_{hg}$  of  $D_{r_h}(\lambda)$  which may be zero or non-zero, but

$$d_h = r_h \lambda_e - \lambda_e r_h = \sigma_h(\lambda_e) r_h - \lambda_e r_h = \lambda_e r_h - \lambda_e r_h = 0.$$

Thus, for  $\lambda = \sum_{g \in G} \lambda_g \in \mathcal{R}$  with  $\lambda_e \neq 0$  and  $D_{r_h}(\lambda) = \sum_{k \in G} d_k$ , we get

$$\#\text{supp}(\lambda) = \#\{s \in G \mid \lambda_s \neq 0\} > \#\{s \in G \mid d_s \neq 0\} = \#\text{supp}(D_{r_h}(\lambda)).$$

Furthermore, note that for all  $r_h \in \mathcal{R}_h$ ,  $I$  is invariant under  $D_{r_h}$  and

$$C_{\mathcal{R}}(\mathcal{R}_H) = \bigcap_{\substack{h \in H \\ r_h \in \mathcal{R}_h}} \ker(D_{r_h}).$$

Now, let  $\lambda = \sum_{g \in G} \lambda_g \in I$  be non-zero. We can assume that  $\lambda_e \neq 0$ . Otherwise there exists some non-zero  $\lambda' = \sum_{g \in G} \lambda'_g \in I$  with  $\lambda'_e \neq 0$ . Indeed,  $\lambda \neq 0$  so there exists  $t \in G$  such that  $\lambda_t \neq 0$ . There exists, as well, some  $j \in \{1, \dots, n_t\}$  such that  $b_{t-1}^{(j)} \lambda_t \neq 0$ , where  $b_{t-1}^{(j)} \in \mathcal{R}_{t-1}$  is as in (D.4), because if  $b_{t-1}^{(i)} \lambda_t = 0, \forall i \in \{1, \dots, n_t\}$ , then

$$\lambda_t = 1_{\mathcal{R}} \cdot \lambda_t = \sum_{i=1}^{n_t} a_t^{(i)} b_{t-1}^{(i)} \lambda_t = 0.$$

Thus, for every non-zero element  $\lambda$  of  $I$  we can have an element  $b_{t-1}^{(j)}\lambda = \lambda' = \sum_{g \in G} \lambda'_g$  of  $I$  with  $\lambda'_e = b_{t-1}^{(j)}\lambda_t \neq 0$ , and  $\# \text{supp}(\lambda) \geq \# \text{supp}(\lambda') \geq 1$ .

We return to the element  $\lambda = \sum_{g \in G} \lambda_g \in I$  with  $\lambda_e \neq 0$ . If  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_H)$  we have nothing to prove. If  $\lambda \notin C_{\mathcal{R}}(\mathcal{R}_H)$ , then there exists  $h \in H$  and  $r_h \in \mathcal{R}_h$  such that  $D_{r_h}(\lambda) \neq 0$ . But  $D_{r_h}(\lambda) \in I$  so we have a new element of  $I$  with smaller support. If we continue in the same way, the procedure must eventually end, because  $\text{supp}(\lambda) < \infty$ . So, there will be a stop of this procedure which gives an element  $\mu = \sum_{g \in G} \mu_g \in I \cap C_{\mathcal{R}}(\mathcal{R}_H)$ , with  $\mu_e \neq 0$ .  $\square$

The following corollary generalizes Theorem D.3.3 to the situation when  $\mathcal{R}_N$  need not necessarily be commutative.

**Corollary D.4.10.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g = \mathcal{R}_e \rtimes_{\sigma'} G$  is a  $G$ -crossed product and both of the following conditions are satisfied*

- (i)  $\mathcal{R}_e$  is commutative
- (ii)  $N$  is a subgroup of  $G$ , such that  $N \subseteq Z(G) \cap \ker(\sigma')$

then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_N) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .

*Proof.* For each  $g \in G$  we may choose  $u_g \in U(R) \cap \mathcal{R}_g$ . It follows from [33, Proposition 1.1.1] that  $(u_g)^{-1} \in \mathcal{R}_{g^{-1}}$ . Clearly  $u_g u_g^{-1} = 1_{\mathcal{R}}$  and following (D.5) we define  $\sigma_g(a) = u_g a u_g^{-1}$  for all  $a \in C_{\mathcal{R}}(\mathcal{R}_e)$ . In particular  $\mathcal{R}_e \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$  since  $\mathcal{R}_e$  is commutative, and it is now clear that the restriction of  $\sigma_g$  to  $\mathcal{R}_e$  is equal to  $\sigma'_g$  for each  $g \in G$ . From Theorem D.4.9 it now follows that each non-zero two-sided ideal in  $\mathcal{R}$  has a non-zero intersection with  $C_{\mathcal{R}}(\mathcal{R}_N)$ .  $\square$

The following corollary generalizes [36, Theorem 2] from  $G$ -crossed products to strongly  $G$ -graded rings.

**Corollary D.4.11.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is commutative, then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .

*Proof.* Consider the subgroup  $\{e\}$  of  $G$ . Clearly  $\{e\} \subseteq Z(G) \cap \ker(\sigma)$  and since  $\mathcal{R}_e$  is commutative it follows from Theorem D.4.9 that  $I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$  for each non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .  $\square$

**Corollary D.4.12.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then*

$$I \cap \mathcal{R}_e \neq \{0\}$$

*for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .*

*Proof.* By the assumption  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , i.e.  $C_{\mathcal{R}}(\mathcal{R}_e) = \mathcal{R}_e$ , and hence the desired conclusion follows immediately from Corollary D.4.11.  $\square$

**Corollary D.4.13.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a twisted group ring, where  $\mathcal{R}_e$  is commutative and  $G$  is abelian, then*

$$I \cap Z(\mathcal{R}) \neq \{0\}$$

*for every non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .*

*Proof.* Since  $\mathcal{R}$  is a twisted group ring, all homogeneous elements commute with  $\mathcal{R}_e$ . Hence,  $\ker(\sigma) = G$  and moreover  $G = Z(G)$  since  $G$  is abelian. Consider the subgroup  $\ker(\sigma) \cap Z(G) = G \cap G = G$  of  $G$  and note that  $C_{\mathcal{R}}(\mathcal{R}_G) = Z(\mathcal{R})$ . By our assumptions  $\mathcal{R}_e$  is commutative and it now follows from Theorem D.4.9 that

$$I \cap Z(\mathcal{R}) \neq \{0\}$$

for each non-zero two-sided ideal  $I$  in  $\mathcal{R}$ .  $\square$

**Remark D.4.14.** It was shown in [43, Theorem 2] that if  $\mathcal{R}$  is a semiprime P.I. ring, then  $I \cap Z(\mathcal{R}) \neq \{0\}$  for each non-zero ideal  $I$  in  $\mathcal{R}$ .

## D.5 Crystalline graded rings

We shall begin this section by recalling the definition of a crystalline graded ring. We would also like to emphasize that rings of this class are in general not strongly graded.

**Definition D.5.1** (Pre-crystalline graded ring). An associative and unital ring  $\mathcal{A}$  is said to be *pre-crystalline graded* if

- (i) there is a group  $G$  (with neutral element  $e$ ),
- (ii) there is a map  $u : G \rightarrow \mathcal{A}$ ,  $g \mapsto u_g$  such that  $u_e = 1_{\mathcal{A}}$  and  $u_g \neq 0$  for every  $g \in G$ ,
- (iii) there is a subring  $\mathcal{A}_0 \subseteq \mathcal{A}$  containing  $1_{\mathcal{A}}$ ,

such that the following conditions are satisfied:

$$(P1) \quad \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_0 u_g ;$$

- (P2) For every  $g \in G$ ,  $u_g \mathcal{A}_0 = \mathcal{A}_0 u_g$  and this is a free left  $\mathcal{A}_0$ -module of rank one ;  
(P3) The decomposition in P1 makes  $\mathcal{A}$  into a  $G$ -graded ring with  $\mathcal{A}_0 = \mathcal{A}_e$ .

**Lemma D.5.2** (see [34]). *With notation and definitions as above:*

- (i) *For every  $g \in G$ , there is a set map  $\sigma_g : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  defined by  $u_g a = \sigma_g(a) u_g$  for  $a \in \mathcal{A}_0$ . The map  $\sigma_g$  is a surjective ring morphism. Moreover,  $\sigma_e = \text{id}_{\mathcal{A}_0}$ .*  
(ii) *There is a set map  $\alpha : G \times G \rightarrow \mathcal{A}_0$  defined by  $u_s u_t = \alpha(s, t) u_{st}$  for  $s, t \in G$ . For any triple  $s, t, w \in G$  and  $a \in \mathcal{A}_0$  the following equalities hold:*

$$\alpha(s, t)\alpha(st, w) = \sigma_s(\alpha(t, w))\alpha(s, tw) \quad (\text{D.7})$$

$$\sigma_s(\sigma_t(a))\alpha(s, t) = \alpha(s, t)\sigma_{st}(a) \quad (\text{D.8})$$

- (iii) *For every  $g \in G$  we have  $\alpha(g, e) = \alpha(e, g) = 1_{\mathcal{A}_0}$  and  $\alpha(g, g^{-1}) = \sigma_g(\alpha(g^{-1}, g))$ .*

A pre-crystalline graded ring  $\mathcal{A}$  with the above properties will be denoted by  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  and each element of this ring is written as a sum  $\sum_{g \in G} r_g u_g$  with coefficients  $r_g \in \mathcal{A}_0$ , of which only finitely many are non-zero. In [34] it was shown that for pre-crystalline graded rings, the elements  $\alpha(s, t)$  are normalizing elements of  $\mathcal{A}_0$ , i.e.  $\mathcal{A}_0 \alpha(s, t) = \alpha(s, t) \mathcal{A}_0$  for each  $s, t \in G$ . For a pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we let  $S(G)$  denote the multiplicative set in  $\mathcal{A}_0$  generated by  $\{\alpha(g, g^{-1}) \mid g \in G\}$  and let  $S(G \times G)$  denote the multiplicative set generated by  $\{\alpha(g, h) \mid g, h \in G\}$ .

**Lemma D.5.3** (see [34]). *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring, then the following assertions are equivalent:*

- (i)  $\mathcal{A}_0$  is  $S(G)$ -torsion free.  
(ii)  $\mathcal{A}$  is  $S(G)$ -torsion free.  
(iii)  $\alpha(g, g^{-1}) a_0 = 0$  for some  $g \in G$  implies  $a_0 = 0$ .  
(iv)  $\alpha(g, h) a_0 = 0$  for some  $g, h \in G$  implies  $a_0 = 0$ .  
(v)  $\mathcal{A}_0 u_g = u_g \mathcal{A}_0$  is also free as a right  $\mathcal{A}_0$ -module, with basis  $u_g$ , for every  $g \in G$ .  
(vi) For every  $g \in G$ ,  $\sigma_g$  is bijective and hence a ring automorphism of  $\mathcal{A}_0$ .

**Definition D.5.4** (Crystalline graded ring). A pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , which is  $S(G)$ -torsion free, is said to be a *crystalline graded ring*.

Examples of crystalline graded rings are given by the algebraic crossed products, the generalized twisted group rings, the Weyl algebras, the quantum Weyl algebra, the generalized Weyl algebras, quantum  $\text{sl}_2$  etc. For more examples we refer to [34].

**Proposition D.5.5.** *Let  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  be a pre-crystalline graded ring,  $H$  a subgroup of  $G$  and consider the subring  $\mathcal{A}_H = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} H$  in  $\mathcal{A}$ . The commutant of  $\mathcal{A}_H$  in  $\mathcal{A}$  is*

$$C_{\mathcal{A}}(\mathcal{A}_H) = \left\{ \sum_{g \in G} r_g u_g \in \mathcal{A} \mid r_{th^{-1}} \alpha(th^{-1}, h) = \sigma_h(r_{h^{-1}t}) \alpha(h, h^{-1}t), \right. \\ \left. r_t \sigma_t(a) = a r_t, \quad \forall a \in \mathcal{A}_0, \forall h \in H, \forall t \in G \right\}.$$

*Proof.* Suppose that  $\sum_{g \in G} r_g u_g \in C_{\mathcal{A}}(\mathcal{A}_H)$ . Clearly  $\mathcal{A}_0 \subseteq \mathcal{A}_H$  and hence for any  $a \in \mathcal{A}_0$ , we have

$$a \left( \sum_{g \in G} r_g u_g \right) = \left( \sum_{g \in G} r_g u_g \right) a \iff \sum_{g \in G} a r_g u_g = \sum_{g \in G} r_g \sigma_g(a) u_g \\ \iff a r_g = r_g \sigma_g(a), \quad \forall g \in G.$$

Furthermore, let  $h \in H$  be arbitrary. Since  $u_h \in \mathcal{A}_H$  we have

$$u_h \left( \sum_{g \in G} r_g u_g \right) = \left( \sum_{g \in G} r_g u_g \right) u_h \iff \\ \sum_{g \in G} \sigma_h(r_g) \alpha(h, g) u_{hg} = \sum_{g \in G} r_g \alpha(g, h) u_{gh} \iff \\ \sum_{t \in G} \sigma_h(r_{h^{-1}t}) \alpha(h, h^{-1}t) u_t = \sum_{t \in G} r_{th^{-1}} \alpha(th^{-1}, h) u_t \iff \\ \sigma_h(r_{h^{-1}t}) \alpha(h, h^{-1}t) = r_{th^{-1}} \alpha(th^{-1}, h), \quad \forall t \in G.$$

Conversely, suppose that the coefficients of an element  $\sum_{g \in G} r_g u_g$  satisfy the following two conditions:

1.  $r_t \sigma_t(a) = a r_t$  for all  $a \in \mathcal{A}_0$  and  $t \in G$ .
2.  $r_{th^{-1}} \alpha(th^{-1}, h) = \sigma_h(r_{h^{-1}t}) \alpha(h, h^{-1}t)$  for all  $h \in H, t \in G$ .

By carrying out calculations similar to the ones presented above, for any  $\sum_{h \in H} b_h u_h \in$

$\mathcal{A}_H$  we get

$$\begin{aligned} \left( \sum_{h \in H} b_h u_h \right) \left( \sum_{g \in G} r_g u_g \right) &= \sum_{h \in H} b_h \left( \sum_{g \in G} r_g u_g \right) u_h \\ &= \sum_{h \in H} \left( \sum_{g \in G} r_g u_g \right) b_h u_h \\ &= \left( \sum_{g \in G} r_g u_g \right) \left( \sum_{h \in H} b_h u_h \right) \end{aligned}$$

which shows that  $\sum_{g \in G} r_g u_g \in C_{\mathcal{A}}(\mathcal{A}_H)$ . This concludes the proof.  $\square$

**Remark D.5.6.** By putting  $H = \{e\}$  respectively  $H = G$  we get expressions for  $C_{\mathcal{A}}(\mathcal{A}_0)$  respectively  $Z(\mathcal{A})$ .

**Theorem D.5.7.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring, where  $\mathcal{A}_0$  is commutative and  $H$  is a subgroup of  $G$  contained in  $Z(G) \cap \ker(\sigma)$ , then*

$$I \cap C_{\mathcal{A}}(\mathcal{A}_H) \neq \{0\}$$

for every non-zero two-sided ideal  $I$  in  $\mathcal{A}$ .

*Proof.* Let  $I$  be an arbitrary non-zero two-sided ideal in  $\mathcal{A}$  and assume that  $\mathcal{A}_0$  is commutative. For each  $g \in G$  we define a map

$$T_g : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{s \in G} a_s u_s \mapsto \left( \sum_{s \in G} a_s u_s \right) u_g.$$

Note that, for each  $g \in G$ ,  $I$  is invariant under  $T_g$ . We have

$$T_g \left( \sum_{s \in G} a_s u_s \right) = \left( \sum_{s \in G} a_s u_s \right) u_g = \sum_{s \in G} a_s \alpha(s, g) u_{sg}$$

for every  $g \in G$ . Suppose that  $\sum_{s \in G} a_s u_s$  is such that  $a_e = 0$  but  $a_p \neq 0$  for some  $p \neq e$ . Then, we get  $T_{p^{-1}} \left( \sum_{s \in G} a_s u_s \right) = \sum_{s \in G} a_s \alpha(s, p^{-1}) u_{sp^{-1}}$ . In particular we see that the coefficient in front of  $u_e$  is given by  $a_p \alpha(p, p^{-1})$  and since  $\mathcal{A}$  is assumed to have no  $S(G)$ -torsion and  $\mathcal{A}_0$  is assumed to be commutative, we see by (iii) in Lemma D.5.3 that  $a_p \alpha(p, p^{-1}) \neq 0$ . It is now clear that for each non-zero element  $c \in \mathcal{A}$  it is always possible to choose some  $g \in G$  and let  $T_g$  operate on  $c$  to end up with an element

where the coefficient in front of  $u_e$  is non-zero. For each  $b \in \mathcal{A}_0$  and  $h \in H$ , we define a map

$$D_{b u_h} : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{s \in G} a_s u_s \mapsto b u_h \left( \sum_{s \in G} a_s u_s \right) - \left( \sum_{s \in G} a_s u_s \right) b u_h.$$

Note that, for each  $b \in \mathcal{A}_0$  and  $h \in H$ ,  $I$  is invariant under  $D_{b u_h}$ . Furthermore, due to the fact that  $H \subseteq Z(G) \cap \ker(\sigma)$ , we have

$$\begin{aligned} D_{b u_h} \left( \sum_{s \in G} a_s u_s \right) &= \left( \sum_{s \in G} b \sigma_h(a_s) \alpha(h, s) u_{hs} \right) \\ &\quad - \left( \sum_{s \in G} a_s \sigma_s(b) \alpha(s, h) u_{sh} \right) \\ &= \sum_{s \in G} (b a_s \alpha(h, s) - a_s \sigma_s(b) \alpha(s, h)) u_{hs} = \sum_{t \in G \setminus \{h\}} d_t u_t \end{aligned}$$

since  $d_h = (b a_e \alpha(h, e) - a_e \sigma_e(b) \alpha(e, h)) = 0$ . It is important to note that

$$C_{\mathcal{A}}(\mathcal{A}_H) = \bigcap_{\substack{b \in \mathcal{A}_0 \\ h \in H}} \ker(D_{b u_h})$$

and hence for any  $\sum_{s \in G} a_s u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_H)$  we are always able to choose  $b \in \mathcal{A}_0$  and  $h \in H$  and the corresponding  $D_{b u_h}$  and have  $\sum_{s \in G} a_s u_s \notin \ker(D_{b u_h})$ . Therefore we can always pick an operator  $D_{b u_h}$  which kills the coefficient  $d_h$  (coming from  $a_e$ ) without killing everything. Hence, if  $a_e \neq 0$ , the number of non-zero coefficients of the resulting element will always be reduced by at least one.

The ideal  $I$  is assumed to be non-zero, which means that we can pick some non-zero element  $\sum_{s \in G} r_s u_s \in I$ . If  $\sum_{s \in G} r_s u_s \in C_{\mathcal{A}}(\mathcal{A}_H)$ , then we are finished, so assume that this is not the case. Note that  $r_s \neq 0$  for finitely many  $s \in G$ . Recall that the ideal  $I$  is invariant under  $T_g$  and  $D_{b u_h}$  for all  $g \in G$ ,  $b \in \mathcal{A}_0$  and  $h \in H$ . We may now use the operators  $\{T_g\}_{g \in G}$  and  $\{D_{b u_h}\}_{b \in \mathcal{A}_0, h \in H}$  to generate new elements of  $I$ . More specifically, we may use the  $T_g$ 's to translate our element  $\sum_{s \in G} r_s u_s$  into a new element which has a non-zero coefficient in front of  $u_e$  (if needed) after which we use the  $D_{b u_h}$  operator to kill this coefficient and end up with yet another new element of  $I$  which is non-zero but has a smaller number of non-zero coefficients. We may repeat this procedure and in a finite number of iterations arrive at an element of  $I$  which lies in  $C_{\mathcal{A}}(\mathcal{A}_H) \setminus \mathcal{A}_0$ , and if not we continue the above procedure until we reach an element in  $\mathcal{A}_0 \setminus \{0\}$ . In particular  $\mathcal{A}_0 \subseteq C_{\mathcal{A}}(\mathcal{A}_H)$  since  $\mathcal{A}_0$  is commutative and hence  $I \cap C_{\mathcal{A}}(\mathcal{A}_H) \neq \{0\}$ .  $\square$

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## **Paper E**

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**E**





## Paper E

# Simple group graded rings and maximal commutativity

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**Abstract.** *In this paper we provide necessary and sufficient conditions for strongly group graded rings to be simple. For a strongly group graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  the grading group  $G$  acts, in a natural way, as automorphisms of the commutant of the neutral component subring  $\mathcal{R}_e$  in  $\mathcal{R}$  and of the center of  $\mathcal{R}_e$ . We show that if  $\mathcal{R}$  is a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then  $\mathcal{R}$  is a simple ring if and only if  $\mathcal{R}_e$  is  $G$ -simple (i.e. there are no nontrivial  $G$ -invariant ideals). We also show that if  $\mathcal{R}_e$  is commutative (not necessarily maximal commutative) and the commutant of  $\mathcal{R}_e$  is  $G$ -simple, then  $\mathcal{R}$  is a simple ring. These results apply to  $G$ -crossed products in particular. A skew group ring  $\mathcal{R}_e \rtimes_{\sigma} G$ , where  $\mathcal{R}_e$  is commutative, is shown to be a simple ring if and only if  $\mathcal{R}_e$  is  $G$ -simple and maximal commutative in  $\mathcal{R}_e \rtimes_{\sigma} G$ . As an interesting example we consider the skew group algebra  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  associated to a topological dynamical system  $(X, h)$ . We obtain necessary and sufficient conditions for simplicity of  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  with respect to the dynamics of the dynamical system  $(X, h)$ , but also with respect to algebraic properties of  $C(X)$  as a subalgebra of  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$ . Furthermore, we show that for any strongly  $G$ -graded ring  $\mathcal{R}$  each nonzero ideal of  $\mathcal{R}$  has a nonzero intersection with the commutant of the center of the neutral component.*

## E.1 Introduction

The aim of this paper is to highlight the important role that maximal commutativity of the neutral component subring plays in a strongly group graded ring when investigating simplicity of the ring itself. The motivation comes from the theory of  $C^*$ -crossed product algebras associated to topological dynamical systems. To each topological dynamical system,  $(X, h)$ , consisting of a compact Hausdorff space  $X$  and a homeomorphism  $h : X \rightarrow X$ , one may associate a  $C^*$ -crossed product algebra<sup>1</sup>  $C(X) \rtimes_{\tilde{h}}^{C^*} \mathbb{Z}$  (see

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<sup>1</sup>To avoid confusion, we let  $C(X) \rtimes_{\tilde{h}}^{C^*} \mathbb{Z}$  denote the  $C^*$ -crossed product algebra in contrast to the (algebraic) skew group algebra, which is denoted  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$ .

e.g. [23]). In the recent paper [22], C. Svensson and J. Tomiyama prove the following theorem.

**Theorem E.1.1.** *The following assertions are equivalent:*

- (i)  $(X, h)$  is topologically free (i.e. the aperiodic points are dense in  $X$ ).
- (ii)  $I \cap C(X) \neq \{0\}$  for each nonzero ideal  $I$  of  $C(X) \rtimes_{\tilde{h}}^{C^*} \mathbb{Z}$ .
- (iii)  $C(X)$  is a maximal commutative  $C^*$ -subalgebra of  $C(X) \rtimes_{\tilde{h}}^{C^*} \mathbb{Z}$ .

This theorem is a generalization (from closed ideals to arbitrary ideals) of a well-known theorem in the theory of  $C^*$ -crossed products associated to topological dynamical system (see e.g. [23] for details). Theorem E.1.1 is very useful when proving the following theorem, which originally appeared in [17].

**Theorem E.1.2.** *Suppose that  $X$  is infinite. Then  $C(X) \rtimes_{\tilde{h}}^{C^*} \mathbb{Z}$  is simple if and only if  $(X, h)$  is minimal (i.e. each orbit is dense in  $X$ ).*

In [19–21] C. Svensson, S. Silvestrov and M. de Jeu proved various analogues of Theorem E.1.1 for (algebraic) skew group algebras which are strongly graded by  $\mathbb{Z}$ . It then became natural to investigate if their results could be generalized to other types of (strongly) graded rings and in [11–14] an extensive investigation of the intersection between arbitrary nonzero ideals of various types of graded rings and certain subrings, was carried out. Given a subset  $S$  of a ring  $\mathcal{R}$  we denote by  $C_{\mathcal{R}}(S)$  the commutant of  $S$  in  $\mathcal{R}$ , i.e. the set of all elements of  $\mathcal{R}$  which commute with each element of  $S$ . In particular  $C_{\mathcal{R}}(\mathcal{R})$ , i.e. the center of  $\mathcal{R}$ , is denoted by  $Z(\mathcal{R})$ . In the recent paper [15], the following theorem was proven.

**Theorem E.1.3.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring, where  $\mathcal{R}_e$  is commutative, then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$$

*for each nonzero ideal  $I$  of  $\mathcal{R}$ .*

This implies that if  $\mathcal{R}$  is a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then each nonzero ideal of  $\mathcal{R}$  has a nontrivial intersection with  $\mathcal{R}_e$ . For skew group rings the following was shown in [14, Theorem 3].

**Theorem E.1.4.** *Let  $\mathcal{R} = \mathcal{R}_e \rtimes_{\sigma} G$  be a skew group ring satisfying either of the following two conditions:*

- $\mathcal{R}_e$  is an integral domain and  $G$  is an abelian group.
- $\mathcal{R}_e$  is commutative and  $G$  is a torsion-free abelian group.

Then the following two assertions are equivalent:

- (i) The ring  $\mathcal{R}_e$  is a maximal commutative subring of  $\mathcal{R}$ .
- (ii)  $I \cap \mathcal{R}_e \neq \{0\}$  for each nonzero ideal  $I$  of  $\mathcal{R}$ .

This theorem can be seen as a generalization of the algebraic analogue of Theorem E.1.1 and it is applicable to the skew group algebra which sits densely inside the  $C^*$ -crossed product algebra  $C(X) \rtimes_{\hbar}^* \mathbb{Z}$ . In the theory of graded rings, one theorem which provides sufficient conditions for a strongly group graded ring to be simple is the following, proven by F. Van Oystaeyen in [25, Theorem 3.4].

**Theorem E.1.5.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring such that the morphism  $G \rightarrow \text{Pic}(\mathcal{R}_e)$ , defined by  $g \rightarrow [\mathcal{R}_g]$ , is injective. If  $\mathcal{R}_e$  is a simple ring, then  $\mathcal{R}$  is a simple ring. (The Picard group,  $\text{Pic}(\mathcal{R}_e)$ , is defined in Section E.2.2.)*

A skew group ring is an example of a strongly graded ring. Given a skew group ring  $\mathcal{R} = \mathcal{R}_e \rtimes_{\sigma} G$ , the grading group  $G$  acts as automorphisms of  $\mathcal{R}_e$ . The results in [3] show that simplicity of a skew group ring  $\mathcal{R}$  is intimately connected to the nonexistence of  $G$ -invariant nonzero proper ideals of  $\mathcal{R}_e$ . Given a strongly  $G$ -graded ring  $\mathcal{R}$ , the grading group  $G$  acts, in a canonical way, as automorphisms of  $C_{\mathcal{R}}(\mathcal{R}_e)$  (see Section E.2.1). This means that for an arbitrary strongly  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ , one may speak of  $G$ -invariant nonzero proper ideals of  $C_{\mathcal{R}}(\mathcal{R}_e)$  and try to relate the nonexistence of such ideals to simplicity of  $\mathcal{R}$ , in a manner similar to the case of skew group rings.

In Section E.2 we give definitions and background information necessary for the understanding of the rest of this paper. In Section E.3 we generalize [15, Corollary 3] and show that in a strongly  $G$ -graded ring  $\mathcal{R}$  each nonzero ideal has a nonzero intersection with  $C_{\mathcal{R}}(Z(\mathcal{R}_e))$  (Theorem E.3.1). Furthermore, we generalize [14, Theorem 3] and show that for a skew group ring  $\mathcal{R}_e \rtimes_{\sigma} G$  where  $\mathcal{R}_e$  is commutative, each nonzero ideal of  $\mathcal{R}_e \rtimes_{\sigma} G$  has a nonzero intersection with  $\mathcal{R}_e$  if and only if  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}_e \rtimes_{\sigma} G$  (Theorem E.3.5).

The main objective of Section E.4 is to describe the connection between maximal commutativity of  $\mathcal{R}_e$  in a strongly group graded ring  $\mathcal{R}$  and injectivity of the canonical map  $G \rightarrow \text{Pic}(\mathcal{R}_e)$ . In Section E.5 we show that if  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a simple crystalline graded ring where  $\mathcal{A}_0$  is commutative, then  $\mathcal{A}_0$  is  $G$ -simple (Proposition E.5.1). In Example E.5.3 we apply this result to the first Weyl algebra. In Section E.6 we investigate simplicity of a strongly  $G$ -graded ring  $\mathcal{R}$  with respect to  $G$ -simplicity and maximal commutativity of  $\mathcal{R}_e$ . In particular we show that if  $\mathcal{R}$  is a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then  $\mathcal{R}_e$  is  $G$ -simple if and only if  $\mathcal{R}$  is simple (Theorem E.6.6). We also show the slightly more general result in one direction, namely that if  $C_{\mathcal{R}}(\mathcal{R}_e)$  is  $G$ -simple (with respect to the canonical action) and  $\mathcal{R}_e$  is commutative (not necessarily maximal commutative!), then  $\mathcal{R}$  is simple (Proposition E.6.5). Thereafter we investigate the simplicity of skew group rings and generalize [3, Corollary 2.1] and [3, Theorem 2.2],

by showing that if  $\mathcal{R}_e$  is commutative, then the skew group ring  $\mathcal{R}_e \rtimes_\sigma G$  is a simple ring if and only if  $\mathcal{R}_e$  is  $G$ -simple and a maximal commutative subring of  $\mathcal{R}_e \rtimes_\sigma G$  (Theorem E.6.13). As an example, we consider the skew group algebra associated to a dynamical system.

In Section E.7 we consider the algebraic crossed product  $C(X) \rtimes_{\bar{h}} \mathbb{Z}$  associated to a topological dynamical system  $(X, h)$ . Under the assumption that  $X$  is infinite, we show that  $C(X) \rtimes_{\bar{h}} \mathbb{Z}$  is simple if and only if  $(X, h)$  is a minimal dynamical system or equivalently if and only if  $C(X)$  is  $\mathbb{Z}$ -simple and maximal commutative in  $C(X) \rtimes_{\bar{h}} \mathbb{Z}$  (Theorem E.7.6). This result is a complete analogue to the well-known result for  $C^*$ -crossed product algebras associated to topological dynamical systems.

## E.2 Preliminaries

Throughout this paper all rings are assumed to be unital and associative and unless otherwise is stated we let  $G$  be an arbitrary group with neutral element  $e$ .

A ring  $\mathcal{R}$  is said to be  $G$ -graded if there is a family  $\{\mathcal{R}_g\}_{g \in G}$  of additive subgroups of  $\mathcal{R}$  such that

$$\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g \quad \text{and} \quad \mathcal{R}_g \mathcal{R}_h \subseteq \mathcal{R}_{gh}$$

for all  $g, h \in G$ . Moreover, if  $\mathcal{R}_g \mathcal{R}_h = \mathcal{R}_{gh}$  holds for all  $g, h \in G$ , then  $\mathcal{R}$  is said to be *strongly  $G$ -graded*. The product  $\mathcal{R}_g \mathcal{R}_h$  is here the usual module product consisting of all finite sums of ring products  $r_g r_h$  of elements  $r_g \in \mathcal{R}_g$  and  $r_h \in \mathcal{R}_h$ , and not just the set of all such ring products. For any graded ring  $\mathcal{R}$  it follows directly from the gradation that  $\mathcal{R}_e$  is a subring of  $\mathcal{R}$ , and that  $\mathcal{R}_g$  is an  $\mathcal{R}_e$ -bimodule for each  $g \in G$ . We shall refer to  $\mathcal{R}_g$  as the *homogeneous component of degree  $g \in G$* , and in particular to  $\mathcal{R}_e$  as the *neutral component*. Let  $U(\mathcal{R})$  denote the group of multiplication invertible elements of  $\mathcal{R}$ . We shall say that  $\mathcal{R}$  is a  $G$ -crossed product if  $U(\mathcal{R}) \cap \mathcal{R}_g \neq \emptyset$  for each  $g \in G$ .

### E.2.1 Strongly $G$ -graded rings

For each  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  one has  $1_{\mathcal{R}} \in \mathcal{R}_e$  (see [9, Proposition 1.1.1]), and if we in addition assume that  $\mathcal{R}$  is a strongly  $G$ -graded ring, i.e.  $\mathcal{R}_g \mathcal{R}_{g^{-1}} = \mathcal{R}_e$  for each  $g \in G$ , then for each  $g \in G$  there exists a positive integer  $n_g$  and elements  $a_g^{(i)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$ , such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}. \quad (\text{E.1})$$

For every  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ , and in particular for every  $\lambda \in Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$ , and  $g \in G$  we define

$$\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}. \quad (\text{E.2})$$

The definition of  $\sigma_g$  is independent of the choice of the  $a_g^{(i)}$ 's and  $b_{g^{-1}}^{(i)}$ 's (see e.g. [15]). For a proof of the following lemma we refer to [15, Lemma 3].

**Lemma E.2.1.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring,  $g \in G$  and write  $\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}$  for some  $n_g > 0$  and  $a_g^{(i)} \in \mathcal{R}_g, b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$  for  $i \in \{1, \dots, n_g\}$ . For each  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  define  $\sigma_g(\lambda)$  by  $\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}$ . The following properties hold:*

(i)  $\sigma_g(\lambda)$  is the unique element of  $\mathcal{R}$  satisfying

$$r_g \lambda = \sigma_g(\lambda) r_g, \quad \forall r_g \in \mathcal{R}_g. \quad (\text{E.3})$$

Furthermore,  $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$  and if  $\lambda \in Z(\mathcal{R}_e)$ , then  $\sigma_g(\lambda) \in Z(\mathcal{R}_e)$ .

(ii) The group  $G$  acts as automorphisms of the rings  $C_{\mathcal{R}}(\mathcal{R}_e)$  and  $Z(\mathcal{R}_e)$ , with each  $g \in G$  sending any  $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$  and  $\lambda \in Z(\mathcal{R}_e)$ , respectively, to  $\sigma_g(\lambda)$ .

(iii)  $Z(\mathcal{R}) = \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_g(\lambda) = \lambda, \forall g \in G\}$ , i.e.  $Z(\mathcal{R})$  is the fixed subring  $C_{\mathcal{R}}(\mathcal{R}_e)^G$  of  $C_{\mathcal{R}}(\mathcal{R}_e)$  with respect to the action of  $G$ .

The map  $\sigma$ , defined in Lemma E.2.1, will be referred to as the *canonical action*.

### E.2.2 The Picard group of $\mathcal{R}_e, \text{Pic}(\mathcal{R}_e)$

We shall now give a brief description of the Picard group of  $\mathcal{R}_e$  in a strongly graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ . For more details we refer to [2].

**Definition E.2.2** (Invertible module). Let  $A$  be a ring. An  $A$ -bimodule  $M$  is said to be *invertible* if and only if there exists an  $A$ -bimodule  $N$  such that  $M \otimes_A N \cong A \cong N \otimes_A M$  as  $A$ -bimodules.

Given a ring  $A$ , the Picard group of  $A$ , denoted  $\text{Pic}(A)$ , is defined as the set of  $A$ -bimodule isomorphism classes of invertible  $A$ -bimodules, and the group operation is given by  $\otimes_A$ .

If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring, the homomorphism of  $\mathcal{R}_g \otimes_{\mathcal{R}_e} \mathcal{R}_h$  into  $\mathcal{R}_{gh}$  sending  $r_g \otimes r_h$  into  $r_g r_h$  for all  $r_g \in \mathcal{R}_g$  and  $r_h \in \mathcal{R}_h$ , is an isomorphism of  $\mathcal{R}_e$ -bimodules, for any  $g, h \in G$  (see [4, p.336]). This implies that  $\mathcal{R}_g$  is an invertible  $\mathcal{R}_e$ -bimodule for each  $g \in G$ . We may now define a group homomorphism  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e), g \mapsto [\mathcal{R}_g]$ , i.e. each  $g \in G$  is mapped to the isomorphism class inside  $\text{Pic}(\mathcal{R}_e)$  to which the invertible  $\mathcal{R}_e$ -bimodule  $\mathcal{R}_g$  belongs.

### E.2.3 Crystalline graded rings

We shall begin this section by recalling the definition of a crystalline graded ring. We would also like to emphasize that rings belonging to this class are in general not strongly graded.

**Definition E.2.3** (Pre-crystalline graded ring). An associative and unital ring  $\mathcal{A}$  is said to be *pre-crystalline graded* if

- (i) there is a group  $G$  (with neutral element  $e$ ),
- (ii) there is a map  $u : G \rightarrow \mathcal{A}$ ,  $g \mapsto u_g$  such that  $u_e = 1_{\mathcal{A}}$  and  $u_g \neq 0$  for every  $g \in G$ ,
- (iii) there is a subring  $\mathcal{A}_0 \subseteq \mathcal{A}$  containing  $1_{\mathcal{A}}$ ,

such that the following conditions are satisfied:

- (P1)  $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_0 u_g$ .
- (P2) For every  $g \in G$ ,  $u_g \mathcal{A}_0 = \mathcal{A}_0 u_g$  is a free left  $\mathcal{A}_0$ -module of rank one.
- (P3) The decomposition in P1 makes  $\mathcal{A}$  into a  $G$ -graded ring with  $\mathcal{A}_0 = \mathcal{A}_e$ .

**Lemma E.2.4** (see [10]). *With notation and definitions as above:*

- (i) For every  $g \in G$ , there is a set map  $\sigma_g : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  defined by  $u_g a = \sigma_g(a) u_g$  for  $a \in \mathcal{A}_0$ . The map  $\sigma_g$  is a surjective ring morphism. Moreover,  $\sigma_e = \text{id}_{\mathcal{A}_0}$ .
- (ii) There is a set map  $\alpha : G \times G \rightarrow \mathcal{A}_0$  defined by  $u_s u_t = \alpha(s, t) u_{st}$  for  $s, t \in G$ . For any triple  $s, t, w \in G$  and  $a \in \mathcal{A}_0$  the following equalities hold:

$$\alpha(s, t) \alpha(st, w) = \sigma_s(\alpha(t, w)) \alpha(s, tw) \tag{E.4}$$

$$\sigma_s(\sigma_t(a)) \alpha(s, t) = \alpha(s, t) \sigma_{st}(a) \tag{E.5}$$

- (iii) For every  $g \in G$  we have  $\alpha(g, e) = \alpha(e, g) = 1_{\mathcal{A}_0}$  and  $\alpha(g, g^{-1}) = \sigma_g(\alpha(g^{-1}, g))$ .

A pre-crystalline graded ring  $\mathcal{A}$  with the above properties will be denoted by  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  and each element of this ring is written as a sum  $\sum_{g \in G} r_g u_g$  with coefficients  $r_g \in \mathcal{A}_0$ , of which only finitely many are nonzero. In [10] it was shown that for pre-crystalline graded rings, the elements  $\alpha(s, t)$  are normalizing elements of  $\mathcal{A}_0$ , i.e.  $\mathcal{A}_0 \alpha(s, t) = \alpha(s, t) \mathcal{A}_0$  for each  $s, t \in G$ . For a pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , we let  $S(G)$  denote the multiplicative set in  $\mathcal{A}_0$  generated by  $\{\alpha(g, g^{-1}) \mid g \in G\}$  and let  $S(G \times G)$  denote the multiplicative set generated by  $\{\alpha(g, h) \mid g, h \in G\}$ .

**Lemma E.2.5** (see [10]). *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a pre-crystalline graded ring, then the following assertions are equivalent:*

- (i)  $\mathcal{A}_0$  is  $S(G)$ -torsion free.
- (ii)  $\mathcal{A}$  is  $S(G)$ -torsion free.
- (iii)  $\alpha(g, g^{-1})a_0 = 0$  for some  $g \in G$  implies  $a_0 = 0$ .
- (iv)  $\alpha(g, h)a_0 = 0$  for some  $g, h \in G$  implies  $a_0 = 0$ .
- (v)  $\mathcal{A}_0 u_g = u_g \mathcal{A}_0$  is also free as a right  $\mathcal{A}_0$ -module, with basis  $u_g$ , for every  $g \in G$ .
- (vi) For every  $g \in G$ ,  $\sigma_g$  is bijective and hence a ring automorphism of  $\mathcal{A}_0$ .

**Definition E.2.6** (Crystalline graded ring). A pre-crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ , which is  $S(G)$ -torsion free, is said to be a *crystalline graded ring*.

**Remark E.2.7.** Note that  $G$ -crossed products are examples of crystalline graded rings. In fact, suppose that  $\mathcal{R}$  is a  $G$ -crossed product and put  $\mathcal{A} = \mathcal{R}$ . For each  $g \in G$ , we may pick some  $u_g \in \mathcal{R}_g \cap U(\mathcal{R})$ . Choose  $u_e = 1_{\mathcal{R}}$  and  $\mathcal{A}_0 = \mathcal{R}_e$ . It is now clear that  $\mathcal{R}_g = \mathcal{A}_0 u_g = u_g \mathcal{A}_0$  for each  $g \in G$ , and that  $\{u_g\}_{g \in G}$  is a basis for  $\mathcal{A}$  as a free left (and right)  $\mathcal{A}_0$ -module. By assumption  $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_0 u_g$  with  $\mathcal{A}_e = \mathcal{A}_0$ . This shows that  $\mathcal{A}$  is pre-crystalline graded. Recall that for each  $g \in G$ ,  $u_g$  is chosen to be a unit in  $\mathcal{R}$  and hence, from Lemma E.2.4 (ii), we get that  $\alpha(s, t) = u_s u_t u_{st}^{-1} \in U(\mathcal{A}_0)$  for all  $s, t \in G$ . This certainly shows that  $\mathcal{A}_0$  is  $S(G)$ -torsion free and hence  $\mathcal{A} = \mathcal{R}$  is a crystalline graded ring.

The notation for  $G$ -crossed products is inherited from the crystalline graded rings, e.g. we shall write  $\{u_g\}_{g \in G}$  for the basis elements. In particular, in the proof of Theorem E.3.5 where we consider a skew group ring, which is a special case of a  $G$ -crossed product, we shall use this notation. However, by custom we shall write  $\mathcal{R}_e \rtimes_{\sigma}^{\alpha} G$  instead of  $\mathcal{R}_e \diamond_{\sigma}^{\alpha} G$ .

### E.3 Ideals of strongly graded rings

In this section we shall improve some earlier results. We begin by making a generalization of Theorem E.1.3 ([15, Corollary 3]). The following proof is based on the same technique as in [15], but we will make it somewhat shorter by doing a proof by contra positivity.

**Theorem E.3.1.** *If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring, then*

$$I \cap C_{\mathcal{R}}(Z(\mathcal{R}_e)) \neq \{0\}$$

*for each nonzero ideal  $I$  of  $\mathcal{R}$ .*



*Proof.* Let  $I$  be an ideal of  $\mathcal{R}$  such that  $I \cap C_{\mathcal{R}}(Z(\mathcal{R}_e)) = \{0\}$ . If we can show that  $I = \{0\}$ , then the desired conclusion follows by contra positivity. Take an arbitrary  $x = \sum_{g \in G} x_g \in I$ . If  $x \in I \cap C_{\mathcal{R}}(Z(\mathcal{R}_e))$ , then  $x = 0$  by the assumption. Therefore, suppose that  $I \setminus C_{\mathcal{R}}(Z(\mathcal{R}_e))$  is not empty. We may choose some  $x \in I \setminus C_{\mathcal{R}}(Z(\mathcal{R}_e))$  such that  $N = \#\text{supp}(x) = \#\{g \in G \mid x_g \neq 0\} \in \mathbb{Z}_{>0}$  is as small as possible. Furthermore, we may assume that  $e \in \text{supp}(x)$ . Indeed, take any  $t \in \text{supp}(x)$  and choose some  $r_{t-1} \in \mathcal{R}_{t-1}$  such that  $x' = r_{t-1}x \neq 0$  and  $e \in \text{supp}(x')$ . It is always possible to choose such an  $r_{t-1}$ , because if  $1_{\mathcal{R}} = \sum_{i=1}^{n_t} a_t^{(i)} b_{t-1}^{(i)}$  as in (E.1), then  $b_{t-1}^{(i)} x_t$  must be nonzero for some  $i \in \{1, \dots, n_t\}$ , for otherwise we would have  $1_{\mathcal{R}} x_t = 0$  which would be contradictory (since  $x_t \neq 0$ ). Note that  $x' \in I$  is nonzero and since  $I \cap C_{\mathcal{R}}(Z(\mathcal{R}_e)) = \{0\}$  we conclude that  $x' \in I \setminus C_{\mathcal{R}}(Z(\mathcal{R}_e))$ . By the assumption on  $N$  we conclude that  $\#\text{supp}(x') = \text{supp}(x) = N$ . Now, take an arbitrary  $a \in Z(\mathcal{R}_e)$ . Then  $x'' = ax' - x'a \in I$  but clearly  $e \notin \text{supp}(x'')$  and hence by the assumption on  $N$  we get that  $x'' \notin I \setminus C_{\mathcal{R}}(Z(\mathcal{R}_e))$ , thus  $x'' = 0$ . Since  $a \in Z(\mathcal{R}_e)$  was chosen arbitrarily we get  $x' \in C_{\mathcal{R}}(Z(\mathcal{R}_e))$  which is a contradiction.  $\square$

**Remark E.3.2.** Note that  $\mathcal{R}_e \subseteq C_{\mathcal{R}}(Z(\mathcal{R}_e))$ . If  $\mathcal{R}_e$  is commutative, then clearly  $\mathcal{R}_e = Z(\mathcal{R}_e)$  and we obtain Theorem E.1.3 as a special case of Theorem E.3.1.

For a crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  we obtain the following result which generalizes [14, Corollary 8].

**Theorem E.3.3.** *If  $\mathcal{A} = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a crystalline graded ring with  $\alpha(g, g^{-1}) \in Z(\mathcal{A}_0)$  for all  $g \in G$ , then*

$$I \cap C_{\mathcal{A}}(Z(\mathcal{A}_0)) \neq \{0\}$$

*for each nonzero ideal  $I$  of  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ .*

*Proof.* Let  $x = \sum_{g \in G} a_g u_g$ , with  $a_g \in \mathcal{A}_0$  for  $g \in G$ , be a nonzero element of the crystalline graded ring  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ . Pick some  $t \in G$  such that  $a_t \neq 0$ . For  $x' = x u_{t-1}$  we have  $e \in \text{supp}(x')$ . Indeed, in degree  $e$  of  $x'$  we have  $(a_t u_t) u_{t-1} = a_t \alpha(t, t^{-1}) = \alpha(t, t^{-1}) a_t$  and, by Lemma E.2.5 (iii), this is a nonzero element of  $\mathcal{A}_0$ . The rest of the proof is analogous to the proof of Theorem E.3.1.  $\square$

For an element  $r$  of a commutative ring  $A$ , the annihilator ideal of  $r$  in  $A$  is defined to be the set  $\text{Ann}(r) = \{b \in A \mid rb = 0\}$ . The following lemma from [11, Corollary 6] applies to  $G$ -crossed products and in particular skew group rings.

**Lemma E.3.4.** *Let  $\mathcal{R}_e \rtimes_{\sigma}^{\alpha} G$  be a  $G$ -crossed product with  $\mathcal{R}_e$  commutative. The subring  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}_e \rtimes_{\sigma}^{\alpha} G$  if and only if, for each pair  $(s, r_s) \in (G \setminus \{e\}) \times (\mathcal{R}_e \setminus \{0\})$ , there exists  $a \in \mathcal{R}_e$  such that  $\sigma_s(a) - a \notin \text{Ann}(r_s)$ .*

The following theorem is a generalization of Theorem E.1.4 ([14, Theorem 3]) and the proof makes use of the same idea as in [14]. However, in this proof we make a crucial observation and are able to make use of an important map.

**Theorem E.3.5.** *Let  $\mathcal{R} = \mathcal{R}_e \rtimes_\sigma G$  be a skew group ring with  $\mathcal{R}_e$  commutative. The following two assertions are equivalent:*

- (i)  $\mathcal{R}_e$  is a maximal commutative subring of  $\mathcal{R}$ .
- (ii)  $I \cap \mathcal{R}_e \neq \{0\}$  for each nonzero ideal  $I$  of  $\mathcal{R}$ .

*Proof.* By Theorem E.3.1 (i) implies (ii) for the (strongly graded) skew group ring  $\mathcal{R}$ . We shall now show that (ii) implies (i). Suppose that  $\mathcal{R}_e$  is not maximal commutative in  $\mathcal{R}$ . If we can show that there exists a nonzero ideal  $I$  of  $\mathcal{R}$ , such that  $I \cap \mathcal{R}_e = \{0\}$ , then by contra positivity we are done. Let  $\{u_g\}_{g \in G}$  be a basis for  $\mathcal{R}$  as a free left (and right)  $\mathcal{R}_e$ -module, as in Section E.2.3. By the assumption and Lemma E.3.4 there exists some  $s \in G \setminus \{e\}$  and  $r_s \in \mathcal{R}_e \setminus \{0\}$  such that  $r_s \sigma_s(a) = r_s a$  for each  $a \in \mathcal{R}_e$ . Let us choose such a pair  $(s, r_s)$  and let  $I$  be the twosided ideal of  $\mathcal{R}$  generated by  $r_s - r_s u_s$ . The ideal  $I$  is obviously nonzero, and furthermore it is spanned by elements of the form  $a_g u_g (r_s - r_s u_s) a_h u_h$  where  $g, h \in G$  and  $a_g, a_h \in \mathcal{R}_e$ . By commutativity of  $\mathcal{R}_e$  and the properties of  $r_s$  we may rewrite this expression.

$$\begin{aligned}
 a_g u_g (r_s - r_s u_s) a_h u_h &= a_g u_g (r_s a_h - \underbrace{r_s \sigma_s(a_h)}_{=r_s a_h} u_s) u_h \\
 &= a_g u_g r_s a_h (1_{\mathcal{R}} - u_s) u_h \\
 &= a_g \sigma_g(r_s a_h) u_g (1_{\mathcal{R}} - u_s) u_h \\
 &= \underbrace{a_g \sigma_g(r_s a_h)}_{:=b} (u_{gh} - u_{gsh}) \\
 &= b u_{gh} - b u_{gsh}
 \end{aligned} \tag{E.6}$$

Each element of  $I$  is a sum of elements of the form (E.6), where  $b \in \mathcal{R}_e$  and  $g, h \in G$ . Define a map

$$\epsilon : \mathcal{R}_e \rtimes_\sigma G \rightarrow \mathcal{R}_e, \quad \sum_{g \in G} a_g u_g \mapsto \sum_{g \in G} a_g.$$

It is clear that  $\epsilon$  is additive and one easily sees that  $\epsilon$  is identically zero on  $I$ . Furthermore,  $\epsilon|_{\mathcal{R}_e}$ , i.e. the restriction of  $\epsilon$  to  $\mathcal{R}_e$ , is injective. Take an arbitrary  $m \in I \cap \mathcal{R}_e$ . Clearly  $\epsilon(m) = 0$  since  $m \in I$  and by the injectivity of  $\epsilon|_{\mathcal{R}_e}$  we conclude that  $m = 0$ . Hence  $I \cap \mathcal{R}_e = \{0\}$ . This concludes the proof.  $\square$

**Remark E.3.6.** It is not difficult to see that the map  $\epsilon$  is multiplicative if and only if the action  $\sigma$  is trivial, i.e.  $\mathcal{R}_e \rtimes_\sigma G$  is a group ring. In that situation the map  $\epsilon$  is commonly referred to as the *augmentation map*. However, note that the preceding proof does not require  $\epsilon$  to be multiplicative.

## E.4 The map $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e)$ and simple strongly graded rings

We begin by recalling a useful lemma.

**Lemma E.4.1** ([15]). *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $a \in \mathcal{R}$  is such that*

$$a \mathcal{R}_g = \{0\} \quad \text{or} \quad \mathcal{R}_g a = \{0\}$$

*for some  $g \in G$ , then  $a = 0$ .*

If we assume that  $\mathcal{R}_e$  is maximal commutative in the strongly  $G$ -graded ring  $\mathcal{R}$ , then we can say the following about the canonical map  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e)$ .

**Proposition E.4.2.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then the map  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e), g \mapsto [\mathcal{R}_g]$ , is injective.*

*Proof.* Let  $\mathcal{R}_e$  be maximal commutative in  $\mathcal{R}$ . Suppose that  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e)$  is not injective. This means that we can pick two distinct elements  $g, h \in G$  such that  $\mathcal{R}_g \cong \mathcal{R}_h$  as  $\mathcal{R}_e$ -bimodules. Let  $f : \mathcal{R}_g \rightarrow \mathcal{R}_h$  be a bijective  $\mathcal{R}_e$ -bimodule homomorphism. By our assumptions  $\mathcal{R}_e = C_{\mathcal{R}}(\mathcal{R}_e)$  and hence we can use the map  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$  defined by (E.2) to write

$$\sigma_h(b) \underbrace{f(r_g)}_{\in \mathcal{R}_h} = f(r_g) b = f(r_g b) = f(\sigma_g(b) r_g) = \sigma_g(b) f(r_g) \quad (\text{E.7})$$

for any  $b \in \mathcal{R}_e$  and  $r_g \in \mathcal{R}_g$ . (It is important to note that  $\sigma_g(b) \in \mathcal{R}_e$  since  $b \in \mathcal{R}_e$ .) The map  $f$  is bijective and in particular surjective. Hence, by (E.7) we conclude that  $(\sigma_h(b) - \sigma_g(b)) \mathcal{R}_h = \{0\}$  for any  $b \in \mathcal{R}_e$ . It follows from Lemma E.4.1 that  $\sigma_h(b) - \sigma_g(b) = 0$  for any  $b \in \mathcal{R}_e$ . Hence  $\sigma_g = \sigma_h$  in  $\text{Aut}(C_{\mathcal{R}}(\mathcal{R}_e)) = \text{Aut}(\mathcal{R}_e)$  and this implies  $\sigma_{g^{-1}h} = \text{id}_{\mathcal{R}_e}$ . Now equation (E.3) shows that the homogeneous component  $\mathcal{R}_{g^{-1}h}$  ( $\neq \mathcal{R}_e$  since  $g \neq h$ ) commutes with  $\mathcal{R}_e$ , and hence  $\mathcal{R}_e$  is not maximal commutative in  $\mathcal{R}$ . We have reached a contradiction and this shows that  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e)$  is injective.  $\square$

The following proposition is a direct consequence of Theorem E.1.4 and we shall therefore omit the proof.

**Proposition E.4.3.** *Let  $\mathcal{R} = \mathcal{R}_e \rtimes_{\sigma} G$  be a skew group ring, where  $\mathcal{R}_e$  is a field and  $G$  is an abelian group. The following assertions are equivalent:*

- (i) *The subring  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ .*
- (ii)  *$\mathcal{R}$  is a simple ring.*

**Example E.4.4.** Consider the group ring  $\mathcal{R} = \mathbb{C}[\mathbb{Z}]$ , which corresponds to the special case of a skew group ring with trivial action. The so called *augmentation ideal*, which is the kernel,  $\ker(\epsilon)$ , of the augmentation map

$$\epsilon : \mathbb{C}[\mathbb{Z}] \rightarrow \mathbb{C}, \quad \sum_{k \in \mathbb{Z}} c_k \bar{k} \mapsto \sum_{k \in \mathbb{Z}} c_k$$

is a nontrivial ideal of  $\mathcal{R}$  and hence  $\mathcal{R} = \mathbb{C}[\mathbb{Z}]$  is not a simple ring. This conclusion also follows directly from Proposition E.4.3. Indeed,  $\mathcal{R} = \mathbb{C}[\mathbb{Z}]$  is commutative and hence  $\mathcal{R}_0 = \mathbb{C}$  is not maximal commutative in  $\mathbb{C}[\mathbb{Z}]$ .

The following proposition shows that in the case when  $\mathcal{R}_e$  is assumed to be commutative, Theorem E.1.5 is equivalent to Corollary E.6.7 (see Section E.6).

**Proposition E.4.5.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $\mathcal{R}_e$  is a field, then the following two assertions are equivalent:*

- (i)  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ .
- (ii) The map  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e)$  is injective.

*Proof.* It follows from Proposition E.4.2 that (i) implies (ii).

To prove that (ii) implies (i), let us assume that  $\mathcal{R}_e$  is not maximal commutative in  $\mathcal{R}$ . We want to show that  $\psi$  is not injective and hence get the desired conclusion by contraposition.

By our assumptions, there exists some nonzero element  $r_g \in \mathcal{R}_g$ , for some  $g \neq e$ , such that  $r_g a = a r_g$  for all  $a \in \mathcal{R}_e$ . Consider the set  $J = r_g \mathcal{R}_{g^{-1}} \subseteq \mathcal{R}_e$ . Since  $r_g$  commutes with  $\mathcal{R}_e$  and  $\mathcal{R}_{g^{-1}}$  is an  $\mathcal{R}_e$ -bimodule,  $J$  is an ideal of  $\mathcal{R}_e$  and as  $r_g \mathcal{R}_{g^{-1}} \neq \{0\}$  (this follows from Lemma E.4.1 since  $r_g \neq 0$ ), we obtain  $r_g \mathcal{R}_{g^{-1}} = \mathcal{R}_e$  since  $\mathcal{R}_e$  is simple. Consequently, we conclude that there exists an  $s_{g^{-1}} \in \mathcal{R}_{g^{-1}}$  such that  $r_g s_{g^{-1}} = 1_{\mathcal{R}}$ . In a symmetrical way we get  $\mathcal{R}_{g^{-1}} r_g = \mathcal{R}_e$  which yields  $w_{g^{-1}} r_g = 1_{\mathcal{R}}$  for some  $w_{g^{-1}} \in \mathcal{R}_{g^{-1}}$ . Clearly  $w_{g^{-1}} = s_{g^{-1}}$ .

From the gradation we immediately conclude that  $\mathcal{R}_e r_g \subseteq \mathcal{R}_g$  and  $\mathcal{R}_g s_{g^{-1}} \subseteq \mathcal{R}_e$ . By the equality  $s_{g^{-1}} r_g = 1_{\mathcal{R}}$  we get  $\mathcal{R}_g \subseteq \mathcal{R}_e r_g$  and hence  $\mathcal{R}_g = \mathcal{R}_e r_g$ . Note that  $r_g$  is invertible and hence a basis for the  $\mathcal{R}_e$ -bimodule  $\mathcal{R}_e r_g$ . This shows that  $\mathcal{R}_g$  and  $\mathcal{R}_e$  belong to the same isomorphism class in  $\text{Pic}(\mathcal{R}_e)$ , and hence the morphism  $\psi : G \rightarrow \text{Pic}(\mathcal{R}_e)$  is not injective. This concludes the proof.  $\square$

**Remark E.4.6.** The previous proof uses the same techniques as the proof of [25, Theorem 3.4].

## E.5 $G$ -simple subrings of crystalline graded rings

If  $A$  is a ring and  $\sigma : G \rightarrow \text{Aut}(A)$  is a group action, then we say that an ideal  $I$  of  $A$  is  $G$ -invariant if  $\sigma_g(I) \subseteq I$  for each  $g \in G$ . Note that it is equivalent to say that

$\sigma_g(I) = I$  for each  $g \in G$ . If there are no nontrivial  $G$ -invariant ideals of  $A$ , then we say that  $A$  is  $G$ -simple. (Not to be confused with the term *graded simple*!)

**Proposition E.5.1.** *Let  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  be a crystalline graded ring, where  $\mathcal{A}_0$  is commutative. If  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is a simple ring, then  $\mathcal{A}_0$  is a  $G$ -simple ring (with respect to the action defined in Lemma E.2.4).*

*Proof.* Note that since  $\mathcal{A}_0$  is commutative, the map  $\sigma : G \rightarrow \text{Aut}(\mathcal{A}_0)$  is a group homomorphism. Let  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  be a simple ring, and  $J$  an arbitrary nonzero  $G$ -invariant ideal of  $\mathcal{A}_0$ . One may verify that  $J \diamond_{\sigma}^{\alpha} G$  is a nonzero ideal of  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ . (This follows from the fact that for each  $g \in G$ ,  $\mathcal{A}_0 u_g$  is a free left  $\mathcal{A}_0$ -module with basis  $u_g$ .) Since  $\mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$  is simple, we get  $J \diamond_{\sigma}^{\alpha} G = \mathcal{A}_0 \diamond_{\sigma}^{\alpha} G$ . Therefore  $\mathcal{A}_0 \subseteq J \diamond_{\sigma}^{\alpha} G$ , and from the gradation it follows that

$$\mathcal{A}_0 \subseteq J \subseteq \mathcal{A}_0$$

and hence  $\mathcal{A}_0 = J$ , which shows that  $\mathcal{A}_0$  is  $G$ -simple.  $\square$

**Corollary E.5.2.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a  $G$ -crossed product, where  $\mathcal{R}_e$  is commutative. If  $\mathcal{R}$  is a simple ring, then  $\mathcal{R}_e$  is a  $G$ -simple ring (with respect to the canonical action).*

**Example E.5.3.** It is well-known that the first Weyl algebra  $\mathcal{A} = \frac{\mathbb{C}\langle x, y \rangle}{(xy - yx - 1)}$  is simple. The first Weyl algebra is an example of a crystalline graded ring, with  $G = (\mathbb{Z}, +)$  and  $\mathcal{A}_e = \mathcal{A}_0 = \mathbb{C}[xy]$  (see e.g. [10] for details). Note that  $\mathbb{C}[xy]$  is not a simple ring. However, by Proposition E.5.1 we conclude that  $\mathcal{A}_0 = \mathbb{C}[xy]$  is in fact  $\mathbb{Z}$ -simple. As a side remark we should also mention that one can show that  $\mathcal{A}_0 = \mathbb{C}[xy]$  is a maximal commutative subring of the first Weyl algebra  $\mathcal{A}$ .

## E.6 $G$ -simple subrings of strongly $G$ -graded rings

In this section we shall describe how simplicity of a strongly  $G$ -graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is related to  $G$ -simplicity of the subrings  $Z(\mathcal{R}_e)$  and  $C_{\mathcal{R}}(\mathcal{R}_e)$ . If  $\mathcal{R}_e$  is commutative, then  $\mathcal{R}_e = Z(\mathcal{R}_e)$ , and hence we have an action  $\sigma : G \rightarrow \text{Aut}(\mathcal{R}_e)$ .

**Proposition E.6.1.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring, where  $\mathcal{R}_e$  is commutative. If  $\mathcal{R}$  is a simple ring, then  $\mathcal{R}_e$  is a  $G$ -simple ring (with respect to the canonical action).*

*Proof.* Let  $J$  be an arbitrary nonzero  $G$ -invariant ideal of  $\mathcal{R}_e$ . Denote by  $J\mathcal{R}$  the right ideal of  $\mathcal{R}$  generated by  $J$ . From the fact that  $J$  is a  $G$ -invariant ideal of  $\mathcal{R}_e$  we conclude that  $J\mathcal{R}$  is also a left ideal of  $\mathcal{R}$ . Indeed, for  $g, h \in G$  and  $c \in J$ ,  $r_h \in \mathcal{R}_h$ ,  $s_g \in \mathcal{R}_g$  we have  $s_g c r_h = \sigma_g(c) s_g r_h \in J\mathcal{R}$ . Furthermore,  $\mathcal{R}$  is unital and hence  $J\mathcal{R}$  must be nonzero. The ring  $\mathcal{R}$  is simple and therefore we conclude that  $J\mathcal{R} = \mathcal{R}$ . In particular we see that  $\mathcal{R}_e \subseteq J\mathcal{R}$ . From the gradation we get

$$\mathcal{R}_e \subseteq J\mathcal{R}_e \subseteq J \subseteq \mathcal{R}_e$$

and hence  $J = \mathcal{R}_e$ . This shows that  $\mathcal{R}_e$  is  $G$ -simple.  $\square$

The preceding proposition is a generalization of Corollary E.5.2. In the following useful lemma,  $\mathcal{R}_e$  is not required to be commutative.

**Lemma E.6.2.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring and  $\mathcal{S}$  a subring of  $C_{\mathcal{R}}(\mathcal{R}_e)$  satisfying the following three conditions:*

- (i)  $1_{\mathcal{R}} \in \mathcal{S}$ .
- (ii)  $\mathcal{S}$  is invariant under  $G$  (with respect to the canonical action).
- (iii)  $\mathcal{S}$  is  $G$ -simple (with respect to the canonical action).

Then  $I \cap \mathcal{S} = \{0\}$  for each proper ideal  $I$  of  $\mathcal{R}$ .

*Proof.* Let  $\mathcal{S}$  be a subring of  $C_{\mathcal{R}}(\mathcal{R}_e)$  satisfying conditions (i)-(iii) of the above, and  $I$  be an ideal of  $\mathcal{R}$  such that  $I \cap \mathcal{S} \neq \{0\}$ . The set  $J = I \cap \mathcal{S}$  is an ideal of  $\mathcal{S}$ . By (ii), for any  $x \in J$  and every  $g \in G$ , we have  $\sigma_g(x) = \sum_{i=1}^{n_g} a_g^{(i)} x b_{g^{-1}}^{(i)} \in I \cap \mathcal{S} = J$ . This shows that  $J$  is a  $G$ -invariant ideal of  $\mathcal{S}$ . By assumption  $J$  is nonzero and hence by (iii),  $J = \mathcal{S}$ . In particular this shows that  $1_{\mathcal{R}} \in J \subseteq I$ , and hence  $\mathcal{R} = I$ .  $\square$

By observing that both  $C_{\mathcal{R}}(\mathcal{R}_e)$  and  $Z(\mathcal{R}_e)$  are subrings of  $C_{\mathcal{R}}(\mathcal{R}_e)$  satisfying conditions (i) and (ii) of Lemma E.6.2 we obtain the following corollary.

**Corollary E.6.3.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $C_{\mathcal{R}}(\mathcal{R}_e)$  (respectively  $Z(\mathcal{R}_e)$ ) is a  $G$ -simple ring (with respect to the canonical action), then  $I \cap C_{\mathcal{R}}(\mathcal{R}_e) = \{0\}$  (respectively  $I \cap Z(\mathcal{R}_e) = \{0\}$ ) for each proper ideal  $I$  of  $\mathcal{R}$ .*

Recall from [5], that a ring  $\mathcal{R}$  is said to be a PI-ring (abbreviation for *polynomial identity ring*) if for some  $n \in \mathbb{Z}_{>0}$  there exists some  $f \in \mathbb{Z}\langle x_1, x_2, \dots, x_n \rangle$ , i.e. the free polynomial ring over  $\mathbb{Z}$  in  $n$  variables, such that  $f(a_1, a_2, \dots, a_n) = 0$  for each  $(a_1, a_2, \dots, a_n) \in \mathcal{R}^n$ . Furthermore, a ring is said to be *semiprime* if  $\{0\}$  is a semiprime ideal [8, Definition 10.8, Definition 10.15].

**Corollary E.6.4.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a semiprime PI-ring which is strongly  $G$ -graded. If either  $Z(\mathcal{R}_e)$  or  $C_{\mathcal{R}}(\mathcal{R}_e)$  is a  $G$ -simple ring (with respect to the canonical action), then  $\mathcal{R}$  is a simple ring.*

*Proof.* Let  $I$  be a nonzero ideal of  $\mathcal{R}$ . It follows from [18, Theorem 2] that  $I \cap Z(\mathcal{R}) \neq \{0\}$ . Clearly  $Z(\mathcal{R}) \subseteq Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$  and hence by Corollary E.6.3 we conclude that  $I = \mathcal{R}$ .  $\square$

As we shall see Theorem E.6.6 requires  $\mathcal{R}_e$  not only to be commutative, but maximal commutative in  $\mathcal{R}$ . We begin by proving the following which applies to the more general situation when  $\mathcal{R}_e$  is not necessarily maximal commutative in  $\mathcal{R}$ .

**Proposition E.6.5.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring, where  $\mathcal{R}_e$  is commutative. If  $C_{\mathcal{R}}(\mathcal{R}_e)$  is a  $G$ -simple ring (with respect to the canonical action), then  $\mathcal{R}$  is a simple ring.*

*Proof.* Let  $I$  be an arbitrary nonzero ideal of  $\mathcal{R}$ . Since  $\mathcal{R}_e$  is commutative it follows from Theorem E.1.3 that  $I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$ . By Corollary E.6.3 we conclude that  $I = \mathcal{R}$  and hence  $\mathcal{R}$  is a simple ring.  $\square$

By combining Proposition E.6.1 and Proposition E.6.5 we get the following theorem.

**Theorem E.6.6.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring. If  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ , then the following two assertions are equivalent:*

- (i)  $\mathcal{R}_e$  is a  $G$ -simple ring (with respect to the canonical action).
- (ii)  $\mathcal{R}$  is a simple ring.

As an immediate consequence of Theorem E.6.6 we get the following corollary, which can also be retrieved from Theorem E.1.5 together with Proposition E.4.2.

**Corollary E.6.7.** *Let  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  be a strongly  $G$ -graded ring where  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$ . If  $\mathcal{R}_e$  is a simple ring, then  $\mathcal{R}$  is a simple ring.*

The following remark shows that the rings considered in Corollary E.6.7 are in fact  $G$ -crossed products.

**Remark E.6.8.** Recall that a commutative and simple ring is a field. If  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  is a strongly  $G$ -graded ring and  $\mathcal{R}_e$  is a field, then  $\mathcal{R}$  is a  $G$ -crossed product. Indeed, for each  $g \in G$ , we have  $\mathcal{R}_g \mathcal{R}_{g^{-1}} = \mathcal{R}_{g^{-1}} \mathcal{R}_g = \mathcal{R}_e$ . Hence, for an arbitrary  $g \in G$  we may fix some nonzero  $a \in \mathcal{R}_g$  and by Lemma E.4.1 choose some nonzero  $b \in \mathcal{R}_{g^{-1}}$  such that  $ab = c \in \mathcal{R}_e \setminus \{0\}$ . This means that  $c$  is invertible in  $\mathcal{R}_e$  and hence  $a$  is right invertible in  $\mathcal{R}$ , with right inverse  $bc^{-1}$ . The other half of Lemma E.4.1 may be used to show that  $a$  also has a left inverse. We conclude that for each  $g \in G$ ,  $\mathcal{R}_g$  contains an invertible element and hence  $\mathcal{R}$  is a  $G$ -crossed product.

One should note that Proposition E.6.5 and Theorem E.6.6 are more general than Theorem E.1.5 in the sense that  $\mathcal{R}_e$  is not required to be simple. On the other hand, this does not come for free. We have to make an additional assumption on  $\mathcal{R}_e$ , namely that it be commutative.

**Remark E.6.9.** Note that Theorem E.6.6 especially applies to  $G$ -crossed products.

One may think that for a simple strongly graded ring  $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$  where  $\mathcal{R}_e$  is commutative and  $G$ -simple, this would imply that  $\mathcal{R}_e$  would be maximal commutative in  $\mathcal{R}$ . In general this is not true, as the following example shows.

**Example E.6.10.** Consider the field of complex numbers  $\mathbb{C} = \mathbb{R} \rtimes^{\alpha} \mathbb{Z}_2$  as a  $\mathbb{Z}_2$ -graded twisted group ring (see e.g. [12] for details). Clearly  $\mathbb{C}$  is simple as is  $\mathbb{R}$ . Hence  $\mathbb{R}$  is also  $\mathbb{Z}_2$ -simple, but it is not maximal commutative in  $\mathbb{C}$ .

The purpose of the following example is to present a strongly group graded ring which is not a crossed product, and to identify a  $G$ -simple subring.

**Example E.6.11** (A strongly group graded, noncrossed product, matrix ring). Let  $\mathcal{R} = M_3(\mathbb{C})$  denote the ring of  $3 \times 3$ -matrices over  $\mathbb{C}$ . By putting

$$\mathcal{R}_0 = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{pmatrix} \quad \text{and} \quad \mathcal{R}_1 = \begin{pmatrix} 0 & 0 & \mathbb{C} \\ 0 & 0 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & 0 \end{pmatrix}$$

one may verify that this defines a strong  $\mathbb{Z}_2$ -gradation on  $\mathcal{R}$ . However, note that  $\mathcal{R}$  is not a crossed product with this grading since the homogeneous component  $\mathcal{R}_1$  does not contain any invertible element of  $M_3(\mathbb{C})$ ! A simple calculation yields

$$Z(\mathcal{R}_0) = \left\{ \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{array} \right) \mid a, b \in \mathbb{C} \right\}$$

and in fact one may verify that  $C_{\mathcal{R}}(\mathcal{R}_0) = Z(\mathcal{R}_0)$ . In order to define an action  $\sigma : \mathbb{Z}_2 \rightarrow \text{Aut}(Z(\mathcal{R}_0))$  we need to make a decomposition of the identity matrix  $I = 1_{\mathcal{R}}$ , in accordance with (E.1). Let  $E_{i,j}$  denote the  $3 \times 3$ -matrix which has a 1 in position  $(i, j)$  and zeros everywhere else. The decomposition in  $\mathcal{R}_0$  is trivial, but in  $\mathcal{R}_1$  we may for example choose

$$I = E_{1,3}E_{3,1} + E_{2,3}E_{3,2} + E_{3,2}E_{2,3}.$$

From these decompositions we are now able to define the map  $\sigma : \mathbb{Z}_2 \rightarrow \text{Aut}(Z(\mathcal{R}_0))$ . One easily sees that  $Z(\mathcal{R}_0)$  has two nontrivial ideals. By calculating, we get

$$\sigma_1(E_{1,1} + E_{2,2}) = E_{3,3}$$

and

$$\sigma_1(E_{3,3}) = E_{1,1} + E_{2,2}.$$

From this we conclude that the two nontrivial ideals of  $Z(\mathcal{R}_0)$  are interchanged by the map  $\sigma_1$ , and therefore they are not invariant under the action of  $\mathbb{Z}_2$ . This shows that for our simple ring  $M_3(\mathbb{C})$ , the subring  $Z(\mathcal{R}_0) = C_{\mathcal{R}}(\mathcal{R}_0)$  is in fact  $\mathbb{Z}_2$ -simple.

**Remark E.6.12.** Proposition E.6.1 shows that in a simple strongly graded ring  $\mathcal{R}$  where  $\mathcal{R}_e$  is commutative, we automatically have that  $\mathcal{R}_e = Z(\mathcal{R}_e)$  is  $G$ -simple. In Proposition E.6.5 we saw that for a strongly graded ring  $\mathcal{R}$  where  $\mathcal{R}_e$  is commutative,  $G$ -simplicity of  $C_{\mathcal{R}}(\mathcal{R}_e)$  implies simplicity of  $\mathcal{R}$ . After seeing Example E.6.11 it is tempting to think that the converse is also true (even for noncommutative  $\mathcal{R}_e$ ), i.e. simplicity of  $\mathcal{R}$  gives rise to  $G$ -simple subrings. The natural questions are:



1. If  $\mathcal{R}$  is strongly group graded and simple, is  $C_{\mathcal{R}}(\mathcal{R}_e)$  necessarily  $G$ -simple?
2. If  $\mathcal{R}$  is strongly group graded and simple, is  $Z(\mathcal{R}_e)$  necessarily  $G$ -simple?

Recall that the center of a simple ring is a field. Thus, if  $G$  is the trivial group, then the answers to both questions are clearly affirmative. Let us therefore consider the case when  $G$  is an arbitrary nontrivial group. Note that if  $\mathcal{R}$  is commutative, then it is trivial to verify that the answers to both questions are affirmative. As we have already mentioned, if  $\mathcal{R}_e$  is commutative then the answer to question no. 2 is affirmative. Furthermore, if  $\mathcal{R}_e$  is maximal commutative, then by Theorem E.6.6 we conclude that the answer to question no. 1 is also affirmative. The case that remains to be investigated is that of a noncommutative ring  $\mathcal{R}$  where  $\mathcal{R}_e$  is not maximal commutative (we may not even assume for it to be commutative) in  $\mathcal{R}$ .

From Example E.6.10 we learnt that simplicity of a strongly graded ring  $\mathcal{R}$  does not immediately imply maximal commutativity of the neutral component  $\mathcal{R}_e$ . However, for skew group rings there is in fact such an implication, as the following theorem shows.

**Theorem E.6.13.** *Let  $\mathcal{R} = \mathcal{R}_e \rtimes_{\sigma} G$  be a skew group ring with  $\mathcal{R}_e$  commutative. The following two assertions are equivalent:*

- (i)  $\mathcal{R}_e$  is a maximal commutative subring of  $\mathcal{R}$  and  $\mathcal{R}_e$  is  $G$ -simple.
- (ii)  $\mathcal{R}$  is a simple ring.

*Proof.* By Theorem E.6.6, (i) implies (ii). Suppose that (ii) holds. It follows from Theorem E.3.5 that  $\mathcal{R}_e$  is maximal commutative in  $\mathcal{R}$  and by Proposition E.6.1 we conclude that  $\mathcal{R}_e$  is  $G$ -simple. This concludes the proof.  $\square$

It follows from [11, Corollary 10] that the assumptions made in [3, Corollary 2.1] force the coefficient ring to be maximal commutative in the skew group ring. By the assumptions made in [3, Theorem 2.2] the same conclusion follows by [3, Proposition 2.2] together with [11, Corollary 6]. This shows that Theorem E.6.13 is a generalization of [3, Corollary 2.1] and [3, Theorem 2.2].

**Remark E.6.14.** Note that, in Theorem E.6.13, the implication from (i) to (ii) holds in much greater generality. Indeed, it holds for any strongly graded ring.

A majority of the objects studied in [19–21] satisfy the conditions of Theorem E.6.13 and hence it applies. We shall show one such example.

**Example E.6.15** (Skew group algebras associated to dynamical systems). Let  $h : X \rightarrow X$  be a bijection on a nonempty set  $X$ , and  $A \subseteq \mathbb{C}^X$  an algebra of functions, such that if  $f \in A$  then  $f \circ h \in A$  and  $f \circ h^{-1} \in A$ . Let  $\tilde{h} : \mathbb{Z} \rightarrow \text{Aut}(A)$  be defined by  $\tilde{h}_n : f \mapsto f \circ h^{\circ(n)}$  for  $f \in A$  and  $n \in \mathbb{Z}$ . We now have a  $\mathbb{Z}$ -crossed system (with trivial  $\tilde{h}$ -cocycle) and we may define the skew group algebra  $A \rtimes_{\tilde{h}} \mathbb{Z}$ . For more details we refer to the papers [19–21], in which this construction has been studied thoroughly.

By Theorem E.6.13 we get the following corollary, since  $\mathbb{C}^X$  is commutative.

**Corollary E.6.16.** *Following Example E.6.15, let  $A \rtimes_{\tilde{h}} \mathbb{Z}$  be the skew group algebra associated to a dynamical system  $(X, h)$ . The following assertions are equivalent:*

- (i)  $A \rtimes_{\tilde{h}} \mathbb{Z}$  is a simple algebra.
- (ii)  $A$  is a maximal commutative subalgebra of  $A \rtimes_{\tilde{h}} \mathbb{Z}$  and  $A$  is  $\mathbb{Z}$ -simple.

## E.7 Application: $\mathbb{Z}$ -graded algebraic crossed products associated to topological dynamical systems

Let  $(X, h)$  be a topological dynamical system, i.e.  $X$  is a compact Hausdorff space and  $h : X \rightarrow X$  is a homeomorphism. The algebra of complex-valued continuous functions on  $X$ , where addition and multiplication is defined pointwise, is denoted by  $C(X)$ . Define a map

$$\tilde{h} : \mathbb{Z} \rightarrow \text{Aut}(C(X)), \quad \tilde{h}_n(f) = f \circ h^{\circ(n)}, \quad f \in C(X)$$

and let  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  be the algebraic crossed product<sup>2</sup> associated to our dynamical system. Recall that elements of  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  are written as formal sums  $\sum_{n \in \mathbb{Z}} f_n u_n$ , where all but a finite number of  $f_n \in C(X)$ , for  $n \in \mathbb{Z}$ , are nonzero. The multiplication in  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  is defined as the bilinear extension of the rule

$$(f_n u_n)(g_m u_m) = f_n \tilde{h}_n(g_m) u_{n+m}$$

for  $n, m \in \mathbb{Z}$  and  $f_n, g_m \in C(X)$ . We now define the following sets:

$$\begin{aligned} \text{Per}^n(h) &= \left\{ x \in X \mid h^{\circ(n)}(x) = x \right\}, \quad n \in \mathbb{Z} \\ \text{Per}(h) &= \bigcup_{n \in \mathbb{Z}_{>0}} \text{Per}^n(h) \\ \text{Aper}(h) &= X \setminus \text{Per}(h) \end{aligned}$$

Elements of  $\text{Aper}(h)$  are referred to as *aperiodic points* of the topological dynamical system  $(X, h)$ . By Urysohn's lemma,  $C(X)$  separates points of  $X$  and hence by [19, Corollary 3.4] we get the following.

**Lemma E.7.1.** *The commutant of  $C(X)$  in  $\mathcal{R} = C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  is given by*

$$C_{\mathcal{R}}(C(X)) = \left\{ \sum_{n \in \mathbb{Z}} f_n u_n \mid \text{supp}(f_n) \subseteq \text{Per}^n(h), \quad f_n \in C(X), \quad n \in \mathbb{Z} \right\}.$$

---

<sup>2</sup>In ring theory literature this would be referred to as a *skew group algebra*, but here we adopt the terminology used in [19–21] which comes from the  $C^*$ -algebra literature. Note however, that this is not a  $C^*$ -crossed product, but an algebraic crossed product.

The topological dynamical system  $(X, h)$  is said to be *topologically free* if and only if  $\text{Aper}(h)$  is dense in  $X$ . Using topological properties of our (completely regular) space  $X$  together with the remarks made in [19], in particular [19, Theorem 3.5], one can prove the following.

**Lemma E.7.2.**  *$C(X)$  is maximal commutative in  $C(X) \rtimes_{\bar{h}} \mathbb{Z}$  if and only if  $(X, h)$  is topologically free.*

If  $I$  is an ideal of  $C(X)$  then we denote

$$\text{supp}(I) = \bigcup_{f \in I} \text{supp}(f)$$

where  $\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$  for  $f \in C(X)$ . Note that a subset  $S \subseteq X$  is  $\mathbb{Z}$ -invariant if and only if  $h(S) = S$ .

**Lemma E.7.3.**  *$C(X)$  is  $\mathbb{Z}$ -simple if and only if there are no nonempty proper  $h$ -invariant closed subsets of  $X$ .*

*Proof.* Suppose that  $C(X)$  is not  $\mathbb{Z}$ -simple. Then there exists some proper nonzero ideal  $I \subsetneq C(X)$  such that  $\text{supp}(I) \neq \emptyset$  is a proper  $h$ -invariant closed subset of  $X$ . Conversely, suppose that there exists some nonempty proper  $h$ -invariant closed subset  $S \subsetneq X$ . Let  $B \subseteq C(X)$  be set of functions which vanish outside  $S$ . Clearly  $B$  is a proper nonzero  $\mathbb{Z}$ -invariant ideal of  $C(X)$  and hence  $C(X)$  is not  $\mathbb{Z}$ -simple.  $\square$

**Definition E.7.4.** A topological dynamical system  $(X, h)$  is said to be *minimal* if each orbit of the dynamical system is dense in  $X$ .

Note that a topological dynamical system  $(X, h)$  is minimal if and only if there are no nonempty proper  $h$ -invariant closed subsets of  $X$ .

**Remark E.7.5.** If  $X$  is infinite and  $(X, h)$  is minimal, then  $(X, h)$  is automatically free and in particular topologically free. Indeed, take an arbitrary  $x \in X$  and suppose that it is periodic. By minimality, the orbit of  $x$  which by periodicity is finite, must be dense in  $X$ . This is a contradiction, since  $X$  is Hausdorff, and hence each  $x \in X$  is aperiodic.

**Theorem E.7.6.** *If  $(X, h)$  is a topological dynamical system with  $X$  infinite, then the following assertions are equivalent:*

- (i)  $C(X) \rtimes_{\bar{h}} \mathbb{Z}$  is a simple algebra.
- (ii)  $C(X)$  is maximal commutative in  $C(X) \rtimes_{\bar{h}} \mathbb{Z}$  and  $C(X)$  is  $\mathbb{Z}$ -simple.
- (iii)  $(X, h)$  is a minimal dynamical system.

*Proof.* (i)  $\iff$  (ii): This follows from Theorem E.6.13.

(iii)  $\Rightarrow$  (ii): Let  $(X, h)$  be minimal. By Remark E.7.5  $(X, h)$  is topologically free and by Lemma E.7.2 this implies that  $C(X)$  is maximal commutative in  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$ . Furthermore, since  $(X, h)$  is minimal there is no nonempty proper  $h$ -invariant closed subset of  $X$  and hence by Lemma E.7.3 it follows that  $C(X)$  is  $\mathbb{Z}$ -simple.

(ii)  $\Rightarrow$  (iii): Suppose that  $(X, h)$  is not minimal. Then there exists some nonempty proper  $h$ -invariant closed subset of  $X$  and by Lemma E.7.3  $C(X)$  is not  $\mathbb{Z}$ -simple.  $\square$

For  $C^*$ -crossed product algebras associated to topological dynamical systems the analogue of the above theorem, Theorem E.1.2, is well-known (see e.g. [1], [17] or [23, Theorem 4.3.3]).

**Example E.7.7** (Finite single orbit dynamical systems). Suppose that  $X = \{x, h(x), h^{\circ(2)}(x), \dots, h^{\circ(p-1)}(x)\}$  consists of a finite  $h$ -orbit of order  $p$ , where  $p$  is a positive integer. One can then show that  $C(X) \rtimes_{\tilde{h}} \mathbb{Z} \cong M_p(\mathbb{C}[t, t^{-1}])$ , i.e. the skew group algebra associated to our dynamical system is isomorphic (as a  $\mathbb{C}$ -algebra) to the algebra of  $p \times p$ -matrices over the ring of Laurent polynomials over  $\mathbb{C}$ . Indeed, let  $\pi : C(X) \rtimes_{\tilde{h}} \mathbb{Z} \rightarrow M_p(\mathbb{C}[t, t^{-1}])$  be the  $\mathbb{C}$ -algebra morphism defined by

$$\pi(f) = \begin{pmatrix} f(x) & 0 & \dots & 0 \\ 0 & f \circ h(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \circ h^{\circ(p-1)}(x) \end{pmatrix}$$

for  $f \in C(X)$ , and

$$\pi(u_1) = \begin{pmatrix} 0 & 0 & \dots & 0 & t \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Calculating, one sees that

$$\pi \left( \sum_{n \in \mathbb{Z}} f_n u_n \right) = \begin{pmatrix} \sum_{n \in \mathbb{Z}} f_{np}(x) t^n & \dots & \sum_{n \in \mathbb{Z}} f_{(n-1)p+1}(x) t^n \\ \sum_{n \in \mathbb{Z}} f_{np+1}(h(x)) t^n & \dots & \sum_{n \in \mathbb{Z}} f_{(n-1)p+2}(h(x)) t^n \\ \sum_{n \in \mathbb{Z}} f_{np+2}(h^{\circ(2)}(x)) t^n & \dots & \sum_{n \in \mathbb{Z}} f_{(n-1)p+3}(h^{\circ(2)}(x)) t^n \\ \vdots & \vdots & \vdots \\ \sum_{n \in \mathbb{Z}} f_{(n+1)p-1}(h^{\circ(p-1)}(x)) t^n & \dots & \sum_{n \in \mathbb{Z}} f_{np}(h^{\circ(p-1)}(x)) t^n \end{pmatrix}$$

and by looking at the above matrix row by row, it is straightforward to verify that  $\pi$  is bijective (see [22, 24] for a similar isomorphism of  $C^*$ -algebras).

Clearly  $(X, h)$  is a minimal dynamical system and by Lemma E.7.3 we conclude that  $C(X)$  is  $\mathbb{Z}$ -simple. However, each element of  $X$  is  $n$ -periodic and hence  $(X, h)$  is not topologically free, which by Lemma E.7.2 entails that  $C(X)$  is not maximal commutative in  $\mathcal{R} = C(X) \rtimes_{\tilde{h}} \mathbb{Z}$ . The ring  $\mathbb{C}[t, t^{-1}]$  is not simple (e.g. by Example E.4.4) and via the isomorphism  $\pi$  we conclude that  $C(X) \rtimes_{\tilde{h}} \mathbb{Z}$  is never simple. From Section E.2.1 it is clear that the action  $\tilde{h}$  extends to an action of  $\mathbb{Z}$  on  $C_{\mathcal{R}}(C(X))$ . Finally, by Proposition E.6.5, we conclude that the commutant of  $C(X)$  is never  $\mathbb{Z}$ -simple for our finite single orbit dynamical system.

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## **Paper F**

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**F**





## Paper F

# Commutativity and ideals in category crossed products

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**Abstract.** *In order to simultaneously generalize matrix rings and group graded crossed products, we introduce category crossed products. For such algebras we describe the center and the commutant of the coefficient ring. We also investigate the connection between on the one hand maximal commutativity of the coefficient ring and on the other hand nonemptiness of intersections of the coefficient ring by nonzero twosided ideals.*

## F.1 Introduction

Let  $R$  be a ring. By this we always mean that  $R$  is an additive group equipped with a multiplication which is associative and unital. The identity element of  $R$  is denoted  $1_R$  and the set of ring endomorphisms of  $R$  is denoted  $\text{End}(R)$ . We always assume that ring homomorphisms respect the multiplicative identities. The center of  $R$  is denoted  $Z(R)$  and by the commutant of a subset of  $R$  we mean the collection of elements in  $R$  that commute with all the elements in the subset.

Suppose that  $R_1$  is a subring of  $R$ , i.e. there is an injective ring homomorphism  $R_1 \rightarrow R$ . Recall that if  $R_1$  is commutative, then it is called a maximal commutative subring of  $R$  if it coincides with its commutant in  $R$ . A lot of work has been devoted to investigating the connection between on the one hand maximal commutativity of  $R_1$  in  $R$  and on the other hand nonemptiness of intersections of  $R_1$  with nonzero twosided ideals of  $R$  (see [2], [3], [6], [7], [10], [12], [13] and [18]). Recently (see [22], [23], [24], [25] and [26]) such a connection was established for the commutant  $R_1$  of the coefficient ring of crossed products  $R$  (see Theorem F.1.1 below). Recall that crossed products are defined by first specifying a crossed system, i.e. a quadruple  $\{A, G, \sigma, \alpha\}$  where  $A$  is a ring,  $G$  is a group (written multiplicatively and with identity element  $e$ ) and  $\sigma : G \rightarrow \text{End}(A)$  and  $\alpha : G \times G \rightarrow A$  are maps satisfying the following four conditions:

$$\sigma_e = \text{id}_A \tag{F.1}$$

$$\alpha(s, e) = \alpha(e, s) = 1_A \tag{F.2}$$

$$\alpha(s, t)\alpha(st, r) = \sigma_s(\alpha(t, r))\alpha(s, tr) \quad (\text{F.3})$$

$$\sigma_s(\sigma_t(a))\alpha(s, t) = \alpha(s, t)\sigma_{st}(a) \quad (\text{F.4})$$

for all  $s, t, r \in G$  and all  $a \in A$ . The crossed product, denoted  $A \rtimes_{\alpha}^{\sigma} G$ , associated to this quadruple, is the collection of formal sums  $\sum_{s \in G} a_s u_s$ , where  $a_s \in A$ ,  $s \in G$ , are chosen so that all but finitely many of them are zero. By abuse of notation we write  $u_s$  instead of  $1_A u_s$  for all  $s \in G$ . The addition on  $A \rtimes_{\alpha}^{\sigma} G$  is defined pointwise

$$\sum_{s \in G} a_s u_s + \sum_{s \in G} b_s u_s = \sum_{s \in G} (a_s + b_s) u_s \quad (\text{F.5})$$

and the multiplication on  $A \rtimes_{\alpha}^{\sigma} G$  is defined by the bilinear extension of the relation

$$(a_s u_s)(b_t u_t) = a_s \sigma_s(b_t) \alpha(s, t) u_{st} \quad (\text{F.6})$$

for all  $s, t \in G$  and all  $a_s, b_t \in A$ . By (F.1) and (F.2)  $u_e$  is a multiplicative identity of  $A \rtimes_{\alpha}^{\sigma} G$  and by (F.3) the multiplication on  $A \rtimes_{\alpha}^{\sigma} G$  is associative. There is also an  $A$ -bimodule structure on  $A \rtimes_{\alpha}^{\sigma} G$  defined by the linear extension of the relations  $a(bu_s) = (ab)u_s$  and  $(au_s)b = (a\sigma_s(b))u_s$  for all  $a, b \in A$  and all  $s, t \in G$ , which, by (F.4), makes  $A \rtimes_{\alpha}^{\sigma} G$  an  $A$ -algebra. In the article [22], the first author and Silvestrov show the following result.

**Theorem F.1.1.** *If  $A \rtimes_{\alpha}^{\sigma} G$  is a crossed product with  $A$  commutative, all  $\sigma_s$ ,  $s \in G$ , are ring automorphisms and all  $\alpha(s, s^{-1})$ ,  $s \in G$ , are units in  $A$ , then every intersection of a nonzero twosided ideal of  $A \rtimes_{\alpha}^{\sigma} G$  with the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$  is nonzero.*

In loc. cit. the first author and Silvestrov determine the center of crossed products and in particular when crossed products are commutative; they also give a description of the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$ . Theorem F.1.1 has been generalized somewhat by relaxing the conditions on  $\sigma$  and  $\alpha$  (see [24] and [25]) and by considering general strongly group graded rings (see [26]). For more details concerning group graded rings in general and crossed product algebras in particular, see e.g. [1], [8] and [19].

Many natural examples of rings, such as rings of matrices, crossed product algebras defined by separable extensions and category rings, are not in any natural way graded by groups, but instead by categories (see [14], [15], [16] and Remark F.2.6). The purpose of this article is to define a category graded generalization of crossed products and to analyze commutativity questions similar to the ones discussed above for such algebras. In particular, we wish to generalize Theorem F.1.1 from groups to groupoids (see Theorem F.4.1 in Section F.4). To be more precise, suppose that  $G$  is a category. The family of objects of  $G$  is denoted  $\text{ob}(G)$ ; we will often identify an object in  $G$  with its associated identity morphism. The family of morphisms in  $G$  is denoted  $\text{mor}(G)$ ; by abuse of notation, we will often write  $s \in G$  when we mean  $s \in \text{mor}(G)$ . The domain and codomain of a morphism  $s$  in  $G$  is denoted  $d(s)$  and  $c(s)$  respectively. We let  $G^{(2)}$  denote

the collection of composable pairs of morphisms in  $G$ , i.e. all  $(s, t)$  in  $\text{mor}(G) \times \text{mor}(G)$  satisfying  $d(s) = c(t)$ . Analogously, we let  $G^{(3)}$  denote the collection of all composable triples of morphisms in  $G$ , i.e. all  $(s, t, r)$  in  $\text{mor}(G) \times \text{mor}(G) \times \text{mor}(G)$  satisfying  $(s, t) \in G^{(2)}$  and  $(t, r) \in G^{(2)}$ . Throughout the article  $G$  is assumed to be small, i.e. with the property that  $\text{mor}(G)$  is a set. A category is called a groupoid<sup>1</sup> if all its morphisms are invertible. By a crossed system we mean a quadruple  $\{A, G, \sigma, \alpha\}$  where  $A$  is the direct sum of rings  $A_e$ ,  $e \in \text{ob}(G)$ ,  $\sigma_s : A_{d(s)} \rightarrow A_{c(s)}$ , for  $s \in G$ , are ring homomorphisms and  $\alpha$  is a map from  $G^{(2)}$  to the disjoint union of the sets  $A_e$ , for  $e \in \text{ob}(G)$ , with  $\alpha(s, t) \in A_{c(s)}$ , for  $(s, t) \in G^{(2)}$ , satisfying the following five conditions:

$$\sigma_e = \text{id}_{A_e} \tag{E.7}$$

$$\alpha(s, d(s)) = 1_{A_{c(s)}} \tag{E.8}$$

$$\alpha(c(t), t) = 1_{A_{c(t)}} \tag{E.9}$$

$$\alpha(s, t)\alpha(st, r) = \sigma_s(\alpha(t, r))\alpha(s, tr) \tag{E.10}$$

$$\sigma_s(\sigma_t(a))\alpha(s, t) = \alpha(s, t)\sigma_{st}(a) \tag{E.11}$$

for all  $e \in \text{ob}(G)$ , all  $(s, t, r) \in G^{(3)}$  and all  $a \in A_{d(t)}$ . Let  $A \rtimes_{\alpha}^{\sigma} G$  denote the collection of formal sums  $\sum_{s \in G} a_s u_s$ , where  $a_s \in A_{c(s)}$ ,  $s \in G$ , are chosen so that all but finitely many of them are zero. Define addition on  $A \rtimes_{\alpha}^{\sigma} G$  by (E.5) and define multiplication on  $A \rtimes_{\alpha}^{\sigma} G$  by (E.6) if  $(s, t) \in G^{(2)}$  and  $(a_s u_s)(b_t u_t) = 0$  otherwise where  $a_s \in A_{c(s)}$  and  $b_t \in A_{c(t)}$ . By (E.7), (E.8) and (E.9) it follows that  $A \rtimes_{\alpha}^{\sigma} G$  has a multiplicative identity if and only if  $\text{ob}(G)$  is finite; in that case the multiplicative identity is  $\sum_{e \in \text{ob}(G)} u_e$ . By (E.10) the multiplication on  $A \rtimes_{\alpha}^{\sigma} G$  is associative. Define a left  $A$ -module structure on  $A \rtimes_{\alpha}^{\sigma} G$  by the bilinear extension of the rule  $a_e(b_s u_s) = (a_e b_s) u_s$  if  $e = c(s)$  and  $a_e(b_s u_s) = 0$  otherwise for all  $a_e \in A_e$ ,  $b_s \in A_{c(s)}$ ,  $e \in \text{ob}(G)$ ,  $s \in G$ . Analogously, define a right  $A$ -module structure on  $A \rtimes_{\alpha}^{\sigma} G$  by the bilinear extension of the rule  $(b_s u_s)c_f = (b_s \sigma_s(c_f)) u_s$  if  $f = d(s)$  and  $(b_s u_s)c_f = 0$  otherwise for all  $b_s \in A_{c(s)}$ ,  $c_f \in A_f$ ,  $f \in \text{ob}(G)$ ,  $s \in G$ . By (E.11) this  $A$ -bimodule structure makes  $A \rtimes_{\alpha}^{\sigma} G$  an  $A$ -algebra. We will often identify  $A$  with  $\bigoplus_{e \in \text{ob}(G)} A_e u_e$ ; this ring will be referred to as the coefficient ring of  $A \rtimes_{\alpha}^{\sigma} G$ . It is clear that  $A \rtimes_{\alpha}^{\sigma} G$  is a category graded ring in the sense defined in [15] and it is strongly graded if and only if each  $\alpha(s, t)$ ,  $(s, t) \in G^{(2)}$ , has a left inverse in  $A_{c(s)}$ . We call  $A \rtimes_{\alpha}^{\sigma} G$  the category crossed product algebra associated to the crossed system  $\{A, G, \sigma, \alpha\}$ .

In Section F.2, we determine the center of category crossed products. In particular, we determine when category crossed products are commutative. In Section F.3, we describe the commutant of the coefficient ring in category crossed products. In Section F.4, we investigate the connection between on the one hand maximal commutativity of the

<sup>1</sup>The term *groupoid* has various meanings in the literature. E.g. in [9], a set with a binary operation is referred to as a *groupoid*.

coefficient ring and on the other hand nonemptiness of intersections of the coefficient ring by nonzero twosided ideals. In the end of each section, we indicate how our results generalize earlier results for other algebraic structures such as group crossed products and matrix rings (see Remarks F.2.6-F.3.7 and Remark F.4.9).

## F.2 The center

For the rest of the article, unless otherwise stated, we suppose that  $A \rtimes_{\alpha}^{\sigma} G$  is a category crossed product. We say that  $\alpha$  is symmetric if  $\alpha(s, t) = \alpha(t, s)$  for all  $s, t \in G$  with  $d(s) = c(s) = d(t) = c(t)$ . We say that  $A \rtimes_{\alpha}^{\sigma} G$  is a monoid (groupoid, group) crossed product if  $G$  is a monoid (groupoid, group). We say that  $A \rtimes_{\alpha}^{\sigma} G$  is a twisted category (monoid, groupoid, group) algebra if each  $\sigma_s, s \in G$ , with  $d(s) = c(s)$  equals the identity map on  $A_{d(s)} = A_{c(s)}$ ; in that case the category (monoid, groupoid, group) crossed product is denoted  $A \rtimes_{\alpha} G$ . We say that  $A \rtimes_{\alpha}^{\sigma} G$  is a skew category (monoid, groupoid, group) algebra if  $\alpha(s, t) = 1_{A_{c(s)}}$ , for  $(s, t) \in G^{(2)}$ ; in that case the category (monoid, groupoid, group) crossed product is denoted  $A \rtimes^{\sigma} G$ . If  $G$  is a monoid, then we let  $A^G$  denote the set of elements in  $A$  fixed by all  $\sigma_s, s \in G$ . We say that  $G$  is cancellable if any equality of the form  $s_1 t_1 = s_2 t_2$ , when  $(s_i, t_i) \in G^{(2)}$ , for  $i = 1, 2$ , implies that  $s_1 = s_2$  (or  $t_1 = t_2$ ) whenever  $t_1 = t_2$  (or  $s_1 = s_2$ ). For  $e, f \in \text{ob}(G)$  we let  $G_{f,e}$  denote the collection of  $s \in G$  with  $c(s) = f$  and  $d(s) = e$ ; we let  $G_e$  denote the monoid  $G_{e,e}$ . We let the restriction of  $\alpha$  (or  $\sigma$ ) to  $G_e^2$  (or  $G_e$ ) be denoted by  $\alpha_e$  (or  $\sigma_e$ ). With this notation all  $A_e \rtimes_{\alpha_e}^{\sigma_e} G_e$ , for  $e \in \text{ob}(G)$ , are monoid crossed products.

**Proposition F.2.1.** *The center of a monoid crossed product  $A \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  satisfying the following two conditions: (i)  $a_s \sigma_s(a) = a a_s$ , for  $s \in G$  and  $a \in A$ ; (ii) for all  $t, r \in G$  the following equality holds  $\sum_{s \in G} a_s \alpha(s, t) = \sum_{t s = r} \sigma_t(a_s) \alpha(t, s)$ .*

*Proof.* Let  $e$  denote the identity element of  $G$ . Take  $x := \sum_{s \in G} a_s u_s$  in the center of  $A \rtimes_{\alpha}^{\sigma} G$ . Condition (i) follows from the fact that  $x a u_e = a u_e x$  for all  $a \in A$ . Condition (ii) follows from the fact that  $x u_t = u_t x$  for all  $t \in G$ . Conversely, it is clear that conditions (i) and (ii) are sufficient for  $x$  to be in the center of  $A \rtimes_{\alpha}^{\sigma} G$ .  $\square$

**Corollary F.2.2.** *The center of a twisted monoid ring  $A \rtimes_{\alpha} G$  is the collection of  $\sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha} G$  satisfying the following two conditions: (i)  $a_s \in Z(A)$ , for  $s \in G$ ; (ii) for all  $t, r \in G$ , the following equality holds  $\sum_{s \in G} a_s \alpha(s, t) = \sum_{t s = r} a_s \alpha(t, s)$ .*

*Proof.* This follows immediately from Proposition F.2.1.  $\square$

**Corollary F.2.3.** *If  $G$  is an abelian cancellable monoid,  $\alpha$  is symmetric and has the property that none of the  $\alpha(s, t)$ , for  $(s, t) \in G^{(2)}$ , is a zerodivisor, then the center of  $A \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  satisfying the following two conditions: (i)  $a_s \sigma_s(a) =$*

$aa_s$ , for  $s \in G$  and  $a \in A$ ; (ii)  $a_s \in A^G$ , for  $s \in G$ . In particular, if  $A \rtimes^\sigma G$  is a skew monoid ring where  $G$  is abelian and cancellable, then the same description of the center is valid.

*Proof.* Take  $x := \sum_{s \in G} a_s u_s$  in  $A \rtimes_\alpha^\sigma G$ . Suppose that  $x$  belongs to the center of  $A \rtimes_\alpha^\sigma G$ . Condition (i) follows from the first part of Proposition F.2.1. Now we show condition (ii). Take  $s, t \in G$  and let  $r = st$ . Since  $G$  is commutative and cancellable, we get, by the second part of Proposition F.2.1, that  $a_s \alpha(s, t) = \sigma_t(a_s) \alpha(t, s)$ . Since  $\alpha$  is symmetric and  $\alpha(s, t)$  is not a zerodivisor, this implies that  $a_s = \sigma_t(a_s)$ . Since  $s$  and  $t$  were arbitrarily chosen from  $G$ , this implies that  $a_s \in A^G$ , for  $s \in G$ . On the other hand, by Proposition F.2.1, it is clear that (i) and (ii) are sufficient conditions for  $x$  to be in the center of  $A \rtimes_\alpha^\sigma G$ . The second part of the claim is obvious.  $\square$

Now we show that the center of a category crossed product is a particular subring of the direct sum of the centers of the corresponding monoid crossed products.

**Proposition F.2.4.** *The center of a category crossed product  $A \rtimes_\alpha^\sigma G$  equals the collection of  $\sum_{e \in \text{ob}(G)} \sum_{s \in G_e} a_s u_s$  in  $\sum_{e \in \text{ob}(G)} Z(A_e \rtimes_{\alpha_e}^{\sigma_e} G_e)$  satisfying  $\sum_{\substack{s \in G_e \\ r s = g}} \sigma_r(a_s) \alpha(r, s) = \sum_{\substack{t \in G_f \\ t r = g}} a_t \alpha(t, r)$  for all  $e, f \in \text{ob}(G)$  with  $e \neq f$ , and all  $r, g \in G_{f,e}$ .*

*Proof.* Take  $x := \sum_{s \in G} a_s u_s$  in the center of  $A \rtimes_\alpha^\sigma G$ . By the equalities  $u_e x = x u_e$ , for  $e \in \text{ob}(G)$ , it follows that  $a_s = 0$  for all  $s \in G$  with  $d(s) \neq c(s)$ . Therefore we can write  $x = \sum_{e \in \text{ob}(G)} \sum_{s \in G_e} a_s u_s$  where  $\sum_{s \in G_e} a_s u_s \in Z(A_e \rtimes_{\alpha_e}^{\sigma_e} G_e)$ , for  $e \in \text{ob}(G)$ . The last part of the claim follows from the fact that the equality  $u_r (\sum_{s \in G_e} a_s u_s) = (\sum_{s \in G_e} a_s u_s) u_r$  holds for all  $e, f \in \text{ob}(G)$ , all  $e \neq f$ , and all  $r \in G_{f,e}$ .  $\square$

**Proposition F.2.5.** *Suppose that  $A \rtimes_\alpha^\sigma G$  is a category crossed product and consider the following five conditions: (0) all  $\alpha(s, t)$ , for  $(s, t) \in G^{(2)}$ , are nonzero; (i)  $A \rtimes_\alpha^\sigma G$  is commutative; (ii)  $G$  is the disjoint union of the monoids  $G_e$ , for  $e \in \text{ob}(G)$ , and they are all abelian; (iii) each  $A_e \rtimes_{\alpha_e}^{\sigma_e} G_e$ , for  $e \in \text{ob}(G)$ , is a twisted monoid algebra; (iv)  $A$  is commutative; (v)  $\alpha$  is symmetric. Then (a) Conditions (0) and (i) imply conditions (ii)-(v); (b) Conditions (ii)-(v) imply condition (i).*

*Proof.* (a) Suppose that conditions (0) and (i) hold. By Proposition F.2.4, we get that  $G$  is the direct sum of  $G_e$ , for  $e \in \text{ob}(G)$ , and that each  $A_e \rtimes_{\alpha_e}^{\sigma_e} G_e$ , for  $e \in \text{ob}(G)$ , is commutative. The latter and Proposition F.2.1(i) imply that (iii) holds. Corollary F.2.2 now implies that (iv) holds. For the rest of the proof we can suppose that  $G$  is a monoid. Take  $s, t \in G$ . By the commutativity of  $A \rtimes_\alpha^\sigma G$  we get that  $\alpha(s, t) u_{st} = u_s u_t = u_t u_s = \alpha(t, s) u_{ts}$  for all  $s, t \in G$ . Since  $\alpha$  is nonzero this implies that  $st = ts$  and that  $\alpha(s, t) = \alpha(t, s)$  for all  $s, t \in G$ . Therefore,  $G$  is abelian and (v) holds.

Conversely, by Corollary F.2.2 and Corollary F.2.3 we get that conditions (ii)-(iv) are sufficient for commutativity of  $A \rtimes_\alpha^\sigma G$ .  $\square$

**Remark F.2.6.** Proposition F.2.4, Corollary F.2.2, Corollary F.2.3 and Proposition F.2.5 generalize Proposition 3 and Corollaries 1-4 in [22] from groups to categories.

**Remark F.2.7.** Let  $A \rtimes G$  be a category algebra where all the rings  $A_e$ , for  $e \in \text{ob}(G)$ , coincide with a fixed ring  $D$ . Then  $A \rtimes G$  is the usual category algebra  $DG$  of  $G$  over  $D$ . Let  $H$  denote the disjoint union of the monoids  $G_e$ , for  $e \in \text{ob}(G)$ . By Proposition F.2.1 and Proposition F.2.4 the center of  $DG$  is the collection of  $\sum_{s \in H} a_s u_s$ , for  $a_s \in Z(D)$ , and  $s \in H$ , in the induced category algebra  $Z(D)H$  satisfying  $\sum_{\substack{s \in H \\ st=r}} a_s = \sum_{\substack{s \in H \\ ts=r}} a_s$  for all  $r, t \in G$ . Note that if  $G$  is a groupoid, then the last condition simplifies to  $a_{rt^{-1}} = a_{t^{-1}r}$  for all  $r, t \in G$  with  $c(r) = c(t)$  and  $d(r) = d(t)$ . This result specializes to two well known cases. First of all, if  $G$  is a group, then we retrieve the usual description of the center of a group ring (see e.g. [27]). Secondly, if  $G$  is the groupoid with the  $n$  first positive integers as objects and as arrows all pairs  $(i, j)$ , for  $1 \leq i, j \leq n$ , equipped with the partial binary operation defined by letting  $(i, j)(k, l)$  be defined and equal to  $(i, l)$  precisely when  $j = k$ , then  $DG$  is the ring of square matrices over  $D$  of size  $n$  and we retrieve the result that  $Z(M_n(D))$  equals the  $Z(D)1_n$  where  $1_n$  is the unit  $n \times n$  matrix.

**Remark F.2.8.** Let  $L/K$  be a finite separable (not necessarily normal) field extension. Let  $N$  denote a normal closure of  $L/K$  and let  $\text{Gal}(N/K)$  denote the Galois group of  $N/K$ . Furthermore, let  $L = L_1, L_2, \dots, L_n$  denote the different conjugate fields of  $L$  under the action of  $\text{Gal}(N/K)$  and put  $F = \bigoplus_{i=1}^n L_i$ . If  $1 \leq i, j \leq n$ , then let  $G_{ij}$  denote the set of field isomorphisms from  $L_j$  to  $L_i$ . If  $s \in G_{ij}$ , then we indicate this by writing  $d(s) = j$  and  $c(s) = i$ . If we let  $G$  be the union of the  $G_{ij}$ , for  $1 \leq i, j \leq n$ , then  $G$  is a groupoid. For each  $s \in G$ , let  $\sigma_s = s$ . Suppose that  $\alpha$  is a map  $G^{(2)} \rightarrow \bigsqcup_{i=1}^n L_i$  with  $\alpha(s, t) \in L_{c(s)}$ , for  $(s, t) \in G^{(2)}$  satisfying (F.2), (F.3) and (F.4) for all  $(s, t, r) \in G^{(3)}$  and all  $a \in L_{d(t)}$ . The category crossed product  $F \rtimes_{\alpha}^{\sigma} G$  extends the construction usually defined by Galois field extensions  $L/K$ . By Proposition F.2.4, the center of  $F \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{e \in \text{ob}(G)} a_e u_e$  with  $a_e = s(a_f)$  for all  $e, f \in \text{ob}(G)$  and all  $s \in G$  with  $c(s) = e$  and  $d(s) = f$ . Therefore the center is a field isomorphic to  $L^{G^{1,1}}$  and we retrieve the first part of Theorem 4 in [14].

### F.3 The commutant of the coefficient ring

**Proposition F.3.1.** *The commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  satisfying  $a_s = 0$ , for  $s \in G$ , with  $d(s) \neq c(s)$ , and  $a_s \sigma_s(a) = a a_s$ , for  $s \in G$  with  $d(s) = c(s)$  and  $a \in A_{d(s)}$ .*

*Proof.* The first claim follows from the fact that the equality  $(\sum_{s \in G} a_s u_s) u_e = u_e (\sum_{s \in G} a_s u_s)$  holds for all  $e \in \text{ob}(G)$ . The second claim follows from the fact that the equality  $(\sum_{s \in G} a_s u_s) a u_e = a u_e (\sum_{s \in G} a_s u_s)$  holds for all  $e \in \text{ob}(G)$  and all  $a \in A_e$ .  $\square$

Recall that the annihilator of an element  $r$  in a commutative ring  $R$  is the collection, denoted  $\text{ann}(r)$ , of elements  $s$  in  $R$  with the property that  $rs = 0$ .

**Corollary F.3.2.** *Suppose that  $A$  is commutative. Then the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  satisfying  $a_s = 0$ , for  $s \in G$  with  $d(s) \neq c(s)$ , and  $\sigma_s(a) - a \in \text{ann}(a_s)$ , for  $s \in G$  with  $d(s) = c(s)$  and  $a \in A_{d(s)}$ . In particular,  $A$  is maximal commutative in  $A \rtimes_{\alpha}^{\sigma} G$  if and only if there for all choices of  $e \in \text{ob}(G)$ ,  $s \in G_e \setminus \{e\}$ ,  $a_s \in A_e$ , there is a nonzero  $a \in A_e$  with the property that  $\sigma_s(a) - a \notin \text{ann}(a_s)$ .*

*Proof.* This follows immediately from Proposition F.3.1.  $\square$

**Corollary F.3.3.** *Suppose that each  $A_e$ ,  $e \in \text{ob}(G)$ , is an integral domain. Then the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  satisfying  $a_s = 0$  whenever  $\sigma_s$  is not an identity map. In particular,  $A$  is maximal commutative in  $A \rtimes_{\alpha}^{\sigma} G$  if and only if for all nonidentity  $s \in G$ , the map  $\sigma_s$  is not an identity map.*

*Proof.* This follows immediately from Corollary F.3.2.  $\square$

**Proposition F.3.4.** *If  $A$  is commutative,  $G$  a disjoint union of abelian monoids and  $\alpha$  is symmetric, then the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$  is the unique maximal commutative subalgebra of  $A \rtimes_{\alpha}^{\sigma} G$  containing  $A$ .*

*Proof.* We need to show that the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$  is commutative. By the first part of Proposition F.3.1, we can assume that  $G$  is an abelian monoid. If we take  $\sum_{s \in G} a_s u_s$  and  $\sum_{t \in G} b_t u_t$  in the commutant of  $A$  in  $A \rtimes_{\alpha}^{\sigma} G$ , then, by the second part of Proposition F.3.1 and the fact that  $\alpha$  is symmetric, we get that

$$\begin{aligned} \sum_{s \in G} a_s u_s \sum_{t \in G} b_t u_t &= \sum_{s, t \in G} a_s \sigma_s(b_t) \alpha(s, t) u_{st} = \sum_{s, t \in G} a_s b_t \alpha(s, t) u_{st} = \\ &= \sum_{s, t \in G} b_t a_s \alpha(t, s) u_{ts} = \sum_{s, t \in G} b_t \sigma_t(a_s) \alpha(t, s) u_{st} = \sum_{t \in G} b_t u_t \sum_{s \in G} a_s u_s \end{aligned}$$

$\square$

**Remark F.3.5.** Proposition F.3.1, Corollary F.3.2, Corollary F.3.3 and Proposition F.3.4 together generalize Theorem 1, Corollaries 5-10 and Proposition 4 in [22] from groups to categories.

**Remark F.3.6.** Let  $A \rtimes G$  be a category algebra where all the rings  $A_e$ ,  $e \in \text{ob}(G)$ , coincide with a fixed integral domain  $D$ . Then  $A \rtimes G$  is the usual category algebra  $DG$  of  $G$  over  $D$ . By Corollary F.3.3, the commutant of  $D$  in  $DG$  is  $DG$  itself. In particular,  $A$  is maximal commutative in  $DG$  if and only if  $G$  is the disjoint union of  $|\text{ob}(G)|$  copies of the trivial group.



**Remark F.3.7.** Let  $L/K$  be a finite separable (not necessarily normal) field extension. We use the same notation as in Remark F.2.8. By Corollary F.3.3, the commutant of  $F$  in  $F \rtimes_{\alpha}^{\sigma} G$  is the collection of  $\sum_{i=1}^n \sum_{s \in G_{ii}} a_s u_s$  satisfying  $a_s = 0$  whenever  $\sigma_s$  is not an identity map. In particular,  $F$  is maximal commutative in  $F \rtimes_{\alpha}^{\sigma} G$  if all groups  $G_{i,i}$ ,  $i = 1, \dots, n$ , are nontrivial; this of course happens in the case when  $L/K$  is a Galois field extension.

## F.4 Commutativity and ideals

In this section, we investigate the connection between on the one hand maximal commutativity of the coefficient ring and on the other hand nonemptiness of intersections of the coefficient ring by nonzero twosided ideals. For the rest of the article, we assume that  $\text{ob}(G)$  is finite. Recall (from Section F.1) that this is equivalent to the fact that  $A \rtimes_{\alpha}^{\sigma} G$  has a multiplicative identity; in that case the multiplicative identity is  $\sum_{e \in \text{ob}(G)} u_e$ .

**Theorem F.4.1.** *If  $A \rtimes_{\alpha}^{\sigma} G$  is a groupoid crossed product such that for every  $s \in G$ ,  $\alpha(s, s^{-1})$  is not a zero divisor in  $A_{c(s)}$ , then every intersection of a nonzero twosided ideal of  $A \rtimes_{\alpha}^{\sigma} G$  with the commutant of  $Z(A)$  in  $A \rtimes_{\alpha}^{\sigma} G$  is nonzero.*

*Proof.* We show the contrapositive statement. Let  $C$  denote the commutant of  $Z(A)$  in  $A \rtimes_{\alpha}^{\sigma} G$  and suppose that  $I$  is a twosided ideal of  $A \rtimes_{\alpha}^{\sigma} G$  with the property that  $I \cap C = \{0\}$ . We wish to show that  $I = \{0\}$ . Take  $x \in I$ . If  $x \in C$ , then by the assumption  $x = 0$ . Therefore we now assume that  $x = \sum_{s \in G} a_s u_s \in I$ ,  $a_s \in A_{c(s)}$ ,  $s \in G$ , and that  $x$  is chosen so that  $x \notin C$  with the set  $S := \{s \in G \mid a_s \neq 0\}$  of least possible cardinality  $N$ . Seeking a contradiction, suppose that  $N$  is positive. First note that there is  $e \in \text{ob}(G)$  with  $u_e x \in I \setminus C$ . In fact, if  $u_e x \in C$  for all  $e \in \text{ob}(G)$ , then  $x = 1x = \sum_{e \in \text{ob}(G)} u_e x \in C$  which is a contradiction. By minimality of  $N$  we can assume that  $c(s) = e$ ,  $s \in S$ , for some fixed  $e \in \text{ob}(G)$ . Take  $t \in S$  and consider the element  $x' := x u_{t^{-1}} \in I$ . Since  $\alpha(t, t^{-1})$  is not a zero divisor we get that  $x' \neq 0$ . Therefore, since  $I \cap C = \{0\}$ , we get that  $x' \in I \setminus C$ . Take  $a = \sum_{f \in \text{ob}(G)} b_f u_f \in Z(A)$  and note that  $Z(A) = \bigoplus_{f \in \text{ob}(G)} Z(A_f)$ . Then  $I \ni x'' := a x' - x' a = \sum_{s \in S} (b_{c(s)} a_s - a_s \sigma_s(b_{d(s)})) u_s$ . In the  $A_e$  component of this sum we have  $b_e a_e - a_e b_e = 0$  since  $b_e \in Z(A_e)$ . Thus, the summand vanishes for  $s = e$ , and hence we get, by the assumption on  $N$ , that  $x'' = 0$ . Since  $a \in Z(A)$  was arbitrarily chosen, we get that  $x' \in C$  which is a contradiction. Therefore  $N = 0$  and hence  $S = \emptyset$  which in turn implies that  $x = 0$ . Since  $x \in I$  was arbitrarily chosen, we finally get that  $I = \{0\}$ .  $\square$

**Corollary F.4.2.** *If  $A \rtimes_{\alpha}^{\sigma} G$  is a groupoid crossed product with  $A$  maximal commutative and for every  $s \in G$ ,  $\alpha(s, s^{-1})$  is not a zero divisor in  $A_{c(s)}$ , then every intersection of a nonzero twosided ideal of  $A \rtimes_{\alpha}^{\sigma} G$  with  $A$  is nonzero.*

*Proof.* This follows immediately from Theorem F.4.1.  $\square$

Now we examine conditions under which the opposite statement of Corollary F.4.2 is true. To this end, we recall some notions from category theory that we need in the sequel (for the details see e.g. [17]). Let  $G$  be a category. A congruence relation  $R$  on  $G$  is a collection of equivalence relations  $R_{a,b}$  on  $\text{hom}(a, b)$ ,  $a, b \in \text{ob}(G)$ , chosen so that if  $(s, s') \in R_{a,b}$  and  $(t, t') \in R_{b,c}$ , then  $(ts, t's') \in R_{a,c}$  for all  $a, b, c \in \text{ob}(G)$ . Given a congruence relation  $R$  on  $G$  we can define the corresponding quotient category  $G/R$  as the category having as objects the objects of  $G$  and as arrows the corresponding equivalence classes of arrows from  $G$ . In that case there is a full functor  $\mathcal{Q}_R : G \rightarrow G/R$  which is the identity on objects and sends each morphism of  $G$  to its equivalence class in  $R$ . We will often use the notation  $[s] := \mathcal{Q}_R(s)$ ,  $s \in G$ . Suppose that  $H$  is another category and that  $F : G \rightarrow H$  is a functor. The kernel of  $F$ , denoted  $\ker(F)$ , is the congruence relation on  $G$  defined by letting  $(s, t) \in \ker(F)_{a,b}$ ,  $a, b \in \text{ob}(G)$ , whenever  $s, t \in \text{hom}(a, b)$  and  $F(s) = F(t)$ . In that case there is a unique functor  $\mathcal{P}_F : G/\ker(F) \rightarrow H$  with the property that  $\mathcal{P}_F \mathcal{Q}_{\ker(F)} = F$ . Furthermore, if there is a congruence relation  $R$  on  $G$  contained in  $\ker(F)$ , then there is a unique functor  $\mathcal{N} : G/R \rightarrow G/\ker(F)$  with the property that  $\mathcal{N} \mathcal{Q}_R = \mathcal{Q}_{\ker(F)}$ . In that case there is therefore always a factorization  $F = \mathcal{P}_F \mathcal{N} \mathcal{Q}_R$ ; we will refer to this factorization as the canonical one.

**Proposition F.4.3.** *Let  $\{A, G, \sigma, \alpha\}$  and  $\{A, H, \tau, \beta\}$  be crossed systems with  $\text{ob}(G) = \text{ob}(H)$ . Suppose that there is a functor  $F : G \rightarrow H$  satisfying the following three criteria: (i)  $F$  is the identity map on objects; (ii)  $\tau_{F(s)} = \sigma_s$ , for  $s \in G$ ; (iii)  $\beta(F(s), F(t)) = \alpha(s, t)$ , for  $(s, t) \in G^{(2)}$ . Then there is a unique  $A$ -algebra homomorphism  $A \rtimes_{\alpha}^{\sigma} G \rightarrow A \rtimes_{\beta}^{\tau} H$ , denoted  $\tilde{F}$ , satisfying  $\tilde{F}(u_s) = u_{F(s)}$ , for  $s \in G$ .*

*Proof.* Take  $x := \sum_{s \in G} a_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  where  $a_s \in A_{c(s)}$ , for  $s \in G$ . By  $A$ -linearity we get that  $\tilde{F}(x) = \sum_{s \in G} a_s \tilde{F}(u_s) = \sum_{s \in G} a_s u_{F(s)}$ . Therefore  $\tilde{F}$  is unique. It is clear that  $\tilde{F}$  is additive. By (i),  $\tilde{F}$  respects the multiplicative identities. Now we show that  $\tilde{F}$  is multiplicative. Take another  $y := \sum_{s \in G} b_s u_s$  in  $A \rtimes_{\alpha}^{\sigma} G$  where  $b_s \in A_{c(s)}$ , for  $s \in G$ . Then, by (ii) and (iii), we get that

$$\begin{aligned} \tilde{F}(xy) &= \tilde{F} \left( \sum_{(s,t) \in G^{(2)}} a_s \sigma_s(b_t) \alpha(s, t) u_{st} \right) = \sum_{(s,t) \in G^{(2)}} a_s \sigma_s(b_t) \alpha(s, t) u_{F(st)} = \\ &= \sum_{(s,t) \in G^{(2)}} a_s \tau_{F(s)}(b_t) \beta(F(s), F(t)) u_{F(s)F(t)} = \tilde{F}(x) \tilde{F}(y) \end{aligned}$$

$\square$

**Remark F.4.4.** Suppose that  $\{A, G, \sigma, \alpha\}$  is a crossed system. By abuse of notation, we let  $A$  denote the category with the rings  $A_e$ , for  $e \in \text{ob}(G)$ , as objects and ring homomorphisms  $A_e \rightarrow A_f$ , for  $e, f \in \text{ob}(G)$ , as morphisms. Define a map  $\sigma : G \rightarrow A$  on objects by  $\sigma(e) = A_e$ , for  $e \in \text{ob}(G)$ , and on arrows by  $\sigma(s) = \sigma_s$ , for  $s \in G$ . By equation (F.4) it is clear that  $\sigma$  is a functor if the following two conditions are satisfied: (i) for all  $(s, t) \in G^{(2)}$ ,  $\alpha(s, t)$  belongs to the center of  $A_{c(s)}$ ; (ii) for all  $(s, t) \in G^{(2)}$ ,  $\alpha(s, t)$  is not a zero divisor in  $A_{c(s)}$ .

**Proposition F.4.5.** Let  $A \rtimes_{\alpha}^{\sigma} G$  be a category crossed product with  $\sigma : G \rightarrow A$  a functor. Suppose that  $R$  is a congruence relation on  $G$  with the property that the associated quadruple  $\{A, G/R, \sigma([\cdot]), \alpha([\cdot], [\cdot])\}$  is a crossed system. If  $I$  is the twosided ideal in  $A \rtimes_{\alpha}^{\sigma} G$  generated by an element  $\sum_{s \in G} a_s u_s$ , where  $a_s \in A_{c(s)}$ , for  $s \in G$ , satisfying  $a_s = 0$  if  $s$  does not belong to any of the classes  $[e]$ , for  $e \in \text{ob}(G)$ , and  $\sum_{s \in [e]} a_s = 0$ , for  $e \in \text{ob}(G)$ , then  $A \cap I = \{0\}$ .

*Proof.* By Proposition F.4.3, the functor  $\mathcal{Q}_R$  induces an  $A$ -algebra homomorphism  $\tilde{\mathcal{Q}}_R : A \rtimes_{\alpha}^{\sigma} G \rightarrow A \rtimes_{\alpha([\cdot], [\cdot])}^{\sigma([\cdot])} G/R$ . By definition of the  $a_s$ , for  $s \in G$ , we get that

$$\begin{aligned} \tilde{\mathcal{Q}}_R \left( \sum_{s \in G} a_s u_s \right) &= \tilde{\mathcal{Q}}_R \left( \sum_{e \in \text{ob}(G)} \sum_{s \in [e]} a_s u_s \right) = \\ &= \sum_{e \in \text{ob}(G)} \sum_{s \in [e]} a_s u_{[s]} = \sum_{e \in \text{ob}(G)} \left( \sum_{s \in [e]} a_s \right) u_{[e]} = 0 \end{aligned}$$

This implies that  $\tilde{\mathcal{Q}}_R(I) = \{0\}$ . Since  $\tilde{\mathcal{Q}}_R|_A = \text{id}_A$ , we therefore get that  $I \cap A = (\tilde{\mathcal{Q}}_R|_A)(A \cap I) \subseteq \tilde{\mathcal{Q}}_R(I) = \{0\}$ .  $\square$

Let  $G$  be a groupoid and suppose that we for each  $e \in \text{ob}(G)$  are given a subgroup  $N_e$  of  $G_e$ . We say that  $N = \bigcup_{e \in \text{ob}(G)} N_e$  is a normal subgroupoid of  $G$  if  $sN_{d(s)} = N_{c(s)}s$  for all  $s \in G$ . The normal subgroupoid  $N$  induces a congruence relation  $\sim$  on  $G$  defined by letting  $s \sim t$ , for  $s, t \in G$ , if there is  $n$  in  $N_{d(t)}$  with  $s = nt$ . The corresponding quotient category is a groupoid which is denoted  $G/N$ . For more details, see e.g. [5]; note that our definition of normal subgroupoids is more restrictive than the one used in [5].

**Proposition F.4.6.** Let  $A \rtimes_{\alpha}^{\sigma} G$  be a groupoid crossed product such that for each  $(s, t) \in G^{(2)}$ ,  $\alpha(s, t) \in Z(A_{c(s)})$  and  $\alpha(s, t)$  is not a zero divisor in  $A_{c(s)}$ . Suppose that  $N$  is a normal subgroupoid of  $G$  with the property that  $\sigma_n = \text{id}_{A_{c(n)}}$ , for  $n \in N$ , and  $\alpha(s, t) = 1_{A_{c(s)}}$  if  $s \in N$  or  $t \in N$ . If  $I$  is the twosided ideal in  $A \rtimes_{\alpha}^{\sigma} G$  generated by an element  $\sum_{s \in G} a_s u_s$ , with  $a_s \in A_{c(s)}$ , for  $s \in G$ , satisfying  $a_s = 0$  if  $s$  does not belong to any of the sets  $N_e$ , for  $e \in \text{ob}(G)$ , and  $\sum_{s \in N_e} a_s = 0$ , for  $e \in \text{ob}(G)$ , then  $A \cap I = \{0\}$ .

*Proof.* By Remark F.4.4,  $\sigma$  is a functor  $G \rightarrow A$  and  $\sim \subseteq \ker(\sigma)$ . Therefore, by the discussion preceding Proposition F.4.3, there is a well defined functor  $\sigma[\cdot] : G/N \rightarrow A$ . Now we show that the induced map  $\alpha([\cdot], [\cdot])$  is well defined. By equation (F.3) with  $s = n \in N_{c(t)}$  we get that  $\alpha(n, t)\alpha(nt, r) = \sigma_n(\alpha(t, r))\alpha(n, tr)$ . By the assumptions on  $\alpha$  and  $\sigma$  we get that  $\alpha(nt, r) = \alpha(t, r)$ . Analogously, by equation (F.3) with  $t = n \in N_{d(r)}$ , we get that  $\alpha(s, t) = \alpha(s, tn)$ . Therefore,  $\alpha([\cdot], [\cdot])$  is well defined. The rest of the claim now follows immediately from Proposition F.4.5.  $\square$

**Proposition F.4.7.** *Let  $A \rtimes^\sigma G$  be a skew category algebra. Suppose that  $R$  is a congruence relation on  $G$  contained in  $\ker(\sigma)$ . If  $I$  is the twosided ideal in  $A \rtimes^\sigma G$  generated by an element  $\sum_{s \in G} a_s u_s$ , where  $a_s \in A_{c(s)}$ , for  $s \in G$ , satisfying  $a_s = 0$  if  $s$  does not belong to any of the classes  $[e]$ , for  $e \in \text{ob}(G)$ , and  $\sum_{s \in [e]} a_s = 0$ , for  $e \in \text{ob}(G)$ , then  $A \cap I = \{0\}$ .*

*Proof.* By Remark F.4.4 and the discussion preceding Proposition F.4.3, there is a well defined functor  $\sigma[\cdot] : G/R \rightarrow A$ . The claim now follows immediately from Proposition F.4.5.  $\square$

**Proposition F.4.8.** *Let  $A \rtimes^\sigma G$  be a skew groupoid ring with all  $A_e$ , for  $e \in \text{ob}(G)$ , equal integral domains and each  $G_e$ , for  $e \in \text{ob}(G)$ , an abelian group. If every intersection of a nonzero twosided ideal of  $A \rtimes^\sigma G$  and  $A$  is nonzero, then  $A$  is maximal commutative in  $A \rtimes^\sigma G$ .*

*Proof.* We show the contrapositive statement. Suppose that  $A$  is not maximal commutative in  $A \rtimes^\sigma G$ . By the second part of Corollary F.3.3, there is  $e \in \text{ob}(G)$  and a nonidentity  $s \in G_e$  such that  $\sigma_s = \text{id}_{A_e}$ . Let  $N_e$  denote the cyclic subgroup of  $G_e$  generated by  $s$ . Note that since  $G_e$  is abelian,  $N_e$  is a normal subgroup of  $G_e$ . For each  $f \in \text{ob}(G)$ , define a subgroup  $N_f$  of  $G_f$  in the following way. If  $G_{e,f} \neq \emptyset$ , then let  $N_f = sN_e s^{-1}$ , where  $s$  is a morphism in  $G_{e,f}$ . If, on the other hand,  $G_{e,f} = \emptyset$ , then let  $N_f = \{f\}$ . Note that if  $s_1, s_2 \in G_{e,f}$ , then  $s_2^{-1}s_1 \in G_e$  and hence  $s_1 N_e s_1^{-1} = s_2 s_2^{-1} s_1 N_e (s_2^{-1} s_1)^{-1} s_2^{-1} = s_2 N_e s_2^{-1}$ . Therefore,  $N_f$  is well defined. Now put  $N = \bigcup_{f \in \text{ob}(G)} N_f$ . It is clear that  $N$  is a normal subgroupoid of  $G$  and that  $\sigma_n = \text{id}_{A_e}$ ,  $n \in N$ . Let  $I$  be the nonzero twosided ideal of  $A \rtimes^\sigma G$  generated by  $u_e - u_s$ . By Proposition F.4.6 (or Proposition F.4.7) it follows that  $A \cap I = \{0\}$ .  $\square$

**Remark F.4.9.** Proposition F.4.1, Corollary F.4.2 and Propositions F.4.5-F.4.8 together generalize Theorem 2, Corollary 11, Theorem 3, Corollaries 12-15 and Theorem 4 in [22] from groups to categories.

By combining Theorem F.4.1 and Proposition F.4.8, we get the following result.

**Corollary F.4.10.** *If  $A \rtimes^\sigma G$  is a skew groupoid ring with all  $A_e$ , for  $e \in \text{ob}(G)$ , equal integral domains and each  $G_e$ , for  $e \in \text{ob}(G)$ , an abelian group, then  $A$  is maximal commutative in  $A \rtimes^\sigma G$  if and only if every intersection of a nonzero twosided ideal of  $A \rtimes^\sigma G$  and  $A$  is nonzero.*

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