Optimal Coordination and Control of Posture and Movements

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Abstract

This paper presents a theoretical model of stability and coordination of posture and locomotion, together with algorithms for continuous-time quadratic optimization of motion control. Explicit solutions to the Hamilton-Jacobi equation for optimal control of rigid-body motion are obtained by solving an algebraic matrix equation. The stability is investigated with Lyapunov function theory and it is shown that global asymptotic stability holds. It is also shown how optimal control and adaptive control may act in concert in the case of unknown or uncertain system parameters. The solution describes motion strategies of minimum effort and variance. The proposed optimal control is formulated to be suitable as a posture and movement model for experimental validation and verification. The combination of adaptive and optimal control makes this algorithm a candidate for coordination and control of functional neuromuscular stimulation as well as of prostheses. Validation examples with experimental data are provided.

Key words: Postural control, Optimization, Dynamics, System Identification

1. Introduction

The quantitative knowledge of biped gait and stance is important both for performance evaluation in basic physiology, neurology, physical therapy and for improvement of functional neuromuscular stimulation and human-limb substitutes [67, 14, 71, 17]. Experimental work has been conducted with several different foci such as purely physical properties (mass, center of gravity, ground reaction forces) and myophysiology [68, 71, 8, 14, 44]. Measurement of mechanical work during walking as a function of speed, step length, frequency is one

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Mathematical Notation

Symbol	Description	Units
	Kinematics—Coordinates	
q	Generalized position coordinates $q = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}^T$	$q \in \mathbb{R}^n$
$\dot{q}$	Generalized velocity coordinates	$\dot{q} \in \mathbb{R}^n$
q_r	Reference value for position	$q_r \in \mathbb{R}^n$
$\widetilde{q}$	Position error $\widetilde{q} = q - q_r$	$\widetilde{q} \in \mathbb{R}^n$
p	Generalized momenta	$p \in \mathbb{R}^n$
$x(t)$	State of motion $x = \begin{bmatrix} \dot{q}^T & q^T \end{bmatrix}^T$	$x \in \mathbb{R}^{2n}$
$x_r(t)$	Reference state of motion $x_r = \begin{bmatrix} \dot{q}_r^T & q_r^T \end{bmatrix}^T$	$x_r \in \mathbb{R}^{2n}$
$\widetilde{x}(t)$	Error state of motion $\widetilde{x} = x - x_r = \begin{bmatrix} \dot{\widetilde{q}}^T & \widetilde{q}^T \end{bmatrix}^T$	$\widetilde{x} \in \mathbb{R}^{2n}$
	Dynamics—Torques, forces, inertias,	
τ	Applied torques or forces	$\tau \in \mathbb{R}^n$
$M(q)$	Moment of inertia $\quad M(q) = M^T(q) > 0$	$M \in \mathbb{R}^{n \times n}$
$C(q, \dot{q})\dot{q}$	Coriolis, centripetal and frictional forces	$C \in \mathbb{R}^{n \times n}$
$G(q)$	Gravitational forces	$G \in \mathbb{R}^n$
$N(q, \dot{q})\dot{q}$	Workless forces of τ	$N \in \mathbb{R}^{n \times n}$
u	Control variable $u = M(q)B^T T_0 \dot{\widetilde{x}} + (\frac{1}{2}\dot{M}(q, \dot{q}) + N(q, \dot{q}))B^T T_0 \widetilde{x}$	$u \in \mathbb{R}^n$
	Energy functions	
$\mathcal{L}$	Lagrangian of mechanical motion	
L	Lagrangian of optimization	
$\mathcal{H}$	Hamiltonian of mechanical motion	
H	Hamiltonian of optimization	
$\mathcal{U}(q)$	Potential energy	
$\mathcal{T}(q, \dot{q})$	Kinetic energy	
$V(\widetilde{x}, t)$	Hamilton principal function of optimization	
$V_X(\widetilde{x}, t)$	Lyapunov function of control and adaptation	
$\mathcal{J}(u)$	Optimization criterion	
	Matrices	
Q	Optimization weighting matrix w.r.t. x	$Q \in \mathbb{R}^{2n \times 2n}$
R	Optimization weighting matrix w.r.t. u	$R \in \mathbb{R}^{n \times n}$
S	Optimization cross weighting matrix w.r.t. x, u	$S \in \mathbb{R}^{n \times 2n}$
S_1	Optimization cross weighting matrix w.r.t. $\dot{q}, u$	$S_1 \in \mathbb{R}^{n \times n}$
S_2	Optimization cross weighting matrix w.r.t. q, u	$S_2 \in \mathbb{R}^{n \times n}$
T_0	State space transformation matrix	$T_0 \in \mathbb{R}^{2n \times 2n}$
	Adaptive control	
θ	Vector of unknown parameters	$\theta \in \mathbb{R}^p$
ψ	Regression matrix	$\psi \in \mathbb{R}^{n \times p}$
$\widetilde{x}(t)$	Error state of motion $\widetilde{x} = \begin{bmatrix} \dot{\widetilde{q}}^T & \widetilde{q}^T & \widetilde{\theta} \end{bmatrix}^T$	$\widetilde{x} \in \mathbb{R}^{2n+p}$

Table 1: Mathematical notation of the optimal control problem

such approach [14]. The elementary reflexes of a muscle to control its force, velocity, and length according to sensory feedback derived from various muscle and tendon receptors have been widely studied [67, 66].

A basic topic of postural control is the capacity to withstand gravitation and disturbances and the dynamics thereof. From a mechanical point of view, a minimal model for postural control must include a model for balancing of the center of mass. A mechanistic abstraction sometimes used is that of a an inverted pendulum, the stability of the unstable equilibrium being maintained by means of neural feedback involving visual, vestibular and somatosensory feedback [50, 54, 72, 81, 15] with elaborations on biomechanical complexity and neuromuscular aspects [33, 32, 36, 37, 40, 41, 44]. In addition to dynamic feedback control, other manifestations of neural feedback should be considered—e.g., adaptation, learning and calibration of 'inverse models' [46, 47].

An important problem is evaluation of multisensory feedback control properties resulting in stable stance and locomotion. System identification methodology for postural feedback assessment have been developed in a series of contributions [50, 52, 56, 57, 58]. Whereas the feedback control is necessary for stable stance, the low error feedback gains observed appear to be insufficient to support voluntary motion and disturbance rejection—*e.g.*, on rough or compliant support surfaces. Based on related methodology and with attention to passive mucscle dynamics and neural feedback latencies, Mergner, Peterka *et al.* summarized some observed properties of multisensory postural control with low-gain feedback combined with integral action and positive feedback control [80, 81, 72, 86]. An important structural observation was that the proportional position control inadequate to maintain upright stance on a tilted support surface was compensated by positive force feedback [80, 81], an idea related to the Hogan principle of 'impedance control' [42].

As compared to the elementary motion reflexes [66], [25], control and coordination strategies of locomotion are incompletely understood [40, 41, 10]. Important contributions with attention to biomechanics were proposed by Hogan *et al.* [22, 41, 42] and Houk *et al.* [43, 44]. A variety of interpretations involving voluntary and reactive behavior is found in the research literature. Mittelstaedt focused on graviception [73]. Grillner suggested central pattern generators for locomotion [29]. Nashner and colleagues [75, 77] made influential contributions with their formulation of 'ankle and hip strategies'. Nashner, Berthoz *et al.* emphasized the kinematic stabilization of the eyes in space—*i.e.*, the notion of the head as a stabilized platform for the eyes and stabilized vision [76, 82].

In order to accomplish coordinated motion in task execution of intended motion, inversion of biomechanical input-output dynamics is required—*e.g.*, transformation of position-velocity trajectories into force and motor commands—which, in turn, suggests neural incorporation of internal (inverse) models instrumenting coordinated control [43, 21, 59, 91, 47]. Whereas calibrated inverse models could execute motor programs, such open-loop control strategies would not be robust with respect to external disturbances and model calibration errors and stabilizing sensory feedback control is necessary to maintain the trajectory during task execution. As decomposition of control into internal inverse models for ('proactive') trajectory generation and sensory feedback for ('reactive') stable task execution and adaptation is essential for motor control, optimality principles should apply to both [95]. This decomposition involving sensory feedback, corrective control and adaptation is illustrated in Fig.1.

In early literature on postural control, the presence of biological optimization criteria was postulated [18], [9]. The linear optimal control solutions thus derived relied on linearized (approximate) equations with regard to a given operating point. Optimality of energy expenditure is an attractive hypothetical principle of motion coordination investigated by Levine, Zajac and colleagues [65], [34]. A reason to presume that biological organisms might adapt to minimization of mechanical work is that such operation would be closely related to the ability of maximum effort and performance, and to thermodynamic equilibrium. However, it has not yet been experimentally established whether human stance and locomotion do indeed obey an optimality principle [12, 23, 48, 7, 95].

Experimental investigation of the integrative action in the mechanisms of motor control must be quantitative and must include both static and dynamic components of the motor response [33], [40]-[44]. A prerequisite for quantitative understanding of integrative aspects is obviously a meticulous mathematical investigation on a form suitable for experimental verification. The need has been stressed of suitable identification models as a necessary basis for progress in the understanding of locomotion control, coordination and adaptation [36], [46]. As yet, however, mathematical modeling has failed to produce experimentally validated, complete models that satisfactorily explains the complexity of coordination, stability, control effort, and equilibrium. The absence of results in this respect is due both to experimental conditions and to the difficulties inherent in control systems modeling [46].

In this context, the subsystems requiring treatment are:

- The mechanical motion of multilinked body segments;
- Control systems modeling of coordination and reflex action;
- The active adaptation of neural control;

A methodological aspect also requires serious attention, namely:

- The model should allow for system identification and model validation with experimental data.

The rigid body mechanics of musculo-skeletal motion is often formulated with the general equations obtained from Lagrangian mechanics (time arguments omitted).

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \quad M(q) = M^T(q) > 0,\, q \in \mathbb{R}^n \tag{1}$$

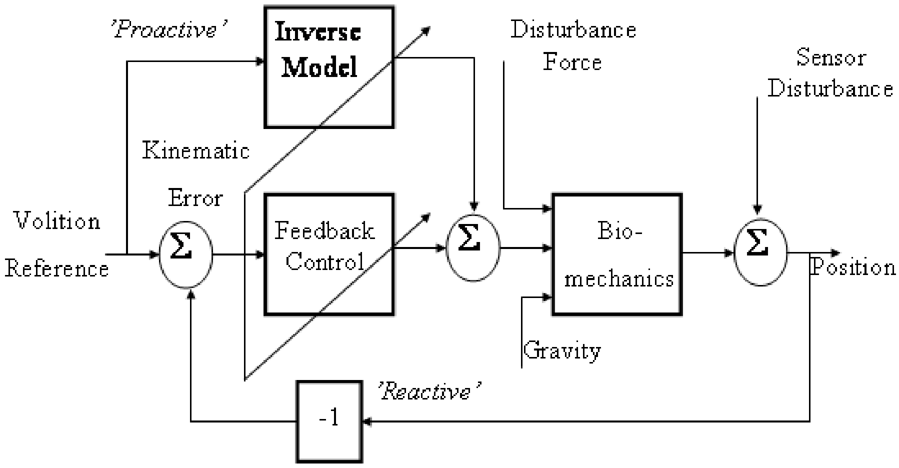

Figure 1: Postural control modeled with components of ('reactive') feedback control and ('proactive') feedforward control using inverse models, the kinematic error as obtained from sensory information being crucial information for feedback control as well as adaptation (*oblique arrows*) of feedforward (inverse model) and feedback control.

The position coordinates $q \in \mathbb{R}^n$ with associated velocities $\dot{q}$ and accelerations $\ddot{q}$ are controlled with the driving torques $\tau \in \mathbb{R}^n$. The (generalised) moment of inertia $M(q)$, the Coriolis, centripetal and frictional forces $C(q, \dot{q})\dot{q}$, and the gravitational forces $G(q)$ all vary along the trajectories. Several models of the type (1), varying in biomechanical complexity have been formulated hitherto: *e.g.*, a four-segment model of Vukobratović and Juricić [97], a five-segment model of Hemami and Farnsworth [40], and a 17-segment model of Hatze [33], [32].

The coordination of muscular forces may be considered either at the level of muscular activation or at the level of joint torque. The control problem formulated in terms of joint torques is as follows: Find the torques (forces) τ so that the linked body segments assume a prescribed final position (or follow a prescribed trajectory), provided that the body mechanics is described by Eq. (1).

Optimal control solutions always rely on the accuracy of the underlying model in order to remain optimal. Contexts of model uncertainty or model changes pose a need of active adaptation to new conditions in order to maintain optimality.

Consider the problem of multilink coordination of torques and kinematics. The aim is to minimize velocity and position errors (state errors) with a minimum both of the applied torques and of the energy consumption. We provide an analytic solution to the optimal motion control problem and formulate the solutions suitable for extensions to adaptive control. The problem how to identify a mathematical model for this type from experimental data is considered in a special section.

2. Problem statement

The following aspects in the modeling of postural control need to be covered in any attempt to describe the integrative coordination of motor control.

- variance of position and velocity errors;
- muscular control effort magnitude;
- mechanical energy consumed by muscular control;
- stability;

Other desirable modeling features:

- The model should explain feedback notions;
- The model should explain quantitative motion coordination;
- The control effort should not tend to violate muscle stiffness;
- The model should allow for adaptation;
- The model should be experimentally verifiable.

The desired reference trajectory for the control object to follow, as generated by some motor program (volition or motion pattern generators), is here assumed to be available as a final desired position or as bounded functions of time in terms of generalized positions $q_r \in \mathcal{C}^1$ in $\mathbb{R}^n$ and, if specified, its corresponding accelerations $\ddot{q}_r$ and velocities $\dot{q}_r$. Define the errors of accelerations, velocities, and positions as

$$\begin{bmatrix} \ddot{\widetilde{q}} \\ \dot{\widetilde{q}} \\ \widetilde{q} \end{bmatrix} = \begin{bmatrix} \ddot{q} - \ddot{q}_r \\ \dot{q} - \dot{q}_r \\ q - q_r \end{bmatrix}; \qquad \widetilde{x} = x - x_r = \begin{bmatrix} \dot{q} - \dot{q}_r \\ q - q_r \end{bmatrix} \tag{2}$$

The control objective is to follow the given, bounded reference trajectory $\dot{q}_r, q_r$ without position errors $\widetilde{q}$, or velocity errors $\dot{\widetilde{q}}$.

Consider an optimization criterion $\mathcal{J}(u)$ where the matrix S is used for weighting of the cross term between $\widetilde{x}$ and u.

$$J(u) = \int_0^\infty L(\widetilde{x}, u)dt; \; L(\widetilde{x}, u) = \frac{1}{2}\widetilde{x}^T(t)Q\widetilde{x}(t) + \frac{1}{2}u^T(t)Ru(t) + u^T(t)S\widetilde{x}(t) \tag{3}$$

The positive definite matrices Q, R and the matrix $S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$ define the weighting compromises of the optimization. The first term of (3) penalizes the variances of position and velocities. The second term of (3) represents the control effort magnitude. A weighted sum of the energy consumed at each joint can be expressed as the integral of $u^T S_1 \dot{\widetilde{q}}$ (the instantaneous power), whereas $u^T S_2 \widetilde{q}$ penalizes control actions that tend to increase local errors and results in enhancement of local reflex actions.

It is the purpose of this paper to present stable, analytic solutions to the problem of quadratic optimal control of motion control with minimization of the applied torques (forces) when velocity and position feedback are available. The optimal control problem is solved with the Hamilton-Jacobi equation, and feedback solutions to the stated optimal motion control problem are presented. The second stage problem of adaptive control is then solved.

3. Dynamics of segmented, articulated bodies

We model the motion dynamics as a set of n rigid bodies connected and described by a set of generalized position coordinates $q \in \mathbb{R}^n$. The derivation of the motion equations (1) in accordance with Lagrange theory [5], [24] involves explicit expression both of kinetic energy $\mathcal{T}$ and potential energy $\mathcal{U}$. The Lagrangian $\mathcal{L}$ of motion in a space with a velocity independent gravitation potential is defined by

$$\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{U}(q) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - \mathcal{U}(q) \tag{4}$$

The Lagrangian $\mathcal{L}$ is the basis for formulation of the Euler-Lagrange equations of motion [24, 5]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau \tag{5}$$

where $\tau \in \mathbb{R}^n$ are the externally applied torques and forces. The standard general equations (1) are obtained from Eq. (5) as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{1}$$

where $M(q)$ is the inertia matrix, $C(q, \dot{q})\dot{q}$ represents Coriolis and centripetal forces. It is assumed that the positions q and velocities $\dot{q}$ but not the accelerations $\ddot{q}$ are available as neural feedback. It is further assumed that the torque vector τ is available as the control input. Already Eq. (1) covers a large model set including the equations of Hatze [32], Hemami and Farnsworth [40], Vukobratovic and Juricic [97]. Further, model extensions with various forms of friction and contact forces as well as holonomic constraints may be expressed in equations of the type (1).

3.1. What control effort should be minimized?

A natural aim is to minimize velocity and position errors (state errors) with a minimum of applied torque and energy consumption. For a velocity-independent potential energy $\mathcal{U}$, the Euler-Lagrange equations give:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{T}}{\partial q} + \frac{\partial \mathcal{U}}{\partial q} = \tau \tag{6}$$

Changes in potential energy due to gravitation are inevitable and can be determined from the start and end points only. Thus, the gravitation-dependent energy expenditure can not be altered and there is little point in trying to optimize the corresponding torques or forces. Consider therefore the applied torques τ_K that selectively affect kinetic energy.

$$\tau_K = \tau - \frac{\partial \mathcal{U}}{\partial q} = \frac{d}{dt}(\frac{\partial \mathcal{T}}{\partial \dot{q}}) - \frac{\partial \mathcal{T}}{\partial q} = M(q)\ddot{q} + \dot{M}(q,\dot{q})\dot{q} - \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}^T M(q)\dot{q}) \tag{7}$$

To investigate the properties of τ_K, we introduce the skew-symmetric matrix $N(q,\dot{q})$ with elements n_{ij} defined from the components m_{ij} of $M(q)$ as

$$n_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial m_{ik}(q)}{\partial q_j} - \frac{\partial m_{jk}(q)}{\partial q_i}\right)\dot{q}_k, \quad n_{ij} = -n_{ji} \tag{8}$$

which verifies

$$N(q,\dot{q})\dot{q} = \frac{1}{2}\dot{M}(q,\dot{q})\dot{q} - \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}^T M(q)\dot{q}) \tag{9}$$

so that

$$\tau_K = M(q)\ddot{q} + \frac{1}{2}\dot{M}(q,\dot{q})\dot{q} + N(q,\dot{q})\dot{q} \tag{10}$$

which contains the force terms associated with inertia (acceleration), centripetal and Coriolis forces. It is a standard result from Lagrangian mechanics that the third term $N(q,\dot{q})\dot{q}$ of Eq. (10) represents the workless forces of the system. It is straightforward to verify that the work done on the system by the applied forces τ_K determines the kinetic energy

$$\int \tau_K^T \dot{q} dt = \int \dot{q}^T \left[M(q)\ddot{q} + \tfrac{1}{2}\dot{M}(q,\dot{q})\right]\dot{q}\, dt = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{11}$$

A modification of Eq. (10) is appropriate in cases where the motion is prescribed not only with respect to the final position but also with respect to its intermediary values. During voluntary motion along a prescribed trajectory ($q_r(t) \neq 0$), it is important that the optimization does not compromise the desired trajectory q_r. Instead, we model the optimization so that the motion of body segments is stabilized to follow the desired trajectory with minimal effort. To minimize the necessary forces (torques), we include the control variable τ_K in the more general definition

$$u = \begin{bmatrix} M(q) & \tfrac{1}{2}\dot{M}(q,\dot{q}) + N(q,\dot{q}) \end{bmatrix} \begin{bmatrix} \dot{\widetilde{z}}_1 \\ \widetilde{z}_1 \end{bmatrix} \tag{12}$$

with $\widetilde{z}$ and T_1 introduced via the following state-space transformation of $\widetilde{x}$

$$\widetilde{z} = \begin{bmatrix} \widetilde{z}_1 \\ - \\ \widetilde{z}_2 \end{bmatrix} = T_0\widetilde{x} = \begin{bmatrix} T_1 \\ - \\ T_2 \end{bmatrix} \begin{bmatrix} \dot{\widetilde{q}} \\ \widetilde{q} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & I_{n\times n} \end{bmatrix} \begin{bmatrix} \dot{\widetilde{q}} \\ \widetilde{q} \end{bmatrix}; \quad T_{11}, T_{12} \in \mathbb{R}^{n\times n} \tag{13}$$

This definition includes forces (torques) affecting kinetic energy (10), reference trajectories (2), and a state-space transformation (13). The control variable u of (13) can be reduced to τ_K of Eq. (10) for $q_r = 0$, $T_{11} = I_{n\times n}$, and $T_{12} = 0$ so that

$$\begin{aligned} u &= M(q)\ddot{q} + (\frac{1}{2}\dot{M}(q,\dot{q}) + N(q,\dot{q}))\dot{q} && (14) \\ &= \tau - G(q) + (\frac{1}{2}\dot{M}(q,\dot{q}) + N(q,\dot{q}) - C(q,\dot{q}))\dot{q} && (15) \end{aligned}$$

where the last term is zero when no friction forces are present.

4. Quadratic optimization

We therefore embed the motion control problem into the following somewhat more general optimization problem. The assumptions made are summarized as follows:

Basic assumptions

A1: The motion equations are $M(q)\ddot{q} + C(\dot{q}, q)\dot{q} + G(q) = \tau$ with coordinates q and external torques (forces) τ.

A2: The reference trajectory given as $q_r, \dot{q}_r, \ddot{q}_r \in L^\infty$, and $q_r \in \mathcal{C}^1$ with the error-state $\widetilde{x} = \begin{bmatrix} \dot{\widetilde{q}}^T & \widetilde{q}^T \end{bmatrix}^T$

A3: A state-space transformation is given

$$\widetilde{z} = T_0 \widetilde{x} = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix} \widetilde{x} \tag{16}$$

A4: The control action to minimize is

$$u = \tau - G(q) \tag{17}$$

or for non-zero reference trajectories q_r as

$$u = (\frac{1}{2}\dot{M}(q,\dot{q}) + N(q,\dot{q}))B^T T_0 \widetilde{x} + M(q) B^T T_0 \dot{\widetilde{x}}, \quad B = \begin{bmatrix} I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \tag{18}$$

A5: Positions and velocities of all segments of rigid link motion are available for measurement

A6: The structure of M, C, G is known

A7: The parameters of M, C, G are known

A8: The optimization criterion of optimal control is

$$\mathcal{J}(u) = \frac{1}{2}\int_{t_0}^{\infty} \begin{bmatrix} \widetilde{x}(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \widetilde{x}(t) \\ u(t) \end{bmatrix} dt,\ Q - S^T R^{-1} S > 0,\ R > 0 \tag{19}$$

the condition $Q - S^T R^{-1} S > 0$ being imposed to render the minimization of $J(u)$ a well-posed optimization problem [64].

□

Given the performance index $J(u)$, we find an optimal control $u = u^*$ that will transfer from an initial state to a desired state. The control $u = u^*$ moves the system from an arbitrary initial state $\widetilde{x}(t_0)$ to the origin of the error-space while minimizing $J(u)$. The control variable u is weighted with the matrix $R = R^T > 0$, and the vector of velocity and position errors $\widetilde{x}$ is weighted with the matrix $Q = Q^T > 0$. The rate of compensation can be adjusted by chosing proper weights Q. The term $u^T R u$ guarantees smoothness of operation.

4.1. The Hamilton-Jacobi equation

Solutions for optimization problems of the type (3) under assumptions (A1-A8) are obtained by solving partial differential equations obtained from Hamilton-Jacobi theory, dynamic programming or the Pontryagin maximum principle [23], [64, 12, 7].

As the Hamiltonian of optimization is defined as

$$H(\widetilde{x}, u, \frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}}) = (\frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}})^T \dot{\widetilde{x}} + L(\widetilde{x}, u) \tag{20}$$

a necessary and sufficient condition for optimality [23], [64], is to choose a value function V that satisfies the Hamilton-Jacobi equation.

$$\frac{\partial V}{\partial t} + \min_u H(\widetilde{x}, u, \frac{\partial V}{\partial \widetilde{x}}) = 0 \tag{21}$$

This minimum is attained for the optimal control $u = u^*$ and the Hamiltonian

$$H^* = \min_u H = \min_u((\frac{\partial V}{\partial \widetilde{x}})^T \dot{\widetilde{x}} + L(\widetilde{x}, u)) = H(\widetilde{x}, u^*, \frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}}) = -\frac{\partial V(\widetilde{x}, t)}{\partial t} \tag{22}$$

The optimal value function V solving Eq. (21) for $u = u^*$ is called the *Hamilton principal function* of the system, the *adjoint* or *co-state* being $\partial V / \partial \widetilde{x}$ [23].

Lemma 1: The following function V composed of $\widetilde{x}$, $q_r(t)$, T_0, M, and a symmetric matrix $K \in \mathbb{R}^{n \times n}$ solves the Hamilton-Jacobi equation

$$V(\widetilde{x}(t), t) = \frac{1}{2} \widetilde{x}^T T_0^T \begin{bmatrix} M(q) & 0 \\ 0 & K \end{bmatrix} T_0 \widetilde{x} \tag{23}$$

for K, T_0 solving the algebraic matrix equation

$$\widetilde{x}^T \left[\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + Q - (S + B^T T_0)^T R^{-1} (S + B^T T_0) \right] \widetilde{x} = 0 \tag{24}$$

The optimal feedback control law $u = u^*$ that minimizes $\mathcal{J}(u)$ is

$$u^*(t) = -R^{-1}(S + B^T T_0)\widetilde{x}(t) \tag{25}$$

The minimum optimization criterion is then obtained as

$$\mathcal{J}(u^*) = \min_u \int_{t_0}^{\infty} L(\widetilde{x}, u)dt = \int_{t_0}^{\infty} L(x, u^*)dt = V(\widetilde{x}(t_0), t_0) - V(\widetilde{x}(\infty), \infty) \tag{26}$$

□

Proof: See Appendix 1.

□

5. Stability and control

All optimal control generated by the solutions (23-25) to the Hamilton-Jacobi equation does not necessarily guarantee stable closed-loop behavior. Only solutions that also guarantee a stable closed-loop behavior are interesting for stance and locomotion. Such a stability condition provides some constraints as to the choice of the weighting matrices Q, R, and S. A sufficient condition for stable, optimal control is that $K = K^T > 0$ in (20) as formulated in the following theorem:

Theorem 1: Let the weighting matrices Q, R with Cholesky factors Q_1, Q_2, R be chosen such that

$$\begin{aligned} Q &= Q^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_1^T Q_1 & Q_{12} \\ Q_{12}^T & Q_2^T Q_2 \end{bmatrix}; \\ 0 &< R = R^T = R_1^T R_1 \\ 0 &< Q_1^T Q_2 + Q_2^T Q_1 - (Q_{12}^T + Q_{12}) \end{aligned} \tag{27}$$

Let T_0, K be chosen as the matrices

$$T_0 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & I_{n\times n} \end{bmatrix} = \begin{bmatrix} R_1^T Q_1 - S_1 & R_1^T Q_2 - S_2 \\ 0 & I_{n\times n} \end{bmatrix} \tag{28}$$

$$K = K^T = \frac{1}{2}(Q_1^T Q_2 + Q_2^T Q_1) - \frac{1}{2}(Q_{12}^T + Q_{12}) > 0 \tag{29}$$

The optimal control solution subject to assumptions (A1-A8) then provides an asymptotically stable and an L^2-stable closed-loop system with the optimal feedback control law $u = u^*$

$$u^*(t) = -R^{-1}(S + B^T T_0)\widetilde{x}(t) \tag{30}$$

The minimal optimization criterion is then obtained as

$$J(u^*) = \min_u \int_{t_0}^{\infty} L(\widetilde{x}, u)dt = \int_{t_0}^{\infty} L(\widetilde{x}, u^*)dt = V(\widetilde{x}(t_0), t_0) \tag{31}$$

where V solves the Hamilton-Jacobi equation (23)

$$V(\widetilde{x}(t), t) = \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} M(q) & 0 \\ 0 & K \end{bmatrix} T_0 \widetilde{x} \tag{23}$$

Moreover, the value function $V(\widetilde{x}, t)$ is a Lyapunov function for the asymptotically stable system. The Lyapunov function derivative $\dot{V} = dV/dt < 0$ for $\|\widetilde{x}\| \neq 0$ and global asymptotic stability holds for $Q - S^T R^{-1} S > 0, R > 0$.

□

PROOF: See Appendix 2.

□

The function $V(\widetilde{x}, t)$ of Eq. (23) can be viewed as an aggregate of kinetic energy and the 'potential energy' from a set of springs with a stiffness matrix K. The controlled motion keeps stable with an equilibrium on the prescribed reference trajectory as long as V does not grow. This physical analogy can be mathematically formalized in a stability proof with a Lyapunov function as stated in the previous theorem.

5.1. The Control Law

The optimal control was given as the feedback control

$$u^*(t) = -R^{-1}(S + B^T T_0)\widetilde{x}(t) \tag{25}$$

The appropriate external torques to apply are then calculated from (18) and (25). This gives the applied torque τ^* which is calculated in accord with assumptions (A1-7), and which is optimal in the sense of (A8).

$$\begin{aligned} \tau^* &= M(q)(\ddot{q}_r - T_{11}^{-1}T_{12}\dot{\widetilde{q}} - T_{11}^{-1}M(q)^{-1}((\frac{1}{2}\dot{M}(q,\dot{q}) + N(q,\dot{q}))B^T T_0\widetilde{x} + u^*)) \\ &+ C(q,\dot{q})\dot{q} + G(q) \end{aligned} \tag{32}$$

The decomposition into an inverse model and optimal feedback is obvious from Eqs. (25) and (32). The torque obtained contains compensation of gravity and Coriolis torques as well as an anticipatory action with respect to $\ddot{q}_r$. The special choices $q_r = 0$, $T_0 = I_{2n\times 2n}$ and the absence of frictional forces provide a simplified interpretation as

$$\tau^* = u^* + G(q) \tag{33}$$

5.2. Stability with Respect to External Persistent Disturbances

The asymptotic stability also implies that the optimal closed-loop system is stable under persistent disturbances [31]. Stability analysis extending Lyapunov theory to system stability with respect to external persistent disturbances can be approached by means of passivity analysis [101, 39]. As shown in stability analysis of Appendix 4, the optimal control is stable in repsonse to external persistent disturbances.

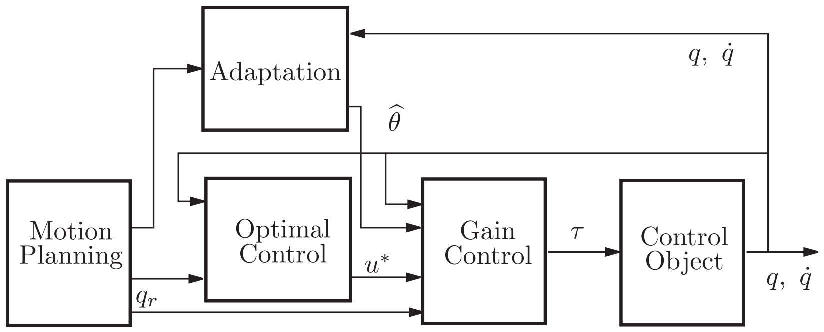

Figure 2: Algorithmic organization of the quadratic optimal feedback control

6. Self-optimizing adaptation

As model based optimal control laws are contingent upon the model accuracy, performance is sensitive to changes of physical parameters or other model changes. In cases with uncertain or time-varying parameters of M, C, G, there is a need of adaptation of the optimal control to the operating conditions. The optimal control algorithm presented here (32-33) is readily modified for self-optimizing adaptive control.

Assume that the matrices M, C, G have a known structure (A6) and consider a case of uncertain parameters (*cf.* A7). Let the optimal control law be expressed in terms of the unknown parameters $\theta \in \mathbb{R}^p$ of M, C, G and the data vector $\psi \in \mathbb{R}^{n\times p}$, $\psi_0 \in \mathbb{R}^n$. The matrix ψ and the vector ψ_0 contain the terms of τ^* that may be computed without reference to unknown or uncertain parameters.

$$\tau^* = M(q)(\ddot q_r - T_{12}\dot{\widetilde q}) - \frac{1}{2}\dot M(q,\dot q)B^T T_0 \widetilde x + C(q,\dot q)\dot q + G(q) + u^* \quad (34)$$
$$= \psi\theta + \psi_0 + u^* \quad (35)$$

The adaptive control law is a modification (28) with θ replaced by $\widehat\theta$

$$\tau = \psi\widehat\theta + \psi_0 + u^* \quad (36)$$
$$= \widehat M(q)(\ddot q_r - T_{12}\dot{\widetilde q}) - \frac{1}{2}\widehat{\dot M}(q,\dot q)B^T T_0 \widetilde x + \widehat C(q,\dot q)\dot q + \widehat G(q) + u^* \quad (37)$$

A prerequisite of successful motor learning and adaptation to a changing environment is that $\widehat\theta$ may be purposely modified.

Theorem 2: Assume that the optimal control u^* is determined according to Theorem 1. Let the optimal control law be expressed in terms of uncertain parameters $\theta \in \mathbb{R}^p$ of M, C, G and the data matrices $\psi \in \mathbb{R}^{n\times p}$, $\psi_0 \in \mathbb{R}^n$. The matrices ψ, ψ_0 contains terms of τ^* that may be computed without reference to

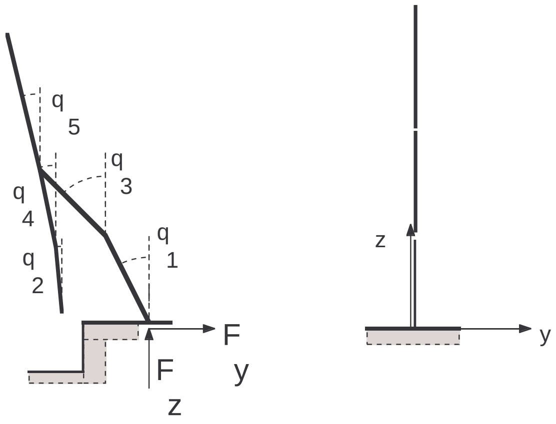

Figure 3: An anthropomorphic five-link model climbing a step. The ground reaction forces are denoted F_y, F_z and act at the foot support point located at the origin of the coordinate system O_{xyz}. The prescribed final erect position is shown (*right*).

unknown or uncertain parameters.

$$M(q)(\ddot{q}_r - T_{12}\dot{\widetilde{q}}) - \frac{1}{2}\dot{M}(q,\dot{q})B^T T_0 \widetilde{x} + C(q,\dot{q})\dot{q} + G(q) = \psi\theta + \psi_0 \tag{38}$$

The adaptive control law with θ replaced by an estimate $\widehat{\theta} \in \mathbb{R}^p$ is

$$\tau = \psi\widehat{\theta} + \psi_0 + u^* \tag{39}$$

$$\dot{\widehat{\theta}} = -K_\theta^{-1}\psi^T B^T T_0 \widetilde{x} \tag{40}$$

The Lyapunov function V_X

$$V_X(\widetilde{x},t) = \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} M(q) & 0 \\ 0 & K \end{bmatrix} T_0 \widetilde{x} + \frac{1}{2}\widetilde{\theta}^T K_\theta \widetilde{\theta}; \qquad K_\theta = K_\theta^T > 0 \tag{41}$$

with the negative semidefinite derivative

$$\dot{V}_X = \dot{V} + \dot{V}_\theta = -\frac{1}{2}\widetilde{x}^T (Q - S^T R^{-1} S + T_0^T B R^{-1} B^T T_0)\widetilde{x} \le 0; \qquad \forall \widetilde{x} \ne 0 \tag{42}$$

then assures that the self-optimizing adaptive control solution (39-40) is L^2-stable and uniformly globally stable in the sense of Lyapunov for constant parameters θ. The solution reaches the the optimal solution for $\widetilde{\theta} = 0$.

□

7. Simulated Examples

The following simulations demonstrate optimal control of the anthropomorphic five-link model (Fig. 3) in a case where only local feedback is available except for the Coriolis and gravity compensations. The following physical parameters were chosen

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 30 \end{bmatrix} [kg]; \qquad \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.4 \\ 1.0 \end{bmatrix} [m] \tag{43}$$

where m_i, l_i denote the mass and length of segment i. The performance index $\mathcal{J}(u)$ for the optimal control was chosen such that $Q - S^T R^{-1} S > 0$.

$$\mathcal{J}(u) = \int_0^\infty \widetilde{x}^T \begin{bmatrix} 90 I_{5\times5} & 324 I_{5\times5} \\ 324 I_{5\times5} & 1305 I_{5\times5} \end{bmatrix} \widetilde{x} + u^T \begin{bmatrix} 3 I_{5\times5} & 12 I_{5\times5} \end{bmatrix} \widetilde{x} + \frac{1}{9} u^T I_{5\times5} u \, dt \tag{44}$$

which results in the control law

$$\tau = G(q) - \begin{bmatrix} 35.0 I_{5\times5} & 120.0 I_{5\times5} \end{bmatrix} \widetilde{x} \tag{45}$$

Three examples are given to demonstrate this methodology.

Example 1

Consider the anthropomorphic five-segment model depicted in Fig. 3. Assume that the support leg in contact with the ground at the origin of the coordinate system O_{xy}. Let all initial angular positions be zero except $q_3(0) \neq 0$. This case simulates the five-link model starting to climb a step. The result is shown in Fig. 4.

□

Example 2

This example shows how the five-link model rises from an initial bending position at rest with the initial position coordinates $q_1(0) = q_2(0) = q_3(0) = q_4(0) = -q_5(0)$, the result being shown in Fig. 5. As compared to Example 1, there is little displacement of the common center of mass initially. As compared to Example 1, the ground reaction shear force is therefore smaller.

□

Example 3

Assume that an additional load of 20 [kg] is attached at the center of mass of segment 5. This situation simulates the presence of a back load on a human climbing a step. The optimal control is now modified by the adaptation which estimates the new segment weight and corrects the feedback and anti-gravity actions. The result is shown in Fig. 6. Note that the adaptation has a fast initial response with a progressively slower response as the error decrease.

□

Examples 1 and 2 show that the anthropomorphic model is capable of stable posture and movement in the presence of antigravity action and local error

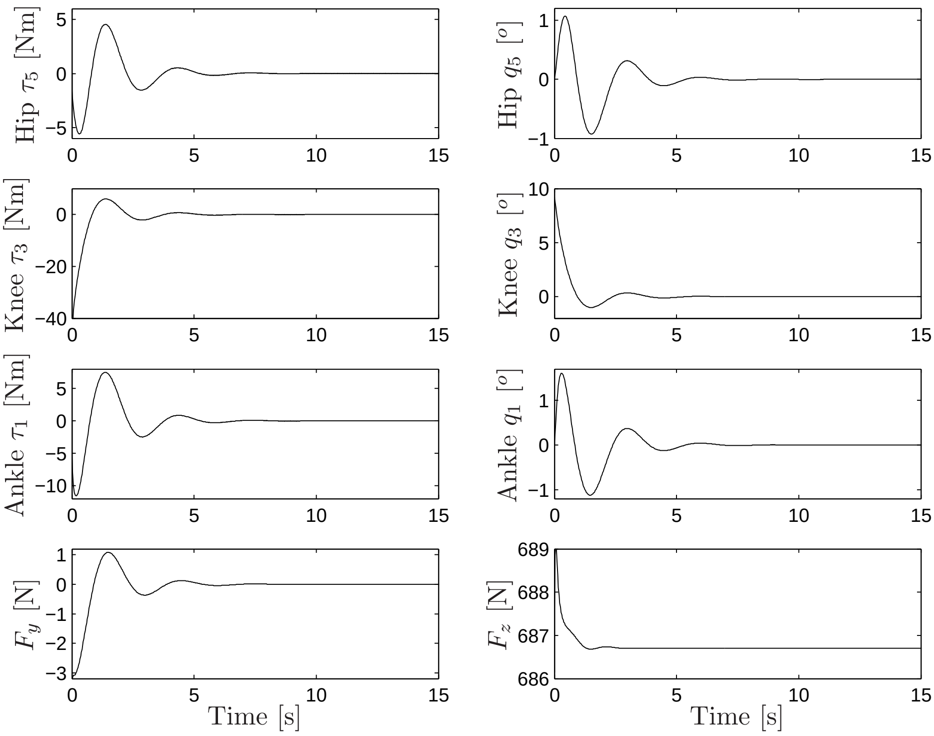

Figure 4: An anthropomorphic five-link model climbing a step. The ground reaction forces are a shear force F_y, and a normal force F_z. The torques at the hip, knee and ankle are presented (*left*) and angular positions (*right*). All graphs *vs.* time [s].

feedback only. Example 3 shows that motor learning and adaptation is feasible in this context. The mass of the torso ($m_5 = 30+20$ [kg]) is also well estimated, although a small bias may persist in cases where the control performance is good. This is in agreement with Eq. (32) where the parameter errors $\widetilde{\theta}$ do not appear on the right hand side.

8. Identification models

It is sometimes overlooked that quantitative modeling must be experimentally verified not only qualitatively but also quantitatively. The explicit solution to Eq. (23) and the associated control law supports the formulation of an identification model similar to Eq. (36). Let the torque equation be formulated in terms of the uncertain parameters Θ as the linear estimation model

$$\tau = \phi\Theta + \phi_0 \tag{46}$$

where ϕ, ϕ_0 contain functions of data $(\ddot{q}_r, \dot{q}_r, q_r, \dot{q}, q)$ computable without reference to the uncertain parameters. It is an easy identification problem to find the unknown Θ provided that observations τ_k, ϕ_k, ϕ_{k0} of (46) at times $k = 1, 2, \ldots N$ are available. The least-squares criterion based on N observations is

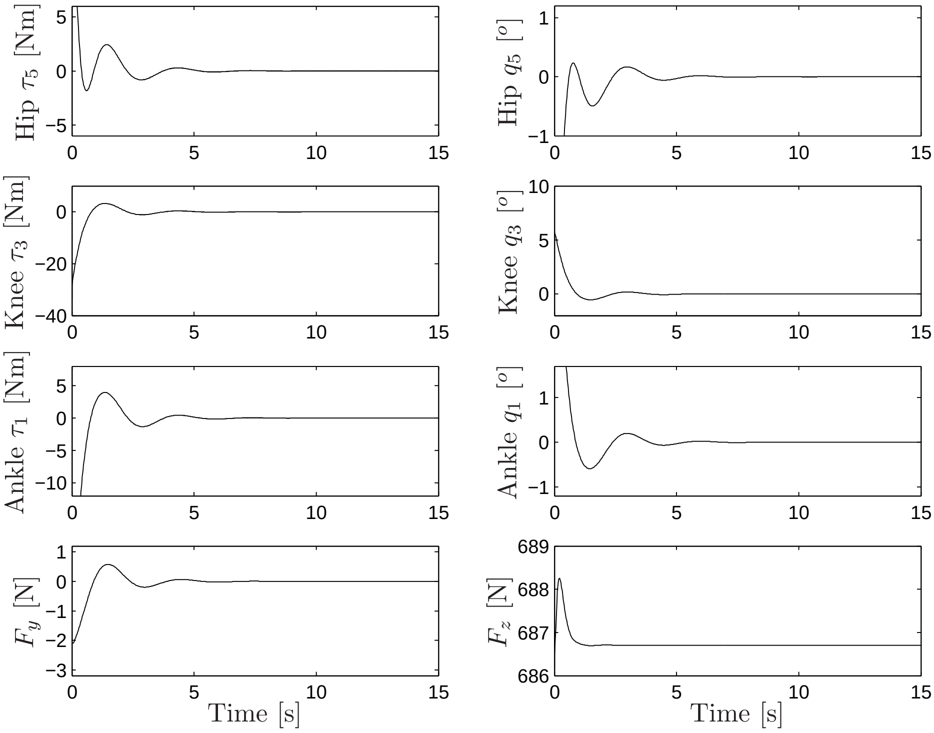

Figure 5: An anthropomorphic five-link model rising from an initial bending position ($q_1(0) = q_3(0) = -q_5(0)$). The ground reaction forces are a shear force F_y, and a normal force F_z. The torques at the hip, knee and ankle are presented (*left*) and angular positions (*right*). All graphs *vs.* time [s].

$$\mathcal{J}_{LS}(\widehat{\Theta}) = \frac{1}{N}\sum_{k=1}^{N} \|\tau_k - (\phi_k\widehat{\Theta} + \phi_{k0})\|^2 = \frac{1}{2}(Y_N - \Phi_N\widehat{\Theta})^T(Y_N - \Phi_N\widehat{\Theta}) \tag{47}$$

where

$$\Phi_N = \begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_N\end{bmatrix}, \qquad Y_N = \begin{bmatrix}\tau_1 - \phi_{10}\\ \tau_2 - \phi_{20}\\ \vdots\\ \tau_N - \phi_{N0}\end{bmatrix}, \qquad \widehat{\Theta} = (\Phi_N^T\Phi_N)^{-1}\Phi_N^T Y_N \tag{48}$$

$\widehat{\Theta}$ being the least-squares solution based on N observations and obtained by completing the squares of (47), *cf.* [92]. This is immediately recognized as a linear regression problem with a least-squares solution subject to statistical hypothesis testing by standard methods of variance analysis (χ^2-tests). This shows that experimental verification of the proposed mathematical modeling is feasible by standard methods.

Hence, the optimization weighting matrices as well as unknown physical or physiologic parameters may be estimated from measurements of τ, q, $\dot{q}$, q_r, $\dot{q}_r$, $\ddot{q}_r$.

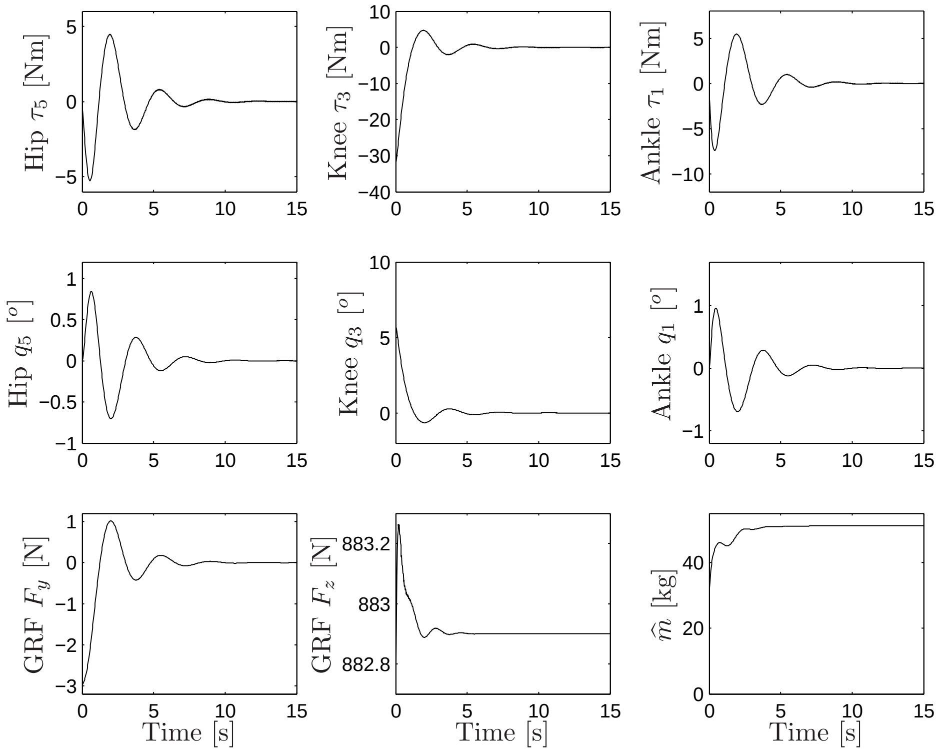

Figure 6: An anthropomorphic five-link model climbing a step while the control law adapts to an extra load (20 [kg]) on the torso of 30 [kg]. The ground reaction forces (GRF) are a shear force F_y (*lower left*), and a normal force F_z (*lower middle*). The torques at the hip, knee and ankle are presented (*upper*) and angular positions (*middle*) with adapting mass estimate (*lower right*). All graphs *vs.* time [s].

9. Experiments

Forces and torques actuated by the feet were recorded with six degrees of freedom (6DOF) by a force platform. Force-platform data were sampled at 50 [Hz] by a computer equipped with an AD converter and a customized program controlled the vibratory and galvanic stimulation, and the sampling of force platform data. The body movements at five anatomical landmarks were measured by a 3D-motion analysis system (Zebris Measuring System) at 50 [Hz]. The first marker (denoted Ankle) was attached to the subject's to the ankle bone (lateral distal fibula head); the second marker (Knee) to the knee (lateral epicondyle of femur); the third (Hip) to the hip bone (Crista Iliaca); the fourth (Shoulder) to the shoulder (Tuberculum Majus); the fifth marker (Ear) behind the ear (Os Mastoideum); and the sixth marker on the forehead (Fig. 7). The marker position data were sampled at 50 Hz and the measurement accuracy of the 3D coordinates was 0.1 [mm].

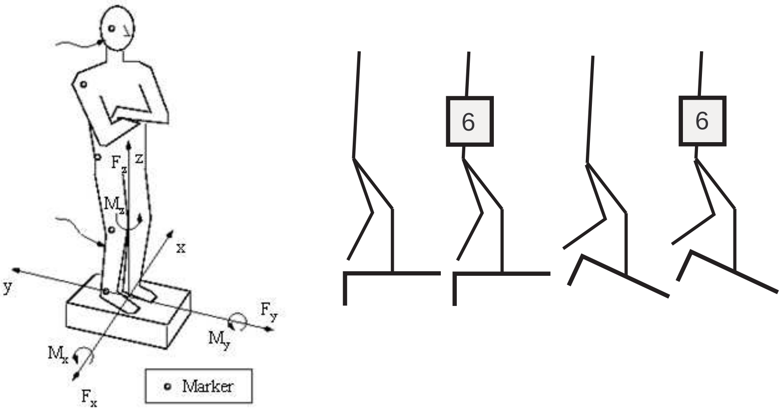

Figure 7: Experimental set-up with a subject stepping onto a force platform, measuring the support forces and multi-segmented kinematics for experiments with horisontal and leaning support surface and 6 [kg] extra weight.

Experimental support was limited to one subject only (body weight 65 [kg]; height 1.65 [m]). While the step response was recorded, the subject was instructed to take a step onto the force platform and resume stable stance on one leg under condition of the *i.* a plane support surface; *ii.* leaning support surface -25^o; *iii.* leaning support surface -25^o, the subject carrying an additional weight 6 [kg]—in all cases with eyes open.

10. Experimental Results

From a qualitative point of view, experimental results were uniformly in good agreement with the behavior predicted by the mathematical model proposed (Figs. 8-9] both in the force responses and the postural responses. Also note the ankle torque steady-state shift providing compensating corrective torque towards upright stance for a leaning support surface.

From a quantitative point of view, application of least-squares identification to ankle torque dynamics using the identification model Eq. (33) gave a good fit withthe estimated ankle stiffness 457.5 [Nm/rad] and damping 15 [Nm/(rad/s)] compatible with optimality model and noisy data (Fig. 10).

11. Discussion

We have solved an optimal control problem of posture and locomotion dynamics with explicit solutions to the Hamilton-Jacobi equation. The optimal solution explains asymptotically stable optimal control, providing both internal model control ('inverse model') and stabilizing feedback. Self-optimization providing globally stable adaptive control has been designed to solve the case of

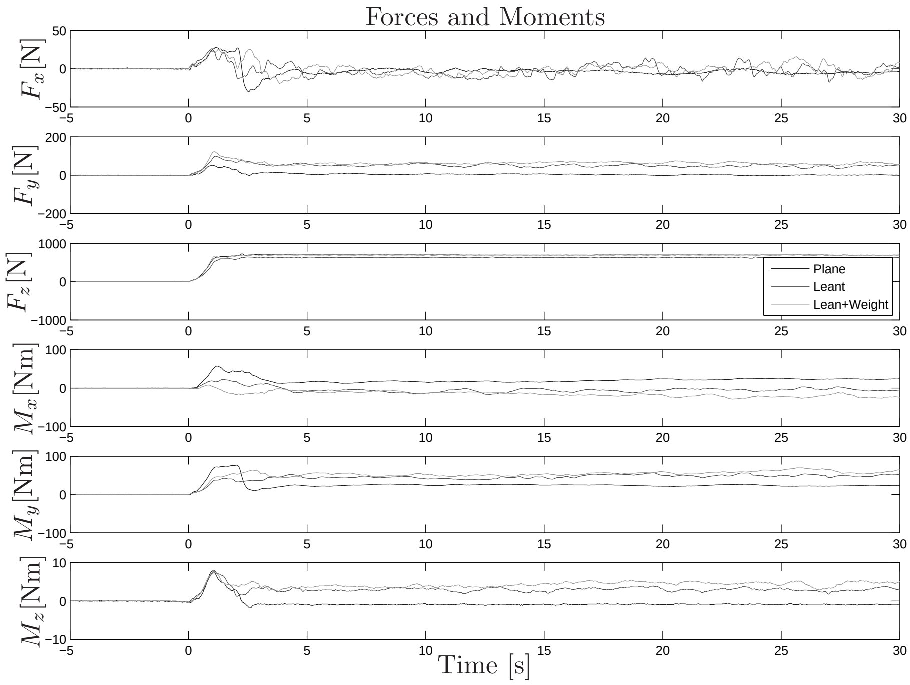

Figure 8: Multi-segmented 6DOF foot-support force response $\{F_x, F_y, F_z, M_x, M_y, M_z\}$ during step onto a plane support surface; leaning surface -25^o; leaning surface -25^o with additional weight 6 [kg] in spatial components x, y, z.

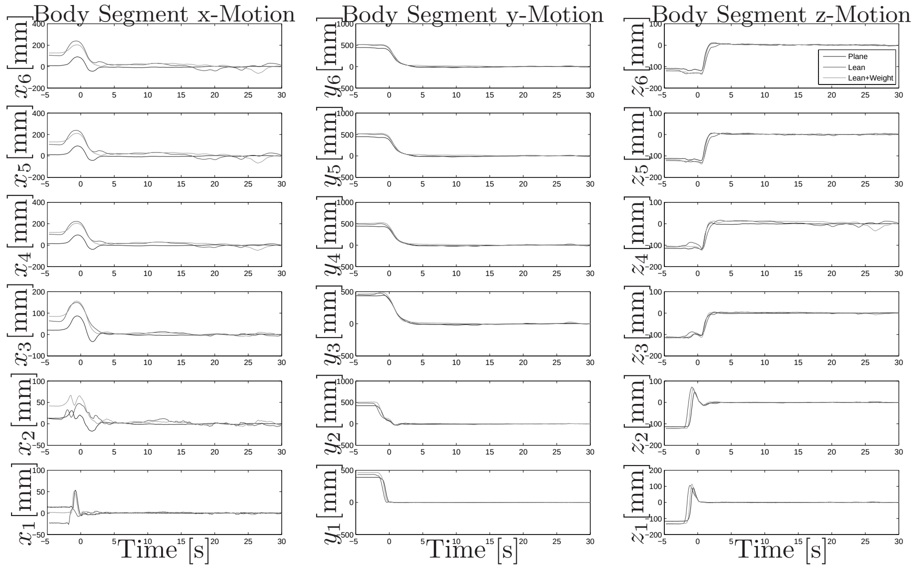

Figure 9: Multi-segmented position response of forehead, ear, shoulder, hip, knee, ankle during step onto a plane support surface, leaning surface -25^o; and leaning surface -25^o with additional weight 6 [kg] in spatial components x (*left*), y (*middle*), z (*right*).

uncertain parameters. The decomposition into an inverse model and optimal feedback is obvious from Eqs. (25) and (32).

The optimal control is globally asymptotically stable, whereas the self-optimizing adaptive control is globally stable in the sense of Lyapunov. The uniform stability in the sense of Lyapunov follows from the existence of a negative semidefinite Lyapunov function derivative as shown in Theorem 1. Finite initial conditions and q_r, $\dot{q}_r \in L^\infty$ mean that the initial value of the Lyapunov function $V(\widetilde{x}(t_0), t_0)$ is bounded. A finite value of the Lyapunov function V implies a finite magnitude of the tracking errors $\widetilde{q}$, $\dot{\widetilde{q}}$. The $L^\infty-$stability follows from the fact that the Lyapunov function is finite and always decreases with time. The asymptotic stability also implies that the optimal closed-loop system is stable under persistent disturbances [31].

Whereas the optimal control algorithm presented here exhibits a certain similarity to the linear quadratic control problem, it is modified to the nonlinear biomechanical conditions of Eq. (1). The closed-loop properties may be effectively determined from the weighting matrices Q, R and S of (A8). Whereas Eq. (24) and the algebraic Riccati equation are similar, the solutions are very different. The Riccati equation solution is positive definite but the present algorithm does not in general provide a symmetric weighting matrix T_0.

From a mechanical point of view there are several interesting aspects. The proposed solutions contribute to the understanding of the close connections between classical mechanics and optimization theory for motion control. The

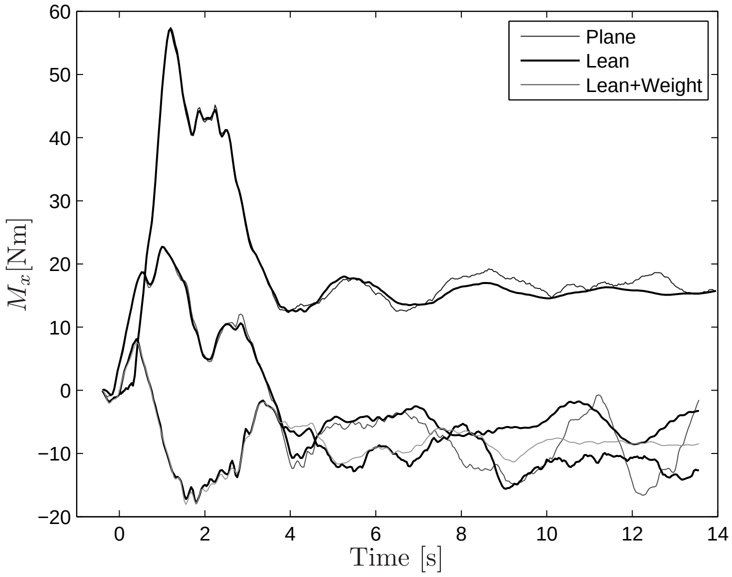

Figure 10: Ankle torque (M_x) step response to plane support surface, leaning surface -25^o; leaning surface -25^o with additional weight 6 [kg]; Fitted linear step-response models (*black*). Note the ankle torque steady-state shift providing compensating corrective torque towards upright stance for a leaning support surface.

matrix K of (24), (27) represents a virtual stiffness around the desired position q_r whereas terms containing the inertia matrix $M(q)$ represent kinetic energy. The Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{U}$ of analytical mechanics may be compared with the Hamilton-Jacobi solution $V(\widetilde{x}, t)$ that represents an aggregate of kinetic energy and the 'potential energy' of a spring action described by a stiffness matrix K. The virtual spring action established by feedback control thus formally replaces gravitation as the source of potential energy. The method offers a description in any set of relevant coordinates, Cartesian space or configuration space. Associated optimization criteria in terms of kinetic energy are invariant to coordinate transformations. Furthermore, a matrix T_0 with $T_{12} \neq 0$ renders the controlled system dissipative—*i.e.*, as the motion is not energy conservative, the system is able to 'absorb' energy of initial conditions and disturbances. Whereas basically of mechanical nature, these properties are biologically relevant inasmuch as energy expenditure issues are relevant in biology

There are several advantages to the analytic solution proposed in this paper, as compared to earlier work [9], [18], [34], [98]. We avoid approximate solutions as well as exact solutions based on approximate models which may exhibit severe degradation of closed-loop control performance as compared to optimal control. The reason is that all model based optimal control is contingent upon the accuracy of the underlying model, *i.e.*, the body segment parameters. A

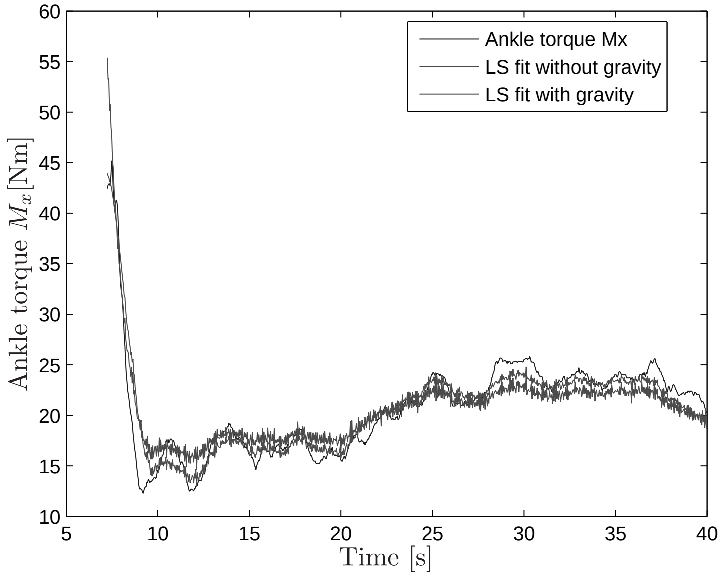

Figure 11: Least-squares estimates of ankle torque (M_x) fitted to angular coordinate and angular velocity of center of mass as approximated by hip position measurement according to model of Eq. (33). Estimates shown without and with gravity regressors, respectively.

small change of system parameters typically results in a severe degradation of performance as compared to optimal conditions. Approximations involved in the linearized or approximated models [18] as well as approximate solutions [85] to the exact problem make from the point of view of parameter uncertainty such optimal control error-prone in nonideal situations. The analytic solution of this paper avoids such problems and is valid also for transient motion with velocities and positions away from particular equilibrium points.

As many tasks of controlled motion must be solved in finite time, it could be argued that the infinite time problem is less relevant for physiologic motion control. However, the problem with an infinite-time optimization criterion may be viewed as a finite-time problem with a performance optimization together with an end point condition at $t = t_f$ on the closed-loop accuracy and stability.

$$V(\widetilde{x}(t_0), t_0) = \int_{t_0}^{\infty} L(\widetilde{x}, u^*)dt = \int_{t_0}^{t_f} L(\widetilde{x}, u^*)dt + V(\widetilde{x}(t_f), t_f) \qquad (49)$$

For finite-time optimization, $V(\widetilde{x}(t_f), t_f)$ has the interpretation of cost-to-go value function at time $t = t_f$ [95]. A similar relation holds for the Lyapunov function V_X of Eq. (41). Notice that the Lagrangian L is positive so that $V(\widetilde{x}(t_f), t_f) \leq V(\widetilde{x}(t_0), t_0)$. This makes the possible adaptation and learning action applicable also as a model of finite-time operation with periodic or iterative motion. The self-optimizing adaptation of an optimal trajectory intended for periodic motion may thus be made in a few repetitive trials.

Only body motion in the form of segmented and articulated rigid links has been explicitly treated here. For example, considerable neural transmission time delay may place a limit upon the applicability of the feedback model. Several models of the type (1) of various biomechanical complexity have been formulated [32], [40], [97]. Elastic deformation and other structural flexibilities that can be modelled by methods of analytical mechanics may be included in the equations (1) and thus in the optimal control solution. Kinematic constraints—*i.e.*, foot-on-ground constraints, multiple contacts—constitute mathematical difficulties in analytical mechanics, requiring constrained optimization or some precalculation involving decomposition into constrained and unconstrained dynamics [60, 89, 45]. This difficulty is inherent to analytic mechanics and Euler-Lagrange equations which require canonical coordinates—*i.e.*, one coordinate for each mechanical degree of freedom [24]. One approach to constraint decomposition of the foot-to-ground kinematic constraints is demonstrated in Eqs. (A5.12-A5.14).

The mathematical description above involves reference trajectories q_r corresponding to intended or instructed motion, motor programs, preprogrammed motion or reference trajectories that are planned by the systems as movement evolves. There is a considerable body of literature on the existence of such reference trajectories and we restrict ourselves to recent contributions in this field. The evidence of motor programs was reviewed by Grillner [27], and supraspinal and spinal mechanisms were considered by Grillner and Dubuc [28]. Zattara and Bouisset [103] support the idea of preprogrammed motion in the context of postural adjustment where anticipatory postural movements appear to counteract the disturbing effects of the forthcoming voluntary motion. Anticipatory postural responses in human subjects were demonstrated by Marsden [69], Haas and Diener [30]. Because of the reproducibility and specificity, the anticipatory postural movements can be considered to be preprogrammed. Grillner and Wallén [29] and Sanes and Jennings [84] treated the control programs underlying the motor behavior. Thorstensson *et al.* [94] demonstrated that trunk movements are generated and controlled by specific patterns of muscle coordination. Anticipatory EMG responses comprising early and late responses, timing and amplitude modulation were analyzed by Lacquaniti and Maioli [62]. Shapovalova *et al.* [90] experimentally investigated the importance of the caudate nucleus in the neural processes preceding motion. Marsden [69] provided evidence for the statement that patients with Parkinson's disease are unable to execute learnt motor strategies involving the selection, sequencing, and initiation of motor programs. The functional role of *substantia nigra* in the initiation and particularly the execution of movements was demonstrated by Viallet *et al.* [96]. Hence, thorough biological experimental evidence exists for motor programs and preprogramming of motor behavior in such context as assumed in the present paper.

The complexity of of the reference trajectory is an open question, and our model allows the specification and inclusion of q_r or q_r, $\dot{q}_r$ or even q_r, $\dot{q}_r$, $\ddot{q}_r$. At least five cases are included in the problem formulation presented:

- Motor programs with preformulated trajectories;
- Reference trajectories generated and planned as movement evolves;
- Goal determination as an end point condition only, or as a sequence of points in position and/or velocity space;
- Goal determination as an end point condition only, to be accomplished at a final time t_f with an additional end point condition of stability;
- Motor planning (trajectory planning) to reach a given end point.

The algorithm provided in this paper may thus solve both the motor planning and the feedback control problems of motion control.

Execution of the motor programs involved requires a sensory feedback of position and velocity obtainable from the visual, vestibular and somatosensory subsystems. Moreover, it has been suggested that a "corollary discharge" exists in motor programs [68]. This is the postulated internal feedback from motor to sensory structures which indicates to the sensory systems that a limb is about to be moved, in order to permit correct interpretation of the sensory consequences of the movement. It is often claimed that such a system must exist, as otherwise we could never be sure whether we had moved or whether the environment had moved us. The interaction between the motor program and sensory feedback is in the present paper modeled as a comparison between the intended motion q_r and the feedback information q that results in an error $\widetilde{q} = q - q_r$. The present 'error feedback' is thus in harmony with ideas of "corollary discharge" although other topographical organization principles are not precluded.

Previously, optimal allocation of forces in redundant biomechanics was studied with respect to task-posture decomposition [60, 89]; and coordination in posture and locomotion [41, 40, 18, 19]. The coordination of muscular forces may be considered either at the level of muscular activations or at the level of joint torques. The relationship between joint torques and muscular action is a research topic in its own right [8], [14]. It should be stressed that there are many degrees of freedom to schedule muscles, a fact which is often described as redundancy or static indeterminacy. The total muscular torque acting on the knee with a certain knee angle is produced by not less than 12 muscles, some of which are diarthric muscles. The force distribution of the n stabilizing forces τ over the m individual muscles ($m > n$) can (formally) be uniquely solved as a quadratic optimization problem as follows:

Assume that the optimal control $\tau \in \mathbb{R}^n$ and the muscular forces of m muscles are described by the vector $F \in \mathbb{R}^m$ where $m > n$. Let $\rho \in \mathbb{R}^{n \times m}$ denote the matrix of anatomical-biometrical data that describes the torque-force relationship $\tau = \rho F$, *i.e.*, the individual muscle action and the load sharing between different muscles as determined by the insertion of muscles and tendons at bone segments ($m > n$). A quadratic optimization of the muscular forces F, *i.e.*, the minimization of $F^T F$ subject to the constraint $\tau = \rho F$ is easily obtained by completing the squares so that

$$\begin{aligned} \min \quad F^T F &= \min(F^T(I - \rho^T(\rho\rho^T)^{-1}\rho)F + \tau^T(\rho\rho^T)^{-1}\tau) = \tau^T(\rho\rho^T)^{-1}\tau \\ \text{subject to} \quad \tau &= \rho F \end{aligned} \tag{50}$$

with the minimum obtained for the unique solution

$$F = F^* = \rho^T(\rho\rho^T)^{-1}\tau \tag{51}$$

which is valid also for position-dependent torque-force relations $\rho = \rho(q)$. (Notice that the matrix ρ is of full row rank n for all well posed problems so that $\rho\rho^T$ is invertible.) In particular, quadratic optimization to obtain the minimum muscle forces that balance the gravitation forces $G(q)$ may be obtained as $F = \rho^T(\rho\rho^T)^{-1}G(q)$.

A similar static optimization problem has been treated with linear programming for *ad hoc* determination of minimum forces [85]-[87]. The problem is statically indeterminate ($m > n$), a circumstance which may cause some frustration in engineering approaches to the evaluation of forces [85]-[87]. It is, however, very difficult to motivate why and how any such restrictions should be imposed. From the point of view of optimization of mechanical energy expenditure, stability and coordination there is no reason to prefer a particular set of forces F to another set of forces F that also satisfies $\tau = \rho F$. The scheduling of individual muscles may also presumably depend on many non-mechanical factors such as the metabolic state. We therefore avoid to suggest any unnecessary restrictions of the solution space that would only limit the explanatory power of the model presented.

Neuromuscular transmission, length and force relationships, and the correspondence between EMG data and muscular forces [8] is a controversial subject, a debate that we hesitate to enter here. By solving the coordination and control problem we support such research by reducing neuromuscular transmission to 'local research topics'. Several problems of muscular physiology may thus be considered at the local level—*e.g.*, inverse myodynamics, the indeterminacy problem, local feedback, metabolic energy consumption, heat production [8].

The proposed solution is sufficient to explain many interesting features of posture control and coordination. The closed-loop properties may be effectively determined over a large range of behaviors from the weighting matrices Q, R and S of (19). Example 1 shows the coordination for climbing of a step based on the information of velocity and position measurements. The control law consists of gravity and Coriolis torque compensations and local error feedback only (corresponding to a diagonal feedback matrix $R^{-1}(S + B^T T_0)$). Several hypotheses on the importance and sufficiency of local feedback for postural control are therefore supported with respect to modeling complexity, *cf.* Houk

[43]. In our examples we reproduce ‘ankle and hip strategies’ of Nashner and colleagues [77]. Example 3 demonstrates the adaptation of step climbing when an additional back load of 20 [kg] is present. The presented model is therefore demonstrated to be of relevance also in the study of motor learning and adaptation.

All model based optimal control rely on the accuracy of the underlying model, *i.e.*, the body segment parameters. A small change of system parameters typically results in a severe degradation of performance, as compared to optimal conditions. It is therefore necessary to consider the problem of adaptation in the context of optimal control, although adaptation is a difficult topic of research [2], [6], [26], [35], [48], [97]-[100].. The adaptation included in this paper is a gradient method with a modification to Lyapunov theory similar to that reported in [78, 20, 51, 52]. The gradient method (40) is also similar to earlier attempts to describe neural learning mechanisms, *cf.* the 'Hebb rule' [35]. The correspondence between the Hebb rule and gradient methods of adaptation has been demonstrated, *e.g.*, [5, p. 492], [102]. Adaptivity in neural mechanisms is usually attributed to higher neural centra—*e.g.*, the cerebellum, with a possible involvement of gating mechanisms at a lower, spinal level. The computational topology (Fig. 2) of the optimal control adaptation does not disagree with known neural tract topography, *cf.* [77].

Experimental verification involves the formulation of adequate postural tests with measurements of joint positions and joint velocities of the articulated and segmental model, as well as of ground reaction forces. Johansson *et al* [50] have reported on identification methodologies, though only ankle strategy dynamics was considered. Statistical hypothesis testing based on the proposed linear regression models (37- 40) is here straightforward. The analysis presented thus supports experimental verification as put forward as a prequisite by Ito [46] and others. Also note the ankle torque steady-state shift in Figs. 8-9 providing compensating corrective torque towards upright stance for a leaning support surface [80, 81, 15, 86].

12. Conclusions

We have solved an optimal control problem of posture and movement dynamics with explicit solutions to the Hamilton-Jacobi equation. The optimal solution explains asymptotically stable optimal control, providing both internal model control ('inverse model') and stabilizing feedback. Self-optimization providing globally stable adaptive control has been designed to solve the case of uncertain parameters. Partial experimental validation was made.

Appendix 1: Proof of Lemma 1

12.1. The Hamilton-Jacobi equation

The Lagrangian is given by (3) and the lemma claims that the Hamilton-Jacobi equation

$$-\frac{\partial V(\widetilde{x},t)}{\partial t} = \min_u \left((\frac{\partial V(\widetilde{x},t)}{\partial \widetilde{x}})^T \dot{\widetilde{x}} + L(\widetilde{x},u) \right) \tag{21'}$$

is satisfied for a function

$$V(\widetilde{x},t) = \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} M(q) & 0 \\ 0 & K \end{bmatrix} T_0 \widetilde{x} \tag{23}$$

The proof contains five steps:

1: A state space description

2: Verification that $V = V(\widetilde{x},t)$

3: Evaluation of partial derivatives of V,

4: Derivation of the u that minimizes H of Eq. (21)

5: Verification that V solves Eq. (21).

12.2. A state-space description

The full error state space representation is found as

$$\widetilde{x}(t) = \begin{bmatrix} \dot{\widetilde{q}}^T(t) & \widetilde{q}^T(t) \end{bmatrix}^T; \quad \widetilde{x} \in \mathbb{R}^{2n} \tag{A1.1}$$

The error dynamics of the linked body segments may be obtained from (1), (2) as a state-space description where the derivative of $\widetilde{x}$ is

$$\dot{\widetilde{x}}(t) = \begin{bmatrix} \ddot{\widetilde{q}}(t) \\ \dot{\widetilde{q}}(t) \end{bmatrix} = \begin{bmatrix} -M^{-1}(q)C(q,\dot{q}) & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix} \widetilde{x}(t)$$

$$+ \begin{bmatrix} -\ddot{q}_r - M^{-1}(q)(G(q) + C(q,\dot{q})\dot{q}_r) \\ 0_{n\times n} \end{bmatrix} + \begin{bmatrix} I_{n\times n} \\ 0_{n\times n} \end{bmatrix} M^{-1}(q)\tau \tag{A1.2}$$

or with shorter notation

$$\dot{\widetilde{x}}(t) = A(q,\dot{q})\widetilde{x}(t) + B_0(\ddot{q}_r, \dot{q}_r, \dot{q}, q) + BM^{-1}(q)\tau \tag{A1.3}$$

where τ is available for assignment of the control law.

$$\dot{\widetilde{x}} = T_0^{-1} \begin{bmatrix} -M(q)^{-1}(\frac{1}{2}\dot{M}(q,\dot{q}) + N(q,\dot{q})) & 0_{n\times n} \\ T_{11}^{-1} & -T_{11}^{-1}T_{12} \end{bmatrix} T_0\widetilde{x} + T_0^{-1} \begin{bmatrix} M(q)^{-1} \\ 0_{n\times n} \end{bmatrix} u \tag{A1.3'}$$

12.3. Verification that $V = V(\widetilde{x}, t)$

First, it is necessary to verify that V, and thus that $M(q)$ is a function of $\widetilde{x}$ and time t only. Notice that the reference value $q_r(t)$ is by definition a function of t only. It is then obvious that

$$M(q) = M(\widetilde{q} + q_r(t)) = M(\widetilde{x}, t) \tag{A1.4}$$

The inertia matrix $M(q)$ is thus a function of the error-state $\widetilde{x}$ and the time t, which implies that $V = V(\widetilde{x}, t)$. The time derivative of the inertia matrix can be expressed as

$$\frac{dM(q)}{dt} = \frac{dM(\widetilde{q} + q_r(t))}{dt} = \sum_{k=1}^{n} \frac{\partial M(q)}{\partial \widetilde{q}_k}\dot{\widetilde{q}}_k + \sum_{k=1}^{n} \frac{\partial M(q)}{\partial q_{r_k}}\dot{q}_{r_k} = \dot{M}(q, \dot{q})$$

and

$$\frac{dM(q)}{dt} = \frac{dM(\widetilde{q} + q_r(t))}{dt} = \frac{dM(\widetilde{x} + x_r(t))}{dt} = \sum_{k=1}^{2n} \frac{\partial M(\widetilde{x}, t)}{\partial \widetilde{x}_k}\dot{\widetilde{x}}_k + \frac{\partial M(\widetilde{x}, t)}{\partial t} \tag{A1.5}$$

Second, partial derivatives of the function V need to be evaluated in order to test the hypothesis that V solves the Hamilton-Jacobi equation. The partial derivative of V with respect to time is

$$\frac{\partial V(\widetilde{x}, t)}{\partial t} = \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} \frac{\partial M(\widetilde{x}, t)}{\partial t} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} T_0 \widetilde{x} \tag{A1.6}$$

$$\frac{\partial V(\widetilde{x}, t)}{\partial t} = (\frac{\partial V(\widetilde{x}, t)}{\partial q_r})^T \frac{\partial q_r(t)}{\partial t} = (\frac{\partial V(\widetilde{x}, t)}{\partial q_r})^T \frac{dq_r(t)}{dt} = (\frac{\partial V(\widetilde{x}, t)}{\partial x_r})^T \frac{dx_r}{dt} \tag{A1.7}$$

The gradient of V with respect to the error-state $\widetilde{x}$ is

$$\frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}} = T_0^T \begin{bmatrix} M(\widetilde{x}, t) & 0_{n\times n} \\ 0_{n\times n} & K \end{bmatrix} T_0 \widetilde{x} + \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} \frac{\partial M(\widetilde{x}, t)}{\partial \widetilde{x}} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} T_0 \widetilde{x} \tag{A1.8}$$

Expression (A1.8) is a function of $\widetilde{x}$ and t only and does not explicitly depend on $\ddot{q}$, $\ddot{\widetilde{q}}$ or u. This gives

$$(\frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}})^T \dot{\widetilde{x}} = \widetilde{x}^T T_0^T \begin{bmatrix} M(\widetilde{x}, t) & 0_{n\times n} \\ 0_{n\times n} & K \end{bmatrix} T_0 \dot{\widetilde{x}} + \frac{1}{2} \sum_{k=1}^{2n} \widetilde{x}^T T_0^T \begin{bmatrix} \frac{\partial M(\widetilde{x}, t)}{\partial \widetilde{x}_k}\dot{\widetilde{x}}_k & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} T_0 \widetilde{x} \tag{A1.9}$$

The state space equation from u to $\widetilde{x}$ of (A1.2) is

$$\dot{\widetilde{x}} = T_0^{-1} \begin{bmatrix} -M(q)^{-1}(\frac{1}{2}\dot{M}(q, \dot{q}) + N(q, \dot{q})) & 0_{n\times n} \\ T_{11}^{-1} & -T_{11}^{-1} T_{12} \end{bmatrix} T_0 \widetilde{x} + T_0^{-1} \begin{bmatrix} M(q)^{-1} \\ 0_{n\times n} \end{bmatrix} u \tag{A1.10}$$

Substitution of $\dot{\tilde{x}}$ in (A1.9) gives

$$(\frac{\partial V(\tilde{x},t)}{\partial \tilde{x}})^T\dot{\tilde{x}} = \frac{1}{2}\tilde{x}^T T_0^T \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ KT_{11}^{-1} & -KT_{11}^{-1}T_{12} \end{bmatrix} T_0\tilde{x} - \tilde{x}^T T_0^T BN(q,\dot{q})B^T T_0\tilde{x} +$$

$$+\tilde{x}^T T_0^T Bu + \frac{1}{2}\tilde{x}^T T_0^T \begin{bmatrix} -\dot{M}(q,\dot{q}) + \sum_{k=1}^{2n} \frac{\partial M(\tilde{x},t)}{\partial \tilde{x}_k}\dot{\tilde{x}}_k & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} T_0\tilde{x} \qquad (A1.11)$$

The second term of (A1.11) disappears due to the anti-symmetric property (8-9) of $N(q,\dot{q})$. The last term of (A1.11) is not explicitly dependent on u, $\ddot{q}$ because $M(\tilde{x},t) = M(q)$ is a function of q.

12.4. Derivation of the u minimizing the value function V

Bearing in mind that the Lagrangian is

$$L(\tilde{x},u) = \frac{1}{2}\tilde{x}^T(t)Q\tilde{x}(t) + \frac{1}{2}u(t)^T Ru(t) + u^T(t)S\tilde{x}(t) \qquad (3')$$

a candidate of the Hamiltonian H (20) is the sum of Eqs. (A1.11) and (3). A fourth step is now to evaluate how H depends on $u \in \mathbb{R}^n$. The $u = u^*$ for which H has its minimum value is obtained from the partial derivatives with respect to u. Only the second terms of (A1.11) and (3) contribute to the partial derivatives.

$$\frac{\partial H}{\partial u} = \frac{\partial}{\partial u}((\frac{\partial V(\tilde{x},t)}{\partial \tilde{x}})^T\dot{\tilde{x}} + L(\tilde{x},u)) = B^T T_0\tilde{x} + Ru + S\tilde{x} \qquad (A1.12)$$

Extremals of the Hamiltonian with respect to u is found by setting the partial derivatives $\partial H/\partial u$ equal to zero. The minimum is obtained for $u = u^*$

$$u^* = -R^{-1}(S + B^T T_0)\tilde{x} \qquad (A1.13)$$

12.5. Verification that V solves the H.J. equation

A fifth step is now to verify that the suggested V satisfies (21). The time derivative of V is composed of (A1.11) and (A1.6-7)

$$\frac{dV(\tilde{x},t)}{dt} = \frac{\partial V(\tilde{x},t)}{\partial t} + (\frac{\partial V(\tilde{x},t)}{\partial \tilde{x}})^T\dot{\tilde{x}}$$

$$\frac{dV(\tilde{x},t)}{dt} = \tilde{x}^T T_0^T \begin{bmatrix} M(q) & 0_{n\times n} \\ 0_{n\times n} & K \end{bmatrix} T_0\dot{\tilde{x}} + \frac{1}{2}\tilde{x}^T T_0^T \begin{bmatrix} \dot{M}(q,\dot{q}) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} T_0\tilde{x} \quad (A1.14)$$

Substitution of $\dot{\tilde{x}}$ of (A1.10) into (A1.14) gives

$$\frac{dV(\tilde{x},t)}{dt} = \frac{1}{2}\tilde{x}^T \begin{bmatrix} 0 & K^T \\ K & 0 \end{bmatrix} \tilde{x} + \tilde{x}^T T_0^T Bu \qquad (A1.15)$$

Application of $u = u^*$ (A1.13) to $\dot{V}$ of (A1.15) gives

$$\frac{dV(\tilde{x},t)}{dt} = \frac{1}{2}\tilde{x}^T \begin{bmatrix} 0 & K^T \\ K & 0 \end{bmatrix} \tilde{x} - \tilde{x}^T(T_0^T BR^{-1}B^T T_0 + T_0^T BR^{-1}S)\tilde{x} \quad (A1.16)$$

Application of $u = u^*$ on the Lagrangian of optimal control

$$L(\widetilde{x}, u^*) = \frac{1}{2}\widetilde{x}^T(Q - S^T R^{-1} S + T_0^T B R^{-1} B^T T_0)\widetilde{x} \qquad (A1.17)$$

The Hamilton-Jacobi equation is satisfied for $u = u^*$ if

$$\begin{aligned} 0 &= \frac{\partial V(\widetilde{x}, t)}{\partial t} + (\frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}})^T \dot{\widetilde{x}} + L(\widetilde{x}, u^*) \\ &= \frac{1}{2}\widetilde{x}^T \left[\begin{bmatrix} 0 & K^T \\ K & 0 \end{bmatrix} + Q - (S + B^T T_0)^T R^{-1}(S + B^T T_0) \right] \widetilde{x} = 0 \end{aligned}$$

It now follows that $V(\widetilde{x}, t)$ is a solution to the Hamilton-Jacobi equation, a *Hamilton's principal function*, for $u = u^*$ and matrices K, T_0 solving the algebraic matrix equation

$$\widetilde{x}^T \left[\begin{bmatrix} 0 & K^T \\ K & 0 \end{bmatrix} + Q - (S + B^T T_0)^T R^{-1}(S + B^T T_0) \right] \widetilde{x} = 0; \qquad \forall \widetilde{x} \qquad (24)$$

This proves Lemma 1.

□

Appendix 2: Proof of Theorem 1

From Lemma 1 it is known that

$$V(\widetilde{x}(t), t) = \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} M(q) & 0 \\ 0 & K \end{bmatrix} T_0 \widetilde{x} \qquad (23)$$

solves the Hamilton-Jacobi equation for $K = K^T$, T_0 solving the algebraic matrix equation

$$\widetilde{x}^T \left[\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + Q - (S + B^T T_0)^T R^{-1}(S + B^T T_0) \right] \widetilde{x} = 0 \qquad (24)$$

where the optimal feedback control law $u = u^*$ minimizing J is

$$u^*(t) = -R^{-1}(S + B^T T_0)\widetilde{x}(t) \qquad (25)$$

Let the weighting matrix Q, R of the Lagrangian be factorized with Cholesky-factorizations Q_1, Q_2, R_1 of (27) so that and chooseT_0, K according to Eqs. (28-29)

$$\begin{aligned} T_0 &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & I_{n\times n} \end{bmatrix} = \begin{bmatrix} R_1^T Q_1 - S_1 & R_1^T Q_2 - S_2 \\ 0 & I_{n\times n} \end{bmatrix} \\ K &= \frac{1}{2}(Q_1^T Q_2 + Q_2^T Q_1) - \frac{1}{2}(Q_{12}^T + Q_{12}) \end{aligned}$$

Application of these factorizations and the conditions of (27) directly show that $K = K^T > 0$. The matrices K, T_0 of (24) solve the algebraic matrix equation of (21)

$$\widetilde{x}^T \left[\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + Q - (S + B^T T_0)^T R^{-1} (S + B^T T_0) \right] \widetilde{x} = 0$$

or with application of (24)

$$\widetilde{x}^T \left[\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} - \begin{bmatrix} Q_1^T Q_1 & Q_1^T Q_2 \\ Q_2^T Q_1 & Q_2^T Q_2 \end{bmatrix} \right] \widetilde{x} = 0 \qquad (A2.1)$$

The Hamilton-Jacobi equation (21) is satisfied because

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t} + \min_u H(\widetilde{x}, u, \frac{\partial V}{\partial \widetilde{x}}) = \frac{\partial V(\widetilde{x}, t)}{\partial t} + (\frac{\partial V(\widetilde{x}, t)}{\partial \widetilde{x}})^T \dot{\widetilde{x}} + L(\widetilde{x}, u^*) = \\ &= \frac{1}{2} \widetilde{x}^T \left[\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + Q - (S + B^T T_0)^T R^{-1} (S + B^T T_0) \right] \widetilde{x} = 0 \end{aligned}$$

Notice that $V \geq 0$ for all positive definite K. The performance index may then be evaluated as

$$\int_{t_0}^{t_f} L(\widetilde{x}, u^*) dt = \int_{t_0}^{t_f} -\dot{V} dt = V(\widetilde{x}(t_0), t_0) - V(\widetilde{x}(t_f), t_f) \leq V(\widetilde{x}(t_0), t_0) \qquad (A2.3)$$

The optimality of the control follows from (A2.2) and it follows that $\widetilde{x} \in L^2(t_0, t_f), \forall t_f \geq t_0$. The claim on L^2-stability follows immediately from (A2.3).

From (16) and (24) follows that $K = K^T > 0$ and the inertia matrix $M(q)$ is positive definite by definition (1). The quadratic function $V(\widetilde{x}, t)$ is a suitable Lyapunov function candidate because it is positive, radially growing with $\|\widetilde{x}\|$ for all $t \geq t_0$. It is continuous, and has a unique minimum at the origin of the error-space. The function V has a unique minimum at the origin.

It remains to show that $\dot{V} < 0$ for all $\|\widetilde{x}\| \neq 0$. From the solution (28) of the Hamilton-Jacobi equation, it follows that $dV/dt + L = \partial V/\partial t + H^* = 0$ is constant for $u = u^*$ so that $\forall t > 0$

$$\frac{dV(\widetilde{x}, t)}{dt} = -L(\widetilde{x}, u^*) = -\frac{1}{2} \widetilde{x}^T (T_0^T B R^{-1} B^T T_0 + Q - S^T R^{-1} S) \widetilde{x} < 0; \quad \widetilde{x} \neq 0. \qquad (A2.4)$$

The time derivative $dV/dt < 0$ because $Q > S^T R^{-1} S$ according to assumption A8. This implies that V is a Lyapunov function for a uniformly, globally asymptotically stable system. The proposition of the theorem then follows directly from the properties of Lyapunov functions [23].

This finishes the proof.

□

Appendix 3: Proof of Theorem 2

The resulting effective control variable u in the case of uncertain parameters can be computed from (30-31) as

$$u = u^* + \psi\widetilde{\theta}; \qquad u^* = -R^{-1}B^T T_0 \widetilde{x} \tag{A3.1}$$

where $\widetilde{\theta}$ denotes the vector of parameter errors $\widetilde{\theta} = \widehat{\theta} - \theta$. This control law is no longer optimal in the sense of Eq. (21) due to the term $\psi\widetilde{\theta}$. Let the parameter error $\widetilde{\theta}$ be included in a new state vector $\widetilde{x}$ that suffices to describe the error dynamics.

$$\widetilde{x} = \begin{bmatrix} \widetilde{x} \\ \widetilde{\theta} \end{bmatrix} \tag{A3.2}$$

The following Lyapunov design of parameter adjustment can make the solution systematically tend toward the optimal solution. Introduce the following Lyapunov function candidate V_X

$$V_X(\widetilde{x}, t) = V(\widetilde{x}, t) + V_\theta(\widetilde{\theta}) = \frac{1}{2}\widetilde{x}^T T_0^T \begin{bmatrix} M(q) & 0 \\ 0 & K \end{bmatrix} T_0 \widetilde{x} + \frac{1}{2}\widetilde{\theta}^T K_\theta \widetilde{\theta};\ K_\theta = K_\theta^T > 0 \tag{A3.3}$$

where V is the solution (27) to the Hamilton-Jacobi equation and V_θ is a quadratic functional of parameter errors. Moreover, V_X is a function of the full error state with a unique minimum at the origin of error state space. The function V_X is thus feasible as a Lyapunov function candidate for the adaptive (sub)optimal system with the derivative

$$\dot{V}_X = \dot{V} + \dot{V}_\theta = -\frac{1}{2}\widetilde{x}^T (Q - S^T R^{-1} S + T_0^T B R^{-1} B^T T_0)\widetilde{x} + \widetilde{x}^T T_0^T B \psi \widetilde{\theta} + \widetilde{\theta}^T K_\theta \dot{\widetilde{\theta}} \tag{A3.3}$$

The following adaptation law

$$\dot{\widehat{\theta}} = -K_\theta^{-1} \psi^T B^T T_0 \widetilde{x} \tag{A3.5}$$

and the control law (31) ensures that $\dot{V}_X$ is equal to $\dot{V}$ of (A2.4) for constant parameters θ.

$$\frac{dV_X(\widetilde{x}, t)}{dt} = -\frac{1}{2}\widetilde{x}^T (Q - S^T R^{-1} S + T_0^T B R^{-1} B^T T_0)\widetilde{x} \tag{A3.6}$$

This proves that the system is globally stable (in the sense of Lyapunov), and adaptation eventually makes the control system optimal. Adaptation thus makes the system work as a self-optimizing control system or an extremum controller. The performance degradation due to the parameter errors can be evaluated as

$$\frac{1}{2}\int_{t_0}^{\infty} \widetilde{x}^T (Q - S^T R^{-1} S + T_0^T B R^{-1} B^T T_0)\widetilde{x}\, dt \leq V_X(\widetilde{x}(t_0), t_0) = \mathcal{J}(u^*) + V_\theta(\widetilde{\theta}(t_0)) \tag{A3.7}$$

The theorem is immediately verified by application of (35-36) under the conditions of constant parameters θ and theorem 1. The solution reaches the optimal solution for $\widetilde{\theta} = 0$. The Lyapunov function derivative is negative semidefinite w.r.t. $\widetilde{x}$ and negative definite w.r.t. $\widetilde{x}$.

Appendix 4: Stability with Respect to External Persistent Disturbances

In the case of external persistent disturbances, stability analysis has to be extended from Lyapunov analysis to passivity analysis. Following [101] and [39], a dynamical system is said to be dissipative if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$, called a storage function such that for all t_0, t_f, $x \in \mathbb{R}^n$ and $u \in \mathbb{U}$, $y \in \mathbb{Y}$, $t_f \geq t_0$ satisfying the inequality

$$V(x(t_0)) + \int_{t_0}^{t_f} w(u, y)dt \geq V(x(t_f)) \tag{A4.1}$$

where $w(u, y)$ is a real-valued function called the supply rate—*i.e.*, $w : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$. Strict dissipativity holds if the inequality (12.5) is a strict inequality. Moreover, the system is said to be *passive* if there is storage function V and coefficients $\epsilon \geq 0, \delta \geq 0, \rho \geq 0$ and supply rate $w = u^T z$ satifying

$$u^T z \geq \frac{\partial V}{\partial x}\frac{dx}{dt} + \epsilon u^T u + \delta z^T z + \rho x^T x, \tag{A4.2}$$

The system is *input strictly passive* if $\epsilon > 0$, *output strictly passive* if $\delta > 0$ and *state strictly passive* if $\rho > 0$ [101, 39].

Without much restriction, we specialize to disturbance entering as force disturbances F entering at a body point $x = f(q)$ in Cartesian space and reflected onto the articulated body-kinematic structure as the disturbance joint torques $\omega = J^T(q)F$ via the Jacobian matrix $J(q) = \partial f(q)/\partial q$

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau + \omega, \qquad \omega = J^T(q)F \tag{A4.3}$$

Using the value function of Eq. (23) as a tentative storage function, we find

$$\frac{dV(\widetilde{x}, t)}{dt} = -L(\widetilde{x}, u) + \widetilde{x}^T T_o B\omega = -L(\widetilde{x}, u) + \widetilde{z}_1^T \omega \tag{A4.4}$$

From Eqs. (A1.10), we find

$$\begin{aligned} \dot{\widetilde{z}}_1 &= -M^{-1}(q)(\frac{1}{2}\frac{dM(q)}{dt} + N(q, \dot{q}))\widetilde{z}_1 + M^{-1}(q)\omega \\ \widetilde{z}_1^T M(q)\dot{\widetilde{z}}_1 &= -\widetilde{z}_1^T(\frac{1}{2}\frac{dM(q)}{dt} + N(q, \dot{q})\widetilde{z}_1 + \widetilde{z}_1^T\omega \end{aligned}$$

By the skew-symmetric properties of $N(q,\dot{q})$, $\widetilde{z}_1^T N(q,\dot{q})\widetilde{z}_1 = 0$, it follows that the energy supply rate from ω to $\widetilde{z}_1$

$$\begin{aligned}
w(\omega,\widetilde{z}_1) &= \widetilde{z}_1^T\omega \\
\int_{t_0}^{t_f} w(\omega,\widetilde{z}_1)dt &= \int_{t_0}^{t_f} \widetilde{z}_1^T\omega dt = \int_{t_0}^{t_f} \frac{1}{2}\widetilde{z}_1^T\frac{dM(q)}{dt}\widetilde{z}_1 + \widetilde{z}_1^T M(q)\dot{\widetilde{z}}_1 dt \\
&= \int_{t_0}^{t_f} \frac{d}{dt}(\frac{1}{2}\widetilde{z}_1^T M(q)\widetilde{z}_1)dt = [\frac{1}{2}\widetilde{z}_1^T M(q)\widetilde{z}_1]_{t_0}^{t_f}
\end{aligned}$$

Let the supplied disturbance energy be denoted

$$V_\omega(t) = \int_{t_0}^{t_f} w(\omega,\widetilde{z}_1)dt = [\frac{1}{2}\widetilde{z}_1^T M(q)\widetilde{z}_1]_{t_0}^{t_f} \tag{A4.6}$$

The resultant dissipation energy balance is

$$\underbrace{V(\widetilde{x}(t_f),t_f) - V(\widetilde{x}(t_0),t_0)}_{\text{Storage Function}} = -\underbrace{\int_{t_0}^{t_f} L(\widetilde{z}_1,u)dt}_{\text{Dissipation Energy}} + \underbrace{V_\omega(t_f) - V_\omega(t_0)}_{\text{Supplied Energy}} \tag{A4.7}$$

which shows that the optimal control system is stable and state strictly passive in the mapping from the disturbance ω to control error $\widetilde{z}_1$ with strict dissipativity. Moreover, the value function $V(\widetilde{x},t)$ of the optimal control problem serves as a storage function in passivity analysis.

Appendix 5: Equations for simulation of Examples 1-3

Consider the anthropomorfic five-link model described by the joint angular coordinates

$$q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}^T \in \mathbb{R}^5 \tag{A5.1}$$

A model where motion of the foot point (*e.g.*, slipping) is possible in the anterior-posterior direction requires the set of coordinates ζ of Cartesian coordinates and joint space coordinates.

$$\zeta = \begin{bmatrix} q^T & y & z \end{bmatrix}^T \in \mathbb{R}^7 \tag{A5.2}$$

where x and y denote the horisontal and vertical in Cartesian coordinates of the position of the foot support. Introduce the following abbreviated notation of trigonometric functions

$$\begin{cases} c_i = \cos q_i(t) \\ s_i = \sin q_i(t) \\ c_{ij} = \cos(q_i(t) - q_j(t)) \\ s_{ij} = \sin(q_i(t) - q_j(t)) \end{cases} \quad \forall i,j \in \{1,2,\dots,5\} \tag{A5.3}$$

Let m_i, l_i, r_i denote the mass of link i, the length of link i, and the distance from the joint i to the center of gravity of link i, respectively. The coordinates

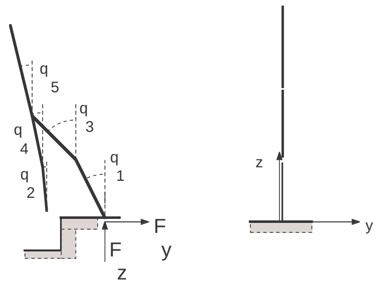

Figure 12: An anthropomorphic five-link model

in Cartesian space of the center of gravity of link i are obtained by elementary geometric considerations

$$\begin{cases} y_1 = -r_1 s_1 + y \\ z_1 = r_1 c_1 + z \end{cases}$$
$$\begin{cases} y_2 = -l_1 s_1 + r_2 s_2 - l_3 s_3 + l_4 s_4 + y \\ z_2 = l_1 c_1 - r_2 c_2 + l_3 c_3 - l_4 c_4 + z \end{cases}$$
$$\begin{cases} y_3 = -l_1 s_1 - r_3 s_3 + y \\ z_3 = l_1 c_1 + r_3 c_3 + z \end{cases}$$
$$\begin{cases} y_4 = -l_1 s_1 - l_3 s_3 + r_4 s_4 + y \\ z_4 = l_1 c_1 + l_3 c_3 - r_4 c_4 + z \end{cases}$$
$$\begin{cases} y_5 = -l_1 s_1 - l_3 s_3 - r_5 s_5 + y \\ z_5 = l_1 c_1 + l_3 c_3 + r_5 c_5 + z \end{cases} \qquad (A5.4)$$

with the corresponding velocities obtained as time derivatives

12.6. Kinetic and potential energy

The total kinetic energy as a sum the kinetic energy of each segment

$$\mathcal{T} = \frac{1}{2}\sum_{i=1}^{5} m_i(\dot{y}_i^2 + \dot{z}_i^2) = \frac{1}{2}\zeta^T M_\zeta(\zeta)\zeta \qquad (A5.5)$$

where we decompose the inertia matrix $M_\zeta(\zeta) \in \mathbb{R}^{7\times 7}$ as follows

$$M_\zeta(\zeta) = \begin{bmatrix} M(q) & M_y & M_z \\ M_y^T & m_{yy} & m_{yz} \\ M_z^T & m_{zy} & m_{zz} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{1y} & m_{1z} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{2y} & m_{2z} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{3y} & m_{3z} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{4y} & m_{4z} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{5y} & m_{5z} \\ m_{y1} & m_{y2} & m_{y3} & m_{y4} & m_{y5} & m_{yy} & m_{yz} \\ m_{z1} & m_{z2} & m_{z3} & m_{z4} & m_{z5} & m_{zy} & m_{zz} \end{bmatrix} \tag{A5.6}$$

with $M(q) \in \mathbb{R}^{5\times 5}$, $M_y, M_z \in \mathbb{R}^5$ and elements

$$\begin{aligned}
m_{11} &= J_1 + m_1 r_1^2 + (m_2 + m_3 + m_4 + m_5) l_1^2 \\
m_{12} &= m_{21} = -m_2 l_1 r_2 c_{12} \\
m_{13} &= m_{31} = m_3 l_1 r_3 c_{13} + (m_2 + m_4 + m_5) l_1 l_3 c_{13} \\
m_{14} &= m_{41} = -m_4 l_1 r_4 c_{14} - m_2 l_1 l_4 c_{14} \\
m_{15} &= m_{51} = m_5 l_1 r_5 c_{15} \\
m_{22} &= J_2 + m_2 r_2^2 \\
m_{23} &= m_{32} = -m_2 l_3 r_2 c_{23} \\
m_{24} &= m_{42} = m_2 l_4 r_2 c_{24} \\
m_{25} &= m_{52} = 0 \\
m_{33} &= J_3 + m_3 r_3^2 + (m_2 + m_4 + m_5) l_3^2 \\
m_{34} &= m_{43} = -m_2 l_3 l_4 c_{34} - m_4 l_3 r_4 c_{34} \\
m_{35} &= m_{53} = m_5 l_3 r_5 c_{35} \\
m_{44} &= J_4 + m_4 r_4^2 + m_2 l_4^2 \\
m_{45} &= m_{54} = 0 \\
m_{55} &= J_5 + m_5 r_5^2 \\
m_{1y} &= m_{y1} = m_1 r_1 c_1 - (m_2 + m_3 + m_4 + m_5) l_1 c_1 \\
m_{2y} &= m_{y2} = m_2 r_2 c_2 \\
m_{3y} &= m_{y3} = -m_3 r_3 c_3 - (m_2 + m_4 + m_5) l_3 c_3 \\
m_{4y} &= m_{y4} = m_2 l_4 c_4 + m_4 r_4 c_4 \\
m_{5y} &= m_{y5} = -m_5 r_5 c_5 \\
m_{1z} &= m_{z1} = m_1 r_1 s_1 - (m_2 + m_3 + m_4 + m_5) l_1 s_1 \\
m_{2z} &= m_{z2} = m_2 r_2 s_2 \\
m_{3z} &= m_{z3} = -m_3 r_3 s_3 - (m_2 + m_4 + m_5) l_3 s_3 \\
m_{4z} &= m_{z4} = m_2 l_4 s_4 + m_4 r_4 s_4 \\
m_{5z} &= m_{z5} = -m_5 r_5 s_5 \\
m_{yz} &= m_{zy} = 0 \\
m_{yy} &= m_1 + m_2 + m_3 + m_4 + m_5 \\
m_{zz} &= m_1 + m_2 + m_3 + m_4 + m_5
\end{aligned}$$

$$(A5.7)$$

The potential energy $\mathcal{U}_i$ of each link i is

$$\begin{bmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \\ \mathcal{U}_3 \\ \mathcal{U}_4 \\ \mathcal{U}_5 \end{bmatrix} = \begin{bmatrix} m_1 g(z + r_1 c_1) \\ m_2 g(z + l_1 c_1 + l_3 c_3 - l_4 c_4 - r_2 c_2) \\ m_3 g(z + l_1 c_1 + r_3 c_3) \\ m_4 g(z + l_1 c_1 + l_3 c_3 - r_4 c_4) \\ m_5 g(z + l_1 c_1 + l_3 c_3 + r_5 c_5) \end{bmatrix} \quad (A5.8)$$

The total potential energy

$$\begin{aligned} \mathcal{U} = \sum_{i=1}^{5} \mathcal{U}_i \quad &= \quad m_1 g r_1 c_1 + (m_2 + m_3 + m_4 + m_5) g l_1 c_1 - \\ &- \quad m_2 g r_2 c_2 + \\ &+ \quad m_3 g r_3 c_3 + (m_2 + m_4 + m_5) g l_3 c_3 - \\ &- \quad m_2 g l_4 c_4 - m_4 g r_4 c_4 + \\ &+ \quad m_5 g r_5 c_5 + \\ &+ \quad (m_1 + m_2 + m_3 + m_4 + m_5) g z \end{aligned}$$

$$(A5.9)$$

12.7. The Euler-Lagrange motion equations

The gravitation torques according to (5-6) are

$$G_\zeta = \frac{\partial \mathcal{U}}{\partial \zeta} = \begin{bmatrix} G(q) \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{U}}{\partial q} \\ \frac{\partial \mathcal{U}}{\partial y} \\ \frac{\partial \mathcal{U}}{\partial z} \end{bmatrix} = \begin{bmatrix} -(m_1 g r_1 + (m_2 + m_3 + m_4 + m_5) g l_1) s_1 \\ m_2 g r_2 s_2 \\ -(m_3 g r_3 + (m_2 + m_4 + m_5) g l_3) s_3 \\ (m_2 g l_4 + m_4 g r_4) s_4 \\ -m_5 g r_5 s_5 \\ 0 \\ (m_1 + m_2 + m_3 + m_4 + m_5) g \end{bmatrix} \quad (A5.10)$$

with $G(q) \in \mathbb{R}^5$ is the gravitation-dependent torques at the joints $i = 1, \ldots, 5$ and with the gravitation constant $g = 9.81\,[\mathrm{m/s^2}]$. The gravitation forces in Cartesian space at the foot support are denoted G_y, G_z, with horisontal and vertical components. The Coriolis and centripetal forces can now be calculated according to (1), (10) and the derivative of $M(q)$

$$\begin{bmatrix} C & C_{qy} & C_{qz} \\ C_{yq} & c_{yy} & 0 \\ C_{zq} & 0 & c_{zz} \end{bmatrix} = \frac{1}{2} \dot{M} + N \quad (A5.11)$$

The Euler-Lagrange equations (1) are thus

$$\begin{bmatrix} M & M_y & M_z \\ M_y^T & m_{yy} & m_{yz} \\ M_z^T & m_{zy} & m_{zz} \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} C & C_{qy} & C_{qz} \\ C_{yq} & c_{yy} & 0 \\ C_{zq} & 0 & c_{zz} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} G \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \tau \\ F_y \\ F_z \end{bmatrix} \quad (A5.12)$$

In cases with no motion in y or z at the support surface—*i.e.*, $\dot{y} = \dot{z} = 0$, $\ddot{y} = \ddot{z} = 0$—gives equations of motion

$$\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) \qquad (A5.13) = (1')$$

and the corresponding ground reaction forces with a horisontal shear force F_y and a vertical force F_z as follows

$$\begin{bmatrix} F_y \\ F_z \end{bmatrix} = \begin{bmatrix} M_y^T \ddot{q} + C_{yq}\dot{q} \\ M_z^T \ddot{q} + C_{zq}\dot{q} + G_z \end{bmatrix} \qquad (A5.14)$$

Elimination of $\ddot{q}$ of (A5.16) provides a relation between the joint torques and the ground reaction forces

$$\begin{bmatrix} F_y \\ F_z \end{bmatrix} = \begin{bmatrix} M_y^T M^{-1} \\ M_z^T M^{-1} \end{bmatrix} \tau + \begin{bmatrix} -M_y^T M^{-1} C + C_{yq} \\ -M_z^T M^{-1} C + C_{zq} \end{bmatrix} + \begin{bmatrix} -M_y^T M^{-1} G \\ -M_z^T M^{-1} G + G_z \end{bmatrix} \qquad (A5.15)$$

with the matrices M_y, M_z obtained from (A5.7)

$$M_y = \begin{bmatrix} m_{1y} \\ m_{2y} \\ m_{3y} \\ m_{4y} \\ m_{5y} \end{bmatrix} = \begin{bmatrix} m_1 r_1 c_1 - (m_2 + m_3 + m_4 + m_5) l_1 c_1 \\ m_2 r_2 c_2 \\ -m_3 r_3 c_3 - (m_2 + m_4 + m_5) l_3 c_3 \\ m_4 r_4 c_4 + m_2 l_4 c_4 \\ -m_5 r_5 c_5 \end{bmatrix} \qquad (A5.16)$$

$$M_z = \begin{bmatrix} m_{1z} \\ m_{2z} \\ m_{3z} \\ m_{4z} \\ m_{5z} \end{bmatrix} = \begin{bmatrix} m_1 r_1 s_1 - (m_2 + m_3 + m_4 + m_5) l_1 s_1 \\ m_2 r_2 s_2 \\ -m_3 r_3 s_3 - (m_2 + m_4 + m_5) l_3 s_3 \\ m_4 r_4 s_4 + m_2 l_4 s_4 \\ -m_5 r_5 s_5 \end{bmatrix} \qquad (A5.17)$$

and the matrices C_{yq}, C_{zq}

$$C_{yq} = \begin{bmatrix} -m_1 r_1 s_1 \dot{q}_1 + (m_2 + m_3 + m_4 + m_5) l_1 s_1 \dot{q}_1 \\ -m_2 r_2 s_2 \dot{q}_2 \\ m_3 r_3 s_3 \dot{q}_3 + (m_2 + m_4 + m_5) l_3 s_3 \dot{q}_3 \\ -m_2 l_4 s_4 \dot{q}_4 - m_4 r_4 s_4 \dot{q}_4 \\ m_5 r_5 s_5 \dot{q}_5 \end{bmatrix} \qquad (A5.18)$$

$$C_{zq} = \begin{bmatrix} m_1 r_1 c_1 \dot{q}_1 - (m_2 + m_3 + m_4 + m_5) l_1 c_1 \dot{q}_1 \\ m_2 r_2 c_2 \dot{q}_2 \\ -m_3 r_3 c_3 \dot{q}_3 - (m_2 + m_4 + m_5) l_3 c_3 \dot{q}_3 \\ m_2 l_4 c_4 \dot{q}_4 + m_4 r_4 c_4 \dot{q}_4 \\ -m_5 r_5 c_5 \dot{q}_5 \end{bmatrix} \qquad (A5.19)$$

12.8. Conditions of simulation

Initial conditions of Examples 1, 2, and 3

$$\begin{bmatrix} \text{left shank} \\ \text{right shank} \\ \text{left thigh} \\ \text{right thigh} \\ \text{torso} \end{bmatrix} : \quad \begin{bmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ q_4(0) \\ q_5(0) \end{bmatrix} = \begin{bmatrix} 5.73 \\ 5.00 \\ 5.73 \\ 5.00 \\ 5.73 \end{bmatrix}, \quad \begin{bmatrix} 5.73 \\ 5.00 \\ 5.73 \\ 5.00 \\ -5.73 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 5.73 \\ 5.00 \\ 5.73 \\ 5.00 \\ 5.73 \end{bmatrix} \quad [\text{deg}] \tag{A5.20}$$

Weights and moments of inertia

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 30 \end{bmatrix} \quad [\text{kg}]; \qquad \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 3 \end{bmatrix} \quad [\text{kg} \cdot \text{m}^2] \tag{A5.21}$$

Geometrical data

$$\begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix} = \begin{bmatrix} 0.40 \\ 0.40 \\ 0.40 \\ 0.40 \\ 1.00 \end{bmatrix} \quad [\text{m}]; \qquad \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} = \begin{bmatrix} 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \\ 0.50 \end{bmatrix} \quad [\text{m}] \tag{A5.22}$$

12.9. Optimal control

The optimization criterion $\mathcal{J}(u)$ for the optimal control was chosen such that $Q - S^T R^{-1} S > 0$.

$$\mathcal{J}(u) = \int_0^\infty \widetilde{x}^T \begin{bmatrix} 90 I_{5\times5} & 324 I_{5\times5} \\ 324 I_{5\times5} & 1305 I_{5\times5} \end{bmatrix} \widetilde{x} + u^T \begin{bmatrix} 3 I_{5\times5} & 12 I_{5\times5} \end{bmatrix} \widetilde{x} + \frac{1}{9} u^T I_{5\times5} u dt \tag{44}$$

which results in the control law

$$\tau = G(q) - \begin{bmatrix} 35.0 I_{5\times5} & 120.0 I_{5\times5} \end{bmatrix} \widetilde{x} \tag{45'}$$

or with more detail, *cf.* (A5.9)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \end{bmatrix} = \begin{bmatrix} -(m_1 g r_1 + (m_2 + m_3 + m_4 + m_5) g l_1) s_1 \\ m_2 g r_2 s_2 \\ -(m_3 g r_3 + (m_2 + m_4 + m_5) g l_3) s_3 \\ (m_2 g l_4 + m_4 g r_4) s_4 \\ -m_5 g r_5 s_5 \end{bmatrix} - 35.0 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} - 120.0 \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \tag{A5.23}$$

As compared to the complexity of the equations of motion, the control law is very simple with only anti-gravitation forces and feedback of position and velocity.

12.10. Adaptive control

An unknown value of m_5 due to the back load suggests that m_5 should be replaced by an estimate $\widehat{\theta}$ and that (A5.23) should be replaced by the adaptive control law

$$\tau = \begin{bmatrix} -gl_1s_1 \\ 0 \\ -gl_3s_3 \\ 0 \\ -gr_5s_5 \end{bmatrix} \widehat{\theta} + \begin{bmatrix} -(m_1gr_1 + (m_2+m_3+m_4)gl_1)s_1 \\ m_2gr_2s_2 \\ -(m_3gr_3 + (m_2+m_4)gl_3)s_3) \\ (m_2gl_4 + m_4gr_4)s_4 \\ 0 \end{bmatrix} -35.0 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} -120.0 \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \tag{A5.24}$$

where $\widehat{\theta}$ (*i.e.* the estimated m_5) adapts according to (34) which gives

$$\dot{\widehat{\theta}} = -K_\theta^{-1} \begin{bmatrix} -gl_1s_1 & 0 & -gl_3s_3 & 0 & -gr_5s_5 \end{bmatrix} \left(\frac{35.0}{9.0} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} + \frac{120.0}{9.0} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \right) \tag{A5.28}$$

with values in the simulated examples chosen as $K_\theta = 0.006$ and with an initial value $\theta(0) = 10$.

Acknowledgement

We wish to thank Dr. John Allum for drawing our attention to this area of research. We would like to express our gratitude to Prof Thomas Mergner for the Neuro-Robotics Symposium, Freiburg, 20-22 July 2008.

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