Model Reduction Using Semidefinite Programming

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Abstract

In this thesis model reduction methods for linear time invariant systems are investigated. The reduced models are computed using semidefinite programming. Two ways of imposing the stability constraint are considered. However, both approaches add a positivity constraint to the program. The input to the algorithms is a number of frequency response samples of the original model. This makes the computational complexity relatively low for large-scale models. Extra properties on a reduced model can also be enforced, as long as the properties can be expressed as convex conditions. Semidefinite program are solved using the interior point methods which are well developed, making the implementation simpler.

A number of extensions to the proposed methods were studied, for example, passive model reduction, frequency-weighted model reduction. An interesting extension is reduction of parameterized linear time invariant models, i.e. models with state-space matrices dependent on parameters. It is assumed, that parameters do not depend on state variables nor time. This extension is valuable in modeling, when a set of parameters has to be chosen to fit the required specifications. A good illustration of such a problem is modeling of a spiral radio frequency inductor. The physical model depends nonlinearly on two parameters: wire width and wire separation. To chose optimally both parameters a low-order model is usually created. The inductor modeling is considered as a case study in this thesis.

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Aivar

Contents

1

Introduction

1.1 Motivation for Model Reduction

There is a need for model reduction algorithms in many applications. All of them have a common problem: an engineer needs to design a system based on another, usually more complicated, one. A good example is controller design. Nowadays more and more controllers are designed using model-based control (e.g. model predictive, $\mathcal{H}_\infty$ control), which builds the control law in consideration to the model of the process. In most applications the calculation capacity of the regulator or the controller is limited, and the processes are usually complex or even nonlinear. Thus the engineer is faced with a problem: install more calculation capacity to a controller or decrease the computational cost. The computational cost may be lowered by finding a simpler in some sense model of the process. An approximation of the model should behave in a similar manner to the original one. In model reduction problems the similarity in behaviour is usually measured by the distance in $\mathcal{H}_\infty$ space (the $\mathcal{H}_\infty$ norm).

A particular application is the design of circuit components. One of the design problems is the choice of parameters to fulfill the required specifications. A physical model of a component that accurately describes the effects is usually complex. Simulation of such a model can be computationally overwhelming. Therefore there is a need to simplify physical models that depend on parameters. This dependence

is not necessarily linear. The simplification of linear model with an arbitrary parameter dependence is another direction pursued in this thesis.

That leaves us with the question "What is a simple model?".

What is a Simple Model?

In this thesis linear time invariant (LTI) models are mainly considered, so one possible answer is: the simplicity of the system is the number of states in the minimal state-space realization (or the McMillan degree) of the system. However, if one considers LTI systems that depend on parameters, then the picture becomes less clear. Even a low-order model with a nonlinear dependence on a parameter can provoke hard computations. A linear or polynomial dependence on a parameter could significantly improve the situation. Thus another simplification need has been described: simplify the dependence on parameters. Reducing the number of parameters by throwing away non-essential ones is certainly another direction worth looking into. However, it is not a part of this thesis.

1.2 How to Read the Thesis

To a reader not familiar with model reduction methods some background is required, which is provided in Chapter 2. Basically three main directions of reduction are distinguished: reduction of LTI models, frequency weighted reduction and reduction of parameterized LTI models. For each direction a chapter of this thesis is dedicated. The basic ideas of LTI model reduction are not repeated in every chapter, therefore the reader is referred to Chapter 3. A more detailed description of each chapter content follows next.

Chapter 2

In this chapter a brief description of existing model reduction methods is made. Singular value decomposition methods such as Hankel model reduction and balanced truncation are described. Only basic ideas of Krylov projection methods are presented, since there exist a great variety of modifications and even a bigger variety of computational improvements. If the reader is familiar with the ideas of these

methods, then he/she can skip this part. References for further reading are also presented. A more detailed introduction is made to the optimization-based $\mathcal{H}_\infty$ model reduction framework. The focus of this thesis is modification and extensions of this approach to different problems. Therefore the reader is strongly advised to familiarize with this work.

Chapter 3

In this chapter multivariable extensions of a recently proposed optimization based method are investigated. If the extensions are reduced to a single variable case, then all the theoretical statements and the method itself are equivalent to ones of the original method. A part of the statements has not been proved in the previous work, such as error bounds for reduction of passive systems. The multivariable relaxation presented in this chapter is referenced later on, however, a brief reminder is given in each chapter. Common problems and approaches are also discussed in this chapter and are referenced if required in the latter chapters.

Chapter 4

In this chapter two multivariable extensions of the original method are presented: frequency-weighted and a partially parameterized model reduction. Examples are presented to show the performance and some common applications of the methods.

Chapter 5

This chapter extends the approach to the set of multivariable fully parameterized models. Another optimization-based method is also developed. It can be regarded as a modification of the previous work or a completely new method. A comparison between the two approaches is performed. The main field of such a comparison is a set of parameterized models. As it turns out it is only for parameterized models that the new approach shows its power. Examples described in this chapter were chosen to capture known drawbacks and advantages of the methods.

Chapter 6

Low order modeling of radio frequency components is required, since a physical model is too complex for simulation and design purposes. A low order model of every instance of a parameter would accelerate the process. In this chapter the methods developed in Chapter 5 are applied to this problem.

Chapter 7

This chapter is dedicated to conclusions and future work directions.

2

Background

2.1 Introduction to Model Reduction

Model simplification has a long history. For nonlinear models the basic tool, which has been known for a while, is linearization around specific trajectories. With a computed linear model around a stationary point one can deduce if a system is stable or not in a vicinity of the point. In the recent years a need for low-complexity models became even more acute. One would want to simulate the model to, possibly, deduce some properties of the behaviour of the simulated model and, given a good approximation, of the modeled system. With the development of control theory and model based control more applications of simplification algorithms came to light. For example, in embedded control systems, real-time systems, MPC control design and many other applications low-order models play a crucial role. In this thesis an investigation is made into model simplification of linear time invariant systems. In this case the simplification is the same as the model order reduction. The problem is posed to preserve the input-output behaviour with an approximation. There were many approaches taken to solve it. Some are focused on approximation of low-scale models, since they require heavy computations. Among those methods balanced truncation (due to simplicity) and optimal Hankel model reduction (due to optimality in some sense) are considered. Other methods, for example, Krylov projection techniques are focused on reduction of large-scale models. These

methods usually provide worse approximations compared to Hankel reduction, hover, the computational cost is much lower. Only the main ideas for such methods are presented.

Another direction in model reduction is matching the frequency samples of the models. One of the major bottlenecks is obtaining a convex formulation of the problem, since the problem is known to be non-convex. One attempt to obtain such a method was made in [Megretski, 2003]. The performance was compared in the real part norm, thus making a comparison to, for example, Hankel model reduction difficult. Recently, a modification was performed (see [Sou *et al.*, 2005]) to form the objective in the $\mathcal{H}_\infty$ norm. There exist many advantages of the method. Firstly, computational complexity is relatively low, since only the frequency samples of the model are required for the optimization. Secondly, the solution is asymptotically optimal in a Hankel-type norm (i.e. when the number of the matched samples is infinite). Thirdly, one can impose extra constraints on the reduced model, as long as they are convex.

2.2 Notation

The following notations will be used throughout the thesis. Denote $\mathbb{D} = \{z \big| |z| \leq 1\}$ a unit disc in the space of complex numbers $\mathbb{C}$. The main spaces used in the thesis are $\mathcal{L}$ and $\mathcal{H}$ spaces. $\mathcal{L}_2$ is a space of square integrable functions on a set. $\mathcal{L}_\infty$ is the space of bounded functions on a set. If it is not stated otherwise the set is the unit circle $\partial\mathbb{D}$. Hardy $\mathcal{H}$ spaces are the subsets of a $\mathcal{L}$ of functions analytic outside $\mathbb{D}$. The Hardy space $\mathcal{H}(\partial\mathbb{D})$ is a space of discrete time transfer functions. $\mathcal{H}^-$ stands for a subspace of $\mathcal{L}$ analytic inside the unit disc $\mathbb{D}$ (antistable transfer functions). For the continuous time transfer functions the unit disc is replaced with $\mathrm{Re}\,(s) > 0$ and the circle with the imaginary axis $\jmath\mathbb{R}$. $G^\sim$ stands for the adjoint in the $\mathcal{H}(\partial\mathcal{D})$ space: $G^\sim(z) = G^T(1/z)$. Consider a polynomial $Q = \sum_{i=0}^{k} Q_i z_i^{-i}$, then $Q^\triangledown$ stands for the reciprocal to Q polynomial, i.e. $Q^\triangledown(z) = Q^\sim(z)/z^k$. $\sigma(G)$ stands for a singular value of a matrix G, $\overline{\sigma}(G)$ is a maximal singular value of matrix G, and $\underline{\sigma}(G)$ is a minimal singular value of matrix G. Finally $\deg(G)$ is a McMillan degree of a transfer function G.

2.3 Theoretical Background.

In this section a background which is required for model reduction methods is provided.

Consider a linear time invariant (LTI) system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{m_2}$, $y \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_2}$, $C \in \mathbb{R}^{m_1\times n}$, $D \in \mathbb{R}^{m_1\times m_2}$. Usually the model is assumed to be asymptotically stable, i.e eigenvalues of A have a restriction $\mathrm{Re}\,(\lambda_i(A)) < 0$ for all i. Using the Laplace transformation a frequency domain description of a model is obtained as $G(s) = C(sI-A)^{-1}B+D$, where s is a complex variable. Here the corresponding stability criterion is that the poles of G lie to the left hand side of the imaginary axis. The space of all the stable transfer functions is denoted $\mathcal{H}_p(\jmath\mathbb{R})$, where $p = 1,\ldots,\infty$. The space of the antistable transfer function, i.e the poles of G lie to the right hand side of the imaginary, is denoted as $\mathcal{H}_p^-(\jmath\mathbb{R})$. To shorten the notation write $G = (A, B, C, D)$, where sometimes D is omitted.

An important concept is passivity. Basically it is a system that does not generate energy. A formal definition can be formulated as:

Definition 2.1
A system:

$$\dot{x} = f(x,u)$$
$$y = g(x,u)$$

is called passive if there exist a continuously differentiable positive semidefinite function $V(x)$ such that:

$$u(t)^T y(t) \leq \frac{\partial V}{\partial t} = \frac{\partial V}{\partial x} f(x,u) \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^p$$

□

For LTI systems an easier object to deal with is positive real functions:

DEFINITION 2.2
A $p \times p$ proper rational transfer function matrix $G(s)$ is called *positive real* if

- poles of all elements of $G(s)$ are in $\mathrm{Re}\,[s] \leq 0$
- for all real ω for which $\jmath\omega$ is not a pole of any element of $G(s)$, the matrix $G(\jmath\omega) + G^{\sim}(\jmath\omega)$ is positive semidefinite, and
- any pole on the imaginary axis of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to \jmath\omega} (s - \jmath\omega) G(s)$ is positive definite Hermitian.

□

Here $G^{\sim}(\jmath\omega) = G^T(-\jmath\omega)$. It turns out that an LTI system is (strictly) passive if its transfer function G is (strictly) positive real (see [Khalil, 2002]). Thus passivity becomes a convex condition in the space of LTI systems. Note, that this condition implies that $D + D^T \geq 0$ for positive realness (and $D + D^T > 0$ for strict positive realness), since $G(\infty) = D$. For reduction purposes only the strict positive realness is considered.

For the analysis of transfer functions in the complex plane one may use a Laurent series expansion of a function in the point $s = \infty$:

$$G(s) = C(sI - A)^{-1}B = \sum_{i=1}^{\infty} h_i s^{-i}$$
$$h_i = CA^{i-1}B, \quad i \geq 1$$

Parameters h_i are called the Markov parameters of the model or an impulse response. A transfer function with a finite impulse response is usually called an FIR (finite impulse response) filter.

In discrete time the definitions are similar:

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{m_2}$, $y \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_2}$, $C \in \mathbb{R}^{m_1 \times n}$,$D \in \mathbb{R}^{m_1 \times m_2}$. The transfer function $G(z)$ now is obtained using the Z-transformation:

$$G(z) = C(zI - A)^{-1}B + D,$$

The stability criterion is changed to $|\lambda_i(A)| < 1$ for all i. The condition in the complex plane is changed too, poles lie inside the unit disc $\mathbb{D} = \{z \Big| |z| < 1\}$. Nothing is changed for passivity and positive realness, i.e. the following condition will be checked:

$$G(e^{j\omega}) + G^{\sim}(e^{j\omega}) > 0, \quad \forall \omega \in [0, \pi]$$

where $\sim$ stands for the adjoint in discrete time $\mathcal{H}_p$ space $G^{\sim}(z) = G^T(1/z)$. The Markov parameters are defined in a similar way, i.e. $h_i = CA^{i-1}B$.

Problem Formulation

The order reduction problem is to find a model $\hat{G} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ that approximates the original one G. The order $\deg(\hat{G}) = k$, i.e. $\hat{A} \in \mathbb{R}^{k \times k}$ and the dimensions of the rest of the matrices are changed correspondingly. Formally one may write the problem as a minimization one:

$$\min_{\deg(\hat{G}) \leq k < n} \|G - \hat{G}\|$$

The minimization norm can be the $\mathcal{H}_2(j\mathbb{R})$ norm:

$$\|G\|_2^2 = \int_0^\infty \operatorname{trace}\left(G(j\omega)G^{\sim}(j\omega)\right) d\omega$$

or the $\mathcal{H}_\infty(j\mathbb{R})$ norm:

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(G(j\omega))$$

The symbol $\overline{\sigma}$ denotes the maximal singular value of a matrix $G(j\omega)$.

The usual way to compare the approximations is $\mathcal{H}_\infty(j\mathbb{R})$ norm, because it has an induced 2-norm meaning:

$$\|G\|_\infty = \min_{u(\cdot)} \left\{ \frac{\|y\|_2}{\|u\|_2} \Big| y = Gu \right\},$$

where $\|\cdot\|_2$ is a 2-norm of an $\mathcal{L}_2$ function. Thus the infinity norm reflects the input-output relationship in an arguably better way.

In the discrete time case the norms are defined in a similar fashion:

$$\|G\|_2^2 = \int_0^\pi \text{trace}\,(G(e^{j\omega})G^{\sim}(e^{j\omega})d\omega$$
$$\|G\|_\infty = \sup_{\omega\in[0,\pi]} \overline{\sigma}(G(e^{j\omega})),$$

And the problem formulation is identical:

$$\min_{\deg(\hat{G})\le k<n} \|G-\hat{G}\|$$

Energy Functions and Gramians

A central concept in balanced truncation and Hankel model reduction is a Gramian. It is tightly related to the observability and controllability energy functions:

$$L_o = \int_0^\infty y(t)^T y(t)dt \quad x(0) = x_0$$
$$L_c = \min_{u(\cdot)} \int_{-\infty}^0 u(t)^T u(t)dt \quad x(-\infty) = 0 \quad x(0) = x_0$$

L_o describes the energy induced by the initial state x_0 to the output signal y and L_c describes the minimal energy required to reach the state x_0 at the zero time. Basically, L_c and L_o represent the mappings from x_0 to signals, and vice versa. A formalization here simplifies things, denote:

$$\Psi_c : \mathcal{L}(-\infty,0) \to \mathbb{R}^n \qquad \Psi_o : \mathbb{R}^n \to \mathcal{L}(-\infty,0)$$

where operator Ψ_o maps x_0 into the future output signals, Ψ_o^* given the future output signals reconstructs the initial state x_0, operator Ψ_c^* maps x_0 into the past input signals, Ψ_c given the past input signals reconstructs the initial state x_0. One can express those operators as:

$$\Psi_c u = \int_{-\infty}^0 e^{-A\tau}Bu(\tau)d\tau \qquad \Psi_o x_0 = Ce^{At}x_0, \quad t \ge 0$$

Rewrite the input and output signals as these mappings and obtain the following expressions:

$$\int_0^\infty y(t)^T y(t)dt = \langle \Psi_o x_0, \Psi_o x_0 \rangle = \langle x_0, \Psi_o^* \Psi_o x_0 \rangle$$
$$\int_{-\infty}^0 u(t)^T u(t)dt = \langle \Psi_c^{-1} x_0, \Psi_c^{-1} x_0 \rangle = \langle x_0, (\Psi_c^* \Psi_c)^{-1} x_0 \rangle,$$

Without going into the technical details of existence and uniqueness of such operators it is safe to say that for a stable transfer function given a minimal state-space realization (A, B, C) such operators can be calculated as:

$$\Psi_o^* \Psi_o = Q \qquad \Psi_c^* \Psi_c = P$$

where P and Q satisfy:

$$A^T Q + QA + C^T C = 0 \qquad AP + PA^T + BB^T = 0$$

The matrices P and Q are called the controllability and the observability Gramians correspondingly. The described equations are called the Lyapunov equations and if all the eigenvalues of A have negative real parts then P, Q are positive semidefinite.

Hankel Operator and Hankel Singular Values

Define the Hankel operator of a stable transfer function G with a state-space realization (A, B, C, D) as:

$$\Gamma_G : \mathcal{L}(-\infty, 0) \to \mathcal{L}(0, \infty)$$
$$\Gamma_G u(t) = \int_{-\infty}^0 Ce^{A(t-\tau)} Bu(\tau)d\tau, \quad \text{for} \quad t \geq 0$$

and Γ_G has an interpretation of an output:

$$y(t) = \Gamma_G u(t) \quad t \geq 0$$

The eigenvalues of the operator $\Gamma_G^* \Gamma_G$ are called the Hankel singular values.

The operator maps past inputs into future outputs. Recall the controllability and observability operators Ψ_o and Ψ_c mentioned earlier and rewrite the Hankel operator as:

$$\Gamma_G = \Psi_o \Psi_c$$

Note that $\Psi_c^* \Psi_c \Psi_o^* \Psi_o = PQ$, where P and Q are the controllability and observability Gramians. Therefore the operator $\Gamma_G^* \Gamma_G$ and the matrix PQ have the same eigenvalues. This expression provides us with a good way of calculating the Hankel singular values and the definition of Hankel norm of a transfer function: $\|G\|_H = \|\Gamma_G\| = \sqrt{\max \lambda(PQ)}$. The relationship between the Hankel and the $\mathcal{H}_\infty$ norms is described by the well-known Nehari's theorem:

THEOREM 2.1—NEHARI
Suppose $G \in \mathcal{H}_\infty$, then

$$\inf_{\Delta_- \in \mathcal{H}_\infty^-} \|G - \Delta_-\| = \|\Gamma_G\| = \|G\|_H$$

where $\mathcal{H}_\infty^-$ is the space of antistable transfer functions and the infinum is achieved. □

The famous AAK (Adamyam-Arov-Krein) theorem may be considered as a generalization of Nehari's theorem. Only a simplified formulation is presented here.

THEOREM 2.2—ADAMYAN-AROV-KREIN
Let $G = G(s)$ be a matrix-valued function bounded on the $\jmath\omega$-axis. Let $\sigma_1 \leq \ldots \sigma_m \leq 0$ be m largest singular values of Γ_G. Then σ_m is the minimum of $\|G - \hat{G}\|_H$ over the set of all stable systems $\hat{G}$ of order less than m. □

As a consequence of the AAK theorem one can rewrite the transfer function G as:

$$G(s) = D + \sigma_1 E_1(s) + \cdots + \sigma_n E_n(s),$$

where $E_i(s)$ are all-pass dilations, i.e. have the infinity norm 1. This representation gives a bound on infinity norm through the Hankel

singular values:

$$\|G\|_\infty \leq 2(\sigma_1 + \cdots + \sigma_n)$$

A tighter bound is given by the next lemma:

LEMMA 2.1
Suppose $G \in \mathcal{H}_\infty$, and $\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of G, then there exist a constant real-valued matrix D_0 such that:

$$\|G - D_0\|_\infty \leq \sigma_1 + \cdots + \sigma_n$$

□

Using the coefficients of impulse response of the system (Markov parameters): $h(z) = h_0 + h_1 z^{-1} + \cdots + h_n z^{-n} + \ldots$ define matrices $M[i,j]$:

$$M[i,j] = \begin{pmatrix} h_i & h_{i+1} & h_{i+2} & \ldots & h_{i+j} \\ h_{i+1} & h_{i+2} & & \cdot & \\ h_{i+2} & & \cdot & & \\ \vdots & \cdot & & & \\ h_{i+j} & & & & h_{i+2j} \end{pmatrix}$$

Note that $M[0,\infty]$ has the same eigenvalues as PQ and thus relates to the Hankel operator Γ_G.

Convex Programming.

Convex programs have nowadays penetrated almost all the fields of engineering due to well-developed theory and tools for solving the problems. The main concepts are presented without going into the details about the solutions. For further reading, see [Boyd and Vandenberghe, 2004].

One can consider only minimization problems, since maximization of a function is a minimization of the same function with a minus sign. Consider a general formulation with $x \in \mathbb{R}^n$:

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, N_1 \\ & f_j(x) = 0 & j = 1, \ldots, N_2 \end{array}$$

DEFINITION 2.3
The minimization problem is called convex if the objective and constraint functions are convex, which means that they satisfy the inequality:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \quad \forall i = 0, \ldots, N_1$$
$$\forall \alpha, \beta \in \mathbb{R}^+ : \alpha + \beta = 1 \quad x, y \in \mathbb{R}^n$$

If the objective function f_0 is a quasiconvex function, i.e. all sub-level sets of f_0, i.e. $A_\gamma = \{x \,|\, f_0(x) \leq \gamma\}$, are convex sets:

$$\forall \gamma \quad \forall x, y \in A_\gamma, \quad \forall \theta \in [0,1] : \quad \theta x + (1-\theta) y \in A_\gamma$$

then the program is called quasiconvex. □

An equivalent convex formulation of a quasiconvex problem may be achieved by introducing an extra constraint $f_0(x) \leq \gamma$ and minimizing γ :

minimize	γ	
subject to	$f_0(x) \leq \gamma$	
	$f_i(x) \leq 0$	$i = 1, \ldots, N_1$
	$f_j(x) = 0$	$j = 1, \ldots, N_2$

In this thesis the problems are formulated as semidefinite ones. A semidefinite program (SDP) is a program with matrix (semi) definite constraints:

minimize	$c^T x$
subject to	$x_1 F_1^{(i)} + \cdots + x_n F_n^{(i)} + G^{(i)} \leq 0$
	$Ax = b$

where $G^{(i)}, F_1^{(i)}, \ldots, F_n^{(i)}, A, G, c, b$ - constant matrices of suitable sizes. In the theory a standard form SDP is formulated as:

minimize	$\text{trace}\,(CX)$
subject to	$\text{trace}\,(A_i X) = b_i$
	$X \geq 0$

Basically any number of LMIs of type $f(x)$, where functions f are affine and matrix valued, can be enforced in a semidefinite program. The great thing about semidefinite programs is that the methods are developed and there exist a number of commercial and general public use products to solve those. The methods usually used for solving such problems are the interior-point ones.

2.4 State-Space Methods

Here the main reduction techniques that use the state-space representation of a model are sketched. Using an established nomenclature two families of methods are distinguished: singular values decomposition (SVD) methods and Krylov methods. The properties of the reduced model vary depending on the method. The SVD methods are usually more accurate, guarantee the stability of the reduced model and provide error bounds depending on the Hankel singular values (see, for example, [Zhou *et al.*, 1996]) of the original model. The moment matching ones, in general, can not guarantee stability, however, the algorithms are significantly computationally cheaper than the ones for SVD.

Singular Value Decomposition (SVD) methods

The main ideas of Hankel model reduction and balanced truncation are presented only in continuous time, however, the methods were also developed for discrete-time systems. They have a similar intuition, so they are skipped, an interested reader may find more details in [Zhou *et al.*, 1996] or [Obinata and Anderson, 2001].

Balanced Truncation [Enns, 1984]. The intuition behind the balanced truncation algorithm is quite simple: reduce poorly observable and at the same time poorly controllable states. Stated in terms of transfer functions: reduce near pole-zero cancellations. Still there is a need in a general approach of identifying such states. Here the energy functions L_c and L_o and the corresponding Gramians P and Q come in handy. Recall their relationships:

$$L_c = \langle x_0, P^{-1}x_0 \rangle \qquad L_o = \langle x_0, Qx_0 \rangle$$

If a state is hard to control, i.e. one needs to spend considerable energy to steer the state in a desired position, then this will be reflected in the corresponding entries of P. On the other hand if the state induces small amount of an energy into the output it is seen in L_o and Q. Now it is clear that the Gramians may be used to determine which states to truncate if the matrices P and Q are diagonal. Simultaneous diagonalization may be performed as:

$$\tilde{Q} = T^{-T}QT^{-1} = \Sigma \quad \tilde{P} = TPT^T = \Sigma, \quad \text{where}$$
$$T = \Sigma^{1/2}U^*P^{-1/2} \quad \text{and} \quad P^{1/2}QP^{1/2} = U\Sigma^2U^*$$

The state-space form (TAT^{-1}, TB, CT^{-1}) is called a balanced realization of G which gave the name to the method. Since a similarity transformation does not effect G, the input-output relationship is still the same. As was mentioned the Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_n$ appear on the diagonal of balanced Gramians Σ :

$$\tilde{P} = \tilde{Q} = \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

Every σ_i reflects how the corresponding state effects the input-output behaviour. Now the algorithm can be formulated.

Let $G = (A, B, C)$ be an asymptotically stable system with $A \in \mathbb{R}^{n\times n}$, and B and C have corresponding sizes. Denote also the reduced model as $\hat{G} = (\hat{A}, \hat{B}, \hat{C})$, with $\hat{A} \in \mathbb{R}^{k\times k}$, and $\hat{B}$ and $\hat{C}$ have corresponding sizes.

ALGORITHM 2.1—BALANCED TRUNCATION

- Solve Lyapunov equations for P and Q :

$$A^TQ + QA + C^TC = 0 \qquad AP + PA^T + BB^T = 0$$

- Calculate matrix $T \in \mathbb{R}^{n\times n,}$ which is a state-space transformation, such that

$$TPT^T = T^{-T}QT^{-1} = \Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\}.$$

- Let $W = T^{-T}\begin{pmatrix} I & 0 \end{pmatrix}$ $V = T\begin{pmatrix} I & 0 \end{pmatrix}$ and obtain the reduced model $(\hat{A}, \hat{B}, \hat{C}) = (W^T AV, W^T B, CV)$

□

Balanced truncation does not take the matrix D into account. Here one can match the static gain $\hat{G}(0)$ to $G(0)$ considering the calculated $\hat{A}, \hat{B}, \hat{C}$ and obtain $\hat{D}$ as an outcome. This approach is called the Schur method. Properties of the reduced model are formulated in, for example, [Obinata and Anderson, 2001]:

THEOREM 2.3
Assume $\hat{G}$ is obtained by balanced truncation of G. Then $\hat{G}$ is asymptotically stable and

$$\|G - \hat{G}\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i,$$

where σ_i are the truncated Hankel singular values of G. □

The error bounds are known not to be tight. Still the Hankel singular values represent how much effect on the input-output relationship the corresponding state has. Therefore an excellent intuition which state to truncate is given and a justification for using the method is provided.

Other Types of Balancing. A way of balancing the model through the Gramians was presented. One can guess that there exist many variations of obtaining a balanced realization for different applications. Here only ways of balancing are formulated without a deep discussion. Given a linear time invariant continuous model:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

State-space transformations W, V can be obtained by the following balancing approaches:

- **LQG Balancing.** Instead of the observability and controllability energy functions (i.e. Gramians) one can diagonalize the solu-

tions to Riccati equations:

$$AP + PA^T + BB^T - PC^TCP = 0$$
$$QA + A^TQ + C^TC - QBB^TQ = 0$$

The truncation with this type of balancing can be applied to unstable systems. The reduced models may be used for a feedback controller design.

- **Stochastic Balancing.** Combining both approaches yields a set of Lyapunov-Riccati equations:

$$AP + PA^T + B_W B_W^T = 0, \quad \text{where } B_W = PC^T + BD^T$$
$$QA + A^TQ + C_W^T C_W = 0, \quad \text{where } C_W = D^{-1}(C - B_W^T Q)$$

The method preserves the minimum-phase property of the model, which is required in the stochastic case. An error bound is given as a relative one:

$$\|G^{-1}(G - \hat{G})\|_\infty \leq \prod_{i=k+1}^{n} \frac{1+\sigma_i}{1-\sigma_i} - 1$$

- **Positive Real Balancing.** This type of balancing preserves the passivity property which is desirable in many applications. The balancing is performed by two Riccati equations:

$$AP + PA^T + (PC^T - B)(D + D^T)^{-1}(PC^T - B)^T = 0$$
$$QA + A^TQ + (C^T - QB)(D + D^T)^{-1}(C^T - QB)^T = 0$$

and the passivity is preserved since $\langle x_0, P^{-1}x_0\rangle$ and $\langle x_0, Qx_0\rangle$ can be used as storage functions for the passivity proof.

- **Cross Gramian Balancing** can be applied only on symmetric square transfer functions. However, it is possible to symmetrize the model by raising the order of the system, for example, as it is done in [Antoulas and Sorensen, 2002]. With this approach one has to solve a Sylvester equation:

$$AX + XA + BC = 0$$

Hankel Model Reduction [Latham and Anderson, 1985]. Even though balance truncation provided an excellent intuition for reducing the states, the approach is somewhat naive and the procedure itself is not optimal in any sense. One would like to have at least some kind of guarantee that the method is trust-worthy.

As the reader may recall, any transfer function can be written as:

$$\hat{G}(s) = D + \sigma_1 E_1(s) + \cdots + \sigma_n E_n(s),$$

using all-pass dilations and Hankel singular values. The idea of the Hankel model reduction boils down to calculating $E_i(s)$ and D_0, then the reduced model $\hat{G}$ is:

$$\hat{G}(s) = D + D_0 + \sigma_1 E_1(s) + \cdots + \sigma_k E_k(s),$$

D_0 is required for a tighter error bound, obtained using the Lemma 2.1.

The detailed description of the algorithm is omitted, since it is complicated and not really relevant to this thesis. However, it is worth mentioning that the algorithm requires solving Lyapunov equations, as balanced truncation.

The properties of the reduced model obtained by Hankel norm minimization are formalized in a statement:

THEOREM 2.4
Suppose G is an asymptotically stable transfer function, $\hat{G}$ is obtained by Hankel norm minimization. Then $\hat{G}$ is asymptotically stable and

$$\sigma_{k+1} = \|G - \hat{G}\|_H \leq \|G - \hat{G}\|_\infty \leq \sum_{i=k+1}^{n} \sigma_i(G),$$

where σ_i are the truncated Hankel singular values of G. □

The obtained $\hat{G}$ with a state-space representation $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is an optimal solution in Hankel norm. However, in $\mathcal{H}_\infty$ norm a better one can be found using already obtained data. Consider an optimization procedure:

ALGORITHM 2.2

- Solve the optimal Hankel model reduction problem and obtain $\hat{G} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$
- Fix $\hat{A}, \hat{B}$ and solve the following optimization problem:

$$\min_{C,D} \|G - C(sI - \hat{A})^{-1}\hat{B} - D\|_\infty$$

□

The constraint on matrices C, D is convex and the minimization can be solved using semidefinite or second order cone programming. A similar technique is used to obtain the reduced model in this thesis. Note that one can also fix the matrix $\hat{C}$ instead of $\hat{B}$ and optimize then over B and D.

Numerical Complexity. All the variations require Lyapunov or even Riccati equations, to acquire the reduced model. Solving Lyapunov equations is usually performed with the Bartels-Stewart or Hammarling methods, if there is no extra information about the matrices. These methods ignore any sparsity or low-rank property in a Lyapunov equation and have the computational complexity $O(n^3)$, where n is the order of the full system. The reduction and transformation do not add extra order in the complexity. The case with sparse or structured state-space matrices is not considered, since then the computational cost for all the methods (including the one considered in the thesis) is lower.

Krylov methods

The Krylov projection techniques are very powerful and the numerical examples usually show an excellent performance, even without any formal guarantee.

Model Order Reduction (MOR) by Krylov projections. Let $G = (E, A, B, C)$ be an asymptotically stable system with a corresponding state-space realization:

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

with a not necessarily invertible matrix $E \in \mathbb{R}^{n\times n}$, and A, B, C have suitable sizes. Denote the reduced model as $\hat{G} = (\hat{E}, \hat{A}, \hat{B}, \hat{C})$ and $\hat{E} \in \mathbb{R}^{k\times k}$. All the Krylov methods can be reduced to a simple algorithm:

ALGORITHM 2.3

- Calculate matrices $W, V \in \mathbb{R}^{n\times k}$ from the given information of the full model.
- Obtain the reduced model as

$$(\hat{E}, \hat{A}, \hat{B}, \hat{C}) = (W^T EV, W^T AV, W^T B, CV)$$

□

The difference in variations of Krylov methods is the way of obtaining W, V, which are called projectors.

If $E = I$ one of the most natural and simple approaches of choosing W, V is to match the Markov parameters. The projectors are obtained as:

$$V = \begin{pmatrix} B & AB & \dots & A^{k-1}B \end{pmatrix} \in \mathbb{R}^{k\times n}$$

$$\bar{W}^T = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{k-1} \end{pmatrix} \in \mathbb{R}^{k\times n}$$

$$W^T = (\bar{W}^T V)^{-1} \bar{W}^T$$

In this case $CA^iB = \hat{C}\hat{A}^i\hat{B}, \forall i = 1, \dots, 2k$. Basically, the algorithm systematically throws away high-order terms in the representation:

$$G(s) = h_0 + h_1 s^{-1} + h_2 s^{-2} + \cdots + h_n s^{-n}$$

Another intuition may be provided by the fact that h_i are the Markov parameters. Those relate to the Hankel operator through the singular values of the matrix $M[0, \infty]$ defined earlier.

An extension can be made to match the moments in particular frequencies λ_i instead of the Markov parameters. Basically the same

idea, but instead of an analytic Laurent series expansion around infinity, perform the expansion around multiple points. The projectors are defined as:

$$V = ((\lambda_1 E - A)^{-1} B \quad (\lambda_2 E - A)^{-1} B \quad \dots \quad (\lambda_k E - A)^{-1} B) \in \mathbb{R}^{k \times n}$$

$$\bar{W}^T = \begin{pmatrix} C(\lambda_1 E - A)^{-1} \\ C(\lambda_2 E - A)^{-1} \\ \dots \\ C(\lambda_k E - A)^{-1} \end{pmatrix} \in \mathbb{R}^{k \times n}$$

$$W^T = (\bar{W}^T V)^{-1} \bar{W}^T$$

then $G(\lambda_i) = \hat{G}(\lambda_i)$ for all i.

A passivity preserving version of the algorithm is also presented in the literature (see, [Odabasioglu *et al.*, 1998] or [Antoulas, 2009]). Basically the passivity comes for free, by choosing the frequency points appropriately.

Parameterized Model Order Reduction (PMOR) by Krylov projections [Daniel and White, 2003; Daniel* et al., *2004] An important extension of the method is reduction of parameterized models. First rewrite an LTI system as a parameter-dependent system:

$$E(\theta)x = Bu$$
$$y = Cy,$$

where state-space matrices depend on a single-parameter θ in an arbitrary manner. Some smoothness is, however, required for a reasonable approximation. θ can have a meaning of the complex Laplace transform variable. As in the non-parameterized case perform a Laurent series expansion up to a certain order k_1, this time expand the matrix E depending on θ :

$$E(\theta) = E_0 + E_1\theta + E_2\theta^2 + E_3\theta^3 + \dots + E_{k_1}\theta^{k_1}$$

Introduce instead of a one parameter problem a k_1 parameter problem

by assuming $\theta_m = \theta^m$, which yields a representation:

$$\left\{ \begin{pmatrix} E_0 & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix} - \theta \begin{pmatrix} -E_1 & -E_2 & \dots \\ I & & \\ & I & \\ & & \ddots \end{pmatrix} \right\} \times \begin{pmatrix} x \\ x_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} B \\ 0 \\ \vdots \end{pmatrix} u,$$

where x_i are fictitious states defined by: $x_1 = \theta x$, $x_2 = \theta x_1$ and so forth. Now apply general Krylov techniques on this system and obtain the projector V, giving the reduced model as $V^T E(\theta) V$. The case with more than one parameter is done in a similar fashion.

Numerical Complexity. The most computationally heavy part in Krylov subspace methods is computing an orthogonal basis of Krylov subspace. Similar problems were encountered in linear algebra and methods were developed, such as Arnoldi process or Padé via Lanczos method. Basically one computes the projectors W, V without an explicit calculation of moments or Markov parameters (see, [Freund, 1995; Freund, 2003a] for details). It requires $O(n^2 k)$ floating point operations, where $k << n$ are the dimensions of Krylov subspace. Given a big variety of Arnoldi and Lanczos algorithms one may lower the cost at the expense of lowering the accuracy, and vice versa. Enforcing extra properties or accuracy may also increase the cost.

2.5 A Quasi-Convex Approach to Model Reduction

The aim of this section is to set up the model reduction problem as a convex minimization procedure. First, formulate the problem as:

$$\min_{\hat{G} \in \mathcal{H}_\infty} \|G - \hat{G}\|_\infty,$$

where $\hat{G}$ has the McMillan degree $\deg(\hat{G})$ lower or equal to k and $\deg(G) > k$. This problem is not convex and to authors best knowledge there exists no optimal solution. A suboptimal solution that is considered in this thesis, was suggested in [Sou *et al.*, 2005].

Consider a discrete-time transfer function G. Similar techniques can be applied for continuous time transfer functions, as it was done in [Sandberg and Murray, 2007]. However, it is more numerically robust to apply the method in discrete time due to a finite frequency interval. Parameterize the reduced model as:

$$\hat{G} = \frac{p}{q} \quad \text{where} \quad p = \sum_{i=0}^{k} p_i z^{-i} \quad q = \sum_{i=0}^{k} q_i z^{-i},$$

q - a minimum phase transfer function

The minimum phase condition means that all the poles and zeros are inside the unit circle $\mathbb{D} = \{z \| z| < 1\}$. Minimum phase here is also equivalent to stability of p/q. Matching will be performed on the unit circle, thus $z = e^{j\omega}$, where $\omega \in [0, \pi]$. With this parameterization p_i, q_i are the decision variables. The infinity norm is reformulated as a minimization with an infinite number of constraints (for every frequency ω):

$$\min \gamma \quad \text{subject to} \quad \left| G(\omega) - \frac{p(\omega)}{q(\omega)} \right| < \gamma \quad \forall \omega,$$

Here $G(\omega)$ denotes the frequency response of G. A similar notation is made for q and p. Multiplication of both sides of the inequality with $|q(\omega)|^2 = q(\omega)q^{\sim}(\omega)$ yields:

$$\min \gamma \quad \text{subject to} \quad \left| G(\omega)|q(\omega)|^2 - p(\omega)q(\omega)^{\sim} \right| < \gamma |q(\omega)|^2 \quad \forall \omega$$

Using the Schur complement properties, obtain an equivalent formulation:

$$\begin{array}{l} \min \gamma \quad \text{subject to} \\ \begin{pmatrix} \gamma |q(\omega)|^2 & G(\omega)|q(\omega)|^2 - p(\omega)q^{\sim}(\omega) \\ * & \gamma |q(\omega)|^2 \end{pmatrix} > 0 \quad \forall \omega \end{array}$$

where the asterisk denotes Hermitian transpose of the upper right element. This program is still not convex in variables p, q. Denote $a =$

$|q|^2$ and relax the structure of the polynomial $pq^{\sim}$, i.e. replace $pq^{\sim}$ with $b = \sum_{i=-k}^{k} b_i z^{-i}$. Now the program becomes convex in the variables a, b for any given value of γ, i.e. quasiconvex:

$$\min \gamma \quad \text{subject to} \quad \begin{pmatrix} \gamma a(\omega) & G(\omega)a(\omega) - b(\omega) \\ * & \gamma a(\omega) \end{pmatrix} > 0 \quad \forall \omega, \qquad (2.1)$$

Note, if the polynomial a is positive then the stability of the reduced model is guaranteed. Positivity is also a convex constraint.

The relaxation of structure needs motivation. Note that b/a as a transfer function has both stable and antistable parts. Consider a decomposition:

$$\frac{b}{a} = \frac{p}{q} + \frac{r}{q^{\nabla}} \qquad (2.2)$$

where r is a polynomial of order less than k. Recall, that the Hankel norm is defined as $\min \|F - F_-\|_\infty = \|F\|_H$, where minimization is performed over all possible antistable terms F_-. Moreover

$$\min_{\Delta_-, p, q} \left\| G - \frac{p}{q} - \Delta_- \right\|_\infty = \min_{p,q} \left\| G - \frac{p}{q} \right\|_H$$

is in fact Hankel model reduction, where $\Delta_- \in \mathcal{H}_\infty^-$. One can consider, that in this method Δ_- is restricted to r/q^{∇}. Thus the obtained optimization problem is a restricted version of Hankel model reduction:

$$\min_{q,p,r} \gamma \quad \text{subject to} \quad \left\| G - \frac{p}{q} - \frac{r}{q^{\nabla}} \right\|_\infty < \gamma$$

There is a one-to-one correspondence between a, b and p, q, r polynomials using the equation 2.2 This fact will be shown for the multivariable case. b/a has both stable and antistable parts. Hence $\hat{G}$ can be found simply as a stable – antistable decomposition of b/a. Obviously there is a question of a feedthrough term: what term should one take in order to obtain the best approximation? Moreover, this reduced model is optimal in a Hankel-type norm, but in the infinity norm it is not. In this case one obtains only the denominator from the relaxed problem (e.g. matrices A and B in state-space representation). The numerator is obtained by means of optimization in $\mathcal{H}_\infty(\jmath\mathbb{R})$ norm. Hence first solve

(2.1) and obtain q_* from $a_* = q_* q_*^{\sim}$, which is a spectral factorization problem. Then given q_*, the numerator p_* is computed as:

$$\min_p \left\| G - \frac{p}{q_*} \right\|_\infty . \tag{2.3}$$

This program does not add to much numerical complexity and provides an optimal numerator given the denominator. Moreover, in [Megretski, 2006] it was shown using the classical Nehari's theorem that if the error $\|G - b_*/a_*\|_\infty$ is small then the error $\|G - p_*/q_*\|_\infty$ must be small as well:

THEOREM 2.5—[MEGRETSKI, 2006]
If γ_*, p_*, q_* are obtained from (2.1, 2.3) then:

$$\sigma_{k+1}(G) \le \gamma_* \le \min_{p,q} \left\| G - \frac{p}{q} \right\|_\infty \le \left\| G - \frac{p_*}{q_*} \right\|_\infty \le (k+1)\gamma_*.$$

Here $\min_{p,q} \left\| G - \frac{p}{q} \right\|_\infty$ is the optimal error of reduction, and $\sigma_{k+1}(G)$ is the $k+1$-th largest Hankel singular value of G. □

The theorem provides not only a relaxation gap bound, but also a bound on the ratio of upper and lower error bounds. This ratio is bounded by the McMillan degree of the reduced model. A similar bound for Hankel model reduction or balanced truncation does not exist.

Implementation

Even though the obtained problem is (quasi-)convex, it has an infinite number of constraints. It can be dealt with by imposing the constraints in a finite number of frequency points. This approach will be extended to a multivariable case, therefore a more profound discussion in Section 3.8 is made.

The implemented program can be represented as:

$$\min \gamma \quad \text{subject to} \quad \begin{pmatrix} \gamma a(\omega) & G(\omega)a(\omega) - b(\omega) \\ * & \gamma a(\omega) \end{pmatrix} > 0 \quad \omega \in \{\omega_i\}_{i=1}^N,$$

However, there is a problem with stability. Now positivity of the polynomial a, is no longer implied, since the constraints are active only

in a finite number of points, and stability may not be achieved. There are number of ways to impose positivity constraint for all frequencies, which will be discussed later on.

Parameterized Model Order Reduction

Consider a parameterized model $G(\omega_0, \theta)$, where $\theta \in \Theta$ is a compact set in $\mathbb{R}^n$ and ω_0 is a frequency variable corresponding to a Z-transformation variable z. Re-parameterize every entry of the vector θ as:

$$\theta_i = \left(\overline{\theta}_i + \underline{\theta}_i + \left(\overline{\theta}_i - \underline{\theta}_i\right)\cos(\omega_i)\right)/2 \quad \omega_i \in [0, \pi] \quad \forall i = 1, \ldots, n$$

where $\overline{\theta}_i, \underline{\theta}_i$ are the maximum and the minimum values of the parameter θ_i, and ω_i is a new variable. Now the frequency variables ω_i are considered, which in many aspects are much easier to handle. Denote $\boldsymbol{\omega}_\theta = \{\omega_1, \ldots, \omega_n\}$ and $\boldsymbol{\omega} = \{\omega_0, \boldsymbol{\omega}_\theta\}$. Formulate the problem of model reduction as:

$$\min_{\hat{G}} \max_{\boldsymbol{\omega}_\theta} \left\| G(\omega_0, \boldsymbol{\omega}_\theta) - \hat{G}(\omega_0, \boldsymbol{\omega}_\theta) \right\|_\infty ,$$

This problem was solved in [Sou *et al.*, 2005]. For every $\theta \in \Theta$ one can obtain a similar program as in the non-parameterized case. Basically the coefficients of polynomials now depend on θ. However, the implementation on a frequency grid, as in the non-parameterized case, brings the problem with stability. The positivity condition of a can not be imposed in a convex manner, if the coefficients of a depend on the parameter freely. Thus the following polynomial parameterization is used:

$$b = \sum_{i=0}^{k} b(\boldsymbol{\omega}_\theta) e^{-\jmath\omega_0 i} \qquad a = \sum_{i=0}^{k} \sum_{\mathbf{j}=-\mathbf{l}}^{\mathbf{l}} a_{i\mathbf{j}} e^{-\jmath\boldsymbol{\omega}_\theta \boldsymbol{j}} e^{-\jmath\omega_0 i}$$

where $\boldsymbol{j} = \{j_1, \ldots, j_n\}$, $\mathbf{l} = \{l_1, \ldots, l_n\}$ are multi-indexes, a notation $e^{-\jmath\boldsymbol{\omega}_\theta \boldsymbol{j}}$ stands for $e^{-\jmath\omega_1 j_1} \cdot \ldots \cdot e^{-\jmath\omega_n j_n}$ and a is called a (pseudo-) polynomial. Instead of the condition "$a(\omega_0, \boldsymbol{\omega}_\theta)$ is positive", the condition "$a(\omega_0, \boldsymbol{\omega}_\theta)$ is a sum of squares" is used. It optimizes the size of constraints.

Formulate the relaxed program, which can be solved in a polynomial time.

$$\min_{a,b,} \gamma \quad \text{subject to} \begin{cases} \begin{pmatrix} \gamma a(\boldsymbol{\omega}) & G(\boldsymbol{\omega})a(\boldsymbol{\omega}) - b(\boldsymbol{\omega}) \\ * & \gamma a(\boldsymbol{\omega}) \end{pmatrix} > 0 \\ \forall \boldsymbol{\omega} \in \{\boldsymbol{\omega}^i\}_{i=0}^N \\ a \quad \text{is a sum-of-squares polynomial} \end{cases} \tag{2.4}$$

In the non-parameterized case the next step would be to solve a spectral factorization problem $a = qq^{\sim}$. In multivariate case, the spectral factorization problem of a positive polynomial $a(\omega_1, \omega_2)$ is known not to have a solution in finite degree polynomials. Instead polynomials a, b are calculated for every instance of a parameter $\hat{\theta}$. Then the reduced model $\hat{G}(\cdot, \hat{\theta})$ is obtained as in the non-parameterized case for every $\hat{\theta}$. Basically one creates a look-up table of $\hat{G}$ for every value of $\hat{\theta}$. This approach provides some difficulties, which will be addressed in Chapter 3.

Frequency Weighted Model Reduction

A frequency weighted model reduction problem can be formulated as:

$$\min_{\hat{G} \in \mathcal{H}_\infty} \left\| W_o \left(G - \hat{G} \right) W_i \right\|_\infty ,$$

The weights W_i, W_o are usually considered proper, but not strictly proper, and minimum phase. Both conditions are common in the model reduction framework. In $\mathcal{H}_\infty$ norm both assumptions can be dealt with. Consider a weight W with poles outside the unit circle. Since the problem is formulated in the $\mathcal{H}_\infty$ norm, one can always replace W with $\hat{W}$ such that $W W^{\sim} = \hat{W} \hat{W}^{\sim}$, where $\hat{W}$ is minimum phase. If W is strictly proper, then replace it with $W z^l$ with suitable choice of a positive scalar l.

In the single variable case both weights W_i, W_o can be replaced with only one w_1/w_2, since then the order of multiplication by the weights does not matter:

$$\min_{\hat{G} \in \mathcal{H}_\infty} \left\| \frac{w_1}{w_2} \left(G - \hat{G} \right) \right\|_\infty$$

A frequency weighted extension was proposed in [Sandberg and Murray, 2007]. Following the scheme of the non-weighted solution, obtain a program:

$$\min \gamma \quad \text{subject to} \quad \begin{pmatrix} \gamma a(\omega) & \frac{w_1}{w_2}\left(G(\omega)a(\omega) - b(\omega)\right) \\ * & \gamma a(\omega) \end{pmatrix} > 0 \quad \forall \omega \tag{2.5}$$

Again the denominator q_* is obtained from $a_* = q_* q_*^{\sim}$. The numerator p_* is a minimizer to the problem:

$$\min_p \left\| \frac{w_1}{w_2}\left(G - \frac{p}{q_*}\right) \right\|_\infty \tag{2.6}$$

An error bound was provided by the next result:

THEOREM 2.6—[SANDBERG AND MURRAY, 2007]
Assume q_*, p_*, γ_* are obtained from solving the program (2.5, 2.6), then the following statements are true:

1. Assume $\hat{G}_1 = p/q$ then:

$$\left\| \frac{w_1}{w_2}(G - \hat{G}_1) \right\|_\infty \leq \gamma \left(1 + 2k \left\| \frac{w_1}{w_2} \right\| \left\| \frac{w_2}{w_1} \right\| \right)$$

2. Define $\hat{G}_2 = \dfrac{p}{q} + \dfrac{r_1}{w_1}$ where:

$$\frac{w_1 r}{w_2 q^{\triangledown}} = \frac{r_1}{w_2} + \frac{r_2}{q^{\triangledown}}$$

   is a stable/ antistable decomposition then:

$$\left\| \frac{w_1}{w_2}\left(G - \hat{G}_2\right) \right\|_\infty < \gamma\,(1 + 2k)$$

3. $\inf \| \frac{w_1}{w_2}\left(G - \hat{G}_1\right) \|_\infty \geq \gamma \geq \sigma_{k+1}(G)$, where $\sigma_{k+1}(G)$ is a $k+1$-th largest Hankel singular value.

□

The frequency weighted extension does not add any special problems in the implementation.

A Multivariable Extension

A heuristic algorithm to extend this approach to the multivariable case was proposed in [Sandberg and Murray, 2007]. Consider a problem:

$$\min_{\hat{G}} \|G - \hat{G}\|_\infty$$

It is assumed that the poles are fixed from the start. Otherwise given a transfer function $G(s) \in H_\infty^{l\times m}(\jmath\mathbb{R})$, the poles can be determined by running the quasi-convex reduction algorithm on every entry $G_{ij}(s)$ of $G(s)$. Given the poles z_j, parameterize $\hat{G}$ as:

$$\hat{G}(s) = \hat{G}_0 + \sum_{j=1}^{k} \frac{1}{s - p_j}\hat{G}_j,$$

where $\hat{G}_j \in \mathbb{R}^{l\times m}$ and $\hat{G}_i^* = \hat{G}_j$, when $p_i = p_j^*$, where $\cdot^*$ denotes a Hermitian transpose. The McMillan degree can be calculated as $\deg(\hat{G}) = \sum_{j=1}^{k} \operatorname{rank}(\hat{G}_j)$. Minimizing the rank or restricting it to a number is a non-convex problem. However, there exist simple and effective heuristics (see [Fazel *et al.*, 2001]). The main idea of heuristics is substituting the rank with a nuclear norm of a matrix:

$$\|\hat{G}_j\|_1 = \sum_{i=1}^{k} \sigma_i(\hat{G}_j),$$

The nuclear norm is a convex envelope for the rank function, thus justifying the method. A convex formulation of the algorithm is:

$$\begin{aligned} &\text{minimize} && \sum_{j=0}^{k} \|\hat{G}_j\|_1 \\ &\text{subject to} && \|G - \hat{G}\|_\infty < \gamma \\ & && \hat{G}_l = \hat{G}_i^*, \quad \text{for} \quad p_l = \overline{p}_i. \end{aligned}$$

And a more convenient semidefinite representation:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j=0}^{k} \operatorname{trace}(Y_j) + \operatorname{trace}(Z_j) \\
\text{subject to} \quad & \begin{pmatrix} Y_j & \hat{G}_j \\ \hat{G}_j^* & Z_j \end{pmatrix} > 0 \quad \forall j = 1, \ldots, k \\
& \begin{pmatrix} \gamma I & G_{\omega_i} - \hat{G}_{\omega_i} \\ (G_{\omega_i} - \hat{G}_{\omega_i})^* & \gamma I \end{pmatrix} > 0 \\
& i = 1, \ldots, N \\
& \hat{G}_l = \hat{G}_i^*, \quad \text{for} \quad p_l = \overline{p}_i.
\end{aligned}
$$

For the frequency weighted extension similar techniques may be applied to the following minimization problem:

$$
\min_{\hat{G}_j} \sum_{j=1}^{k} \|\hat{G}_j\|_1 \quad \text{subject to} \quad \|W_o(G - \hat{G})W_i\|_\infty < \gamma
$$

Note that the accuracy of the approximation is fixed in advance, which may sometimes be convenient. The upper bound on the McMillan degree of $\hat{G}$ is $k \cdot \min\{l, m\}$. One can always obtain less states, however, the accuracy usually is worse. Therefore there will be a trade-off between accuracy and McMillan degree.

One can see that the method has some drawbacks. Firstly, there is no relaxation gap bound. After obtaining the poles there is no way of predicting the accuracy of the approximation. It does not look like a crucial problem since the accuracy is fixed in advance, then the degree of the system, however, will be unknown. To authors best knowledge, there is no systematic way of solving optimally the problem of trade-off between accuracy and degree. Secondly, the problem of obtaining poles is crucial, since there is no guarantee that the chosen poles are optimal in any way. It may in fact be the case that for obtaining an approximation of lower order one needs to shift some of the poles. With this approach one has to run the method once more. And finally, so far an extension for the parameterized model reduction seems unlikely.

2.6 Conclusion

The methods described in this section have many advantages and but also drawbacks. The SVD methods have a good accuracy, guarantee stability and error bounds, but they are computationally heavy. Krylov methods are computationally more efficient than SVD methods, however, they do not, in general, guarantee stability and a good $\mathcal{H}_\infty$ performance. One would like to obtain a method that has the advantages of both approaches. An attempt was made and the quasiconvex optimization approach was developed. The natural continuation of this work is a multivariable extension.

3

Multivariable Optimization-Based Model Reduction

This chapter is based on [Sootla *et al.*, 2009]. In this chapter multivariable or multi-input-multi-output (MIMO) extensions of the single-input-single-output (SISO) model reduction method are discussed. The SISO method was proposed in [Sou *et al.*, 2005]. The algorithm is based on matching of the frequency samples in the original and the reduced models. For large scale systems the most computationally heavy part is calculating the frequency samples. At the same time the algorithm guarantees the stability of the system, and if required the passivity. The multivariable extensions have all the properties of the original method. A relaxation gap similar to one from [Megretski, 2006] is also obtained.

3.1 Introduction

Model reduction problems have received considerable attention in the past and several approaches have been developed. Balanced truncation and Hankel optimal model reduction are recognized in the control literature, (see, for example [Zhou *et al.*, 1996; Obinata and Anderson, 2001]), due to well-developed theory and *a priori* error bounds. They are called singular value decomposition (SVD) methods. Both methods

rely on the solution of Lyapunov equations whose sizes are determined by the order of the original models. Therefore reducing large scale models becomes a hard task, due to a heavy calculations associated with solving Lyapunov equations. In addition, imposing extra constraints on a reduced model is a problem. For large scale systems Krylov-subspace based methods (see, [Freund, 2003b]) can be used. A big advantage of these methods is their applicability on descriptor systems and low computational cost comparing to SVD methods. However, there are open issues regarding the stability of the reduced model. The main aspects of the methods re presented in Section 2.4.

In [Sou *et al.*, 2005] a new approach to model reduction was developed. A relaxation was proposed that allows a formulation of model reduction as a convex optimization problem. The method can be applied to exact or frequency-sampled models. The computational cost for large-scale systems, when frequency samples as input are used, is comparable to Krylov subspace methods. At the same time, the algorithm matches the frequency samples, providing a bound on the overall $(\mathcal{H}_\infty)$ performance. The program minimizes the error in Hankel-type norm, thus the methods performance should be close to one of Hankel model reduction. Roughly speaking, the proposed method is almost as good as Krylov subspace methods in terms of computational cost and almost as good as SVD methods in terms of $\mathcal{H}_\infty$ error. In [Megretski, 2006], a guaranteed sub-optimality bound is obtained. In [Sandberg and Murray, 2007] a multivariable extension was proposed, which uses rank-minimization heuristics from [Fazel *et al.*, 2001].

3.2 Preliminaries

A basic reduction problem can be formulated as:

$$\min_{\hat{G}\in\mathcal{H}_\infty} \|G-\hat{G}\|_\infty \tag{3.1}$$

where $G \in \mathcal{H}_\infty$ is a $m_1 \times m_2$ rational transfer function with $m_1, m_2 \geq 1$, a high or infinite McMillan degree $\deg(G)$ and $\deg(\hat{G})$ is fixed and less than $\deg(G)$. This problem is not convex and the goal is to obtain a suboptimal convex formulation. Here the case $m_1 = m_2 = m$ is con-

sidered. It is done for simplicity, however the same techniques can be applied for a general ($m_1 \neq m_2$) case.

Sketch of the Solution

Assume $m > 1$. The reduced model is assumed to have a form $\hat{G} = PQ^{-1}$, where Q, P are matrix polynomials in variable z^{-1} of degree k :

$$P = \sum_{i=0}^{k} P_i z^{-i} \quad Q = \sum_{i=0}^{k} Q_i z^{-i}$$

The inverse of polynomial Q should provide a stable transfer function. Hence, Q is a minimum phase transfer function (all the poles and the zeros are inside the unit disc $\mathbb{D} = \{z \Big| |z| \leq 1\}$) and the feed through term (Q_0) has to be invertible.

As the matching is performed on the unit circle $\partial\mathbb{D}$ denote $G(\omega)$ the frequency response of G. A similar notation is used for all transfer functions. The same scheme as in the single variable case is used. Consider the problem:

$$\min \gamma \quad \text{subject to} \quad (G(\omega) - P(\omega)Q^{-1}(\omega))(G(\omega) - P(\omega)Q^{-1}(\omega))^{\sim} < \gamma^2 I$$

Rewrite the LMI in the minimization problem with a notation $A = QQ^{\sim}$ as:

$$(G(\omega)A(\omega) - P(\omega)Q^{\sim}(\omega))A^{-2}(\omega)(G(\omega)A(\omega) - P(\omega)Q^{\sim}(\omega))^{\sim} < \gamma^2 I$$

Now introduce B as a substitute to $PQ^{\sim}$, (i.e. relax the structure of the last polynomial). A, B are parameterized as:

$$A = \sum_{i=-k}^{k} A_i z^i \quad B = \sum_{i=-k}^{k} B_i z^i$$

Some structural restrictions are placed on B_i and A_i, more details about it in section 3.3. Thus a constraint is obtained:

$$(G(\omega)A(\omega) - B(\omega))A^{-2}(\omega)(G(\omega)A(\omega) - B(\omega))^{\sim} < \gamma^2 I \qquad (3.2)$$

This change of variables will be referred to as the antistable relaxation. Unlike the single variable case, here a convex LMI condition can not be derived directly from (3.2). Our goal is to obtain an LMI similar to:

$$\begin{pmatrix} \gamma A(\omega) & G(\omega)A(\omega) - B(\omega) \\ * & \gamma A(\omega) \end{pmatrix} > 0 \tag{3.3}$$

Note, that (3.3) is convex in the variables A, B, and is always equivalent to

$$(G(\omega)A(\omega) - B(\omega))^{\sim} A(\omega)^{-1} (G(\omega)A(\omega) - B(\omega)) < \gamma^2 A(\omega) \tag{3.4}$$

due to the Schur complement properties. The difference of (3.4) to (3.2) is the right-hand side polynomial A which can not carried to the left-hand side in order to obtain (3.2). Assume, that $A = aI$, where a is a scalar polynomial, then $\|G - Ba^{-1}\|_\infty < \gamma$ is equivalent to (3.4) and (3.3). This case will be referred to, as the trivial MIMO extension or the scalar denominator case.

The structure $A = aI$ may not be the best solution for a parameterization of reduced system. We still want A to be an arbitrary positive definite matrix. Consider the convex constraints:

$$\begin{pmatrix} \gamma f(\omega) I & G(\omega)A(\omega) - B(\omega) \\ * & \gamma A(\omega) \end{pmatrix} > 0, \tag{3.5}$$

$$0 < f(\omega) I \le A(\omega) \quad \forall \omega \in [0, \pi] \tag{3.6}$$

where $f(\cdot)$ is a bounded, scalar function and it is obtained by means of optimization. It will be shown that from (3.5) and (3.6) the inequality $\|G - BA^{-1}\|_\infty < \gamma$ follows. To distinguish the two relaxations, this one is called a multivariable relaxation. The minimization program will be referred to as a matrix denominator case. Thus a (quasi-)convex program have been obtained. γ is minimized over B, A and f subject to (3.5, 3.6). This procedure is used only to obtain the polynomial A_*. After solving the spectral factorization problem for $A_* = Q_* Q_*^{\sim}$, one can proceed with computing the numerator. It is obtained analogously to the single variable case:

$$\min_P \|G - PQ_*^{-1}\|_\infty$$

3.3 Parameterization

In this section a detailed description of the parameterization is provided. Assume $\hat{G} = PQ^{-1} \in \mathcal{H}_\infty$ where

$$Q = \sum_{i=0}^{k} Q_i z^{-i}, \quad P = \sum_{i=0}^{k} P_i z^{-i}$$

are matrix polynomials in z^{-1}, with Q minimum phase transfer function. and $Q_i, P_i \in \mathbb{R}^{m\times m}\ \forall i$.

In Section 3.2 a relaxation have been introduced. The transfer function PQ^{-1} had only a stable part, whereas BA^{-1} has both stable and antistable parts and

$$A = \sum_{i=-k}^{k} A_i z^i, \quad B = \sum_{i=-k}^{k} B_i z^i.$$

The motivation for this relaxation is the same as in the single variable case described in section 2.5. Here also

$$\min_{\Delta_-,P,Q} \|G - PQ^{-1} - \Delta_-\|_\infty = \min_{P,Q} \|G - PQ^{-1}\|_H$$

is Hankel model reduction, where $\Delta_- \in \mathcal{H}_\infty^-$. Again in the proposed method Δ_- is restricted to $RT^{-\triangledown}$. Where R, T are matrix polynomials of order less than k and equal to k correspondingly, T is minimum phase. $T^\triangledown$ denotes a reciprocal polynomial to T, namely $T^\triangledown(z) = T^\sim(z)/z^k$, where k is the order of T. To follow the SISO scheme T is chosen such that $T^\sim T = QQ^\sim$. Thus an optimization problem is obtained:

$$\min_{Q,T,P,R} \gamma \quad \text{subject to} \quad \|G - PQ^{-1} - RT^{-\triangledown}\|_\infty < \gamma$$

Now, given the polynomials P, Q, R and T a parameterization of A, B has to be chosen which satisfies:

$$A = QQ^\sim = T^\sim T \tag{3.7}$$

$$B = (PQ^{-1} + RT^{-\triangledown})A \tag{3.8}$$

For the general form A and B polynomials the following lemma holds:

LEMMA 3.1
There exists a one to one correspondence between Q, T, P, R and A, B. □

Proof. If Q, T, P, R are known, then one can always calculate A, B. On the other hand, with known A, B (3.7) is a spectral factorization problem. Hence the polynomials Q, T are uniquely obtained from A. Since $Q, T^{\sim}$ are coprime, (3.8) can be solved uniquely for given B_i (e.g. using Sylvester-type equations). ■

The given parameterization is too general. Some restrictions on A and B have to enforced. First make a generalization from the single-variable case. Parameterize A, as $A > 0, \quad A_i = A_{-i} = A_i^T, \quad A_0 > 0$ in this case A is a real valued function for any matrices and imposing such constraints guarantees the existence of solution to the spectral factorization problem (see, [Sayed and Kailath, 2001]). Instead of the last condition, $A_0 > I$ will be used in order to normalize the calculations. Together with conditions $\overline{B}_i = \overline{B}_{-i}, \overline{C}_i = \overline{C}_{-i}$ a trigonometric polynomial parameterization is obtained:

$$A = \sum_{i=0}^{k} A_i (z^i + z^{-i})$$

$$B = \sum_{i=0}^{k} \overline{B}_i (z^i + z^{-i}) + \sum_{i=0}^{k-1} \overline{C}_i (z^i - z^{-i})$$

Now the polynomials depend on cosine and sine functions which appears to be numerically robust. However, one may consider relaxing a structure of A to:

$$A = \sum_{i=1}^{k} A_i z^i + A_0 + \sum_{i=1}^{k} A_i^T z^{-i}, \quad A_0 > I$$

It creates more variables and thus more freedom. One also may bound $A_0 < cI$, where c is a constant. It may be necessary to limit the growth of A. Extra freedom and extra restrictions may create numerical problems. The choice of parameterization is left to the designer, however all the theoretical results are valid as long as the parameterization creates a convex problem.

3.4 Multivariable Extensions

Scalar Denominator Case.

In the scalar denominator case there is no need to perform the multivariable relaxation, since the denominator is scalar. Thus all the ideas are quite similar to the single variable case. Assume $\hat{G} = P/q$, and:

$$P = \sum_{i=0}^{k} P_i z^{-i}, \quad q = \sum_{i=0}^{k} q_i z^{-i},$$
$$B = \sum_{i=-k}^{k} B_i z^{i} \quad a = \sum_{i=0}^{k} a_i (z^{i} + z^{-i}).$$

where q_i, a_i are scalars, P_i, B_i are matrices of suitable size. Now the algorithm can be summarized:

ALGORITHM 3.1—SCALAR DENOMINATOR ALGORITHM

1. Solve a (quasi-)convex optimization problem

$$\min \gamma \quad \text{subject to} \quad a > 0 \tag{3.9}$$
$$\begin{pmatrix} \gamma a(\omega) I & G(\omega)a(\omega) - B(\omega) \\ * & \gamma a(\omega) I \end{pmatrix} > 0 \quad \forall \omega \tag{3.10}$$

2. Obtain q_* from $a = q_* q_*^{\sim}$. Solving the spectral factorization problem in this case is equivalent to computing the stable zeros of a_*.

3. Calculate the numerator P_* given the denominator q_* as:

$$\min_P \|G - P/q_*\|_\infty \tag{3.11}$$

□

The reduced model in this case is stable and the McMillan degree is less or equal to km ($\deg(\hat{G}) < km$ if there is a pole-zero cancellation).

Matrix Denominator Case

For the matrix denominator case consider $\hat{G} = PQ^{-1}$ with:

$$P = \sum_{i=0}^{k} P_i z^{-i}, \quad Q = \sum_{i=0}^{k} Q_i z^{-i},$$
$$B = \sum_{i=-k}^{k} B_i z^{i} \quad A = \sum_{i=-k}^{k} A_i z^{i}.$$

Now Q, A are matrix polynomials, as well, as P, B.

First the algorithm is presented and then it will be shown that the algorithm provides a low-order approximation with desired properties.

Algorithm 3.2—Matrix Denominator Case

1. Consider the (quasi-)convex program:

$$\min_{A,B} \gamma \quad \text{subject to} \quad A > 0 \quad f(\omega) I \leq A(\omega), \tag{3.12}$$

$$\begin{pmatrix} \gamma f(\omega) I & G(\omega)A(\omega) - B(\omega) \\ * & \gamma A(\omega) \end{pmatrix} > 0, \quad \forall \omega \in [0, \pi] \tag{3.13}$$

   where $f(\cdot) \in \mathcal{L}_\infty[0, \pi]$ is a scalar function, and it is obtained by means of optimization.

2. The polynomial $Q_*(z)$ is obtained from (3.12) by solving the spectral factorization problem for $A_* = Q_* Q_*^{\sim}$.

3. Proceed with computing the numerator P_*, which is obtained analogously to the previous cases:

$$\min_{P} \| G - PQ_*^{-1} \|_\infty \tag{3.14}$$

□

The step 2 in Algorithm 3.2 has a unique solution if and only if $A > 0$ and $A_0 > 0$ (see, for example, [Sayed and Kailath, 2001]). The scalar and the matrix denominator cases are similar in terms of the properties of the reduced models. Since A_* is positive and Q_* is a spectral

factor, the direct feedtrough term of Q_* is invertible thus implying the invertibility of the transfer function Q_*. One can regard $\{z^k P_*, z^k Q_*\}$ as the matrix fraction description of a transfer function. This fact is used to calculate the McMillan degree of $\hat{G} = P_* Q_*^{-1}$.

LEMMA 3.2—[KAILATH, 1980]
If $\{\tilde{P}, \tilde{Q}\}$ is an irreducible matrix fraction description of a transfer function and the highest-column-degree coefficient matrix of $\tilde{Q}$ has full rank then the McMillan degree of $\tilde{P}\tilde{Q}^{-1}$ is $\deg \tilde{P}\tilde{Q}^{-1} = \deg(\tilde{Q}) = km$. □

The conditions seem to be complicated, but in this case it is easy to show that they are fulfilled. In the matrix fraction description $z^k Q_*$ the term Q_0 consists of highest degree coefficients. Since the feedthrough term Q_0 is invertible, the rank is full. Irreducibility of fractions is not necessarily implied and $\deg P_* Q_*^{-1} = \deg \tilde{P}\tilde{Q}^{-1}$ can be smaller than km for some P_*. One can interpret this fact as a pole-zero cancellation. Though it is highly unlikely, since P_* and Q are obtained using optimization procedures.

The next lemma will be used later on for deriving error bounds. It justifies in some way the multivariable relaxation, since a bound on $\|G - BA^{-1}\|_\infty$ similar to one in the scalar denominator and the single-variable cases is achieved.

LEMMA 3.3
If γ, A, B, f satisfy the conditions in (3.13), and $fI \leq A$, for all frequencies, then $\|G - BA^{-1}\|_\infty \leq \gamma$ □

Proof. Rewrite the LMI condition in (3.13) for A, B, f, γ using the Schur complement:

$$\gamma^2 A^2 \geq \gamma^2 A f > (GA - B)^{\sim}(GA - B) \Rightarrow$$
$$\Rightarrow \gamma > \overline{\sigma}(G - BA^{-1})$$

Now the result follows after taking the supremum over all ω of both hand sides of the last inequality. ■

It is clear that the scalar denominator case always provides less accurate approximations than the matrix denominator case, due to

fewer degrees of freedom. However, there is no estimates on how much the matrix denominator case approximation is better. One should also keep in mind extra variables that the matrix denominator brings to the optimization program. For higher order approximations it may be wise to use a scalar denominator due to fewer decision variables.

3.5 Error Bounds

Here the error bounds for the reduced system are estimated. They depend only on γ and the order of the reduced model. That allows an *a priori* guarantee of the relaxation gap. Moreover, the relaxation gap is similar to one in the single-variable case. The bounds are shown only for the matrix denominator case. Error bounds for the scalar denominator case follow as a less general case.

THEOREM 3.1
Assume γ_*, A_*, B_*, P_* and Q_* are obtained from the Algorithm 3.2, then:

1. $\gamma_* \geq \sigma_{km+1}(G)$
2. $\gamma_* \leq \min\limits_{P,Q} \|G - PQ^{-1}\|_\infty \leq \|G - P_*Q_*^{-1}\|_\infty$
3. $\|G - P_*Q_*^{-1}\|_\infty \leq (km+1)\gamma_*$

where $\min\limits_{P,Q} \|G - PQ^{-1}\|_\infty$ is the optimal error of reduction and $\sigma_{km+1}(G)$ is the $(km+1)$-th largest Hankel singular value of G. □

Proof. The proof below is analogous to one in [Megretski, 2006].
1) By Lemma 3.3 the LMI in (3.13) yields:

$$\|G - B_*A_*^{-1}\|_\infty < \gamma_* \tag{3.15}$$

Then apply Lemma 3.1 and obtain a unique stable antistable decomposition of $B_*A_*^{-1}$ as $B_*A_*^{-1} = P_0Q_*^{-1} + R_0T_*^{-\triangledown}$, assuming also that $A_* = Q_*Q_*^{\sim} = T_*^{\sim}T_*$ and $T_*^{\triangledown} = T_*^{\sim}/z^k$, where k is the order of polynomial T_*. Now consider the inequalities:

$$\min_{P,R,Q,T} \|G - PQ^{-1} - RT^{-\triangledown}\|_\infty \leq \gamma \tag{3.16}$$

$$\sigma_{km+1}(G) \leq \min_{P,R,Q,T} \|G - PQ^{-1} - RT^{-\triangledown}\|_\infty \tag{3.17}$$

where $P_0 Q_*^{-1}, R_0 T_*^{-\nabla}$ have the McMillan degree km. The upper bound (3.16) is satisfied by construction and Lemma 3.1. The proposed optimization procedure is less general than Hankel model reduction, since the antistable term is restricted to $R_0 T_*^{-\nabla}$. Thus (3.17) follows as the lower bound of Hankel model reduction (see, for example [Zhou *et al.*, 1996]). Hence the statement is proved.
2) Since $\gamma = \min_{P,Q,R,T} \|G - PQ^{-1} - RT^{-\nabla}\|$, by setting $R = 0$ this minimization becomes restricted. Therefore the first inequality is shown. The second inequality is satisfied by construction.
3) The inequality (3.16) implies that the Hankel norm of $R_0 T_*^{-\nabla}$ is less than γ_*. Since the order of $R_0 T_*^{-\nabla}$ is less than or equal to km, by Lemma 2.1 there exists a real matrix K such that:

$$\|R_0 T_*^{-\nabla} - K\|_\infty < km\gamma_* \tag{3.18}$$

Combining (3.18), (3.16) and the triangle inequality yields:

$$\|G - (P_0 + KQ_*)Q_*^{-1}\|_\infty < (km + 1)\gamma_*$$

Since $\hat{G}$ is obtained by minimizing over the numerator, the result follows. ■

3.6 Passive Model Reduction

In many applications one may require preserving the passivity in the reduced model. There exist many methods that preserve passivity, e.g. PRIMA ([Odabasioglu *et al.*, 1998]), or balancing using Riccati equation as described in Section 2.4. The proposed setup does not require the original model to be passive. One can approximate a stable model with a passive one.

Positive Real Constraint

The positive real constraint on $G(z)$ can be described as

$$\mathrm{Re}\,(G(\omega)) = (G(\omega) + G(\omega)^{\sim})/2 > 0$$

for all $\omega \in [0,\pi]$. For a scalar G, it means that the phase is restricted as $|\arg(G)| < \pi/2$. If the obtained model is close enough to the original then the positive real constraint will be met by the relaxed model BA^{-1}. Since an error bound for the relaxation have been derived one may hope that PQ^{-1} will be positive real, as well. However, it is hard to predict if the positive real restriction, in fact, will be met. The idea of a solution is quite simple: enforce the positive real constraint on the relaxed model BA^{-1} in the program (3.12, 3.13). Here assume that $A = aI$, i.e. consider only the scalar denominator case. A constraint $\mathrm{Re}\,(Ba^{-1}) > 0$, is equivalent to positivity constraints $\mathrm{Re}\,(B) > 0, \quad a > 0$. To guarantee the positive realness of Pq^{-1} add the constraint $Pq_*^{\sim} + P^{\sim}q_* > 0$ to the program (3.14).

ALGORITHM 3.3—POSITIVE REAL MODEL REDUCTION

1. Solve a (quasi-)convex minimization program:

$$\min_{f,a,B} \gamma \quad \text{subject to} \quad \mathrm{Re}\,(B) > 0 \quad a > 0 \tag{3.19}$$

$$\begin{pmatrix} \gamma a(\omega) I & G(\omega)a(\omega) - B(\omega) \\ * & \gamma a(\omega) I \end{pmatrix} > 0 \forall \omega \in [0,\pi] \tag{3.20}$$

2. Calculate q_* from $a_* = q_* q_*^{\sim}$.

3. Obtain the numerator P_* given q_* :

$$\min_{P} \|G - Pq_*^{-1}\|_\infty \quad \text{subject to} \quad Pq_*^{\sim} + P^{\sim}q_* > 0 \tag{3.21}$$

□

In this work, a relaxation gap, that is valid for the SISO case, is also estimated:

THEOREM 3.2

If $a_*, B_*, P_*, q_*, \gamma_*$ are obtained from the Algorithm 3.3 given G then the following inequalities are true:

$$\sigma_{k+1}(G) \le \gamma_* \le \min_{P,q} \|G - Pq^{-1}\|_\infty \le \|G - P_* q_*^{-1}\|_\infty < (2km+1)\gamma_*, \tag{3.22}$$

where $\min_{P,q} \|G - Pq^{-1}\|_\infty$ is the optimal error of reduction and $\sigma_{km+1}(G)$ is the $(km+1)$-th largest Hankel singular value of G. □

Proof. Recall that $B_* a_*^{-1}$ has a stable and an antistable part. Consider P_0 and R_0 such that:

$$B_* a_*^{-1} = P_0 q_*^{-1} + R_0 q_*^{-\nabla} \tag{3.23}$$

$$\|R_0 q_*^{-\nabla}\|_\infty \leq km\gamma_* \tag{3.24}$$

$$\|G - P_0 q_*^{-1}\|_\infty \leq (km+1)\gamma_* \tag{3.25}$$

Inequalities are shown in the proof of the Theorem 3.1. Given the constraint $\mathrm{Re}\,(B_* a_*^{-1}) > 0$ and a bound (3.23) yields:

$$\mathrm{Re}\,(P_0 q_*^{-1}) + \mathrm{Re}\,(R_0 q_*^{-\nabla}) > 0$$

Now (3.24) implies that $\overline{\sigma}\,(\mathrm{Re}\,(R_0 q_*^{-\nabla})) \leq km\gamma_* I$, since a norm of a matrix is always bigger than a norm of its real part. The following assertion follows from the described above inequalities:

$$\left.\begin{array}{l} \mathrm{Re}\,(P_0 q_*^{-1}) + \mathrm{Re}\,(R_0 q_*^{-\nabla}) > 0 \\ \overline{\sigma}\,(\mathrm{Re}\,(R_0 q_*^{-\nabla})) \leq km\gamma_* \\ \overline{\sigma}\,(\mathrm{Re}\,(R_0 q_*^{-\nabla}))\, I - \mathrm{Re}\,(R_0 q_*^{-\nabla}) \geq 0 \end{array}\right\} \Rightarrow \mathrm{Re}\,\left(P_0 q^{-1}\right) + km\gamma_* I > 0$$

And as a result $\mathrm{Re}\,((P_0 + km\gamma_* q_*) q_*^{-1}) > 0$. (3.25) states that:

$$\|G - (P_0 + km\gamma_* q_*) q_*^{-1} - km\gamma_* I\|_\infty \leq (km+1)\gamma_*$$

and finally

$$\|G - (P_0 + km\gamma_* q_*) q_*^{-1}\|_\infty \leq (2km+1)\gamma_*$$

Since P_* is obtained by means of optimization and $(P_0 + km\gamma_* q_*) q_*^{-1}$ is positive real the statement is valid. ■

Bounded Real Constraint

The bounded real condition for the transfer function is described as:

$$\|G(z)\|_\infty < 1 \quad \Leftrightarrow \quad \overline{\sigma}(G(e^{j\omega})) < 1 \quad \forall \omega \in [0, \pi]$$

Following a "rule of thumb" from above impose the bounded real constraint on BA^{-1} in the relaxed program, as well as, on PQ_*^{-1} in the step 3. Let us take a closer look at the condition $\|BA^{-1}\|_\infty < 1$:

$$\|BA^{-1}\|_\infty < 1 \quad \Leftrightarrow \quad B(\omega)A(\omega)^{-1}(B(\omega)A(\omega)^{-1})^\sim < I \quad \forall \omega$$

Here assume that $A = aI$, i.e. consider only the scalar denominator case. Thus obtain the conditions:

$$\begin{pmatrix} a(\omega)I & B(\omega) \\ B(\omega)^\sim & a(\omega)I \end{pmatrix} > 0 \quad \forall \omega$$

This LMI is treated as a positivity constraint.

ALGORITHM 3.4—BOUNDED REAL MODEL REDUCTION

1. Solve a (quasi-)convex minimization program:

$$\min_{f,a,B} \gamma \quad \text{subject to} \quad a > 0 \tag{3.26}$$

$$\begin{pmatrix} \gamma a(\omega)I & G(\omega)a(\omega) - B(\omega) \\ * & \gamma a(\omega)I \end{pmatrix} > 0 \tag{3.27}$$

$$\begin{pmatrix} a(\omega)I & B(\omega) \\ B^\sim(\omega) & a(\omega)I \end{pmatrix} > 0 \quad \forall \omega \in [0, \pi] \tag{3.28}$$

2. Calculate q_* from $a_* = q_* q_*^\sim$.
3. Obtain the numerator P_* given q_* :

$$\min_P \|G - Pq_*^{-1}\|_\infty \quad \text{subject to} \quad \|Pq_*^{-1}\|_\infty < 1 \tag{3.29}$$

□

3.7 Positivity Constraint

Enforcing a positivity constraint for all frequencies in one finite dimensional LMI is a known problem and has a number of solutions. Here solutions are listed that can be used both in the single and multi--variable cases.

Parameterization Using the KYP lemma

Notice that, A polynomial can be rewritten as:

$$A(z) = v\Pi v^{\sim} \tag{3.30}$$

where $v = \begin{pmatrix} I & Iz & \cdots & Iz^k \end{pmatrix}$ and matrix Π is chosen such that it fulfills the equality, for example:

$$\Pi = \begin{pmatrix} A_0/2 & A_1 & \cdots & A_k \\ A_1 & & & \\ \vdots & & \mathbf{0} & \\ A_k & & & \end{pmatrix}$$

The following formulation of the KYP lemma can be applied directly to (3.30).

THEOREM 3.3—[RANTZER, 1996]
Given real-valued matrices A, B, M with $\det(e^{j\omega} I - A) \neq 0$ for $\omega \in \mathbb{R}$ and (A, B) controllable, the following two statements are equivalent:

(i)

$$\begin{pmatrix} (e^{j\omega} I - A)^{-1} B \\ I \end{pmatrix}^{*} M \begin{pmatrix} (e^{j\omega} I - A)^{-1} B \\ I \end{pmatrix} \leq 0 \quad \forall \omega$$

(ii) There exist a matrix $P \in \mathbb{R}^{n \times n}$ such that $P = P^T$ and

$$M + \begin{pmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B \end{pmatrix} \leq 0$$

□

Gram Matrix Parameterization

Here the equation (3.30) is used, as well. But one can derive a condition on the matrix Π explicitly. First define the block trace operator TR as:

$$\mathrm{TR}[A] = \sum_{j=0}^{p-1} A_{jj} \quad A = [A_{jl}]_{j,l=0:p-1}, A_{jl} \in \mathbb{C}^{m\times m}$$

THEOREM 3.4—[DUMITRESCU, 2007]
A trigonometric polynomial A is positive on the unit circle if and only if there exists a matrix Π such that:

$$\mathrm{TR}(\Gamma_i \Pi) = A_i, \quad \Pi > 0 \tag{3.31}$$

where Γ_i is the elementary block Toeplitz matrix with unit matrices on the i-th block diagonal and zeros elsewhere. □

Thus to enforce the positivity constraint $A > 0$ the conditions (3.31) have to be added to the program (3.12, 3.13).

Gram Pair Parameterization

The reader may notice that the described above positivity conditions require LMIs of size proportional to the order of the reduced model. The number of variables associated with the positivity constraint can be decreased by half. In order to do so substitute the condition "A is non-negative" with "A is a sum-of squares". Here only the solution for the single variable case is presented, though the same ideas are valid for the multivariable case.

THEOREM 3.5—[DUMITRESCU, 2007]
The trigonometric polynomial $A(z)$ of degree k is a sum-of-squares, if and only if there exist matrices $R \in \mathbb{R}^{k+1\times k+1}$ and $S \in \mathbb{R}^{k\times k}$ such that

$$A_i = \mathrm{TR}[\Phi_i R] + \mathrm{TR}[\Lambda_i S], \quad R, S \geq 0 \tag{3.32}$$

where $\Phi_i \in \mathbb{R}^{k+1\times k+1}$ and $\Lambda_i \in \mathbb{R}^{k\times k}$ are constant matrices. □

R, S matrices are called a Gram pair associated with A. All the details (including the way to calculate Φ_i, Λ_i) the reader may find in [Dumitrescu, 2007].

For our use the condition "A is non-negative" is not satisfactory, however, the condition "$A - \varepsilon$ is non-negative, given small $\varepsilon > 0$" will be sufficient.

3.8 Implementation and Examples

The method was implemented using the LMI solver SeDuMi [Sturm, 1999] with YALMIP [Löfberg, 2004].

Polynomial Time Algorithm

The convex minimization program (3.12, 3.13) has infinite number of constraints, since they are imposed for all ω. In order to solve this problem the constraints are imposed on a frequency-grid only. Consider only a finite number of frequency samples $G(\omega)$ where $\omega \in \{\omega_i\}_{i=1}^{N}$.

ALGORITHM 3.5

1. Solve a (quasi-)convex minimization program:

$$\min_{f,A,B} \gamma^N \quad \text{subject to} \quad A > 0 \quad \forall \omega \tag{3.33}$$

$$\begin{pmatrix} \gamma^N f(\omega) I & G(\omega)A(\omega) - B(\omega) \\ * & \gamma^N A(\omega) \end{pmatrix} > 0 \tag{3.34}$$

$$f(\omega) I \leq A(\omega) \quad \forall \omega \in \{\omega_i\}_{i=0}^{N} \tag{3.35}$$

2. Calculate Q_* from $A = Q_* Q_*^{\sim}$.
3. Obtain the numerator P_* given Q_* :

$$\min_{P} \|G - P Q_*^{-1}\|_\infty \tag{3.36}$$

□

The algorithm for the scalar denominator case is similar. To obtain it substitute both f and A with a scalar polynomial a, and Q_* with scalar polynomial q_*, multiplied with identity matrix if required.

Stability guarantee is not implied in the finite dimensional setup, since the condition $A > 0$ is imposed only in a finite number of points.

Imposing a positivity constraint $A > 0$ for all frequencies is discussed in Section 3.7.

Denote $\gamma^{\infty}(G)$ solution of an infinite dimensional program. The infinum $\gamma^{N}(G)$ is clearly a lower bound for $\gamma^{\infty}(G)$. If $\{\omega_i\}_{i=1}^{N}$ are the first samples of a sequence $\{\omega_i\}_{i=1}^{\infty}$ dense in $\mathbb{R}$, then

$$\lim_{N\to\infty} \gamma^{N} = \gamma^{\infty}$$

Numerical experiments show, that γ^{N} becomes a good approximation of γ^{∞}, when $N > (km)^2$, where k is the order of the polynomials P, Q, and m is the number of inputs of PQ^{-1}. Since only a partial information about the original system is used, we do not expect a formal guarantee for the upper bound obtained as an outcome of minimizing (3.33, 3.35).

Numerical Complexity

For large scale systems the main contribution of complexity is computing the frequency samples, which is equal to $O(n^3)$ for each frequency point with n being the order of the full model. However, in most of the applications the cost may be lowered. For example, the cost can be lowered to $0(n\log(n))$, if the frequency response is obtained by accelerated solvers (see, for example, [Zhu *et al.*, 2003; Kapur and Long, 1998; Moselhy *et al.*, 2007]). The cost of the optimization algorithm does not exceed $O(N_1^2 N_2(N_1+N_2))$ with N_1 being the number of decision variables and N_2 the number of constraints, when is solved with SeDuMi. In this case $N_1 = O((km)^2 + Nm^2)$ and $N_2 = Nm$, where k is the order of polynomials, m is the number of inputs and N is the number of samples.

EXAMPLE 3.1—COMPUTATIONAL TIME VERSUS COMPLEXITY OF THE ORGINAL MODEL.

The main advantage of the presented approach is that less information about the original system is used than in the case of Hankel model reduction or balanced truncation. There is always a computational restriction on solving high-dimensional Lyapounov equations. In this example the computational time of different reduction procedures for various orders of an original model is studied. The case study is a

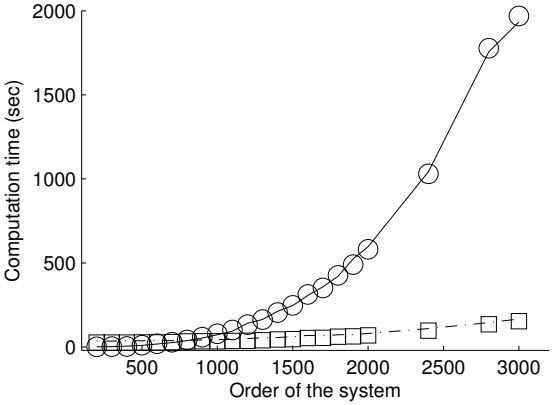


Figure 3.1 Computation time (in seconds) depending on the number of states in the model. Balanced truncation (solid line), Hankel model reduction (circles), matrix denominator case (dashed line) and scalar denominator case (squares).

mass-spring-damper system.

$$A = \begin{pmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 3 \cdot I_1^T \end{pmatrix} \quad C = \begin{pmatrix} 0 & 280 \cdot I_1 \end{pmatrix}$$

$$I_1 = \begin{pmatrix} 0 & \dots & 0 & \\ 0 & \dots & 0 & I \end{pmatrix} \qquad I_2 = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

And $M = 2/nI$, $K = 400I_2$, $D = 300I_2$, where n is the number of masses in the system. Our goal is to obtain the reduced models of order 20 and compare the results for different methods.

The number of points in the uniform grid is 100. Since there is a structure in the model, the cost for calculating the frequency response is $O(n\log(n))$, for each frequency sample. The computational cost for solving Lyapunov equations is $O(n^3)$, considering the special structure and sparsity the cost may be decrease to $O(n^2)$. Where n is the order of the model. These effects are illustrated in figure 3.1. □

Table 3.1 Reduction errors for shell oil fractionator benchmark.

Number of states in the reduced model	6	9	12	15
Hankel model reduction	7.65%	4.33%	3.28%	2.29%
Matrix denominator	7.93%	5.15%	3.29%	2.31%
Scalar denominator	7.97%	6.05%	3.67%	2.79%

EXAMPLE 3.2—SHELL OIL FRACTIONATOR MODEL.
The main goal of this example is to show that the proposed method has a comparable performance to Hankel model reduction. Surely, as the proposed method is a more restrictive version of the Hankel model reduction, in general, one can not and will not get a better approximation with the proposed approach. However, in many cases not too much in accuracy is lost.

The model is described in [Prett and Morari, 1987], but a simplified version is studied. It is given in [J.M.Maciejowski, 2002]. Only the dynamics without disturbances are considered. It yields a 3-input-3-output discrete-time model with 80 states and the sampling time 4.

The proposed algorithm is compared to Hankel model reduction for 6-th, 9-th, 12-th, 15-th order approximations. This is done since the algorithm can be consider as a restricted version of optimal Hankel reduction. 50 frequency in the grid were calculated. It was enough to obtain a reasonably low ratio between the actual error and the obtained value of γ. Hankel model reduction was performed using MATLAB^{TM} HANKELMR routine.

As expected matrix denominator case provides an approximation close to an optimal Hankel one. The scalar denominator provides a worse approximation, however the computation time is lower. □

EXAMPLE 3.3—DEFORMABLE MIRROR MODEL
The following model was studied in [Giselsson, 2006]. Modeling was performed using finite element modeling approach, that result in a system of second-order differential equations:

$$M\ddot{x} + C\dot{x} + Kx = Fu,$$

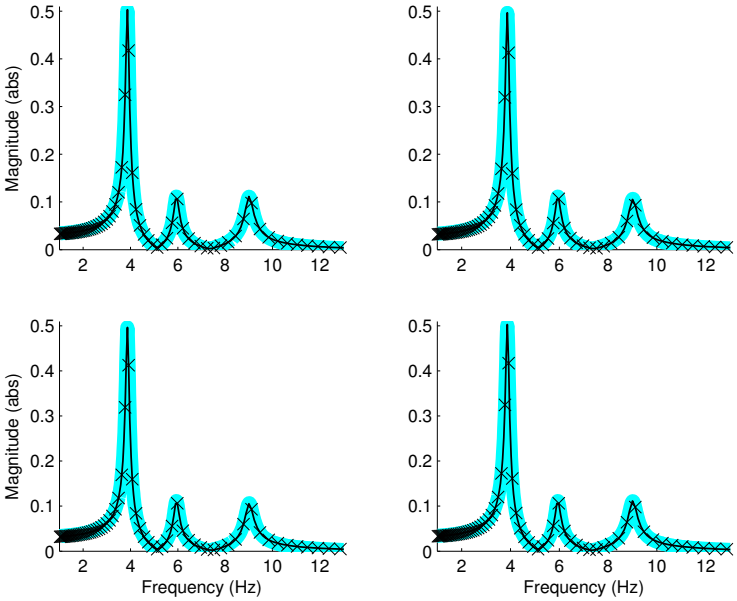


Figure 3.2 Reduction of Deformable Mirror Model. Crosses - frequency response of the original model, thick solid - frequency response of the Hankel approximation, thin solid - frequency response of the optimization based (QCO) approximation.

where matrices M, K, C have dimension more than 10000. Calculating the frequency response for such systems is possible, but problematic on a desktop computer. We also want to compare the method to state-space methods, what would be practically impossible. In [Giselsson, 2006] heuristic reduction was performed and a model with 2000 states was obtained. The investigated model has 420 sensors and actuators. A 2×2 transfer function is approximated, since the controllers used in the system are designed in a distributed fashion. The acquired model is discretized using bilinear transformation. The frequencies are considered in the range $[0, 25\text{Hz}]$, since for the rest of the frequencies G has a close zero frequency response. The model is truncated to 20 states by means of the described quasi-convex optimization technique (QCO

method), Hankel model reduction.

The QCO method is implemented on the frequency grid with 84 samples. The optimization together with calculating the frequency samples took 74 seconds and the resulting approximation error is $2.9 \cdot 10^{-5}$. Hankel model reduction took around 20 minutes providing the error $7.98 \cdot 10^{-5}$. Results, see in the figure 3.2. For a given frequency interval the QCO method provided a better model than Hankel reduction. It was expected, since the optimization was subjected only to this segment. This example shows, that for large/medium scale systems there is a significant win in the computational time and the approximation quality is not lost. □

3.9 Conclusion

In this chapter multi-input-multi-output extensions of [Megretski, 2006] have been discussed, where convex optimization is used to search for low order models. It have been shown that the same approximation gap bound for MIMO extension stands as in SISO methods. The method can be very useful for large scale systems, since it is rational fit algorithm with stability guarantee and relatively low computational complexity.

4

Extensions to an Optimization-Based Model Reduction

This chapter is based on [Sootla and Rantzer, 2009]. In this chapter two extensions of an optimization based model reduction method are considered: frequency-weighted and parameter-dependent model reduction. The performed extensions are simple and natural, this fact illustrates the advantages of the optimization based method. Also, the methods may be applied using an exact model or frequency samples of a model. Numerical examples are constructed that illustrate both extensions.

4.1 Introduction

The first extension of the optimization-based reduction framework considered is the frequency-weighted model reduction, which is a very important class of problems. For process models a good approximation is required only along some trajectories, i.e. around some frequencies. A controller or a plant reduction can be done in a robust way with frequency-weighting (see, [Obinata and Anderson, 2001]). In the unweighted case there exist well-developed and recognized methods, like balanced truncation and optimal Hankel model reduction (see, [Obinata and Anderson, 2001]). For the frequency-weighted extensions,

however, there exists many open issues (see, [Zhou, 1995; Enns, 1984]). In [Sandberg, 2008] a new state-space method was proposed, that may turn out to be very useful for reduction of structured system. However, to author best knowledge there exist no *a priori* guarantee, that the methods would work in the general case.

Instead of the state-space information of a model one can use a frequency domain information. The problem formulation in $\mathcal{H}_\infty$ space is not convex. In [Megretski, 2006] a relaxation that makes the problem convex in $\mathcal{H}_\infty$ space was proposed. One can use this method given an exact model or a finite number of frequency samples of a model. With a sufficient number of points one can get as close as required to the solution of the optimization procedure, that uses the exact model. The error bounds for this method were obtained in [Sou *et al.*, 2005].

In [Sandberg and Murray, 2007] a frequency-weighted extension was proposed. It consists of the method for single-input-single-output (SISO) models and an algorithm for multivariable (or MIMO) systems, that uses rank-minimization heuristics from [Fazel *et al.*, 2001].

The second framework extension is reduction of parameter-dependent models. First define a notion of a simplified parameter-dependent system. One can reduce the number of parameters, the McMillan degree of the system or simplify the way how the parameters are introduced into the dynamics. The focus of this chapter will be on the last two problems. In this chapter only a direction for future work is defined. The extension is introduced to show, how convenient the optimization-based reduction framework is.

4.2 Frequency Weighted Model Reduction

Problem formulation

A frequency weighted problem can be formulated as an optimization problem:

$$\min_{\hat{G}\in\mathcal{H}_\infty} \|W_o(G-\hat{G})W_i\|_\infty,$$

where $G, \hat{G}$ is a m-input m-output system, the McMillan degree of $\hat{G}$ is fixed and $\deg(G) \geq \deg(\hat{G})$. Weights W_o, W_i are considered proper,

stable with stable inverses. It is a common assumption in the literature (see, for example [Obinata and Anderson, 2001]).

Algorithm

Follow the same scheme as in the non-weighted case. The reduced model is parameterized with matrix polynomials of degree k $\hat{G} = PQ^{-1}$. Define P, Q, A, B as in the non-weighted case, i.e:

$$P = \sum_{i=0}^{k} P_i z^{-i} \quad Q = \sum_{i=0}^{k} Q_i z^{-i}$$
$$B = \sum_{i=-k}^{k} B_i z^{i} \quad A = \sum_{i=-k}^{k} A_i z^{i}$$

with some structural restrictions on B_i and A_i (see, Section 3.3) The matching will be performed on the unit circle $z = e^{j\omega}$. Hence the performance of G is required only on the unit circle. It is denoted as $G(\omega)$. The same notation applies to A, B, P, Q. Following the same pattern as in the non-weighted case obtain a (quasi-) convex algorithm:

ALGORITHM 4.1

1. Solve a (quasi-)convex minimization program:

$$\min_{A,B} \gamma \quad \text{subject to} \quad A > 0 \tag{4.1}$$
$$\begin{pmatrix} \gamma f(\omega) I & W_o(\omega)(G(\omega)A(\omega) - B(\omega))W_i(\omega) \\ * & \gamma W_i^{\sim}(\omega) A(\omega) W_i(\omega) \end{pmatrix} > 0 \tag{4.2}$$
$$W_i(\omega) f(\omega) W_i^{\sim}(\omega) \leq A(\omega) \quad \forall \omega \tag{4.3}$$

2. Calculate Q_* from $A_* = Q_* Q_*^{\sim}$.

3. Obtain the numerator P_* given Q_* :

$$\min_{P} \| W_o \left(G - P Q_*^{-1} \right) W_i \|_{\infty} \tag{4.4}$$

□

This algorithm can be made a finite dimensional one by imposing constraints (4.2, 4.3) on a finite frequency grid as in the non-weighted case. The constraint $A > 0$ (stability constraint) is treated as in Section 3.7, one can also enforce the passivity of the reduced model if required as in Section 3.6.

Error bounds

The error bounds will be shown, again, in two steps. First a statement similar to Lemma 3.2 is proved:

LEMMA 4.1
If γ, A, B, f, G, W_o, and W_i satisfy the conditions in (4.2, 4.3), then:

$$\|W_o(G - BA^{-1})W_i\|_\infty \leq \gamma.$$

□

Proof. First show that from LMIs (4.2, 4.3) the next one will follow:

$$(W_o(G - BA^{-1})W_i)(W_o(G - BA^{-1})W_i)^\sim < \gamma^2 I \quad \forall\omega$$

After taking the supremum over all ω, then result will follow. Skip ω in A, B, and so forth to simplify the notation. Using the Schur complement on the LMI in (4.2) yields:

$$\begin{aligned}&(W_o(GA - B)W_i)^\sim W_o(GA - B)W_i < \gamma^2 W_i^\sim A W_i f \quad \Leftrightarrow \\ &W_i^\sim A(W_o(G - BA^{-1}))^\sim W_o(G - BA^{-1})AW_i < \gamma^2 W_i^\sim A W_i f\end{aligned}$$

Multiply from the left hand side with $W_i^\sim A$ and from the right hand with AW_i :

$$(W_o(G - BA^{-1}))^\sim W_o(G - BA^{-1}) < \gamma^2 A^{-1} f$$

Now multiplying from the left hand side with $W_i^\sim$ and from the right hand with W_i yields:

$$W_i^\sim(W_o(G - BA^{-1}))^\sim W_o(G - BA^{-1})W_i < \gamma^2 W_i^\sim A^{-1} f W_i$$

The inequality $W_i^{\sim} A^{-1} f W_i \leq I$ follows directly form (4.3) first by inverting and then multiplying with $W_i^{\sim}$ and W_i from different sides. Hence:

$$W_i^{\sim}(W_o(G - BA^{-1}))^{\sim} W_o(G - BA^{-1})W_i < \gamma^2 I$$

and the statement is proved. ■

Now it is left to show that the reduction error is bounded, if the weights are suitable and there is a solution to minimization the problem (4.1—4.3). Consider a stable - antistable decomposition $BA^{-1} = PQ^{-1} + RT^{-\nabla}$, where $T^{\sim}T = A$, and R has a degree $k-1$.

THEOREM 4.1
Assume $P_*Q_*, A_*, B_*, \gamma_*$ are obtained from the Algorithm 4.1. Then the following statements are true:

1. Here $\hat{G} = P_* Q_*^{-1}$ then:

$$\|W_o(G - \hat{G})W_i\|_\infty \leq \gamma\,(1 + km \cdot C(W_o, W_i))$$
$$\text{where } C(W_o, W_i) = \frac{\sup \underline{\sigma}(W_i)\underline{\sigma}(W_o)}{\inf \underline{\sigma}(W_o)\underline{\sigma}(W_i)}$$

2. Define a stable transfer function $\hat{G}_2$ as $W_o\hat{G}_2W_i = W_oR_0T_*^{-\nabla}W_i - R_2T_*^{-\nabla}$ as a result of stable – antistable decomposition of $W_oR_0T_*^{-\nabla}W_i$, then:

$$\|W_o(G - \hat{G} - \hat{G}_2)W_i\|_\infty \leq \gamma\,(1 + km)$$

3. $\min\limits_{\hat{G}} \|W_o(G - \hat{G})W_i\|_\infty \geq \gamma \geq \sigma_{km+1}(G)$, where $\sigma_{km+1}(G)$ is a $km+1$-th largest Hankel singular value.

□

Proof. The main idea of this proof is taken from [Sandberg and Murray, 2007].

1. One can bound from above $\|W_o(G - B_*A_*^{-1})W_i\|_\infty$ using the singular value property $\overline{\sigma}(JK) \geq \overline{\sigma}(J)\underline{\sigma}(K)$ and the inequality for supremums:

$$\sup_\omega(h(\omega)g(\omega)) \geq \sup_\omega(h(\omega)) \inf_\omega(g(\omega)).$$

First use twice the singular value property on each of the weights yielding:

$$\overline{\sigma}(W_o(G - B_*A_*^{-1})W_i) \geq \overline{\sigma}(G - B_*A_*^{-1})\underline{\sigma}(W_o)\underline{\sigma}(W_i)$$

and then use the supremum property:

$$\sup \overline{\sigma}(W_o(G - B_*A_*^{-1})W_i) \geq \sup(\overline{\sigma}(G - B_*A_*^{-1})) \inf(\underline{\sigma}(W_o)\underline{\sigma}(W_i))$$

obtaining:

$$\inf_\omega[\underline{\sigma}(W_i)\underline{\sigma}(W_o)]\|G - B_*A_*^{-1}\|_\infty \leq \|W_o(G - B_*A_*^{-1})W_i\|_\infty$$

Since $B_*A_*^{-1} = P_0Q_*^{-1} - R_0T_*^{-\nabla}$, where $A_* = T_*^{\sim}T_*$:

$$\inf_\omega[\underline{\sigma}(W_i)\underline{\sigma}(W_o)]\|G - P_0Q_*^{-1} - R_0T_*^{-\nabla}\|_\infty \leq \gamma_*$$

$$\|G - P_0Q_*^{-1} - R_0T_*^{-\nabla}\|_\infty \leq \gamma_* \left(\inf_\omega[\underline{\sigma}(W_i)\underline{\sigma}(W_o)]\right)^{-1} \quad \text{and}$$

$$\|R_0T_*^{-\nabla}\|_H \leq \gamma_* \left(\inf_\omega[\underline{\sigma}(W_i)\underline{\sigma}(W_o)]\right)^{-1}$$

Now, by Lemma 2.1 there exists a real matrix K such that:

$$\|R_0T_*^{-\nabla} - K\|_\infty \leq km\gamma_* \left(\inf_\omega[\underline{\sigma}(W_i)\underline{\sigma}(W_o)]\right)^{-1},$$

where km is the McMillan degree of $R_0T_*^{-\nabla}$. The following inequality:

$$\|W_o(R_0T_*^{-\nabla} - K)W_i\|_\infty \leq \sup\left(\overline{\sigma}(W_o)\overline{\sigma}(W_i)\right)\|R_0T_*^{-\nabla} - K\|_\infty$$

is valid due to the property of singular values:

$$\overline{\sigma}(JK) \leq \overline{\sigma}(J)\overline{\sigma}(K)$$

and a similar expression for supremum:

$$\sup(f(\omega)g(\omega)) \leq \sup(f(\omega))\sup(g(\omega))$$

Hence:

$$\begin{aligned}\|W_o(R_0T_*^{-\nabla} - K)W_i\|_\infty &\leq km\gamma_* \frac{\sup_\omega[\overline{\sigma}(W_i)\overline{\sigma}(W_o)]}{\inf_\omega[\underline{\sigma}(W_i)\underline{\sigma}(W_o)]} \\ &= km\gamma_* C(W_o, W_i)\end{aligned}$$

Combining this inequality with $\|W_o(G - B_*A_*^{-1})W_i\| \leq \gamma_*$ yields:

$$\begin{aligned}&\|W_o(G - (P_0 + KQ_*)Q_*^{-1})W_i\|_\infty \leq \\ &\leq (km C(W_i, W_o) + 1)\gamma_*,\end{aligned}$$

Since the numerator P_* is obtained by means of minimization, the first assertion is shown.

2. In the first statement Lemma 2.1 was applied without considering the weights. Rewrite $\|W_o(G - B_*A_*^{-1})W_i\| \leq \gamma_*$, after performing the stable – antistable decomposition of $W_oR_0T_*^{-\nabla}W_i$:

$$\|W_o(G - P_0Q_*^{-1} - \hat{G}_2)W_i - R_2T_*^{-\nabla}\|_\infty \leq \gamma_*$$

Applying the same techniques as above proves the result.

3. The result follows by applying Lemma 3.1 and noting that the relaxation is related to Hankel model reduction.

■

In the second part of the theorem the error bound is obtained, which does not depend on the weights. However the reduced system now has the McMillan degree $km + d$, where d is the McMillan degree of $\hat{G}_2$. The results should be considered as a theoretical justification for the method. The derived error bounds are conservative, since, for example, the sub multiplicative property of matrix norm is used.

EXAMPLE 4.1
This example is based on the examples in [Zhou, 1995]. Consider

$$W_o = \begin{pmatrix} W_{0.01} & 0 \\ 0 & W_{0.1} \end{pmatrix} \quad G = \begin{pmatrix} \dfrac{1}{a_1(s)} & 0 \\ 0 & \dfrac{b_2(s)}{a_2(s)} \end{pmatrix}$$

where

$$a_1(s) = s^6 + 7.4641s^5 + 3.8637s^4 + 9.1416s^3 + 7.4641s^2 + 3.8637s + 1$$
$$a_2(s) = (s^2 + 0.2s + 4.04)(s^2 + 0.2s + 16.02)$$
$$b_2(s) = (s^2 + 0.2s + 1.01)(s^2 + 0.2s + 9.01)$$
$$W_\alpha = \frac{(s-1)^2}{s^2 - 2\alpha s + 1}$$

and the input weight is identity. Enns' method is frequency weighted

Table 4.1 Approximation errors for entry-wise reduction.

Entry	$W_{0.1}b_2/a_2$		$W_{0.01}/a_1$
Reduction order	2	3	4
Lower bounds	2.7037	2.5261	0.0248
Enns	5.128	4.993	0.0584
LA	20.08	11.94	0.0860
AI	4.827	8.20	0.0448
AII	4.822	3.946	0.0256
Proposed Method	4.6686	3.8409	0.0253

balanced truncation (see, [Enns, 1984]), LA is a frequency-weighted Hankel model reduction algorithm by Latham and Anderson (see, [Latham and Anderson, 1985]), AI and AII are the modifications of the LA algorithm (see, [Zhou, 1995]). In AI, after the reduced system is obtained, the minimization over constant terms D of G_r is performed. In AII, obtain the denominator by Hankel model reduction, and then

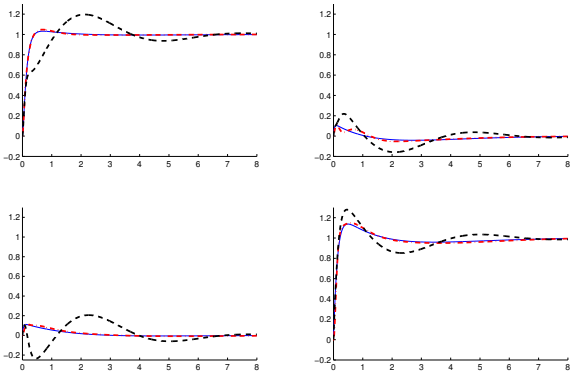


Figure 4.1 Entry-wise step responses: solid - for the loop with original controller, dash-dotted - the loop with controller reduced by means of proposed method (coincides with the solid one), dashed - the loop with controller reduced by means of Hankel model reduction.

optimize over the numerator. Lower bounds for the methods are the $l+r+1$ Hankel singular values of the stable part of $W(s)G(s)$, where l is the order of the stable part of $W(s)$, and r is the order of the reduced function. There is $N = 100$ points in the grid for all the experiments. It can be seen that entry-wise the proposed method acts almost as the AII algorithm, which is also optimization-based, but can not be used for the large-scale systems due to use of Lyapounov equations.

Now the multivariable model will be reduced. In this case due to increase in the decision variables the grid will consist of 150 points. The reduction of G to order $2k$ is equivalent to reducing both entries to order k. The approximation error in this case will be the maximum of two approximations errors, the slight change may occur due to compensation in non-diagonal entries. G is reduced to orders $4, 6$ and 8, what corresponds to reducing both entries to orders $2, 3$ and 4. In the first two cases the approximation error will be the error of reducing b_2/a_2 in the third – $1/a_1$. The approximation errors for orders $4, 6, 8$ are $4.3916, 3.8091, 0.0267$ correspondingly. □

EXAMPLE 4.2—CONTROLLER REDUCTION
This example is considered in [Safonov *et al.*, 1988]. The plant P is a NASA HiMAT aircraft model, which is a 2-input-2-output system with 6 states with 2 unstable ones. The controller K is designed in [Chiang and Safonov, 1992], is a 2-input-2-output model with 16 states. It was reduced to a 9 state model using balanced truncation without considering the weights, i.e. the plant. The developed frequency-weighted setup will be used to reduce the controller. The usual way to apply frequency-weighted reduction to a controller simplification problem is solving (see, [Obinata and Anderson, 2001]):

$$\min_{\hat{K}} \|W_o(K - \hat{K})\|_\infty,$$

where $W_o = (I + PK)^{-1}P$. The quasi-convex optimization approach can not be applied directly, since there are integrator states in the controller K Therefore the controller is decoupled first to $K = K_i + K_c$, where K_i is the integrator and K_c is the asymptotically stable part of the controller. Now the developed method can be applied to the obtained problem:

$$\min_{\hat{K}_c} \|W_o(K_c - \hat{K}_c)\|_\infty$$

K_c is reduced to 6 states, so $\hat{K} = K_i + \hat{K}_c$ has 8 states, using $N = 100$ points in the grid. The system is also reduced using frequency-weighted Hankel model reduction for the minimum phase weight $\tilde{W}_o$. Controller reduction by means of balanced truncation provided unstable step responses. See, figure 4.1 for results. □

4.3 Partially Parameterized Model Reduction

In this section we will focus on reduction of the McMillan degree of the parameter-dependent system and the simplification of the parameter dependence. In the proposed setup one does not have to decouple these problems, and one can reduce both the order, and the parameter dependence at the same time. The problem can be formulated as:

$$\min_{\hat{G}(\omega,\cdot)\in\mathcal{H}_\infty} \max_{\theta} \|G(\omega,\theta) - \hat{G}(\omega,\theta)\|_\infty$$

where $\theta \in \Theta$, the set Θ is compact in $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider $\hat{G}(\omega,\theta) = P(\omega,\theta)Q^{-1}(\omega)$, where the parameter dependence is introduced only in the numerator and $P(\omega,\theta) = \sum_{i=0}^{k} P_i(\theta)e^{-ij\omega}$, $Q(\omega) = \sum_{i=0}^{k} Q_i e^{-ij\omega}$. In general $P_i(\theta)$ can be let to be an integrable function of θ. It does provide a better accuracy, however, it is not as convenient and efficient in some aspects as having polynomial dependencies on θ. Thus consider a parameterization:

$$P(\omega,\theta) = \sum_{i=0}^{k}\sum_{j=0}^{l} P_{ij}\theta^{-j}e^{-ij\omega}$$

with which the parameter dependence may be simplified as well. Follow the same pattern as earlier and obtain an algorithm:

Algorithm 4.2—Partially Parameterized Model Reduction

1. Solve a (quasi-)convex minimization problem:

$$\min_{f,A,B} \gamma \quad \text{subject to} \quad A > 0 \tag{4.5}$$

$$\begin{pmatrix} \gamma f(\omega)I & G(\omega,\theta)A(\omega) - B(\omega,\theta) \\ * & \gamma A(\omega) \end{pmatrix} > 0 \tag{4.6}$$

$$f(\omega)I \le A(\omega) \quad \forall\omega \in [0,\pi], \quad \theta \in \Theta \tag{4.7}$$

2. Obtain Q_* from $A_* = Q_* Q_*^{\sim}$.

3. The numerator $P_*(\omega,\theta)$ of $\hat{G}(\omega,\theta)$ is calculated from:

$$\min_{P(\omega,\theta)} \|G(\omega,\theta) - P(\omega,\theta)Q_*^{-1}\|_\infty \tag{4.8}$$

□

Since the denominator is not parameterized the stability is enforced in a similar manner.

THEOREM 4.2
Assume $A_*, B_*, P_*Q_*, \gamma_*$ are obtained from Algorithm 4.2. Then the following statements are true:

1. $\|G(\theta) - P_*(\theta)Q_*^{-1}\|_\infty \leq \gamma_*(1+km) \quad \forall\theta \in \Theta$
2. $\min\limits_{P,Q} \|G - PQ^{-1}\|_\infty \geq \gamma_* \geq \sigma_{km+1}(G)$.

Here $\sigma_{km+1}(G)$ is a $km+1$-th largest Hankel singular value of G, and $\min\limits_{P,Q} \|G - PQ^{-1}\|_\infty$ is the optimal reduction error. □

Proof. Rewrite $\|G(\theta) - B_*(\theta)A_*^{-1}\|_\infty \leq \gamma_*$ as:

$$\|G(\theta) - P_0(\theta)Q_*^{-1} - R_0(\theta)T_*^{-\nabla}\|_\infty \leq \gamma_* \quad \forall\theta$$

Minimizing this expression over P_0, T_*, Q_*, R_0 provides us with Hankel model reduction for any fixed parameter θ, thus $\gamma_* \geq \sigma_{km+1}(G)$ and the second statement follows. By Lemma 2.1 there exists a real matrix $K(\theta)$ such that:

$$\|R_0(\theta)T_*^{-\nabla} - K(\theta)\|_\infty \leq km\gamma_*$$

where km is the McMillan degree of $R_0(\theta)T_*^{-\nabla}$. Combining these inequalities yields:

$$\|G(\theta) - (P_0(\theta) + K_0(\theta)Q_*)Q_*^{-1}\|_\infty \leq (km+1)\gamma_* \quad \forall\theta$$

Since $\hat{G}_\theta$ is obtained for every parameter θ by minimizing over the numerator $P(\theta)$, with fixed Q_* the statement follows. ■

EXAMPLE 4.3—REDUCTION OF PARAMETER DEPENDED SYSTEM
A mass-spring-damper system G with 6 masses is considered, where $m_i = 0.3$, $i = 1,\ldots,5,7,8$ $k = 1.7$, $d = 0.9$ $k_y = k_m = 1$. Mass m_6 is a parameter varying as: $\theta \in [0.1, 0.5]$.

$$A = \begin{pmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ k_m \cdot M^{-1}I_1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & k_y \cdot M^{-1}I_1^T \end{pmatrix} \qquad I_1 = \begin{pmatrix} 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$$

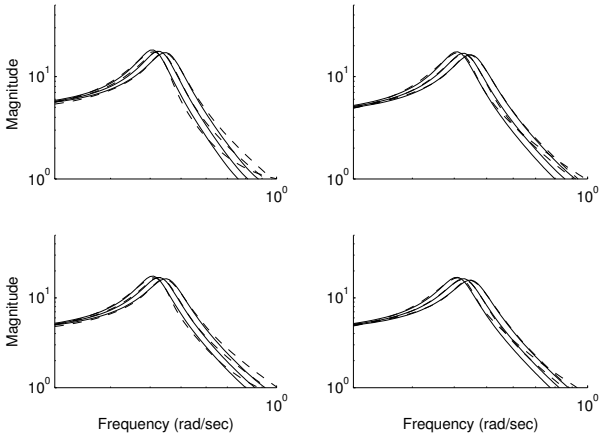


Figure 4.2 Bode plots for parameter-dependent system reduction example. Dashed lines - reduced systems for different values of parameters, solid lines - original systems for different values of parameters.

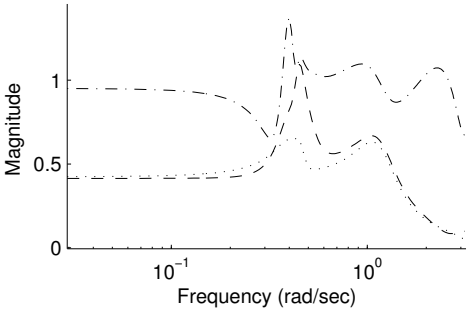


Figure 4.3 Dashed line - $\max \sigma(G(\omega,\theta_i) - \hat{G}(\omega,\theta_i))$ for $\theta_1 = 0.1$. Dotted line - $\max \sigma(G(\omega,\theta_i) - \hat{G}(\omega,\theta_i))$ for $\theta_2 = 0.3$. Dash-dotted line - $\max \sigma(G(\omega,\theta_i) - \hat{G}(\omega,\theta_i))$ for $\theta_3 = 0.5$.

$$I_2 = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \qquad M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}$$

And $K = kI_2, \quad D = dI_2$. The parameter grid consists only of three points $\theta = \{0.1, 0.3, 0.5\}$ for simplicity. The frequency grid is uniform and consists of 60 points. The reduced parameterized model has order 6 and the parameter dependence is:

$$\hat{G}(\omega, \theta) = (P_0 + P_1\theta + P_2\theta^{-1})Q^{-1}.$$

The obtained error value for the parameter-depended system is $\gamma = 1.19$. A relatively big error occurs due to the fixed A polynomial for the different values of parameters (see, fig 4.3). □

4.4 Conclusion

Two extensions for the optimization based model reduction framework have been presented: frequency-weighted and parameter-dependent. The biggest advantage of this framework is the ability to perform the optimization using frequency samples of the model. That may be useful for large-scale systems. It has been shown that the frequency-weighted extension is competitive in comparison with frequency-weighted state space methods. It also have been shown the application of the method in problems where it is important to preserve the structure of the system.

5

Model Reduction Methods for Parameterized Systems

This chapter is based on [Sootla and Sou, 2010]. In this chapter a multivariable extension of a recently proposed model simplification method for linear time invariant parameterized models is presented. A modification of the same method is also introduced. It is able to obtain models with explicit parameter dependence. The methods are based on convex optimization with semidefinite constraints. They match the frequency samples of the original and the reduced models, hence making it possible to apply the methods to large-scale models. Numerical examples showing the drawbacks and the advantages of the methods are also presented.

5.1 Introduction

Parameterized model order reduction is a well known problem and reduced order models are required in many applications. For example, the design and simulation of radio frequency (RF) inductors, resistor inductance capacitor (RLC) lines, or micro-electromechanical resonators is expensive due to high-order models, obtained by discretization of integral-differential equations. The low-order models would significantly reduce the simulation time. There are many approaches in the literature to obtain low-order approximations. However, most of them fail to guarantee one or more of the desired properties: stability,

passivity, rational fit, and so forth.

One of the most common approaches to parameterized reduction is moment matching which is based on Krylov subspace projection techniques. The major bottleneck is preserving stability of the reduced models. It can be done, but the computational cost may be not satisfactory. Preserving passivity, however, can be accomplished efficiently using the PRIMA algorithm ([Odabasioglu *et al.*, 1998]) and its extension to parameterized reduction ([Daniel *et al.*, 2004]). Also only the moments of models are matched, therefore the overall ($\mathcal{H}_\infty$) performance can be far from the original model. Rational fit algorithms are also presented in the literature ([Beyene and Schutt-Aine, 1998; Coelho *et al.*, 1999; Coelho *et al.*, 2001]). However, there are problems with enforcing extra constraints without extra *a priori* information. In the case of [Beyene and Schutt-Aine, 1998] the objective is not minimizing the norm of the error.

A recently proposed optimization-based algorithm ([Sou *et al.*, 2005]) addresses these problems. It is based on matching of frequency samples, and thus provides a low order model close to the original one in $\mathcal{H}_\infty$. With a convenient parametrization, stability and passivity become convex constraints. However, the output of the method is a look up table of transfer functions for every instance of parameters. An explicit parameterized model is hardly obtainable due to multivariate spectral factorization of positive trigonometric polynomials, which is known generally not to have a solution in finite degree polynomials. For simplicity a quasi-convex optimization approach from [Sou *et al.*, 2005] will be denoted as a QCO method.

In this chapter a multivariable extension of the algorithm in [Sou *et al.*, 2005] is presented. The extension is performed using the multivariable relaxation discussed earlier (see, section 3.4). An error bound for the method is also formulated and proved. It is similar to the non-parameterized case for single-variable and multivariable systems. The continuity of the obtained solution is investigated as well.

In this chapter also a modification of [Sou *et al.*, 2005] is proposed, that bypasses the spectral factorization problem. In order to do so, the search is restricted to models with positive real denominators which also guarantees stability of the obtained model. This restriction may result in a higher order approximation than one obtained by the QCO method, but the obtained model will be simplified with respect to pa-

rameter dependence, as well. These issues are illustrated on applications, such as the reduction of a highly-resonant deformable mirror model and a construction of an RF inductor parameterized model. The method presented in this paper will be referred to as the Positive Real Denominator (PRD) method.

A number of applications and directions of future work can be specified. The simplification methods for parameterized models, hopefully, can be applied to a number of different problems. A most straight forward application is a reduction of linear multivariate (or multidimensional) models, i.e. instead of one complex domain variable z, one has a vector of frequency variables $z_1, \ldots, z_n$. Such systems are met in signal processing. The method is probably applicable to reduction and modeling of linear parameter varying (LPV) systems. One could consider a linearly parameterized model as an LPV model with slowly varying parameters. However, in both cases a further investigation is required, since stability for such systems has different criteria than for linear time invariant systems.

5.2 Multivariable Parameterized Quasi-Convex Optimization Approach.

Assume the original model G depends on parameter $\theta = (\theta_1, \ldots, \theta_n)$ where $\underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i$ for all $i = 1, \ldots, n$. Formulate the reduction problem as:

$$\min_{P,Q} \max_{\theta} \left\| G(z,\theta) - P(z,\theta)Q(z,\theta)^{-1} \right\|_{\infty},$$

where P and Q are polynomials in z^{-1} as earlier with coefficients P_i, Q_i depending on θ. The dependence on θ will be clear later. The full model G is assumed to be an LTI model for every θ. Assume also that the coefficients of transfer function G depend on θ continuously. It is done for regularity conditions, but the method itself does not require this assumption. Such a continuity is not a strict requirement, since in many applications parameters do change continuously.

Since every θ_j is bounded, one can re-parameterize it as $\theta_j = \frac{\overline{\theta}_j + \underline{\theta}_j}{2} + \frac{\overline{\theta}_j - \underline{\theta}_j}{2}\cos(\omega_j)$, where $\omega_j \in [0, \pi], \forall j = 1, \ldots, n$. Denote $\boldsymbol{\omega} = \{\omega_0, \ldots, \omega_n\}$.

Using multi-indexes $\mathbf{j} = \{j_1, \ldots, j_n\}$ simplify the notation in (pseudo) polynomials $P_{j_0,\mathbf{j}} = P_{j_0,j_1,\ldots,j_n}$.

According to the notation in the multivariable reduction framework, introduce polynomials A and B :

$$A(\boldsymbol{\omega}) = \sum_{i_0=-k_0}^{k_0} \sum_{\mathbf{i}=-\mathbf{k}}^{\mathbf{k}} A_{i_0,\mathbf{i}} \cos(i_1\omega_1)\cdots\cos(i_n\omega_n)e^{-ji_0\omega_0}$$

$$B(\boldsymbol{\omega}) = \sum_{i_0=-k_0}^{k_0} \sum_{\mathbf{i}=-\mathbf{k}}^{\mathbf{k}} B_{i_0,\mathbf{i}} \cos(i_1\omega_1)\cdots\cos(i_n\omega_n)e^{-ji_0\omega_0}$$

The dependence of $B(\boldsymbol{\omega})$ on $\cos(\omega_j)$ is not required, one can use different basis functions, e.g. one can choose a polynomial dependence on θ without any transformation of variables.

The reduction algorithm is a direct generalization of a non-parameterized algorithm:

ALGORITHM 5.1—QCO PMOR

1. Consider a (quasi-)convex program:

$$\min_{f,A,B} \gamma \quad \text{subject to} \quad A > 0, \tag{5.1}$$

$$\begin{pmatrix} \gamma f(\boldsymbol{\omega})I & G(\boldsymbol{\omega})A(\boldsymbol{\omega}) - B(\boldsymbol{\omega}) \\ * & \gamma A(\boldsymbol{\omega}) \end{pmatrix} > 0, \tag{5.2}$$

$$f(\boldsymbol{\omega})I \le A(\boldsymbol{\omega}) \quad \forall \boldsymbol{\omega} \in [0,\pi]^{n+1} \tag{5.3}$$

   where $f(\cdot) \in \mathcal{L}_\infty([0,\pi]^{n+1} \to \mathbb{R})$ is obtained by means of optimization.

2. Fix an arbitrary θ, which corresponds to $\{\hat{\omega}_1, \ldots, \hat{\omega}_n\}$ Obtain the mapping Q_θ, which is a polynomial in $e^{-J\omega_0}$ for every θ, by solving the spectral factorization problem:

$$A_\theta^0 = Q_\theta^0 (Q_\theta^0)^\sim$$

3. Proceed with computing the numerator P_θ^0, which is obtained from:

$$B_\theta^0 = \left(P_\theta^0 (Q_\theta^0)^{-1} + (R_\theta^0)(T_\theta^0)^{-\triangledown}\right) A_\theta^0 \tag{5.4}$$

for every value of the parameter θ. Where $A_\theta^0 = Q_\theta^0 (Q_\theta^0)^\sim = (T_\theta^0)^\sim T_\theta^0$ and R_θ^0 is a polynomial in $e^{j\omega_0}$ of degree $k_0 - 1$ and defined uniquely by (5.4) for given B^0, A^0, Q^0, T^0.

□

As the output of the algorithm two mappings are obtained:

$$P_\theta : \quad \{\hat{\omega}_1, \ldots, \hat{\omega}_n\} \to P(\omega_0, \hat{\omega}_1, \ldots, \hat{\omega}_n)$$
$$Q_\theta : \quad \{\hat{\omega}_1, \ldots, \hat{\omega}_n\} \to Q(\omega_0, \hat{\omega}_1, \ldots, \hat{\omega}_n)$$

such that $P(\omega_0, \hat{\omega}_1, \ldots, \hat{\omega}_n), Q(\omega_0, , \hat{\omega}_1, \ldots, \hat{\omega}_n)$ are polynomials in the variable $e^{j\omega_0}$. In realization mappings P_θ, Q_θ can be stored as look-up tables for required values of θ or one can store polynomials A and B, and obtain the required P_θ, Q_θ when needed.

One could try to obtain a better model in step 3 of the algorithm. For example, given Q_θ, A_θ and B_θ for every $\theta \in \Theta$, compute P_θ as:

$$\min_{P_\theta} \left\| B_\theta A_\theta^{-1} - P_\theta Q_\theta^{-1} \right\|_\infty$$

Clearly for every instance of θ the obtained model would be more accurate than the one from Algorithm 5.1. However, in this case it would be extremely difficult to guarantee any kind of continuity with respect to parameters. The error bounds would hold and, in fact, could be improved.

Error Bounds and Regularity

As was mentioned earlier the algorithm can guarantee the error bounds and the regularity of the solution.

THEOREM 5.1
Assume the mappings A_θ^0, B_θ^0, P_θ^0, Q_θ^0, and the scalar γ^0 are obtained from Algorithm 5.1. Then the following statements are true:

1. $\max_\theta \|G - P_\theta^0 (Q_\theta^0)^{-1}\|_\infty \le \gamma^0 (1 + 2km)$
2. $\inf_{P,Q} \max_\theta \|G - P_\theta Q_\theta^{-1}\|_\infty \ge \gamma^0 \ge \min_\theta \sigma_{km+1}(G_\theta)$, where $\sigma_{km+1}(G_\theta)$ is the $km+1$-th largest Hankel singular value of G_θ and $\inf_{P,Q} \max_\theta \|G - P_\theta Q_\theta^{-1}\|_\infty$ is the optimal error bound.

□

Proof. Fix an arbitrary θ, rewrite $\|G_\theta - B^0_\theta (A^0_\theta)^{-1}\|_\infty \le \gamma^0$ using (5.4) as:

$$\|G_\theta - P^0_\theta (Q^0_\theta)^{-1} - R^0_\theta (T^0_\theta)^{-\nabla}\|_\infty \le \gamma^0 \tag{5.5}$$

It is possible for a fixed θ as it was possible in non-parameterized case. Minimizing (5.5) over $R^0_\theta, P^0_\theta, T^0_\theta$ and Q^0_θ provides us with Hankel model reduction for any fixed parameter θ, thus $\gamma^0 \ge \sigma_{km+1}(G_\theta)$.

Now notice that:

$$\inf_{P,Q} \max_\theta \|G - P_\theta Q_\theta^{-1}\|_\infty \ge \inf_{P,Q,R,T} \max_\theta \|G - P_\theta Q_\theta^{-1} - R_\theta T_\theta^{-1}\|_\infty = \gamma$$

Now the second statement follows.

Since

$$\|R^0_\theta (T^0_\theta)^{-\nabla}\|_H \le \gamma^0$$

one can bound also the infinity norm (see, Chapter 2 for details):

$$\|R^0_\theta (T^0_\theta)^{-\nabla}\|_\infty \le 2km\gamma^0$$

where km is the McMillan degree of $R^0_\theta (T^0_\theta)^{-\nabla}$. Combining these inequalities yields:

$$\|G_\theta - P^0_\theta (Q^0_\theta)^{-1}\|_\infty \le (2km+1)\gamma^0$$

Therefore one can maximize over θ and obtain:

$$\max_\theta \|G - P^0_\theta (Q^0_\theta)^{-1}\|_\infty \le (2km+1)\gamma^0$$

■

The error bounds are not enough in this situation. Since the explicit mapping Q_θ was not obtained, we are interested in its properties. With a (pseudo) polynomial parameterization of A the following convergence holds:

$$\|A_{\theta^k} - A_{\theta^0}\|_{\mathcal{H}_\infty} \longrightarrow 0 \quad \text{when} \quad \|\theta^k - \theta^0\|_2 \overset{k\to\infty}{\longrightarrow} 0$$

where θ^k is a sequence of parameters converging to the element θ^0 in a 2-norm of a vector in $\mathbb{R}^n$. If $\|\theta^k - \theta^0\|_2 \longrightarrow 0$, then the coefficients of A_{θ^k} approximate coefficients of A_{θ^0}, since the coefficients depend on the parameters smoothly. Similar argument applies to B. However, Q_θ is not, in general, a finite degree (pseudo) polynomial, so even the continuity is not implied. The same applies to polynomial P_θ.

THEOREM 5.2—REGULARITY OF THE SOLUTION
Assume A, B, P, Q are obtained from Algorithm 5.1 and therefore are continuous in variable θ, then the mappings

$$\begin{aligned} P_\theta : &\quad \{\hat{\omega}_1, \ldots, \hat{\omega}_n\} \to P(\omega_0, \hat{\omega}_1, \ldots, \hat{\omega}_n) \\ Q_\theta : &\quad \{\hat{\omega}_1, \ldots, \hat{\omega}_n\} \to Q(\omega_0, \hat{\omega}_1, \ldots, \hat{\omega}_n) \end{aligned}$$

are also continuous. □

In the proof of the Theorem 5.2 the following statement is required:

THEOREM 5.3—[ANDERSON, 1985]
Let $A_i(e^{j\omega}) = I + M_i(e^{j\omega}), \quad i = 1, 2$ be two positive definite Hermitian matrices defined on $-\pi \leq \omega \leq \pi$ with

$$\|M_i(e^{j\omega})\|_\infty \leq q < 1$$

Let Q_1, Q_2 be the associated normalized, minimum phase, stable spectral factors. Assume in addition that $dM_i(e^{j\omega})/de^{j\omega} \in \mathcal{L}_2^{n\times n}$ for $i = 1, 2$. Then there exists some $\widetilde{K}$ dependent only on q and $\|dM_i(e^{j\omega})/d\omega\|_2$ such that:

$$\|Q_1 - Q_2\|_\infty \leq \widetilde{K}\|M_1 - M_2\|_\infty^{1/2}$$

□

Proof of the Theorem 5.2.. The conditions of the Theorem 5.3 are fulfilled for a positive trigonometric in variable $e^{-j\omega_0}$ polynomial A_θ. Hence the Theorem states that in this case:

$$\|Q_{\theta^k} - Q_{\theta^0}\|_{\mathcal{H}_\infty} \longrightarrow 0 \quad \text{when} \quad \|\theta^k - \theta^0\|_2 \stackrel{k\to\infty}{\longrightarrow} 0$$

The same argument is valid for T_θ as a right spectral factor: $A_\theta = T_\theta^\sim T_\theta$. Recall that B was parameterized in such a way that for every θ the following holds:

$$B_\theta = P_\theta Q_\theta^\sim + R_\theta T_\theta e^{-j\omega_0 k_0}$$

As it was shown Q_θ is continuous in θ, if A_θ is. Thus if B_θ depends continuously on θ so do the mappings P_θ, R_θ, as the solutions to a linear Sylvester equation. ■

5.3 A Positive Real Denominator Approach

Non-Parameterized Setup.

The non-parameterized model order reduction problem can be formulated as a minimization problem in $\mathcal{H}_\infty$ space:

$$\min \left\| G - \frac{p}{q} \right\|_\infty, \quad \text{with} \quad p = \sum_{i=0}^{k} p_i z^{-i}, q = \sum_{i=0}^{k} q_i z^{-i}, \tag{5.6}$$

$$q \text{ is a minimum phase transfer function.} \tag{5.7}$$

where q_i, p_i are the decision variables. Minimum phase condition means that all the poles and zeros of q lie inside the unit circle $\mathbb{D}$. Hence it is a stability constraint for p/q. However, this formulation is known to be not convex in decision variables (due to stability constraint in the first place) and there exists no polynomial time method to obtain an optimal solution. While one may consider a relaxation, as in [Sou *et al.*, 2005], to obtain an approximation, one can go another way: make a restriction on q, that allows a convex problem formulation.

First reformulate the original problem, denoting $G(\omega)$ behaviour of G on the unit circle $z = e^{j\omega}$ with $\omega \in [0, \pi]$. A similar notation is made for p and q :

$$\min \gamma \quad \text{subject to} \quad |G(\omega)q(\omega) - p(\omega)| < \gamma |q(\omega)| \quad \forall \omega \in [0, \pi] \tag{5.8}$$

Then restrict $q(\omega)$ to strictly positive real transfer functions. This restriction guarantees the asymptotic stability and it is convex in decision

variables. As the reader may notice the right hand side now is the only reason why the problem is not convex. If $q(\omega) > 0 \ \forall \omega$, then a second order cone program for every value of γ can be derived. Since the denominator q is already restricted to positive real functions, it seems reasonable to substitute $|q(\omega)|$ with $\mathrm{Re}\,(q(\omega))$, i.e. solve instead of (5.6) the following problem:

$$\min \gamma \quad \text{subject to} \begin{cases} \mathrm{Re}\,(q(\omega)) > 0 \\ |G(\omega)q(\omega) - p(\omega)| < \gamma \mathrm{Re}\,(q(\omega)) \end{cases} \quad \forall \omega \in [0, \pi] \tag{5.9}$$

This substitution is valid since the minimal γ will be an upper bound on the optimal solution of the original problem. This is true since $|\mathrm{Re}\,(q(\omega))| \leq |q(\omega)|, \ \forall q, \omega$. Consider p^*, q^*, γ^* obtained from (5.9), then the inequality $\|G - p^*/q^*\|_\infty < \gamma^*$ follows.

In order to simplify the implementation, obtain a semi-definite formulation, using the Schur complement:

$$\begin{array}{l} \min \gamma \quad \text{subject to} \\ \begin{pmatrix} \gamma \mathrm{Re}\,(q(\omega)) & G(\omega)q(\omega) - p(\omega) \\ * & \gamma \mathrm{Re}\,(q(\omega)) \end{pmatrix} > 0 \quad \forall \omega \in [0, \pi], \end{array} \tag{5.10}$$

where the asterisk stands for a complex conjugate of upper right corner.

This formulation still has an infinite number of constraints. In order to obtain a finite dimensional formulation the constraints will be imposed on a finite frequency grid like in the previous chapters. One may run into problems with the constraint $\mathrm{Re}\,(q) > 0$, since enforcing that on a grid does not ensure positive realness and stability. The solution to this problem is provided in Section 5.5.

Multivariable Extension.

Assume now, $\hat{G} = PQ^{-1}$, where $P = \sum_{i=0}^{k} P_i z^{-i}, Q = \sum_{i=0}^{k} Q_i z^{-i}$, $P_i \in \mathbb{R}^{m_1 \times m_2}$, $Q_i \in \mathbb{R}^{m_2 \times m_2}$ for all $i = 1, \ldots, k$, i.e. the model has m_2 inputs and m_1 outputs, and $Q(z)$ is strictly positive real. Depending on the problem one can also parameterize $\hat{G} = Q^{-1}P$. In this case $Q_i \in \mathbb{R}^{m_1 \times m_1}$.

In the multivariable case the real part of Q is also used. The real part of a matrix is uniquely determined using the Cartesian decomposition $Q(\omega) = \mathrm{Re}\,(Q(\omega)) + \jmath \mathrm{Im}\,(Q(\omega))$, and $\mathrm{Re}\,(Q(\omega)) = (Q(\omega) +$

$Q(\omega)^{\sim})/2$, $\mathrm{Im}\,(Q(\omega)) = (Q(\omega) - Q(\omega)^{\sim})/(2\jmath)$. Matrices $\mathrm{Re}\,(Q(\omega))$ and $\mathrm{Im}\,(Q(\omega))$ are Hermitian by definition.

A multivariable extension can be performed as in [Sootla *et al.*, 2009], and a semidefinite program can be obtained:

$$\begin{array}{l} \min \gamma \quad \text{subject to} \quad f(\omega) I \leq \mathrm{Re}\,(Q(\omega)) \\ \begin{pmatrix} \gamma f(\omega) I & G(\omega)Q(\omega) - P(\omega) \\ * & \gamma \mathrm{Re}\,(Q(\omega)) \end{pmatrix} > 0, \quad \forall \omega \in [0, \pi] \end{array}, \tag{5.11}$$

where $f \in \mathcal{L}_\infty[0, \pi]$ is a scalar measurable function and it is obtained from the optimization procedure.

A justification for this extension follows from the next lemma:

LEMMA 5.1
Assume G, P, Q are transfer functions, f is a measurable function and γ is a positive scalar. Then the following statement holds:

$$\left.\begin{array}{l} f(\omega) I \leq \mathrm{Re}\,(Q(\omega)) \\ \begin{pmatrix} \gamma f(\omega) I & G(\omega)Q(\omega) - P(\omega) \\ * & \gamma \mathrm{Re}\,(Q(\omega)) \end{pmatrix} > 0 \qquad \forall \omega \in [0, \pi] \end{array}\right\} \Rightarrow$$
$$\Rightarrow \|G - PQ^{-1}\|_\infty \leq \gamma$$

□

Proof.

$$\begin{pmatrix} \gamma f I & GQ - P \\ * & \gamma \mathrm{Re}\,(Q) \end{pmatrix} > 0 \overset{(a)}{\Longrightarrow}$$
$$\gamma^2 f I > (GQ - P)(\mathrm{Re}\,(Q))^{-1}(GQ - P)^{\sim} \Rightarrow$$
$$\gamma^2 I > (GQ - P)(\mathrm{Re}\,(Q) f)^{-1}(GQ - P)^{\sim} \overset{(b)}{\geq}$$
$$\geq (GQ - P)(Q^{\sim}Q)^{-1}(GQ - P)^{\sim} =$$
$$= (G - PQ^{-1})(G - PQ^{-1})^{\sim}$$

The conjecture (a) is valid due to the Schur complement. To prove (b) one have to show that $\mathrm{Re}\,(Q) f \leq Q^{\sim}Q$. Consider inequality, that is valid for all ω :

$$(Q - fI)^{\sim}(Q - fI) \geq 0$$

Now with simple transformations the result is achieved:

$$Q^{\sim}Q - \underbrace{(fQ + fQ^{\sim})}_{=2f\mathrm{Re}\,(Q)} + f^2 I \geq 0$$
$$Q^{\sim}Q - f\mathrm{Re}\,(Q) \geq f(\mathrm{Re}\,(Q) - fI) \geq 0$$

The inequality $\mathrm{Re}\,(Q) \geq fI$ is one of the condition of this lemma, therefore the assertion is valid. ■

An interesting feature became available in model reduction by this approach. Given this multivariable extension, one can enforce a structure on the reduced model. Consider, $G(z)$, such that it has a triangular structure, i.e.:

$$G = \begin{pmatrix} g_1 & 0 & 0 \\ g_2 & g_3 & 0 \\ g_4 & g_5 & g_6 \end{pmatrix}$$

where $g_i(z)$ are scalar rational transfer functions. Now parameterize $P(z)$ and $Q(z)$ as:

$$P = \begin{pmatrix} p_1 & 0 & 0 \\ p_2 & p_3 & 0 \\ p_4 & p_5 & p_6 \end{pmatrix} \quad Q = \begin{pmatrix} q_1 & 0 & 0 \\ q_2 & q_3 & 0 \\ q_4 & q_5 & q_6 \end{pmatrix}$$

The resulting reduced model $\hat{G} = PQ^{-1}$ also has a similar triangular structure, since the product of two triangular matrices is triangular. One can perform similar structurizations with the QCO method, however, with less freedom in the denominator, i.e. $Q = \mathrm{diag}\{q_1, \ldots, q_m\}$. It is not clear if structurizations provide any advantages at all, except obviously fewer decision variables, this makes it an interesting future research direction.

Parameterized Model Simplification.

In this setup simplification is performed in two directions: the order of the system is reduced, and the parameter dependence is simplified.

Assume the original model G depends on a parameter $\theta = [\theta_1, \ldots, \theta_n]$ where $\underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i$, $\forall i = 1, \ldots, n$. Now formulate the reduction problem as:

$$\min_{p,q} \max_{\theta} \left\| G(z_0, \theta) - \frac{p(z_0, \theta)}{q(z_0, \theta)} \right\|_\infty ,$$

where $p(z_0, \theta) = \sum_{i=0}^{k_0} p_i(\theta) z_0^{-i}$, $q(z_0, \theta) = \sum_{i=0}^{k_0} q_i(\theta) z_0^{-i}$ and q is a minimum phase transfer function in z_0^{-1} variable (all the poles and zeros are inside the unit circle $\mathbb{D}$) for all possible θ.

Assume that p, q are polynomials both in θ and z_0. q has to be a (trigonometric) polynomial in θ_j (ω_j) in order to enforce the positivity constraints. One can assume, however, more general structures on p. Since every θ_j is bounded, re-parameterize it as $\theta_j = \frac{\overline{\theta}_j + \underline{\theta}_j}{2} + \frac{\overline{\theta}_j - \underline{\theta}_j}{2} \cos(\omega_j)$, where $\omega_j \in [0, \pi], \forall j = 1, \ldots, n$. Thus obtain multivariate trigonometric (pseudo) polynomials:

$$q(\boldsymbol{\omega}) = \sum_{i_0=0}^{k} q_{i_0}(\omega_1, \ldots, \omega_n) e^{-\jmath \omega_0 i_0},$$
$$p(\boldsymbol{\omega}) = \sum_{i_0=0}^{k} p_{i_0}(\omega_1, \ldots, \omega_n) e^{-\jmath \omega_0 i_0},$$

where $q_{i_0} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{k}} q_{i_0,\mathbf{i}} \prod_{j=1}^{n} \cos(i_j \omega_j)$, $p_{i_0} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{k}} p_{i_0,\mathbf{i}} \prod_{j=1}^{n} \cos(i_j \omega_j)$, and multi--indexes $\mathbf{i} = \{i_1, \ldots, i_n\}$, $\mathbf{k} = \{k_1, \ldots, k_n\}$.

Again slightly abuse the notation by denoting $p(\boldsymbol{\omega})$ frequency response of p to $\boldsymbol{\omega} = \{\omega_0, \ldots, \omega_n\}$. Following the same pattern as in non-parameterized case, a semidefinite program is obtained:

ALGORITHM 5.2—PRD PMOR

$$\min \gamma \quad \text{subject to}$$
$$\begin{pmatrix} \gamma \mathrm{Re}\,(q(\boldsymbol{\omega})) & G(\boldsymbol{\omega}) q(\boldsymbol{\omega}) - p(\boldsymbol{\omega}) \\ * & \gamma \mathrm{Re}\,(q(\boldsymbol{\omega})) \end{pmatrix} > 0 \qquad \forall \boldsymbol{\omega} \in [0, \pi] \tag{5.12}$$

□

For a finite dimensional formulation enforce the constraint on a finite frequency grid of $\boldsymbol{\omega}$. Once again to ensure stability the positive real

constraint on q has to be imposed. This problem is addressed in Section 5.5. Multivariable extension is performed as in non-parameterized case.

5.4 Theoretical Limitations and Advantages

The main limitation when applying this method is a quite restrictive property $\mathrm{Re}\,(q(\omega)) > 0$. The property implies that the phase of the denominator is restricted to the interval $[-\pi/2, \pi/2]$. It is possible to construct numerical examples, where the proposed method fails to fit the original model, without any reduction, i.e. the order of approximation is the same as the one of the original model. The main attribute of such models is a fast change in phase of G. For the simplification of large scale models it means that the reduced model will be of a higher degree, than one provided by QCO. However, one can always approximate a rational function with an FIR filter of sufficiently big order. And an FIR filter, in its turn, can always be approximated by the PRD method (simply by letting $q \equiv 1$).

The main problem is fitting heavy oscillating (poorly damped) systems with a fast phase change. Therefore one can first identify the poles and zeros lying near the unit circle. Then apply the PRD method incorporating this information as a weight into the optimization procedure, as it is done in [Sou *et al.*, 2005]. This also improves the robustness of optimization procedure and should be considered when implementing the method.

When the PRD method was applied to parameter-dependent models a major advantage was noticed. The parameterization of the PRD method is more efficient than the one of QCO. Recall, that the mentioned method solves the problem:

$$\min_{a,b} \left\| G - \frac{b}{a} \right\|_\infty ,$$

where $a > 0, b$ are polynomials and correspond to p and q as:

$$a = qq^\sim \qquad \frac{b}{a} = \frac{p}{q} + \frac{r}{q^\sim}$$

where the reduced model is found as p/q. Thus if p and q are parameterized as polynomials of degree $\mathbf{n} = \{n_0, \ldots, n_k\}$, then a, b have to be at least of order $2\mathbf{n} = \{2n_0, \ldots, 2n_k\}$. The PRD method requires the polynomials of order $\mathbf{n} = \{n_0, \ldots, n_k\}$. The number of decision variables related to a and $\mathrm{Re}\,(q)$ is the same and only the parameterized part of polynomials is affected.

Thus both the QCO and the PRD methods have some advantages and some limitations, moreover the numerical experiments show that neither of them performs in general better than the other with respect to $\mathcal{H}_\infty$ error.

5.5 Implementation and Examples

Algorithms

As in the previous cases obtain finite-dimensional versions of Algorithms 5.1 and 5.2, by enforcing the constraints on a grid. The positivity constraints, that guarantee stability or passivity, are enforced for all the frequencies. Hence there is a need in a multivariate and numerically robust version of the positivity constraint, which will be discussed in the next section.

In the parameterized case it is very important to have a numerically robust algorithm. It is clear that the poles and zeros near the unit circle have a great impacts on frequency response in the form of peaks and troughs. Surely in the approximation these poles and zeros should be preserved. Hence it feels natural to keep them out of the reduction procedure and reduce only the rest of the model. This idea can be formalized in the algorithm that can be applied to any reduction scheme.

Algorithm 5.3—Dominant Pole Algorithm

1. Identify dominant poles and zeros of the system G, for example, by running a non-parameterized version of Algorithm 5.1. Denote $G_d(z, \theta)$ the transfer function with all the dominant poles and zeros.

2. Run the Algorithm 5.1 (or the Algorithm 5.2) on one of the fol-

lowing problems:

$$\min_{\hat{G}} \|G - G_d\hat{G}\|_\infty \quad \text{or} \quad \min_{\hat{G}} \|GG_d^{-1} - \hat{G}\|_\infty$$

The choice of the problem should be made depending on the identified G_d.

□

By identifying these poles and zeros the speed of algorithm is also improved, since now a smaller in dimensions positivity constraint is needed. It is also easier to impose the stability constraint on G_d, since each pole is considered individually. Surely the overall accuracy may be an issue, thus there is a need in an accurate procedure to identify the dominant part of the model. Also should be noted, that it is not always possible to identify those or they do not exist. Basically, the decision of such a identification is left to user and performed by inspection of the frequency response.

Stability Constraint

The condition $\mathrm{Re}\,(q(\boldsymbol{\omega})) > 0$ enforced for all frequencies guarantees positive realness of the denominator and stability of the reduced model. The most efficient way to impose the positivity constraint for multivariate polynomials is to replace it with a sum of squares constraint. And as a justification the following theorem is used:

THEOREM 5.4—[DRITSCHEL, 2004]
Let $q(z_0, \ldots, z_n)$ be a multivariate Laurent polynomial which is strictly positive on the multivariate torus $|z_0| = \cdots = |z_n| = 1$, where $n \geq 1$ is an integer. Then $q(z_0, \ldots, z_n)$ can be expressed as a sum of square of magnitudes of polynomials in $z_0, \ldots, z_n$ □

For more details about the problem, see [Geronimo and Lai, 2006]. Imposing the sum-of-squares condition for all the frequencies can be efficiently solved using the following theorem:

THEOREM 5.5—[DUMITRESCU, 2007]
The trigonometric polynomial $q(z)$ is the sum-of-squares with factors of degree n, if and only if there exist positive semidefinite matrices

$R \in \mathbb{R}^{Nc \times Nc}$ and $S \in \mathbb{R}^{Ns \times Ns}$ such that

$$\widetilde{q}_{\mathbf{i}} = \mathrm{tr}[\Phi_{\mathbf{i}} R] + \mathrm{tr}[\Lambda_{\mathbf{i}} S] \tag{5.13}$$

where $\Phi_{\mathbf{i}} \in \mathbb{R}^{Nc \times Nc}$ and $\Lambda_{\mathbf{i}} \in \mathbb{R}^{Ns \times Ns}$ are constant matrices. □

R, S matrices are called a Gram pair. Depending on the parity of the orders $k_0, \ldots, k_n$ set $N_c = \frac{1+\Pi_{i=0}^{n}(k_i+1)}{2}$, $N_s = N_c - 1$ or $N_c = N_s = \frac{\Pi_{i=0}^{n}(k_i+1)}{2}$. The polynomial q is parameterized in the Theorem 5.5 as:

$$q = \sum_{\mathbf{i} \in \mathcal{M}} \widetilde{q}_{\mathbf{i}} \cos(\mathbf{i}^T \omega)$$

where, $\mathcal{M}$ is a halfspace. As the reader may recall in the proposed reduction algorithm q is parametrized as

$$q = \sum_{\mathbf{i}=0}^{\mathbf{k}} q_{i_0,\mathbf{i}} \prod_{j=1}^{n} \cos(i_j \omega_j) e^{-J\omega_0 i_0}$$

and the multi-index $\mathbf{i}$ has only positive entries (i.e. belongs to the quaterspace and not a halfspace as in Theorem 5.5). Thus in order to apply the theorem we need a mapping $q_{\mathbf{i}} \to \widetilde{q}_{\mathbf{i}}$, which is linear and obtained straight forward. For details (including the way to calculate $\Phi_{\mathbf{i}}, \Lambda_{\mathbf{i}}$) see [Dumitrescu, 2007].

Numerical Complexity

There are two main contributers to the complexity of the PRD method. The first one is computing the frequency samples which is $O(m^3)$ for each frequency point, with m being the order of the original model. In some case it can be lowered to $O(m \log(m))$ even for the dense models. The second one is the cost of the optimization algorithm. It does not exceed $O(N_1^2 N_2^2)$ with N_1 being the number of decision variables and N_2 the number of constraints, when the method is implemented with SeDuMi. In non-parameterized reduction $N_1 = 0(k^4)$ (if the stability constraints are imposed with KYP lemma) and $N_2 = 2N$, where k is the order of polynomials and N is the number of points in the grid.

In parameterized reduction the positive real constraint adds many more decision variables. Indeed the number of variables N_1 will be

$O((k_0+1)\cdots(k_n+1))^2$. So the computational complexity rises significantly with the order of the reduced system, thus being the heaviest part. The number of constraints is the same. Then the total cost will be:

$$O(m^3N) + O(((k_0+1)\cdots(k_n+1))^4N^2)$$

The conventional PRIMA method ([Odabasioglu *et al.*, 1998]) has computational complexity $O(m^2)$, since the most computationally heavy part, an orthogonal basis of the Krylov subspace, requires $O(m^2)$ floating point operations for calculating every column of the basis. In non-parameterized case, the total cost will not be bigger than $O(m^2)$. In the parameterized case, the parameters have to be taken into account. Then the total cost is equal to $O(m^2(k_0+1)\cdots(k_n+1))$. Note, however, that the passivity will not be enforced for all instances of parameters.

Numerical Examples

In this section, several applications and numerical examples are presented to describe the advantages and limitations of the proposed method. The algorithm is implemented by the cutting plane method (for more details, look [Sou *et al.*, 2008]) on the computer with the Core Duo 2.13 GHz processor and 2 GB RAM. The method can also be implemented using the optimization toolbox SeDuMi ([Sturm, 1999]). In order to reduce a continuous time model it is discretized using a bilinear transformation, and after the reduced model is obtained it is mapped back to continuous time.

Example 5.1—Reduction of a Deformable Mirror Model.
The following model was studied in [Giselsson, 2006] and obtained by means of a finite element modeling approach that resulted in a system of second-order differential equations:

$$M\ddot{x} + D\dot{x} + Kx = Fu,$$

where matrices M, K, D have high dimensions which results in a state-space model with more than 20000 states.

In [Giselsson, 2006] heuristic reduction have been performed and the obtained model has 420 sensors and actuators and 2000 states. In the process the system becomes mass-normalized, i.e. $M = I$, and

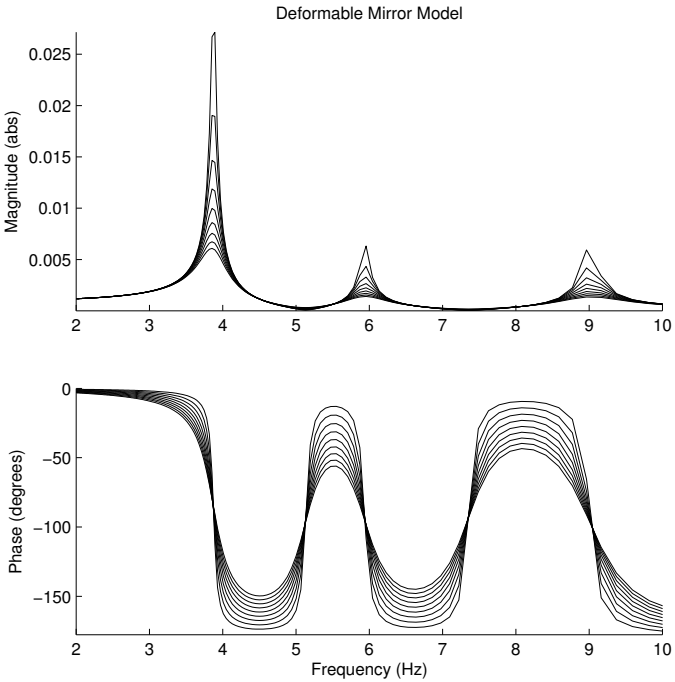


Figure 5.1 Deformable mirror model for values of friction coefficient $\alpha = 0.01$, 0.02, 0.03, 0.04, 0.05.

$K = \Lambda^2$, where Λ is a diagonal matrix. Damping is chosen to be a modal one, i.e. $D = \alpha\Lambda$, where $\alpha = 0.02$. Only one entry of the transfer function will be investigated, since the controllers used for regulating the system are designed in a distributed fashion. The frequencies in the range $[0, 60\text{Hz}]$ are considered. The frequency response of the models is depicted in figure 5.1

Non-parameterized reduction. The QCO and PRD methods are implemented on the frequency grid with 100 samples. It was possible to reduce the model to order 10 on the mentioned frequency interval by means of the QCO method. The PRD method did not provide a good 10-th order approximation on this interval. Supposedly the restriction on the denominator is activated in this example. Nevertheless, by letting

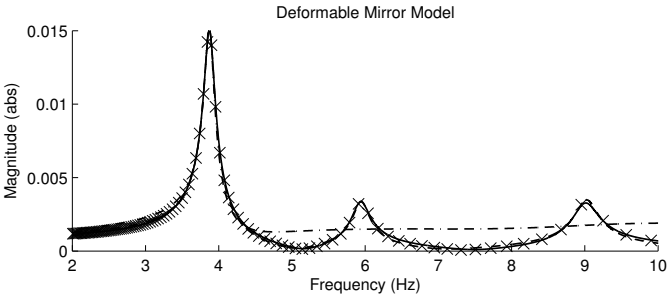


Figure 5.2 Non-parameterized Reduction. Solid - reduced by PRD to order 15 (coincides with the solid line), dotted - reduced by QCO to order 10, dash-dotted - reduced by PRD to order 10, crosses - original frequency data.

the reduction order be 15, the same quality approximation is achieved as in the case of the 10-th order QCO approximation. Results, see in the figure 5.2.

Parameterized reduction. Consider the coefficient α that defines friction. Let α take values 0.01, 0.02, 0.03, 0.04, 0.05. Calculating 100 frequency samples for every parameter takes about 20 seconds. Thus the orders of reduction are set to $k_0 = 15,\ k_1 = 1$. The order for parameter dependence is chosen to 1, since a slight change in the friction coefficient will not affect the frequency response radically. The results are somewhat surprising. The QCO method could not provide a matching frequency response, even though in the non-parameterized case the order 10 was sufficient to do that. The PRD, however, did better. Results, see in figure 5.3. The overall error for PRD is around 1% and the QCO 5%.

Identifying the dominant part of the model. After a careful consideration it was decided to identify 6 dominant poles and zeros and remove them from the modeling procedure i.e. those, that correspond to peaks shown in all the figures. The dominant poles are identified by running the non-parameterized QCO algorithm for every parameter in the training grid $\alpha = 0.01, 0.02, 0.03, 0.04, 0.05$ and setting the order of approximation to 15. The obtained approximations had errors around $10^{-6}\%$. Also assume, that poles are not intersecting. Here it is a valid assumption, since the parameter-dependence is not compli-

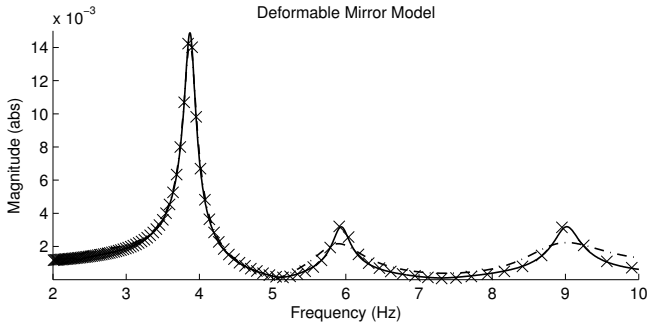


Figure 5.3 Parametrized Reduction. Approximations orders $k = [15, 1]$, Dash-dotted - reduced by QCO, solid - reduced by PRD, crosses - original frequency data. The friction coefficient $\alpha = 0.02$

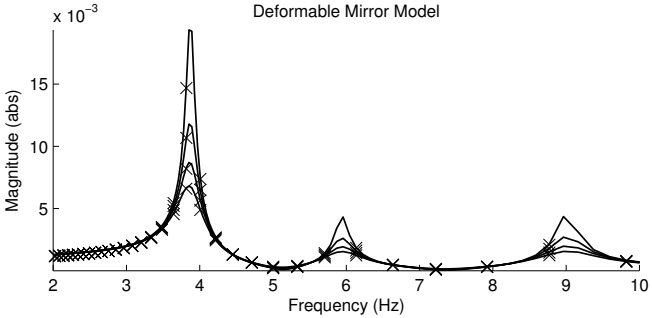


Figure 5.4 Parametrized Reduction by PRD using dominant pole algorithm. Approximations orders $k = [10, 2]$, The friction coefficient $\alpha = 0.015,\ 0.025,\ 0.035,\ 0.045$.

cated. However, in general one should keep in mind this problem. The following scheme is used to calculate $\hat{G}$:

$$\min_{\hat{G}} \|G - G_d \hat{G}\|_\infty$$

The results of modeling with the non-dominant part with order $k = [4, 0]$ are shown in figure 5.4. The overall error is around 1%, so we did not lose anything in accuracy. The parameter dependence is trans-

ferred to the dominant poles and zeros. The actual dependence on the parameter is linear thus extra accuracy is not gained by this transfer.

From the above, it is safe to say that the PRD method should be mainly considered in a parameterized framework. In non parameterized reduction case the PRD method has no advantages in comparison with the QCO method. Numerical experiments show that for simple LTI models the PRD method may provide a slightly better approximation, but, in general, this is not the case. □

Example 5.2—Wind Farm Modeling. Multivariable Distributed Non-Parameterized Reduction.
This example is meant to investigate the possibility of enforcing the structure on a multivariable transfer function. A wind farm model is described in [Knudsen *et al.*, 2009]. Only a part of the farm is considered, i.e. the system is consisting of two wind-mills in one row. The first wind-mill is affecting the second one by creating the wind turbulence, a corresponding entry is $(1,2)$. Therefore the model has a structure:

$$G = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

where the asterisks stand for non-zero entries. The QCO and PRD algorithm are applied without considering the structure. This is referred to as algorithm I. Algorithm II is enforcing the following structure on the decision variables explicitly:

$$P_{prd} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \qquad Q_{prd} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$
$$P_{qco} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \qquad Q_{qco} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

The results are shown in the table 5.2. The order of approximation is chosen 10. As the reader may see in the PRD case, unlike the QCO case, accuracy was not lost by enforcing the structure. However, there is a win in the number of decision variables and hence the computational time. □

Table 5.1 Approximation errors in percent.

Methods and Orders of Reduction	I	II
QCO 10-th order	3.06%	2.77%
PRD 10-th order	2.52%	2.52%

5.6 Conclusion

A multivariable extension of a known method have been presented. Also a new method for model reduction of linear systems have been proposed. The advantages of the new method can be seen only in the parameterized setup, e.g. one can obtain models with an explicit parameter dependence or simplify the parameter dependence. The positive real denominator assumption in some cases can make the solution rather restrictive. As a result the designer is forced to raise the order of approximation with respect to frequency variables in order to achieve the same level of approximation as with other methods. However, the parametrization is more efficient than in analogue methods and the $\mathcal{H}_\infty$ performance for parameterized model simplification can be in fact better than the performance of a QCO approximation. Known advantages and drawbacks of the method were illustrated on industrial models.

6

Case Study. Low-Order Modeling of an Integrated Spiral RF Inductor

In this chapter the developed techniques are applied to a modeling problem. The goal is stated as obtaining a low order parameterized model of a spiral radio-frequency (RF) inductor. The physical model consists of a set of integral-differential equations. The systems describes the effects of the inductor in three dimensions, moreover the dependence on the chosen parameters is nonlinear in nature. This makes choosing an appropriate set of parameters complicated. In order to overcome these difficulties a discretized with respect to space dimensions model is considered. This makes the model linear time invariant, and parameterized model reduction techniques can be applied as developed in this thesis.

6.1 Introduction

One of designers main focus is wireless connectivity, which often includes radio-frequency (RF) subsystems. Almost all RF subsystems include inductors, thus making modeling of RF inductors an important problem. The dependence on design parameters, such as metal width and spacing, is nonlinear. Moreover the physical models are described

by Maxwell's equation on the three-dimensional geometry making the calculations expensive for design purposes.

Generation of parameterized reduced-order models is a well-known problem and a number of methods exists in the literature. A modeling procedure for an RC(L) interconnect was developed in [Liu *et al.*, 1999; Heydari and Pedram, 2001] and it is based on statistical performance. Resistor-capacitor interconnects turned-out to depend on width and spacing parameters linearly, for simplified models, thus making the application of a projection approach possible (see, [Silveira *et al.*, 1996; Odabasioglu *et al.*, 1998]). A multi-parameter version of projection techniques to reduction of RC interconnect was investigated in [Daniel *et al.*, 2004], and an extension to RF Inductors was performed in [Daniel and White, 2003].

In this chapter a low-order parameter dependent linear time invariant (LTI) model is obtained. A discretized model from [Daniel and White, 2003] is used. Parameterized model reduction is based on techniques developed in [Sou *et al.*, 2005; Sootla and Sou, 2010]. This work should be considered as a case study for the above mentioned methods, the discretization techniques are not investigated in this chapter. Both the Quasi-Convex Optimization (QCO) and Positive Real Denominator (PRD) approaches are applied to the modeling problem. The main encountered difficulty was complexity of matching. For RF inductor modeling it is required to match not only the frequency response, but also the characteristics of an inductor, such as quality factor and inductance. The model is highly resonant which yields in high magnitude peaks. Hence approximating the response for low frequencies, which is required for a good match in quality factor, is extremely difficult. At the same time some accuracy can be sacrificed around the peaks, since quality factor does not depend on the high frequency response. Incorporating a bound on a quality factor error as a constraint in the minimization program is possible, but may influence the robustness of the procedure. Different ways of solving this problem were proposed and the corresponding results are presented.

6.2 Physical Modeling of an RF Inductor

Consider the physical model of an RF inductor obtained in [Daniel and

White, 2003]. In this section the main points of the modeling procedure are described. A three dimensional model of an inductor, mentioned in [Kamon *et al.*, 1997; Ruehli, 1974], captures the most relevant effects, as the impact of the distributed capacitance, skin depth, and proximity effects. The model can be described as set of integral-differential equations:

$$\begin{aligned}
\frac{\boldsymbol{J}(\boldsymbol{r})}{\sigma} + \jmath\omega\frac{\mu}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}' &= -\nabla\phi, \\
\frac{\mu}{4\pi\varepsilon}\int_S \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}' &= -\phi(\boldsymbol{r}), \\
\nabla\cdot\boldsymbol{J}(\boldsymbol{r}) &= 0, \\
\hat{\boldsymbol{n}}\cdot\boldsymbol{J}(\boldsymbol{r}) &= \jmath\omega\rho(\boldsymbol{r}),
\end{aligned}$$

where V and S are the union of conductor volumes and surfaces, $\boldsymbol{J}$ is the volume current distribution, ρ is the conductor surface charge, ϕ is the scalar potential. The constants μ, ε, σ are the free space permeability, the free space permittivity, and the conductor conductivity respectively. ω is the angular frequency of the conductor excitation.

This set of equations is then discretized using Galerkin method along the variables V and S in order to simplify the evaluation of the unknown parameters $\boldsymbol{J}, \rho$. The main idea is to use piece-wise constant basis functions to represent V and S. Then perform the integration over V, S by integrating the piece-wise constant functions. Usually a high-order basis is chosen in order to obtain an accurate discretization. This leads to a set of algebraic equations in the Laplace transformation domain:

$$\begin{pmatrix} M_f & M_p \end{pmatrix} \begin{pmatrix} R_f + sL_f & 0 \\ 0 & \frac{P}{s} \end{pmatrix} \begin{pmatrix} M_f^T \\ M_p^T \end{pmatrix} I_m(s) = V_m(s)$$

where s is the Laplace transform variable, R_f, L_f and P are the resistance, partial inductance and coefficient of potential matrices respectively. $I_m(s)$ is the vector of unknown mesh currents, and $V_m(s)$ is the vector of known mesh voltages. M_f and M_p matrices are mostly zero and represent Kirchhoff voltage law equations for corresponding basis

blocks. The state variables now can be defined as I_m and $\phi = PM_p^T I_m/s$, and the state-space form can be obtained as:

$$sEx = -Ax + Bu$$
$$y = B^T x$$

where

$$x = \begin{pmatrix} I_m \\ \phi \end{pmatrix} \qquad E = \begin{pmatrix} M_f L_f M_f^T & 0 \\ 0 & P^{-1} \end{pmatrix}$$
$$B = \begin{pmatrix} V_{m,s} \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} M_f R_f M_f^T & M_p \\ -M_p^T & 0 \end{pmatrix}$$

The usual way to evaluate the performance of inductors is by using the quality factor QF and the inductance L:

$$QF = \frac{\mathrm{Im}\,(G)}{\mathrm{Re}\,(G)}, \quad L = \frac{\mathrm{Im}\,(G)}{\omega},$$

where $G = B^T(sE + A)^{-1}B$ is the transfer function of the inductor.

6.3 Parameterized Model Reduction Methods

The methods used for low-order modeling in this work are those that have been developed earlier in the thesis, namely the Positive Real Denominator (PRD) and Quasi-Convex Optimization (QCO) methods. The goal is to compare the performance of the methods to each other. A comparison of QCO to PRIMA was made in [Sou *et al.*, 2008].

Consider a reduction problem of a parameterized LTI system in the discrete time:

$$\min_{\hat{G}} \max_{\boldsymbol{\omega}_\theta} \|G(\boldsymbol{\omega}) - \hat{G}(\boldsymbol{\omega})\|_\infty$$

where $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots \omega_n)$, and $\boldsymbol{\omega}_\theta = (\omega_1, \ldots, \omega_n)$, G is the original model and $\hat{G}$ is the reduced one. The parameter $\boldsymbol{\omega}_\theta$ is assumed to lie

in the set $[0,\pi]^n$. It is not restrictive since one can always map any bounded parameter $\underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i$, with known constants $\underline{\theta}_i$, $\overline{\theta}_i$ to the interval $[0,\pi]$, using, e.g. a transformation:

$$\theta_i = \frac{\overline{\theta}_i + \underline{\theta}_i}{2} + \frac{\overline{\theta}_i - \underline{\theta}_i}{2}\cos(\omega_i)$$

Also a frequency weighted version of each algorithm is used:

$$\min_{\hat{G}} \max_{\boldsymbol{\omega}_\theta} \|W(\boldsymbol{\omega})(G(\boldsymbol{\omega}) - \hat{G}(\boldsymbol{\omega}))\|_\infty$$

A formal description of the frequency-weighted version of the non-parameterized case of the QCO method is given in [Sootla and Rantzer, 2009]. A similar extension can be made to the parameterized case of QCO and also to the parameterized PRD method.

The goal is to obtain a model that is accurate for low frequencies while providing a good match in quality factor and inductance. A good overall approximation is important, however, around the peaks some accuracy may be sacrificed in order to obtain a better match in the quality factor.

ALGORITHM 6.1—QCO PMOR

1. Consider a (quasi-)convex program:

$$\begin{aligned} &\min_{A,B} \gamma \quad \text{subject to} \quad A(\boldsymbol{\omega}) > 0 \quad \forall \boldsymbol{\omega}, \\ &\begin{pmatrix} \gamma f(\boldsymbol{\omega}) I & W(\boldsymbol{\omega})(G(\boldsymbol{\omega})A(\boldsymbol{\omega}) - B(\boldsymbol{\omega})) \\ * & \gamma A(\boldsymbol{\omega}) \end{pmatrix} > 0, \\ &f(\boldsymbol{\omega}) I \leq A(\boldsymbol{\omega}) \quad \forall \boldsymbol{\omega} \in \{\boldsymbol{\omega}\}_{i=1}^N \subset [0,\pi]^{n+1} \end{aligned}$$

   where $f(\cdot) \in \mathcal{L}_\infty([0,\pi]^{n+1} \to \mathbb{C})$ is obtained by means of optimization.

2. Fix an arbitrary θ, which corresponds to $\{\hat{\omega}_1, \ldots, \hat{\omega}_n\}$ Obtain the mapping Q_θ, which is a polynomial in $e^{-J\omega_0}$ for every θ, by solving the spectral factorization problem:

$$A_\theta = Q_\theta \tilde{Q_\theta}$$

3. For the required parameter values θ solve a minimization problem:

$$\begin{aligned} &\min_{P} && \|B_\theta A_\theta^{-1} - P Q_\theta^{-1}\|_\infty \\ &\text{subject to} && \operatorname{Re}(P Q^{-1}) > 0 \end{aligned}$$

□

In step 3 passivity is enforced on reduce models. It is done in order to avoid complications in the quality factor approximation, since non-passive model will have jumps in this curve.

ALGORITHM 6.2—PRD PMOR

$$\begin{aligned} &\min \gamma \quad \text{subject to} \\ &\operatorname{Re} Q(\boldsymbol{\omega}) > 0 \quad \forall \boldsymbol{\omega} \in [0,\pi]^{n+1} \\ &\begin{pmatrix} \gamma f(\boldsymbol{\omega}) I & W(\boldsymbol{\omega})(G(\boldsymbol{\omega})Q(\boldsymbol{\omega}) - P(\boldsymbol{\omega})) \\ * & \gamma \operatorname{Re}(Q(\boldsymbol{\omega})) \end{pmatrix} > 0 \\ &f(\boldsymbol{\omega}) I \le \operatorname{Re}(Q(\boldsymbol{\omega})) \quad \forall \boldsymbol{\omega} \in \{\boldsymbol{\omega}\}_{i=1}^{N} \subset [0,\pi]^{n+1} \end{aligned}$$

□

For the PRD algorithm a passivity constraint can not imposed, however, with a good match passivity comes for free. The main drawback of the method is restrictive, in comparison with a stability one, positive real constraint. One can avoid some difficulties by using Algorithm 6.3 or by raising the order of approximations. In the parametrized case there are also advantages in comparison with the QCO, namely a more efficient parametrization and an explicit parameter dependence (for details, see [Sootla and Sou, 2010]).

Note that the conditions $\operatorname{Re}(Q) > 0$ and $A > 0$ require a special treatment, since they have to be enforced for all the frequencies $\boldsymbol{\omega}$. Meaning for all the parameters and the frequency variable, as well (for details, see [Sootla and Sou, 2010]). In the single variable case $f = A$ (or $f = \operatorname{Re}(Q)$) and it is a scalar (pseudo) polynomial.

As shown in the previous chapter pulling the dominant poles out of the optimization procedure can be useful due to numerical considerations. Thus the following procedure will be also used in the modeling:

ALGORITHM 6.3—DOMINANT POLE ALGORITHM

1. Identify dominant poles and zeros of the system G, for example, by running a non-parametrized version of algorithm 6.1, for some values of parameters. Denote $G_d(\boldsymbol{\omega})$ the transfer function with all the dominant poles and zeros.
2. Run the algorithm 6.1 (or the algorithm 6.2) incorporating the obtained data in the procedure, i.e. considering a new original model as $G_{new} = GG_d^{-1}$ or considering a new parameterization of the reduced model $\hat{G}_{new} = G_d PQ^{-1}$.

□

The choice of the parametrization should be made depending on an identified G_d. For some problems weighing the reduced model $\hat{G}$ as $G_d\hat{G}$ may create numerical problems. Similar problems may occur while weighing the original model G as GG_d^{-1}. Note that the parameterization $\hat{G}_{new} = \hat{G}PQ^{-1}$ for QCO algorithm will result in a minimization problem:

$$\min_{A,B} \|G - G_d G_d^{\sim} BA^{-1}\|_{\infty}$$

The choice of weight is left to a designer. By default consider $W = I$.

In this work all the methods were implemented using a cutting plane ellipsoidal method. A detailed description of the implementation may be found in [Sou, 2008].

6.4 Low-Order Modeling of an RF Inductor

Two modeling parameters are considered: wire separation D and wire width W. The original system G depends on these parameters in a nonlinear fashion. The input data to the algorithms consists of frequency samples calculated in advance. Two sets of data are distinguished: training and validation. The reduced model is calculated using the training data. The validation or testing data will be used for validation of the obtained model. The training grid is chosen as $W = [1, 2, 3, 4, 5]\mu m$, $D = [1, 2, 3, 4, 5]\mu m$ and, for every instance of parameters, 25 frequency samples are calculated. The testing grid is chosen as $W =$

$[1.5, 2.5, 3.5, 4.5]\mu m$, $D = [1.5, 2.5, 3.5, 4.5]\mu m$ and, for every instance of parameters, 20 frequency samples are computed. The comparison of results is done not only in frequency response, but also in quality factor QF, and inductance L. Rewrite the parameters W and D as the frequency variables ω_1 and ω_2 respectively, in order to achieve the desired problem formulations:

$$\min_{\hat{G}} \max_{\omega_1,\omega_2} \|G(\boldsymbol{\omega}) - \hat{G}(\boldsymbol{\omega})\|_\infty$$
$$\min_{\hat{G}} \max_{\omega_1,\omega_2} \|W(\boldsymbol{\omega})(G(\boldsymbol{\omega}) - \hat{G}(\boldsymbol{\omega}))\|_\infty,$$

with $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2)$. Both algorithms have a better performance in discrete time and the model is given in continuous time. Hence perform the discretization warping around ω_c, which is a frequency corresponding to the biggest derivative of the frequency response. Thus the frequencies are mapped to the interval $[0, \pi]$ using the following transformation:

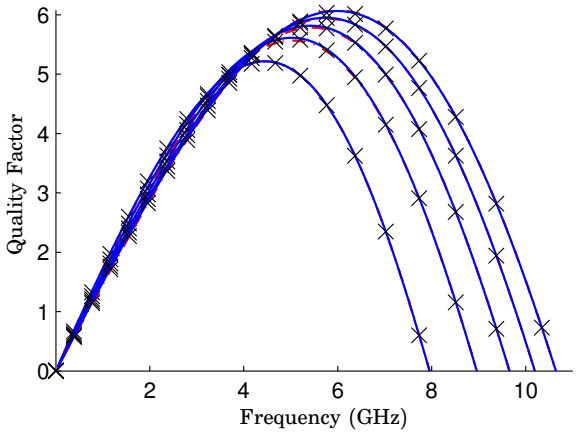
$$\omega_0 = 2\pi \operatorname{atan} \frac{2\pi f}{\omega_c},$$

where f is frequency in Hz. After the reduction procedure map the model back to continuous time. The warping procedure is done for numerical robustness, since it is desirable to have the changes in the frequency response to be as slow as possible.

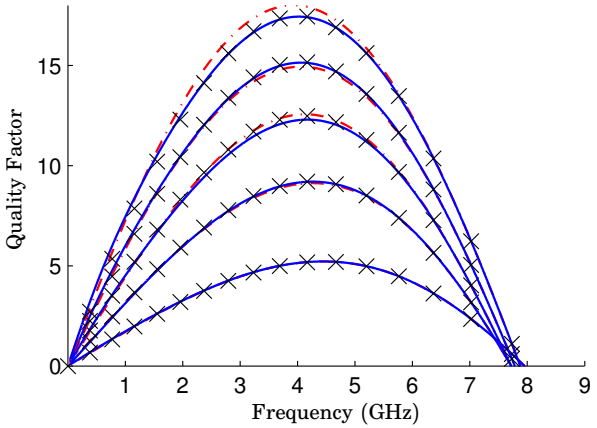
A Straight-Forward Algorithm Application

Here the algorithms 6.1 and 6.2 are applied directly to the training data. The QCO approximation procedure with orders $k_0 = 4$, $k_1 = 4$, and $k_2 = 4$ took a couple of hours and the resulting γ was 0.3%. Percents are taken with respect to the largest magnitude of the frequency response on the training grid. The PRD method provided a γ bound that was the same as for the QCO algorithm, i.e. 0.3%. The optimization was finished in around a half an hour. Although the value of γ for the QCO method is the same as for the PRD method, the actual performance of a PRD approximation could be better, since it is still required to calculate a stable model from the obtained QCO data.

The results on the training grid are not really surprising. Both the PRD and the QCO approximations give an almost exact fit in qual-

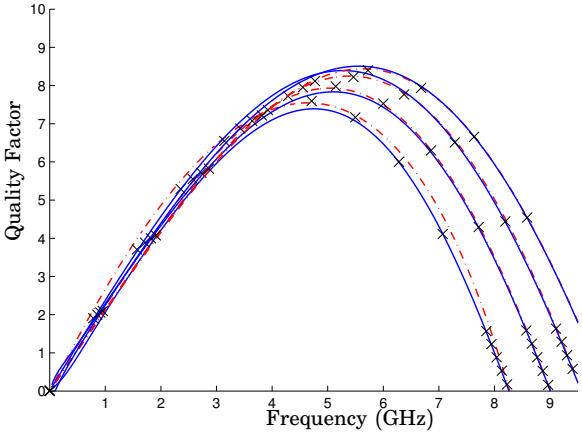


(a) Quality factor on the testing grid for $W = 1\mu m$, $D = 1$, 2, 3, 4,5μm.

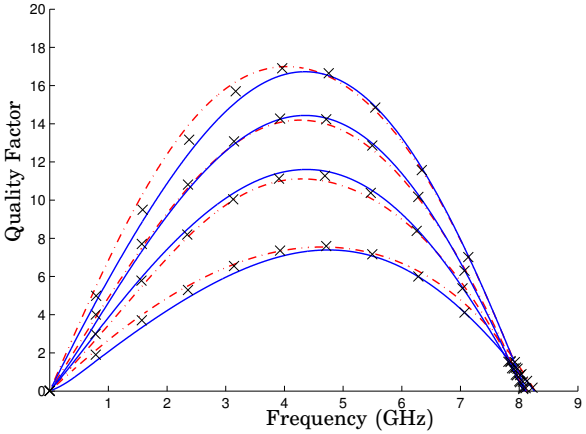


(b) Quality factor on the testing grid for $W = 1.5\mu m$, $D = 1.5$, 2.5, 3.5, 4.5μm.

Figure 6.1 Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

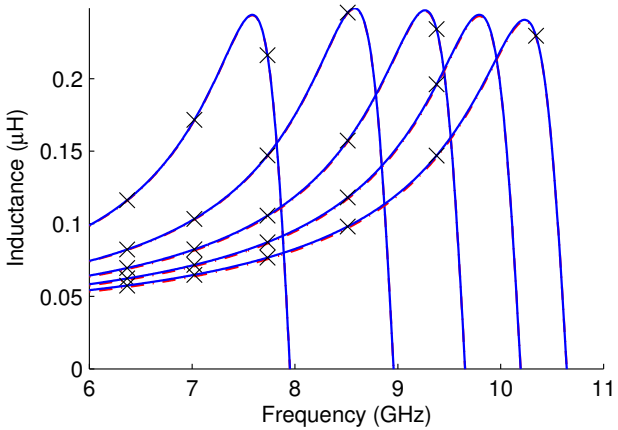


(a) Quality factor on the testing grid for $W = 1\mu m$, $D = 1, 2, 3, 4, 5\mu m$.

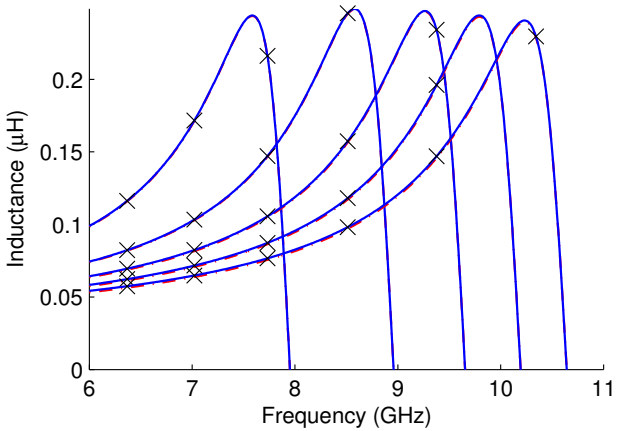


(b) Quality factor on the testing grid for $D = 1\mu m$, $W = 1, 2, 3, 4, 5\mu m$.

Figure 6.2 Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

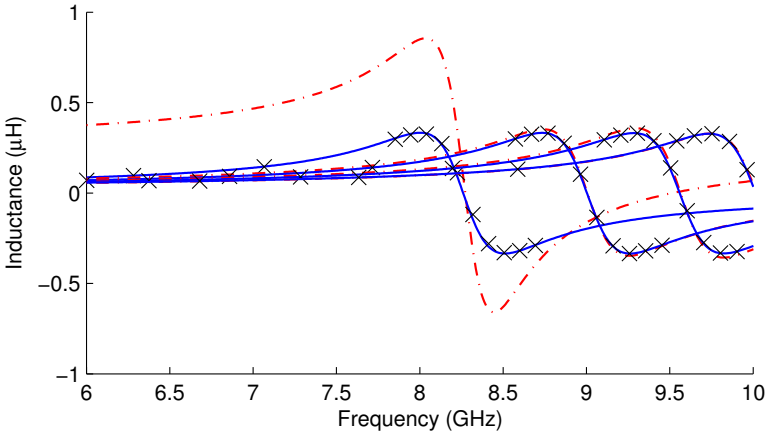


(a) Inductance on the training grid for $D = 1\mu m$, $W = 1, 2, 3, 4, 5\mu m$.

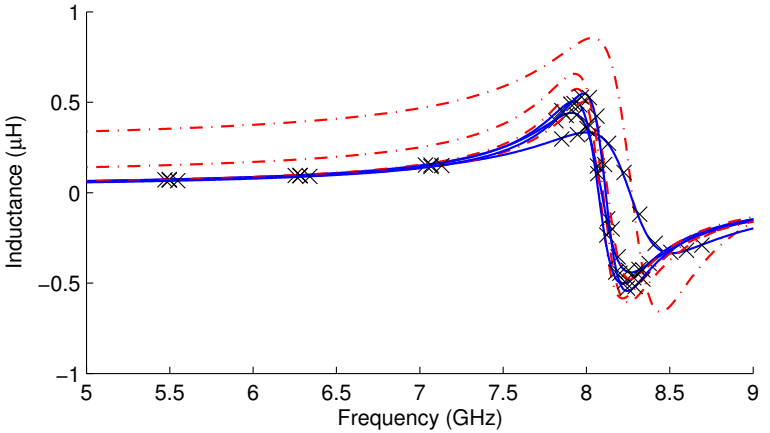


(b) Inductance on the training grid for $W = 1\mu m$, $D = 1, 2, 3, 4, 5\mu m$.

Figure 6.3 Inductance on the training grid. Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones (coincides with solid lines almost everywhere).



(a) Inductance on the testing grid for $W = 1.5\mu m$, $D = 1.5, 2.5, 3.5, 4.5\mu m$.



(b) Inductance on the testing grid for $D = 1.5\mu m$, $W = 1.5, 2.5, 3.5, 4.5\mu m$.

Figure 6.4 Inductance on the testing grid. Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

ity factor and inductance, which is expected given the good transfer function approximation in the training points (see figures 6.1, 6.3).

The most interesting is comparing the data on the validation or testing grid. In figure 6.2 the quality factors on the testing grid are presented. A similar plot to figure 6.3 (figure 6.4) shows a considerable difference in inductance between the PRD approximation (solid line) and the QCO (dash-dotted line). The reader may also notice, that the PRD approximation follows the testing data, which is represented by crosses, better than the QCO one.

In all the figures one can see that the QCO method does not provide a better approximation. It is supposed that the efficient parameterization in this problem contributes much more than the restriction on the denominator takes away. Even though there is a reasonable match in inductance and quality factor for all validation models the overall error is about 10% even for the PRD approximations. For some of the parameters the fit around the peak is bad. In order to get a better fit the orders of polynomials in approximation had to be raised. However, it takes considerable time, hence identifying first the dominant part is a reasonable option.

Identifying the Dominant Part of the Model

Now the dominant pole algorithm (algorithm 6.3) is applied to the RF inductor modeling problem. Using the non-parameterized version of the QCO algorithm the frequency response for every parameter in the training grid is modeled. The order of approximation is 12, yielding the approximation error $10^{-8}\%$ of the maximal magnitude in training data. The frequency data provides us with only one pair of dominant poles (i.e. there is only one peak). In this application identifying the dominant zeros did not improve the fit. Consider the dominant factor:

$$G_d = \frac{Kz^2}{(z - P_d)(z - \overline{P}_d)}$$

The gain and the poles are identified with a given data. Thus the gain K, the magnitude of pole $|P_d|$ and the phase of the pole $\angle(P_d)$ each are modeled with a rational function. The denominator and the numerator are polynomials of order 4 and 4 correspondingly. Here the goal is to achieve a good approximation on the testing grid.

Four versions for each of the PRD and the QCO algorithms are implemented.

***Minimization problem** I.* An easy to implement and probably a more numerically robust way to calculate the reduced models is to approximate the non dominant part of the original model G, i.e. approximating GG_d^{-1} instead:

$$\min_{\hat{G}} \|GG_d^{-1} - \hat{G}\|_\infty$$

***Minimization problem** II.* Another implementation which seems natural, is to substitute a part of PQ^{-1} with G_d.

$$\min_{\hat{G}} \|G - G_d\hat{G}\|_\infty$$

This formulation is more suitable if errors in the identification of the dominant part are anticipated. Since the original model and its dominant part are incorporated as different entities in the optimization procedure, the program can counteract to errors by changing $\hat{G}$.

***Minimization problem** III.* The main focus of this problem is to weight the low frequencies in such a way that the quality factor matching is improved. It corresponds to weighting the low frequencies. The *Butterworth* filter is chosen for a low-pass filter design. The design was performed using the MATLAB™ routine BUTTER. The cutoff frequency was fixed to the frequency of the corresponding dominant pole, i.e. the cutoff frequency and thus the weight depends on parameters. Denote a Butterworth filter for every instance of parameter in the training grid as $W_b(\boldsymbol{\omega})$. Weight only the magnitude and leave the phase unchanged. Therefore use $W(\omega_0, \omega_1, \omega_2) = |W_b(\omega_0, \omega_1, \omega_2)|$.

$$\min_{\hat{G}} \|W(GG_d^{-1} - \hat{G})\|_\infty$$

***Minimization problem** IV.* In this problem the same weight is used as in the previous one, but with another way of incorporating the dominant part into the simplification algorithm.

$$\min_{\hat{G}} \|W(G - G_d\hat{G})\|_\infty$$

Table 6.1 Approximation Errors

Methods	Approximation on the training grid		Quality Factor on the testing grid	
	QCO	PRD	QCO	PRD
Problem I	$20.1 \cdot 10^{-2}$	$6.7 \cdot 10^{-2}$	1.19	1.94
Problem II	$3.7 \cdot 10^{-2}$	$0.3 \cdot 10^{-2}$	6.12	0.63
Problem III	$13.1 \cdot 10^{-2}$	$6.7 \cdot 10^{-2}$	1.80	1.12
Problem IV	$13.1 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$	6.11	0.83

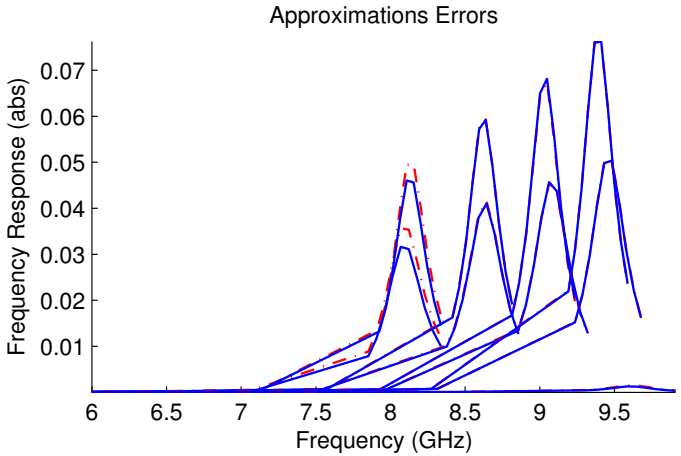


Figure 6.5 Approximation errors on the testing grid in %, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

The orders for parameter dependence were chosen as $k_1 = 4$, $k_2 = 3$, $k_3 = 3$. Since some parameter dependence is used to fit the dominant poles, these orders should be sufficient.

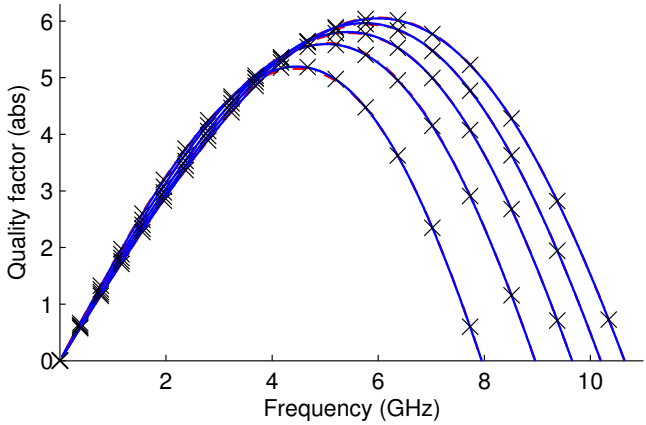
The errors of frequency response on the testing grid for all the method stays around 7%, even with higher order approximations. The errors occur around the resonant peak for some of the parameters (see, figure 6.5). Most likely due to an error in identification of the

dominant part. To raise accuracy even more more data is required for a better approximation of the dominant poles. However, even with this deficiency the obtained models are valuable, since the low-frequency approximations are precise and the characteristics, such as quality factor and inductance are accurately followed by the reduced models.

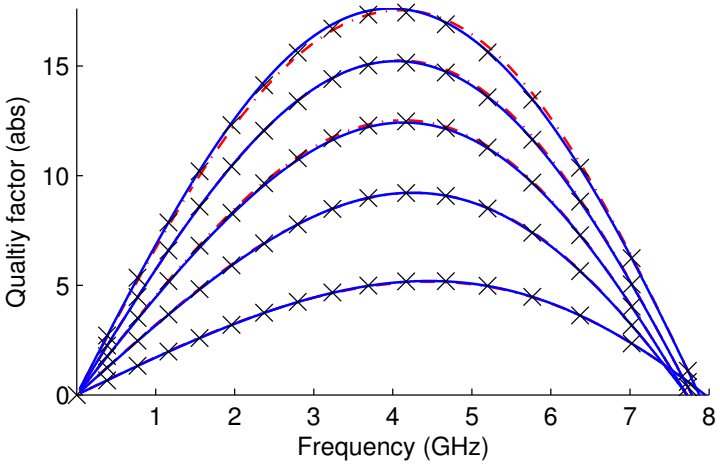
The models obtained by solving the problems I and III using QCO algorithm provided a poor quality factor matching. The accuracy was extremely bad for the frequencies less than 2 GHz, probably due to the problem formulation. It seems that the dominant part has a bad influence on the low frequencies. Hence the problem formulations I and III are not suitable for the QCO algorithm. The best results are depicted in figures 6.6, 6.7, 6.8, 6.9. That is the problem III QCO models and problem II PRD models.

6.5 Conclusion

In this chapter a number parameter-dependent model of an RF inductor are obtained. A direct application of used methods requires heavy calculations, in order to avoid those first the dominant part of the original model have been identified and only then the methods have been applied. The information about the dominant part is incorporated in the minimization program using different approaches. As a main result of the chapter low order models with a reasonable match in quality factor and inductance were obtained.

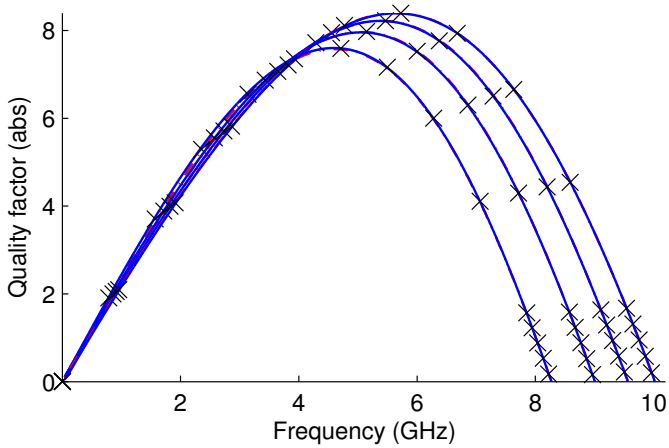


(a) Quality factor on the testing grid for $W = 1\mu m$, $D = 1, 2, 3, 4{,}5\mu m$.

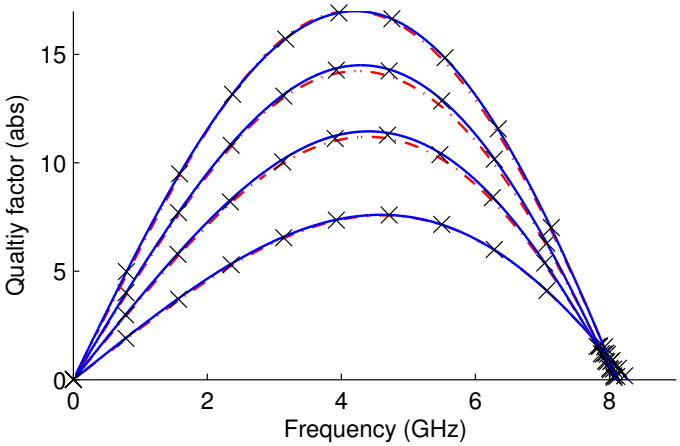


(b) Quality factor on the testing grid for $W = 1.5\mu m$, $D = 1.5, 2.5, 3.5, 4.5\mu m$.

Figure 6.6 Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

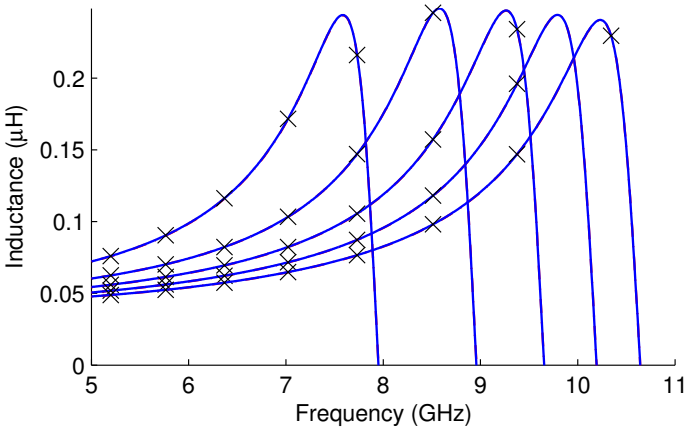


(a) Quality factor on the testing grid for $W = 1\mu m$, $D = 1, 2, 3, 4, 5\mu m$.

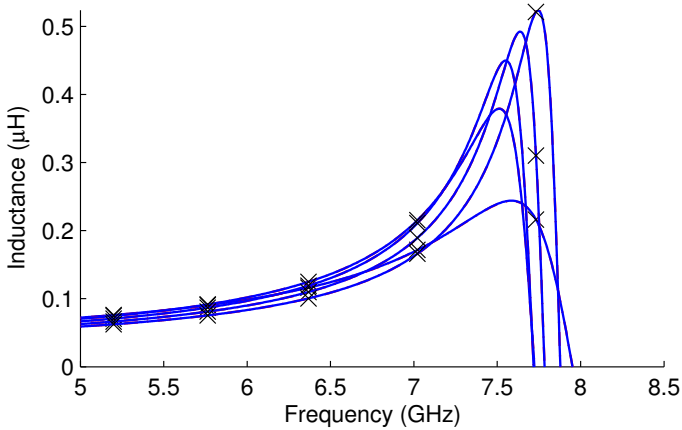


(b) Quality factor on the testing grid for $D = 1\mu m$, $W = 1, 2, 3, 4, 5\mu m$.

Figure 6.7 Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

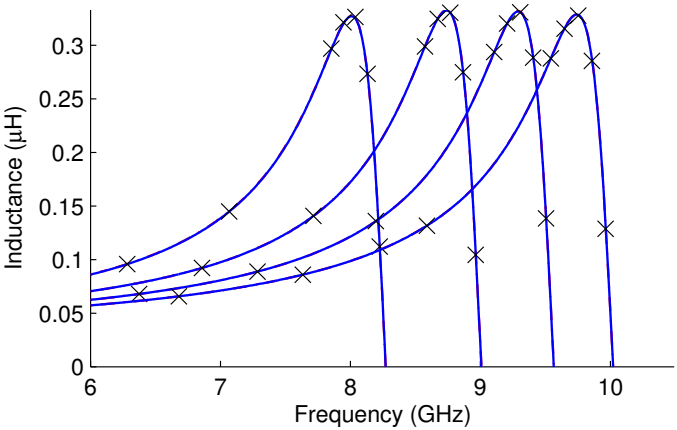


(a) Inductance on the training grid for $D = 1\mu m$, $W = 1, 2, 3, 4, 5\mu m$.

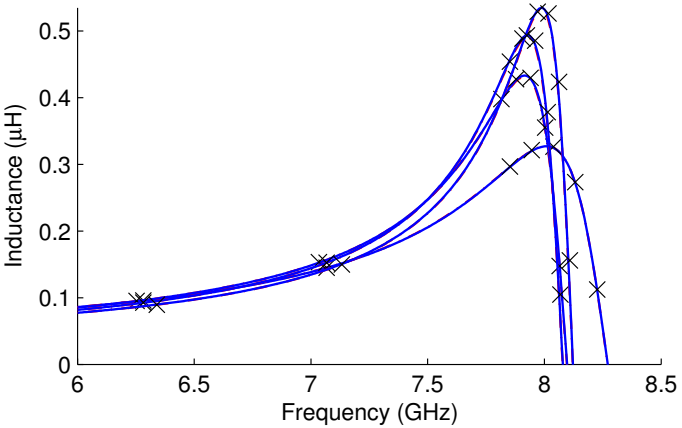


(b) Inductance on the training grid for $W = 1\mu m$, $D = 1, 2, 3, 4, 5\mu m$.

Figure 6.8 Inductance on the training grid. Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones (coincides with solid lines almost everywhere).



(a) Inductance on the testing grid for $W = 1.5\mu m$, $D = 1.5, 2.5, 3.5, 4.5\mu m$.



(b) Inductance on the testing grid for $D = 1.5\mu m$, $W = 1.5, 2.5, 3.5, 4.5\mu m$.

Figure 6.9 Inductance on the testing grid. Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

7

Conclusion and Future Work

In this thesis model reduction algorithms based on semidefinite programming were studied. Multivariable extensions to a recently proposed method with theoretical results justifying the relaxations were presented. Considerable focus was on the reduction of parameterized linear models. For a multivariable version of a parameterized model reduction the continuity of solutions was shown. Another reduction method based on optimization was developed. It can be used for the reduction of linear time invariant models, but its advantages are mostly revealed in the reduction of parameterized models.

There is quite a few interesting topics for future research. Considering the presented methods some questions were not answered, such as:

Problem 7.1—Global Error Bound
Given frequency response of the model G in points $\{\omega_i\}_{i=1}^N$ and the level of approximation γ provide an error bound between the frequency points. □

Discussion. This problem is interesting from the modeling perspective. It is important to know how close our model is to the actual system based on the frequency response or possibly other data. Some work in a similar direction have been already done in, for example, [Helmersson, 1994] or [den Hamer *et al.*, 2009]. However, application of these results is not straight-forward.

PROBLEM 7.2—CONSERVATISM OF MIMO EXTENSION
The reduction order can be set only to $k \cdot m$, where m is the number of outputs (or inputs) and k is the order of a polynomial approximation. □

Discussion. There is some conservatism in the multivariable relaxation itself, since a matrix is replaced with a scalar. For transfer functions with large number of inputs and outputs the method can fail. Solving the problem in a different manner, that does not have this relaxation is not yet developed. Enforcing the order not a multiple of m is easier in terms of understanding, however, the efficient ways to do that in this framework are still not clear.

PROBLEM 7.3—POSITIVE REAL DENOMINATOR APPROXIMATION
Consider a stable function G of degree n :

$$G = \frac{p_0 + p_1 z^{-1} + \cdots + p_n z^{-n}}{q_0 + q_1 z^{-1} + \cdots + q_n z^{-n}},$$

i.e. all the poles are inside the unit circle. Given the tolerance level $\varepsilon << 1$, identify an order $k > n$, such that a transfer function $\hat{G}$ of order k

$$\hat{G} = \frac{\hat{p}_0 + \hat{p}_1 z^{-1} + \cdots + \hat{p}_k z^{-k}}{\hat{q}_0 + \hat{q}_1 z^{-1} + \cdots + \hat{q}_k z^{-k}},$$

satisfies the following properties:

$$\|G - \hat{G}\|_\infty < \varepsilon \quad \text{and}$$
$$\mathrm{Re}\,(\hat{q}_0 + \hat{q}_1 z^{-1} + \cdots + \hat{q}_k z^{-k}) > 0, \forall z \notin \mathbb{D}$$

□

Discussion. The existence of such a k and ε can be shown. Since any transfer function can be approximated with an FIR filter. And an FIR filter can be approximated with the PRD method by setting $q \equiv 1$. A more interesting question is if there exist a uniform on G, ε bound on k, for a given n. Another related problem can be also formulated: *How many poles should one add to the transfer function, so that the denominator would become positive real.*

Regarding the development of the framework a couple of applications come in mind.

Application 7.1—Reduction of Multidimensional Systems
In signal processing and some other applications transfer functions that depend on two (or more) frequency variables are used, i.e. $G(z_1, z_2)$, with $z_1, z_2 \in \mathbb{C}$. The parameterized reduction framework can be applied to this kind of problems. There is a question of stability of G, which is complicated, since even the causality can be defined in many different manners. □

Application 7.2—Reduction of Linear Parameter Varying Models
Consider a system:

$$\begin{aligned} x(k+1) &= A(p)x(k) + B(p)u(k) \\ y(k) &= C(p)x(k) + D(p)u(k), \end{aligned}$$

where p is a parameter vector. Parameters can be of a great variety of nature and can be even time-dependent. A transfer function $G(z, p)$ depending on p can indeed be obtained, however, again there is a problem with stability. Stability of LPV systems is a complicated problem and to authors best knowledge can not be expressed as a convex condition of coefficients of G. As final remark it can be stated that a linear parameterized model is an LPV with slowly varying parameters, but the generalization requires a rigorous research. □

8

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Abstract

In this thesis model reduction methods for linear time invariant systems are investigated. The reduced models are computed using semidefinite programming. Two ways of imposing the stability constraint are considered. However, both approaches add a positivity constraint to the program. The input to the algorithms is a number of frequency response samples of the original model. This makes the computational complexity relatively low for large-scale models. Extra properties on a reduced model can also be enforced, as long as the properties can be expressed as convex conditions. Semidefinite program are solved using the interior point methods which are well developed, making the implementation simpler.

A number of extensions to the proposed methods were studied, for example, passive model reduction, frequency-weighted model reduction. An interesting extension is reduction of parameterized linear time invariant models, i.e. models with state-space matrices dependent on parameters. It is assumed, that parameters do not depend on state variables nor time. This extension is valuable in modeling, when a set of parameters has to be chosen to fit the required specifications. A good illustration of such a problem is modeling of a spiral radio frequency inductor. The physical model depends nonlinearly on two parameters: wire width and wire separation. To chose optimally both parameters a low-order model is usually created. The inductor modeling is considered as a case study in this thesis.

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