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# Physical Modeling of MIMO Antennas and Channels by Means of the Spherical Vector Wave Expansion

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## Abstract

In this paper we propose a new physically motivated model that allows to study the interaction between the antennas and the propagation channel for Multiple-Input Multiple-Output (MIMO) systems. The key tools employed in the model are the expansion coefficients of the electromagnetic field in spherical vector waves and the scattering matrix representation of the properties of the antenna. We derive the expansion of the MIMO channel matrix,  $\mathbf{H}$ , in spherical vector wave modes of the electromagnetic field of the antennas as well as the propagation channel. We also introduce the channel scattering dyadic,  $\mathcal{C}$ , with a corresponding correlation model for co- and cross-polarized elements and introduce the concept of mode-to-mode channel mapping, the  $\mathbf{M}$ -matrix, between the receive and transmit antenna modes. The  $\mathbf{M}$ -matrix maps the modes excited by the transmitting antenna to the modes exciting the receive antennas and vice versa. The covariance statistics of this  $\mathbf{M}$ -matrix are expressed as a function of the double-directional power-angular spectrum (PAS) of co- and cross-polarized components of the electromagnetic field. Our approach aims at gaining insights into the physics governing the interaction between antennas and channels and it is useful for studying the performance of different antenna designs in a specified propagation channel as well as for modeling the propagation channel. It can furthermore be used to quantify the optimal properties of antennas in a given propagation channel. We illustrate the developed methodology by analyzing the interaction of a  $2 \times 2$  system of slant polarized half-wavelength dipole antennas with some basic propagation channel models.

## 1 Introduction

In the last two decades, communication systems with multiport antenna systems at both the receiver and the transmitter, Multiple-Input Multiple-Output (MIMO) systems, have attracted much attention [6, 12, 28, 37, 38, 41]. These systems have the potential to provide higher bandwidth efficiency and greater robustness to fading in wireless systems due to their intrinsic ability to exploit the spatial and polarization domains. The antennas are fundamental elements of the physical layer and play an essential role in maximizing system performance for a given propagation channel. Therefore, a thorough understanding of the physics of the interaction between the antennas and the propagation channel is essential for analyzing and optimizing MIMO systems.

The first theoretical investigations of MIMO systems [12, 38, 41] were based on the transfer function matrix, the H-matrix, which contains as its elements the transfer function from each transmit antenna element to each receive antenna element<sup>1</sup> and therefore lumps the antenna properties together with the propagation channel. This approach does not allow studying the antenna-channel interaction and the antenna array optimization. To alleviate this problem, [32] introduced the "double-directional channel model" that describes the "directions-of-departure" (DoD) and

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<sup>1</sup>More specifically, from each transmit antenna connector to each receive antenna connector.

“directions-of-arrival” (DoA) of the multi-path components (MPCs) or plane waves. While this expansion has been used extensively in the past, e.g., [2, 9, 42] and is “natural” for propagation models, it is not compact (i.e., can require a large number of terms) and does not give straightforward insights of the interaction of channels especially with small antennas. We are therefore interested in an alternative, compact and physically tractable description of the joint properties of channels and antennas.

Fortunately, both the propagation channel and the antennas can be described in terms of the electromagnetic field and thus a homogeneous characterization is feasible. More precisely, it is possible to use the expansion of the electromagnetic field in *spherical vector waves* [18], together with the *scattering matrix*<sup>2</sup> representation of an antenna [17] to get a unified description. The spherical vector wave expansion is a natural way to express the polarization, angle, and spatial properties of MIMO systems. By expressing the channel directly in the spherical vector wave modes [3] it is possible to determine the characteristics of a multipoint antenna system for wireless transmission of information, in that same propagation channel. Our goal is to formulate a theoretical framework to study the mechanisms governing the interaction between antennas and channels in order to determine the optimum information transmission over wireless channels with multipoint antenna systems.

Previous theoretical studies employed spherical modes to represent the propagation channel in terms of scalar fields, see e.g., [15, 16, 19, 23, 29, 30, 33, 34]. However, the physics of electromagnetic fields is naturally described in terms of vector fields, where the polarization plays an important role. The application of the spherical *vector* wave expansion of electromagnetic waves as well as the modal expansion in guiding structures for *deterministic* MIMO channels has been intuitively outlined in [24]. There some initial insights into the electromagnetic MIMO channel capacity are also provided. We recently introduced the spherical vector wave mode expansion of the field and the scattering matrix representation of the antenna to quantify the interaction between the antennas and the propagation channel in a more physically meaningful way. For example in [13, 14, 26] we studied the spectral efficiency of MIMO antennas based on antenna theory and broadband matching theory in the isotropic channel (or 3D uniform channel). In [3] our focus was on the spatio-polar characterization of an arbitrary receive multipoint antenna in a random electromagnetic field. However, the physics of the interaction between receive antennas, the transmit antennas, *and* the propagation channel in MIMO systems was not addressed in previous research. In this work we extend the framework for analysis of antenna-channel interaction to MIMO channels and multipoint antenna systems at Tx and Rx. This is accomplished by expressing both the *stochastic* channel and *deterministic* antennas in a unified way. The main contributions of the paper can be summarized as follows:

1. We introduce the concept of mode-to-mode channel matrix, the **M**-matrix, to describe the coupling between the modes excited by the transmit and the

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<sup>2</sup>It is worthwhile to notice that all properties of multi-port antennas can be derived from the scattering matrix representation, [17], inclusive mutual coupling between antennas elements of the same multi-port antenna and/or between antennas belonging to different antenna systems and spatially separated as outlined in [22].

receive antennas. The  $\mathbf{M}$ -matrix contains all relevant information about the channel on a fundamental level, information about the spherical vector wave modes that are the most likely to be excited by the propagation channel. The  $\mathbf{M}$ -matrix also provides a mapping of modes excited at the receiver and transmitter, respectively.

2. Using the correlation model for the amplitudes incident at the receive antenna [3], we derive a general correlation model for the double-directional channel. More specifically, we study the correlation between the components of the *channel scattering dyadic*<sup>3</sup>  $\mathcal{C}$ . The dyadic  $\mathcal{C}$  maps the field radiated by the transmitting antenna to that impinging at the receive antenna by superposition of plane waves<sup>4</sup>. We show that the formulation is equivalent to the single-scatterer process, where the scatterer represents the propagation channel<sup>5</sup>.
3. We expand the channel transfer function matrix, the  $\mathbf{H}$ -matrix, in spherical vector wave modes, i.e., the  $\mathbf{M}$ -matrix, using the derived correlation model for the elements of the *channel scattering dyadic*. We also provide results for first and second-order statistics of the expansion coefficients based on the assumption that the dyadic elements are independently distributed Gaussian variables.

The derived equations allow us to establish a relationship between  $\mathbf{H}$  and  $\mathbf{M}$ , and therefore to describe the spatial, directional and polarization properties of the channel and the antennas for MIMO systems in spherical vector waves.

The remainder of the paper is organized as follows. In Sec. 2 we present a brief introduction to the spherical vector wave expansion of the electromagnetic field and the antenna scattering matrix. Here we also introduce the  $\mathbf{M}$ -matrix concept. Sec. 3 introduces the channel scattering dyadic,  $\mathcal{C}$ , and some of its properties, and derives a model for the correlation between the dyadic components. In Sec. 4, we provide the expansion of the channel matrix  $\mathbf{H}$  in spherical vector waves. We further show some properties of the expansion coefficients and provide a brief discussion with some specific examples. Sec. 5 provides simulation results for a  $2 \times 2$  MIMO slant-antenna polarization system with half-wavelength dipoles orthogonally placed with respect to each other, in a generic propagation channel. Finally, a summary with conclusions is in given Sec. 6.

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<sup>3</sup>This is basically the double-directional channel transfer function introduced in [32] in combination with the physical representation in [35].

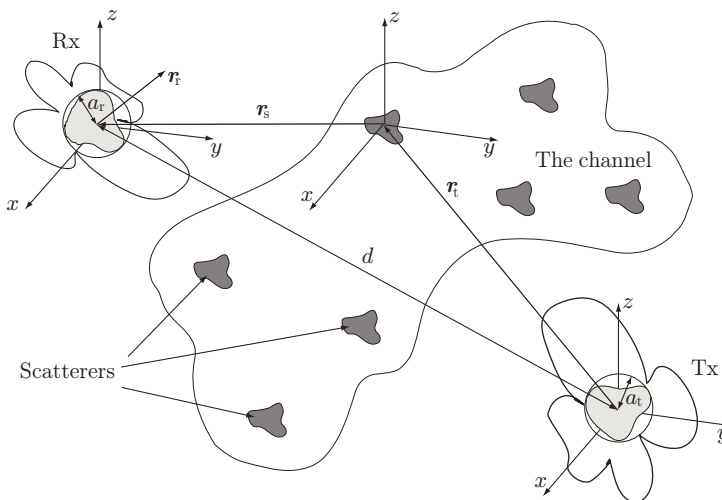
<sup>4</sup>This is a good approximation when both transmitting and receiving antennas are in each others' far-field region as well as when the distance from the antennas to the scatterers are much larger than the size of the scatterers. This approximation is required for the scattering approach assumed in this paper.

<sup>5</sup>It is worthwhile to notice that this equivalence only holds in the narrowband case.

## 2 Mode-to-Mode MIMO channel matrix, $\mathbf{M}$

In this section we present a straightforward treatment that aims at interconnecting the signals from the receive antenna, the field from the transmit antenna and the field impinging at the receive antenna. Building on the approach of [3], we define two main mathematical tools that describe the interaction between antennas and channels. This approach provides a complete description of reciprocal antennas by using the scattering matrix of the antennas of each multi-port antenna system involved in the communications.

Consider two multi-port antennas separated by a distance  $d$  as shown in Fig. 1 where one of them acts as a transmitter (Tx) and one acts as a receiver (Rx). Further assume that each antenna phase center coincides with the origin of their own coordinate system and that there is no mutual coupling between the Tx and Rx antennas<sup>6</sup> though we do allow for mutual coupling between the elements of the TX antenna (and similarly for the RX antenna).



**Figure 1:** Schematic representation of the propagation channel with non interacting scatterers and the antennas.

The electric field emitted by the Tx antenna,  $\mathbf{E}^{(t)}(\mathbf{r}_t)$ , can be expanded into *outgoing spherical vector waves*  $\mathbf{u}_\kappa^{(2)}(k\mathbf{r}_t)$ , where  $\mathbf{r}_t$  is the coordinate vector with origin in the phase center of the Tx and  $\kappa \rightarrow (\tau, m, l)$  is a multi-index that is also calculated as<sup>7</sup>  $\kappa = 2(l^2 + l - 1 + m) + \tau$ , see Appendices A and B. Then, the expansion of the transmitted electric field in a source-free region (all sources are inside the sphere with radius  $a_t$ ) [18], can be expressed as

$$\mathbf{E}^{(t)}(\mathbf{r}_t) = k\sqrt{2\eta} \sum_{\kappa} b_{\kappa} \mathbf{u}_{\kappa}^{(2)}(k\mathbf{r}_t), \text{ for } |\mathbf{r}_t| \geq a_t. \quad (2.1)$$

<sup>6</sup>This is fulfilled when  $d$  is larger than a few wavelengths, i.e., in most practically relevant cases.

<sup>7</sup> $\kappa = 2(l^2 + l - 1 + m) + \tau$  is computed for  $l = 1 \dots l_{max}$ ,  $m = -l \dots l$  and  $\tau = 1, 2$ .

where  $\eta$  is the free-space impedance,  $k$  is the wave-number and the  $b_k$  are the expansion coefficients. Similarly, the electric field sensed by the receive antenna,  $\mathbf{E}^{(r)}(\mathbf{r}_r)$ , can be expanded in *incoming spherical vector waves*  $\mathbf{u}_\iota^{(1)}(k\mathbf{r}_r)$

$$\mathbf{E}^{(r)}(\mathbf{r}_r) = k\sqrt{2\eta} \sum_{\iota} a_{\iota} \mathbf{u}_{\iota}^{(1)}(k\mathbf{r}_r), \text{ for } |\mathbf{r}_r| \geq a_r, \quad (2.2)$$

where  $\iota \rightarrow (t, \mu, \lambda)$  is the multi-index notation for the Rx antenna, which is computed as  $\iota = 2(\lambda^2 + \lambda - 1 + \mu) + t$ .

The scattering matrix<sup>8</sup> of an  $N$ -port antenna provides a full description of all its properties [17], i.e., the incoming signals,  $\mathbf{v} \in \mathbf{C}^{N \times 1}$  and waves,  $\mathbf{a} \in \mathbf{C}^{\infty \times 1}$ , the outgoing signals,  $\mathbf{w} \in \mathbf{C}^{N \times 1}$  and waves  $\mathbf{b} \in \mathbf{C}^{\infty \times 1}$ , the matrix containing the complex antenna reflection coefficients,  $\mathbf{\Gamma} \in \mathbf{C}^{N \times N}$ , the matrix containing the antenna receiving coefficients,  $\mathbf{R} \in \mathbf{C}^{N \times \infty}$ , the matrix containing the antenna transmitting coefficients,  $\mathbf{T} \in \mathbf{C}^{\infty \times N}$  and the matrix containing the antenna scattering coefficients  $\mathbf{S} \in \mathbf{C}^{\infty \times \infty}$  [17]

$$\begin{pmatrix} \mathbf{\Gamma} & \mathbf{R} \\ \mathbf{T} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ \mathbf{b} \end{pmatrix}. \quad (2.3)$$

Based on the spherical wave expansion above and the scattering matrix representation of the antenna<sup>9</sup> we know that a transmitting antenna is characterized by  $\mathbf{a} = \mathbf{0}$  and  $\mathbf{w} = \mathbf{0}$ . Consequently, the transmitted signals,  $\mathbf{v}$ , are mapped into the outgoing spherical vector wave expansion coefficients,  $\mathbf{b}$ , by the transmission matrix,  $\mathbf{T}$ , as

$$\mathbf{b} = \mathbf{T}\mathbf{v}. \quad (2.4)$$

On the other hand, at the receiving side, setting  $\mathbf{b} = \mathbf{0}$  and  $\mathbf{v} = \mathbf{0}$ , the incoming spherical vector wave expansion coefficients,  $\mathbf{a}$ , are mapped into the received signals,  $\mathbf{w}$ , through the antenna matrix,  $\mathbf{R}$ , as

$$\mathbf{w} = \mathbf{R}\mathbf{a}. \quad (2.5)$$

In order to establish a relationship between input-output signals at the transmit and the receive antennas, i.e., transmitted signals,  $\mathbf{v}$ , and received signals,  $\mathbf{w}$ , we need first to establish a mapping between the outgoing waves at the transmit antennas,  $\mathbf{b}$  and the incoming waves at the receive antenna,  $\mathbf{a}$ . We do this by using a mode-to-mode mapping  $\mathbf{M}$  [14]

$$\mathbf{a} = \mathbf{M}\mathbf{b}. \quad (2.6)$$

The mode-to-mode MIMO channel matrix,  $\mathbf{M}$ , is a stochastic matrix that describes the properties of the wireless channel in terms of the multimode expansion coefficients of the electromagnetic field. Hence, combining (2.4)-(2.6) we arrive at the

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<sup>8</sup>From here and on we will use the term scattering matrix to denote different mathematical objects, whose meaning will follow from the context.

<sup>9</sup>It is worthwhile to notice that transmit and the receive antennas are each characterized by a scattering matrix (see (2.3)), though with different parameters. Moreover, we use the same notation for both the transmit and receive antennas. However, through the paper,  $\mathbf{a}$ ,  $\mathbf{w}$ ,  $\mathbf{R}$  and  $\mathbf{b}$ ,  $\mathbf{v}$ ,  $\mathbf{T}$  are used to identify the receive and transmit antennas, respectively.



following linear relationship for the MIMO channel

$$\mathbf{w} = \mathbf{RMT}\mathbf{v}. \quad (2.7)$$

Denoting the MIMO channel transfer function by

$$\mathbf{H} = \mathbf{RMT}, \quad (2.8)$$

we then arrive at the classical model for the input-output relation between the transmitted,  $\mathbf{x} = \mathbf{v}$ , and the received signals in a noisy channel,  $\mathbf{y} = \mathbf{w} + \mathbf{n}$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \quad (2.9)$$

where  $\mathbf{n}$  is the Additive White Gaussian Noise (AWGN) component.

Expressions (2.4)-(2.9) establish a full chain of relationships that enables the analysis of the interaction between the antennas and the propagation channel and their impact on the communication link by simple relationships. The classical transfer function matrix  $\mathbf{H}$  is a linear function of the physical properties of both the antennas,  $\mathbf{R}$  and  $\mathbf{T}$  and the propagation channel  $\mathbf{M}$ .

### 3 The Channel Scattering $\mathcal{C}$ -Dyadic

In the previous section we introduced a mapping, the  $\mathbf{M}$ -matrix, between the modes excited by the transmit antennas to the modes excited by the receive antenna. The spatio-polar selectivity nature of the wireless propagation channels has been extensively investigated for stochastic propagation channels, we know therefore that the properties of the  $\mathbf{M}$ -matrix should be a function of the polarimetric double-directional impulse response (if we are considering a single, deterministic, channel) or the Power Angular Spectrum (PAS) of each of the orthogonal polarizations of the electromagnetic waves, as well as the depolarization of transmitted waves. The dependence of the  $\mathbf{M}$ -matrix on the propagation channel need to be studied experimentally to fully establish its properties.

In the following, we show from first principles, that the multiple-scattering processes taking place in the wireless propagation channel can be described by a scattering dyadic. Its representation is similar to the scattering dyadic when only a single-scattering process is considered. This is of course an *effective* (or equivalent) description of the actual scattering process since multiple-scattering actually takes place within this *effective scatterer*, i.e., the channel. Backscattering from the channel to the transmitting antenna as well as from the receiving antenna to the channel is neglected. Hence, we can model the channel with a "black box" where the energy that is transmitted in direction  $\hat{\mathbf{k}}_t$  (the symbol  $(\cdot)$  denotes a unit vector) traverses the channel by means of an arbitrary number of interactions with physical objects (the interactions may be due to scattering, specular reflection, diffraction, and others) and arrives at the receiver from one or multiple directions  $\hat{\mathbf{k}}_r$ . In the following, we first derive some properties of the  $\mathbf{M}$ -matrix for some special cases

(only one scatterer present; only single-scattering processes), and then give a more general description.

Consider now the situation where there is a single linear scatterer present at a position in space defined by the radius-vector  $\mathbf{r}_t$  defined in the coordinate system of the Tx antenna, see Fig. 2. The field radiated by the transmitting antenna port  $j$ ,  $\mathbf{E}_j(\mathbf{r}_t)$ , is defined in the far-field region by the far-field amplitude,  $\mathbf{F}_j(\hat{\mathbf{r}}_t)$  radiated in direction  $\hat{\mathbf{r}}_t = \mathbf{r}_t/r_t$

$$\mathbf{E}_j(\mathbf{r}_t) = \mathbf{F}_j(\hat{\mathbf{r}}_t) \frac{e^{-ikr_t}}{kr_t} + \mathcal{O}(r_t^{-2}) \text{ as } r_t \rightarrow \infty, \quad (3.1)$$

where  $\mathcal{O}(x^n)$  is the "big-O" notation standing for "order of" asymptotics, i.e.,  $|\mathcal{O}(x^n)/x^n| < C$  as  $x \rightarrow \infty$ , and  $r_t = |\mathbf{r}_t|$ .

An electric field  $\mathbf{E}_j$  impinges on a scatterer from direction  $\hat{\mathbf{r}}_t$ . In the far-field region, the scattered electric field  $\mathbf{E}_s$  is fully described by the far-field amplitude  $\mathbf{F}_s$  scattered in direction  $\hat{\mathbf{r}}_s = \mathbf{r}_s/r_s$  as

$$\mathbf{E}_s(\mathbf{r}_s) = \mathbf{F}_s(\hat{\mathbf{r}}_s) \frac{e^{-ikr_s}}{kr_s} + \mathcal{O}(r_s^{-2}) \text{ as } r_s \rightarrow \infty, \quad (3.2)$$

where  $\mathbf{F}_s$  can be expressed in terms of the scattering dyadic  $\mathcal{S}(\hat{\mathbf{r}}_s, \hat{\mathbf{r}}_t)$

$$\mathbf{F}_s(\hat{\mathbf{r}}_s) = \mathcal{S}(\hat{\mathbf{r}}_s, \hat{\mathbf{r}}_t) \cdot \mathbf{E}_j(\mathbf{r}_t), \quad (3.3)$$

where we have assumed that the amplitude of the plane wave incident at the scatterer from direction  $\hat{\mathbf{r}}_t$  is given by  $\mathbf{E}_j(\hat{\mathbf{r}}_t)$ . Hence, from (3.1)-(3.3), the scattered field can be expressed as

$$\mathbf{E}_s(\mathbf{r}_s) \approx \mathcal{S}(\hat{\mathbf{r}}_s, \hat{\mathbf{r}}_t) \cdot \mathbf{F}_j(\hat{\mathbf{r}}_t) \frac{e^{-ik(r_s+r_t)}}{k^2 r_s r_t} \text{ as } r_s, r_t \rightarrow \infty \quad (3.4)$$

In the far-field region the properties of the field are those of a plane wave, therefore the scattered field in the far-field region can be described by a plane wave

$$\mathbf{E}_s(\mathbf{r}_r) \approx \mathbf{E}_0 e^{-ik\hat{\mathbf{r}}_s \cdot \mathbf{r}_r}, \quad (3.5)$$

with the complex amplitude given by

$$\mathbf{E}_0 = \frac{e^{-ikr_s}}{kr_s} \mathcal{S}(\hat{\mathbf{r}}_s, \hat{\mathbf{r}}_t) \cdot \mathbf{E}_j(\mathbf{r}_t). \quad (3.6)$$

In wireless communications channels, it is seldom the case that there is only a single scatterer interacting with the receive and transmit antennas. Many scatterers have to be considered in order to completely define the propagation channel. This is hard due to the fact that multiple-scattering propagation that takes place and the shape and electromagnetic properties of each scatterer are not exactly known. However, in many applications the exact physical properties of the channel scatterers

are superfluous<sup>10</sup> rather, we are mainly concerned with the statistical description of the channel, which is a widely used approach [5, 25, 42].

We next consider a situation where many scatterers are present between the transmit and the receive antennas. We assume that: 1) all the scatterers are in the far-field regions of both the receiver and the transmitter, and 2) scatterers are grouped into a *group of scatterers*<sup>11</sup>, where the maximum dimension of a group of scatterers is much smaller than the distances to both the receiver and the transmitter.

It can be shown that, if the assumption 2) above holds, then the scattered field  $\mathbf{E}_s$  can be written in a similar way as the single scatterer case (3.3)-(3.4), where the *effective scattering dyadic*<sup>12</sup>  $\mathcal{S}_c^e(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  replaces  $\mathcal{S}$  for each group of scatterers.

The key difference is that the contributions from the scatterers in each group of scatterers are superimposed, leading to fading - in other words, the field arising from each group of scatterers can be modeled as a stochastic process. We can then express the total scattered far-field as a superposition of the scattered fields from each group of scatterers. Hence, we arrive at the following relationship for  $\mathbf{E}_s$

$$\mathbf{E}_s(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) = \mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \cdot \mathbf{F}_j(\hat{\mathbf{k}}_t), \quad (3.7)$$

where  $\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  is the *channel scattering dyadic*, [32, 35]

$$\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) = \sum_c \frac{e^{-ik(r_{rc}+r_{tc})}}{k^2 r_{rc} r_{tc}} \mathcal{S}_c^e(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t), \quad (3.8)$$

where the summation is over the group of scatterers,  $r_{rc}$  and  $r_{tc}$  are the distances between a reference position within the group of scatterers and the receiver and the transmitter, respectively.

Since the scatterers are assumed to be linear it follows that  $\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  satisfies an identity similar to (7.38) in Appendix E for reciprocal channels

$$\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) = \mathcal{C}^T(-\hat{\mathbf{k}}_t, -\hat{\mathbf{k}}_r). \quad (3.9)$$

As a result of the assumptions made, one can further state that the received field, i.e., the field incident at the receiver can be obtained as a linear transformation of the transmitted field. Components of the electric field generated by the transmitter antenna and the electric field available at the receiver are connected through a matrix that could be interpreted as a "scattering" dyadic,  $\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$ , which is basically the

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<sup>10</sup>Of course, if there are scatterers in the near-field region of the antenna this assumption is not valid. For a handheld terminal the user's head, hands or body will in general have a large impact on the performance. Similarly, for a base station scenario structures surrounding the base station antenna will affect its radiation pattern.

<sup>11</sup>The definition of a group of scatterers here is related to both the classical definition of a cluster and the concept of multipath-groups. However, strictly speaking it doesn't fit either of them. A thorough description of both cluster and multipath group concepts can be found in [8-10].

<sup>12</sup>Observe that in order to introduce a more general framework we from now on express the directionality of the channels and the antennas in terms of wave vectors,  $\hat{\mathbf{k}}$ , instead of radius vectors  $\hat{\mathbf{r}}$ .

channel transfer function in the “angular domain”,  $(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$ , or equivalently, the double-directional impulse response or transfer function [32, 42].

It is worthwhile to observe that the superposition of the scattered fields from different group of scatterers is true only when there is no interaction or coupling between each group of scatterers; in reality the coupling between groups of scatterers always exists. This is an interesting question that needs further investigations, but it is outside the scope of this paper.

### 3.1 Correlation model for the channel scattering dyadic, $\mathcal{C}$

In order to further study the statistical properties of the  $\mathbf{M}$ -matrix we need to introduce a correlation model for co- and cross-polarized elements of the channel scattering dyadic. We write the stochastic matrix  $\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  as

$$\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) = \begin{pmatrix} \mathcal{C}_{\theta\theta} & \mathcal{C}_{\theta\phi} \\ \mathcal{C}_{\phi\theta} & \mathcal{C}_{\phi\phi} \end{pmatrix}, \quad (3.10)$$

where  $\mathcal{C}_{\theta\theta}$  and  $\mathcal{C}_{\phi\phi}$  are co-polarized components,  $\mathcal{C}_{\theta\phi}$  and  $\mathcal{C}_{\phi\theta}$  are the cross-polarized components, with  $\theta$  and  $\phi$  denoting two orthogonal polarizations. We further postulate that each entry of  $\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  is a zero-mean complex Gaussian<sup>13</sup> variable

$$\langle \mathcal{C}_{\alpha\beta} \rangle = 0. \quad (3.11)$$

where  $\alpha = \{\theta, \phi\}$  and  $\beta = \{\theta, \phi\}$  and  $\langle \cdot \rangle$  denotes the ensemble average see, e.g., [27], p.285).

The cross-covariance is given by

$$\langle \mathcal{C}_{\alpha\beta} \mathcal{C}_{\alpha'\beta'}^* \rangle = \mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \delta^2(\hat{\mathbf{k}}_r - \hat{\mathbf{k}}_r') \delta^2(\hat{\mathbf{k}}_t - \hat{\mathbf{k}}_t') \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \quad (3.12)$$

where  $\delta^2(\hat{\mathbf{k}}) = \delta(\theta)\delta(\phi)/\sin(\theta)$  denotes the Dirac-delta in spherical coordinates defined on the sphere of unit radius, the asterisk  $(.)^*$  denotes complex conjugate, and  $\mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  denotes the double-directional power angular spectrum (DD-PAS) between polarization  $\alpha$  at the receiver and polarization  $\beta$  at the transmitter. See, e.g., [2, 7, 31, 32] for further details on the DD-PAS.

The DD-PAS of both co- and cross-polarized components can be expressed in terms of joint probability distributions of the angle of arrivals (AoAs) and angle of departures (AoDs) following the convention presented in [3]

$$\mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) = P_{\alpha\beta} p_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t), \quad (3.13)$$

where  $P_{\alpha\beta}$  denotes the co- or cross-coupling power between polarizations along  $\alpha$  and  $\beta$ . The joint probability density function  $p_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  satisfies the normalization

$$\int \int p_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) d\Omega_r d\Omega_t = 1. \quad (3.14)$$

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<sup>13</sup>The Gaussian assumption is valid if the number of scatterers in each group of scatterers is large and none of them are dominating [10].

Hence,  $P_{\alpha\beta} = \int \int \mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) d\Omega_r d\Omega_t$ , where we have expressed  $\mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$  in spherical coordinates.

The covariance model (3.12) is, as we show in Appendix C, a direct consequence of the correlation model for the incident field presented in [3] and the reciprocity of the wireless propagation channel. Here, for the sake of clarity, we restrict our analysis to the Rayleigh fading environment. Extension to the more general Rice case is straightforward following the exposition presented here and in [3]. The main assumptions for the model of the Rayleigh case are<sup>14</sup>:

1. The phases of the co-polarized waves are independent for different DoAs  $\hat{\mathbf{k}}$  and  $\hat{\mathbf{k}}'$ .
2. The phases of the cross-polarized waves are independent for any DoAs  $\hat{\mathbf{k}}$  and  $\hat{\mathbf{k}}'$ .

Let  $E_\alpha, E'_\beta$  denote the complex amplitudes of the random incident electric field in  $\alpha$  and  $\beta$  polarizations, respectively, and  $\tilde{E}_\alpha$  and  $\tilde{E}_\beta$  denote the corresponding components of the plane wave spectrum, then we summarize the above postulates

$$\langle \tilde{E}_\alpha \tilde{E}'_{\beta*} \rangle = \langle E_\alpha E'_\alpha \rangle \delta^2(\hat{\mathbf{k}} - \hat{\mathbf{k}}') \delta_{\alpha\beta}, \quad (3.15)$$

where  $\delta_{\alpha\beta}$  denotes the Kronecker-delta function.

For the characterization we also need the cross polarization ratio (XPR) that characterizes the power imbalance between the two orthogonal polarizations. It is defined as the ratio between the power in the  $\theta$ -polarization and the power in the  $\phi$ -polarization

$$\chi = \frac{P_{\theta\theta} + P_{\theta\phi}}{P_{\phi\theta} + P_{\phi\phi}}, \quad (3.16)$$

where the power of the co- and cross-polarized components have been defined above<sup>15</sup>.

## 4 Spherical vector wave expansion of the double-directional MIMO channel

Our main focus now is to derive the correlation properties of the mode-to-mode channel, the  $\mathbf{M}$ -matrix, given the PAS of the double-directional channel. The incident field can be described in either plane waves or spherical vector waves. This section gives the transformation between the two.

The expansion of an arbitrary electromagnetic field  $\mathbf{E}(\mathbf{r})$  at the observation point in space  $\mathbf{r}$  in regular spherical vector waves,  $\mathbf{v}_l(k\mathbf{r})$ , (see Appendix A) can be written as

$$\mathbf{E}(\mathbf{r}) = k\sqrt{2\eta} \sum_l f_l \mathbf{v}_l(k\mathbf{r}). \quad (4.1)$$

<sup>14</sup>Note that these assumptions are a straightforward generalization of the generalized WSSUS (Wide Sense Stationary Uncorrelated Scattering) assumption [11, 20].

<sup>15</sup>In many practical cases,  $P_{\theta\phi} \approx P_{\phi\theta} \ll P_{\theta\theta}, P_{\phi\phi}$  and the XPR reduces to  $\chi = \frac{P_{\theta\theta}}{P_{\phi\phi}}$ .

The expansion coefficients  $f_\iota$  [3] are given by

$$f_\iota = \frac{4\pi(-i)^{\lambda-t+1}}{k\sqrt{2\eta}} \int \mathbf{A}_\iota^*(\hat{\mathbf{k}}_r) \cdot \tilde{\mathbf{E}}_0(\hat{\mathbf{k}}_r) d\Omega_r, \quad (4.2)$$

where  $\tilde{\mathbf{E}}_0(\hat{\mathbf{k}}_r)$  denotes the amplitude of the (random) complex plane-wave spectrum (PWS) in the direction  $\hat{\mathbf{k}}_r$ , and  $\mathbf{A}_\iota^*(\hat{\mathbf{k}}_r)$  is the complex conjugated spherical vector harmonic with index set  $\iota \rightarrow \{t\mu\lambda\}$ . The relationship between the incident field at the antenna and the antenna field of the receiver is given by (2.5) and (4.2) as we outlined in [3]. Next we relate the incident field to the transmit field, as given by (2.4)-(2.6). Observing that the PWS of the incident field can be obtained from the integral over the AoD, (3.7)

$$\tilde{\mathbf{E}}_0(\hat{\mathbf{k}}_r) = \int \mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \cdot \mathbf{F}_j(\hat{\mathbf{k}}_t) d\Omega_t. \quad (4.3)$$

We further expand the far-field amplitude of the transmit field in spherical vector waves in the corresponding coordinate system

$$\mathbf{F}_j(\hat{\mathbf{r}}_t) = k\sqrt{2\eta} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l i^{l+2-\tau} T_{\kappa j} v_j \mathbf{A}_\kappa(\hat{\mathbf{k}}_t). \quad (4.4)$$

By combining (4.2)-(4.4) we arrive at the following expression for the expansion coefficients of the incoming field in regular spherical vector waves,  $f_{\iota j}$ ,

$$\begin{aligned} f_{\iota j} &= 4\pi \sum_{\kappa}^{\infty} i^{1+l-\lambda-\tau+t} \dots \\ &\dots \int \int \mathbf{A}_\iota^*(\hat{\mathbf{k}}_r) \cdot \mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \cdot \mathbf{A}_\kappa(\hat{\mathbf{k}}_t) T_{\kappa j} v_j d\Omega_r d\Omega_t, \end{aligned} \quad (4.5)$$

where the index  $j$  denotes the contribution from the corresponding port at the transmit antenna.

The expansion coefficients  $a_{\iota j}$ , of the incoming waves are related to the expansion coefficients  $f_{\iota j}$  of the regular waves with multipole index  $\iota$  as

$$2a_{\iota j} = f_{\iota j}. \quad (4.6)$$

This result follows from the properties of the spherical vector wave functions and the fact that the outgoing and incoming waves carry the same power in free space (empty minimal sphere), i.e.,  $\|\mathbf{a}\|_F^2 = \|\mathbf{b}\|_F^2$ , where the scattering matrix  $\mathbf{S} = \mathbf{I}$ . Thus, combining (2.5), (4.5) and (4.6) we arrive at the following expression

$$\begin{aligned} H_{ij} &= 2\pi \int \int \sum_{\iota} \sum_{\kappa} i^{1+l-\lambda-\tau+t} \dots \\ &\dots R_{i\iota} \mathbf{A}_\iota^*(\hat{\mathbf{k}}_r) \cdot \mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \cdot \mathbf{A}_\kappa(\hat{\mathbf{k}}_t) T_{\kappa j} d\Omega_r d\Omega_t. \end{aligned} \quad (4.7)$$

This expression describes the mapping of the outgoing (output) signal from the receive antenna port  $i$  and the incoming (input) signals at the transmit antenna

port  $j$ . Hence, (4.7) is the expansion of the transfer function matrix,  $\mathbf{H}$ , in spherical vector waves under the assumptions provided above. The elements of the channel matrix  $H_{ij}$  can also be expressed by expanding (2.8) in terms of the elements of matrices  $R_{i\iota}$ ,  $M_{\iota\kappa}$  and  $T_{\kappa j}$

$$H_{ij} = \sum_{\iota} \sum_{\kappa} R_{i\iota} M_{\iota\kappa} T_{\kappa j}, \quad (4.8)$$

where  $M_{\iota\kappa}$  are the elements of the matrix coupling the spherical vector wave modes at the receiver  $\iota \rightarrow (t, \mu, \lambda)$  and the transmitter  $\kappa \rightarrow (\tau, m, l)$  respectively. Now, by comparing (4.7) and (4.8) we obtain that  $M_{\iota\kappa}$  is given by the double integral

$$M_{\iota\kappa} = 2\pi i^{1+l-\lambda-\tau+t} \dots \int \int \mathbf{A}_{\iota}^*(\hat{\mathbf{k}}_r) \cdot \mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \cdot \mathbf{A}_{\kappa}(\hat{\mathbf{k}}_t) d\Omega_r d\Omega_t. \quad (4.9)$$

The entries of the  $\mathbf{M}$ -matrix, i.e., the matrix that maps the modes excited at the receiver with the modes excited at the transmitter, are thus directly related to the properties of the channel scattering dyadic  $\mathcal{C}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t)$ .

We now proceed to study the statistical properties of the mode-to-mode channel, the  $\mathbf{M}$ -matrix, as well as their implications on the statistics of the "classical" channel matrix, the  $\mathbf{H}$ -matrix. The elements of the correlation matrix for the entries of the  $\mathbf{H}$ -matrix can be readily obtained from (4.8) as

$$\mathcal{R}_{ij}^{i'j'} = \sum_{\iota} \sum_{\kappa} \sum_{\iota'} \sum_{\kappa'} R_{i\iota} R_{i'\iota'}^* \mathcal{R}_{\iota\kappa}^{i'\kappa'} T_{\kappa,j} T_{\kappa',j'}^*, \quad (4.10)$$

where  $\mathcal{R}_{ij}^{i'j'} = \langle H_{ij} H_{i'j'}^* \rangle$  denotes the covariance between any two elements of the  $\mathbf{H}$ -matrix, and  $\mathcal{R}_{\iota\kappa}^{i'\kappa'} = \langle M_{\iota\kappa} M_{\iota'\kappa'}^* \rangle$ , denotes the elements of the correlation matrix of the  $\mathbf{M}$ -matrix.

It is worthwhile to notice that (4.10) contains the full characterization of the joint correlation properties of the channel and the antennas at both communication link ends. Therefore, if  $\mathcal{R}_{\iota\kappa}^{i'\kappa'}$  is available, or an estimate thereof, we can design the communication system that, e.g., maximizes the system capacity.

In the following we study some general properties of the  $\mathbf{M}$ -matrix. The following proposition summarizes our main result on the computation of the correlation matrix as function of the PAS of the double directional channel.

**Proposition 1.** *In a multipath propagation environment characterized by a Gaussian unpolarized field component only (Rayleigh fading), the "double directional" expansion coefficients, or the  $\mathbf{M}$ -matrix entries, are Gaussian variates with zero mean,  $\langle M_{\iota\kappa} \rangle = 0$ , and variance<sup>16</sup>*

$$P_{\iota\kappa} = \langle |M_{\iota\kappa}|^2 \rangle = 4\pi^2 \sum_{\alpha} \sum_{\beta} \int \int \dots \dots |A_{\iota,\alpha}(\hat{\mathbf{k}}_r)|^2 \mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) |A_{\kappa,\beta}(\hat{\mathbf{k}}_t)|^2 d\Omega_r d\Omega_t. \quad (4.11)$$

<sup>16</sup>Usually, the total power in the co- and cross-polarized components satisfies the normalization  $P_{\theta\theta} + P_{\phi\phi} + P_{\phi\theta} + P_{\theta\phi} = 1$ . Then normalization of the total multi-pole power,  $\sum_{\tau=1}^2 \sum_{t=1}^2 \sum_{m=-l}^l \sum_{\mu=-l}^l \sum_{l=1}^L \sum_{\lambda=1}^L P_{t\mu\lambda,\tau ml} = \frac{L^2(L+2)^2}{8}$ , which directly follows from the addition theorem of spherical vector waves in Appendix A.

where the summation is over  $\alpha = \{\theta, \phi\}$  and,  $\beta = \{\theta, \phi\}$ . Moreover, the entries of the correlation matrix are functions of the joint power angular spectrum of the co- and cross-polarized components,  $\mathcal{R}_{\iota\kappa}^{i'\kappa'} = \langle M_{\iota\kappa} M_{\iota'\kappa'}^* \rangle$ ,

$$\begin{aligned} \mathcal{R}_{\iota\kappa}^{i'\kappa'} &= 4\pi^2 i^{l-l'-\lambda+\lambda'-\tau+\tau'+t-t'} \sum_{\alpha} \sum_{\beta} \int \int \dots \\ &\dots A_{l,\alpha}^*(\hat{\mathbf{k}}_r) A_{l',\alpha}(\hat{\mathbf{k}}_r) \mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \dots \\ &\dots A_{\kappa,\beta}(\hat{\mathbf{k}}_t) A_{\kappa',\beta}^*(\hat{\mathbf{k}}_t) d\Omega_r d\Omega_t. \end{aligned} \quad (4.12)$$

The derivation follows directly from the Gaussianity preservation property of affine transformations; details are given in Appendix D.

As we can see from Proposition 1 the correlation depends on the joint distributions of co-polarized and cross-polarized components. Several well-known channels can be interpreted in the framework of this model. For example, if isotropic PAS are assumed at both the receiver and the transmitter

$$\mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) = 1/16\pi^2 \quad (4.13)$$

we obtain that

$$\mathcal{R}_{\iota\kappa}^{i'\kappa'} = \delta_{i'i'} \delta_{\mu\mu'} \delta_{\lambda\lambda'} \delta_{\tau\tau'} \delta_{mm'} \delta_{ll'}. \quad (4.14)$$

Hence, all modes are uncorrelated in the isotropic case, this is the most random field that can be encountered in wireless channels. The covariance of the  $\mathbf{H}$ -matrix is then

$$\mathcal{R}_{ij}^{i'j'} = \sum_{\iota} \sum_{\kappa} R_{i\iota} R_{i',\iota}^* T_{\kappa j} T_{\kappa j'}^* \quad (4.15)$$

In this case the correlation properties of the  $\mathbf{H}$ -matrix are, as we should expect, completely defined by the properties of the antennas used.

## 5 Numerical Examples

The increasing demand for smaller devices for wireless communication has led to the search for methods to reduce the size of antennas in devices such as cellular phones. A well-known way to keep the volume occupied by antennas down, while still achieving or even increasing signal diversity, is to exploit polarization diversity. Polarization diversity has been the objective of extensive theoretical and practical studies, see [39] and references therein.

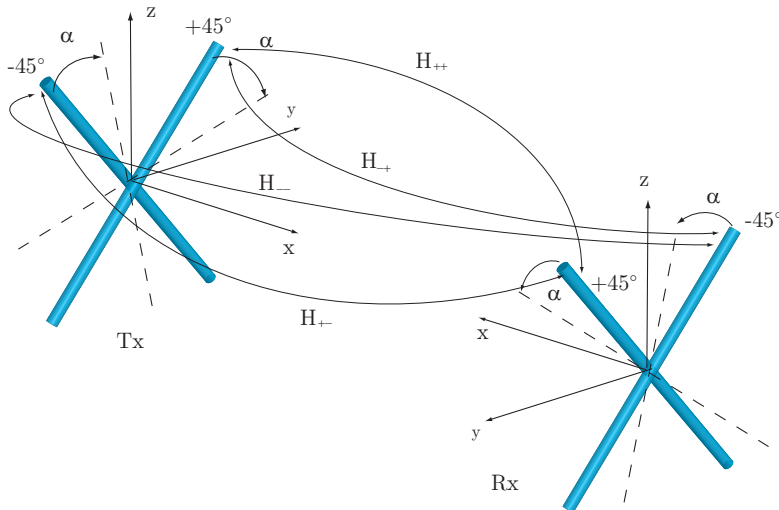
In the following example<sup>17</sup> we apply our approach to a MIMO system based on polarization diversity. We consider a  $2 \times 2$  MIMO system with cross-polarized antennas at both ends. Two half-wavelength dipoles are used, one is tilted  $45^\circ$  from the z-axis towards the positive side of the y-axis, while the second dipole is also tilted

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<sup>17</sup>This example is an oversimplified antenna configuration that does not take mutual coupling or matching into account. However, it serves as a straightforward example demonstrating the main points in the paper.



45° but in the opposite direction. Thus we denote them as the +45°- dipole and the -45°- dipole respectively (see Fig. 2). In order to illustrate how the multimode expansion can be used to analyze the interaction between the antennas and the channel we rotate both antenna pairs around the x-axis, towards the positive y-axis, in their respective coordinate systems. The rotation angles are denoted by  $\alpha_r$  and  $\alpha_t$  for the receive and the transmit antenna pairs, respectively.



**Figure 2:** Geometry of the cross polarized MIMO system.

We denote the transfer function matrix, the  $\mathbf{H}$ -matrix, of the system by

$$\mathbf{H} = \begin{pmatrix} H_{++} & H_{+-} \\ H_{-+} & H_{--} \end{pmatrix}, \quad (5.1)$$

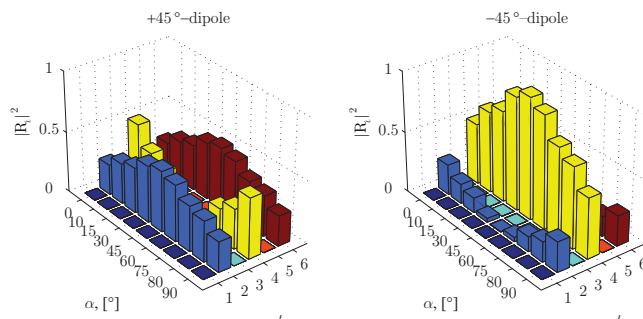
where for instance,  $H_{+-}$  denotes the matrix element coupling the +45° antenna at the receiver with the -45° antenna at the transmitter. The notation of the remaining matrix elements follows the same principle. We keep the same notation for the rotated antennas.

The polarization sensitivity of the multipole antenna system as a function of the rotation angle can be directly derived from the behavior of the multipoles. Fig. 3 shows the squared absolute values of the transmission (or reception) coefficients<sup>18</sup>,  $|R_l|^2$ , for the +45°- dipole and the -45°- dipole antennas as function of the rotation angle  $\alpha$  for the six lowest multipole multi-indices,  $l$ , i.e.,  $l = 1$ . As expected only the TM dipoles, i.e.,  $l = 1$  and  $\tau = 2$ , are excited while the antennas are rotated<sup>19</sup>. Their magnitudes clearly change, indicating that the coupling to the different modes changes as the antenna is rotated. For example, at the initial position, when  $\alpha = 0^\circ$ , the positions of the +45°- and the -45°- dipole antennas are equivalent due to symmetry and therefore the magnitudes of the multipoles are the same, with half

<sup>18</sup>For clarity, the transmission coefficients in Fig. 3 are normalized as  $\sum_l |R_l|^2 = 1$ .

<sup>19</sup>From the properties of spherical functions it follows that rotation conserves the power in the  $l$ -index and in the  $\tau$ -index.

of the power distributed equally between the two “horizontal”<sup>20</sup> dipole modes with  $\iota = 2 \rightarrow \{2, -1, 1\}$  and  $\iota = 6 \rightarrow \{2, 1, 1\}$ . The other half of the power goes into the vertical dipole modes with  $\iota = 4 \rightarrow \{2, 0, 1\}$ . As we increase the rotation angle to  $\alpha = 10^\circ$  the tilt angle of the  $+45^\circ$ - dipole increases to  $55^\circ$ . At this angle the half-wavelength dipole senses powers in the horizontal and vertical polarizations in a similar manner [36]. In this case the antenna coupling into the three dipole modes is equal as shown in the left plot of Fig. 3. Any further increase of the rotation angle implies an increase of the power into the “horizontal” dipoles with the proportional decrease of the power into the “vertical” dipole, which reaches their respective maxima and minimum for  $\alpha = 45^\circ$ . On the contrary, for the  $-45^\circ$ -dipole, as the rotation angle increases, the power coupled into the “vertical” dipole increases, while the power into the “horizontal” dipoles decreases. Due to symmetry the reverse process is observed as the rotation angle is increased towards  $\alpha = 90^\circ$ .



**Figure 3:** Squared absolute values of the transmission (or reception) coefficients,  $|R_l|^2$  of the half-wavelength dipole antenna tilted from  $+45^\circ$  to  $+135^\circ$  from the z-axis towards the y-axis (left plot) and the same but for tilting angles from  $-45^\circ$  to  $+45^\circ$  (right plot).

The correlation between the two antenna branches can also be explained from the multipole behavior. For example, for  $\alpha = 0^\circ$ , the correlation between the two antennas should be the highest since, as explained earlier, both antennas excite the same modes equally. On the other hand for  $\alpha = 45^\circ$ , the two antennas are “purely orthogonal” since they excite orthogonal modes, i.e., one antenna, the  $+45^\circ$ - dipole, is oriented along the y-axis, hence only “horizontal” dipoles are excited while the other antenna is oriented along the z-axis.

We now proceed to specify the channel models we use for the analysis of the MIMO system. It is well-known that an antenna performs differently depending on the propagation environment where it is deployed. Thus, if we by some means can gain knowledge of the properties of the propagation channel we can, in principle, improve the performance in that channel by *reconfiguring* the antenna so that it matches the channel characteristics. Here we are, however, just going to *analyze* the

<sup>20</sup>In this paper we use the complex exponential convention of the spherical vector waves, [13, 17], therefore no distinct association between the actual physical orientation of the horizontal dipoles and the dipole order can be made.

performance to illustrate how well (or badly) our simple system performs in terms of the properties of the channel  $\mathbf{H}$ -matrix and how the channel properties depend upon the interaction between the multipoles of the antenna and that of the channel.

We consider two simple but widely used models. In both of them it is assumed that the joint probability density functions are the same for both co- and cross-polarized components, and that they are independent in azimuth and elevation

$$p_{\alpha\beta}(\theta_r, \phi_r, \theta_t, \phi_t) = p_{\theta_r\alpha\beta}(\theta_r)p_{\phi_r\alpha\beta}(\phi_r)p_{\theta_t\alpha\beta}(\theta_t)p_{\phi_t\alpha\beta}(\phi_t), \quad (5.2)$$

where  $\alpha = \{\theta, \phi\}$  and  $\beta = \{\theta, \phi\}$  stands for either of  $\hat{\boldsymbol{\theta}}$  or  $\hat{\boldsymbol{\phi}}$ , polarizations. We also assume that the powers of the cross-polarized components are much lower than the powers of the co-polarized components, i.e.,  $P_{\theta\phi} \approx P_{\phi\theta} \ll P_{\theta\theta}, P_{\phi\phi}$  and therefore the XPR is completely defined by  $\chi = \frac{P_{\theta\theta}}{P_{\phi\phi}}$ .

Model-A describes a highly isotropic channel with a balanced polarization ( $\chi = 0\text{dB}$ ), where

$$p_{\theta_r\alpha\beta}(\theta) = p_{\theta_t\alpha\beta}(\theta) = A_\theta e^{-\frac{\sqrt{2}|\theta-\mu_\theta|}{\sigma_\theta}} \sin \theta, \theta \in [0, \pi] \quad (5.3)$$

$$p_{\phi_r\alpha\beta}(\phi) = p_{\phi_t\alpha\beta}(\phi) = A_\phi e^{-\frac{\sqrt{2}|\phi-\mu_\phi|}{\sigma_\phi}}, \phi \in [-\pi, \pi), \quad (5.4)$$

with parameters  $\sigma = \sigma_\theta = \sigma_\phi = 10$  rad and  $\mu = \mu_\theta = \mu_\phi = 0$  rad.

Model-B emulates the propagation in a macro-cell deployed in an urban environment as outlined in [9]<sup>21</sup>. We assume that

$$p_{\theta_r\alpha\beta}(\theta) = \sqrt{2} \sin \theta, \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \quad (5.5)$$

$$p_{\phi_r\alpha\beta}(\phi) = \frac{1}{2\pi}, \phi \in [-\pi, \pi) \quad (5.6)$$

$$p_{\theta_t\alpha\beta}(\theta) = A_\theta e^{-\frac{\sqrt{2}|\theta-\mu_\theta|}{\sigma_\theta}} \sin \theta, \theta \in [0, \pi] \quad (5.7)$$

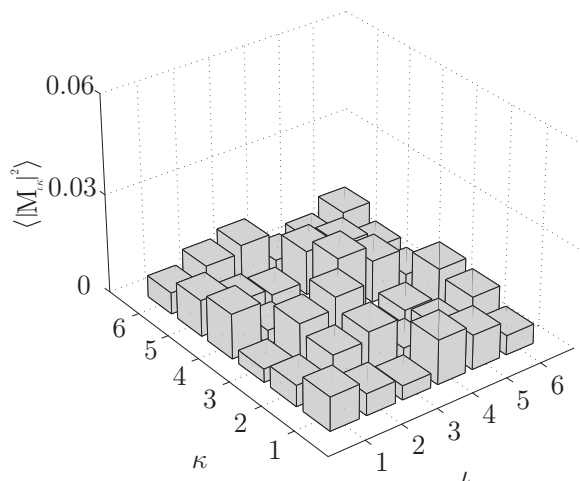
$$p_{\phi_t\alpha\beta}(\phi) = A_\phi e^{-\frac{\sqrt{2}|\phi-\mu_\phi|}{\sigma_\phi}}, \phi \in [-\pi, \pi), \quad (5.8)$$

with parameters  $\sigma = \sigma_\theta = \sigma_\phi = 0.1$  rad and  $\mu = \mu_\theta = \mu_\phi = 0$  rad. We also assume unbalanced polarization in favor to the vertical polarization, i.e.,  $\chi = 10\text{dB}$ . Here the mobile terminal is the receiver, while the transmitter is the base station. Hence, the angle-spread around the Rx is much higher than the angle-spread around the Tx.

In both models the resulting channels produce a Rayleigh probability density function for the envelopes of the elements of the  $\mathbf{H}$ -matrix.

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<sup>21</sup>Here we use the shape of the distributions provided in [9]. However, the parametrization is generic.

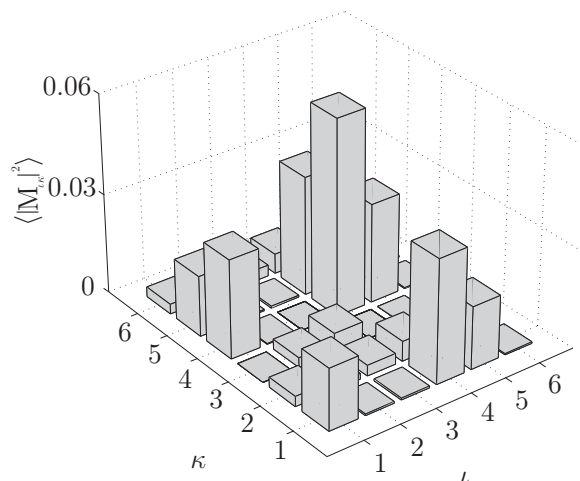


**Figure 4:** Average mode to mode power,  $\langle |M_{\iota\kappa}|^2 \rangle$ , as function of the multipole indices at the receiver,  $\iota$ , and the transmitter,  $\kappa$ , for Model-A.

Figs. 4 and 5 show the average mode-to-mode power,  $\langle |M_{\iota\kappa}|^2 \rangle$ , as a function of the multipole indices at the Rx,  $\iota$ , and the Tx,  $\kappa$ , for Model-A and Model-B, respectively. The more uniform distribution of the power among the multimodes of Model-A is a direct result of the uniform distribution of the AoA and the AoD at the receiver and the transmitter, respectively. The fact that the power into the vertical and horizontal polarizations is the same, also supports this uniform power distribution among the modes. On the other hand, spatial selectivity, as we have at the transmitter for Model-B, together with polarization power imbalance also implies selectivity in modes, i.e., some multipole-multipole interactions are stronger than others as shown in Fig. 5. More specifically, we see from Fig. 5 that the coupling between the power into the “vertical” dipole modes,  $\iota = \kappa = 4 \rightarrow \{2, 0, 1\}$ , is much stronger than the power coupling between other modes, since the XPR of the channel is such that the power into the power of the vertical polarization is much stronger than the power of the horizontal polarization,  $\chi = 10\text{dB}$ . It can also be observed that the coupling between electric and magnetic dipoles is quite strong since the propagation at the receiver takes place mostly on the horizontal plane.

Figs. 6 and 7 show the absolute values of the correlation matrix,  $|R_{\iota\kappa}|$ , as function of the multipole indices at the receiver,  $(\iota, \kappa)$ ,<sup>22</sup> and the transmitter,  $(\iota', \kappa')$ , for Model-A and Model-B, respectively. As we see from Fig. 6 the multimodes become uncorrelated in the case of a uniform distribution of AoA and AoD, while the correlation becomes noticeably higher for the spatially selective channel provided by Model-B. It should be noted that in the case when the power of co- and cross-polarized components is the same, i.e.,  $P_{\theta\phi} = P_{\phi\theta} = P_{\theta\theta} = P_{\phi\phi} = \frac{1}{4}$  and the joint pdf of the AoA and AoD is uniform, i.e.,  $p_{\theta\theta}(\Omega_r, \Omega_t) = p_{\phi\phi}(\Omega_r, \Omega_t) = p_{\phi\theta}(\Omega_r, \Omega_t) = p_{\theta\phi}(\Omega_r, \Omega_t) = \frac{1}{16\pi^2}$ , the multimodes have identical powers given by the diagonal

<sup>22</sup>The index pair  $(\iota, \kappa)$  denotes a multi-index calculated as  $\iota \otimes \kappa$ , where  $\otimes$  denotes the Kronecker product.



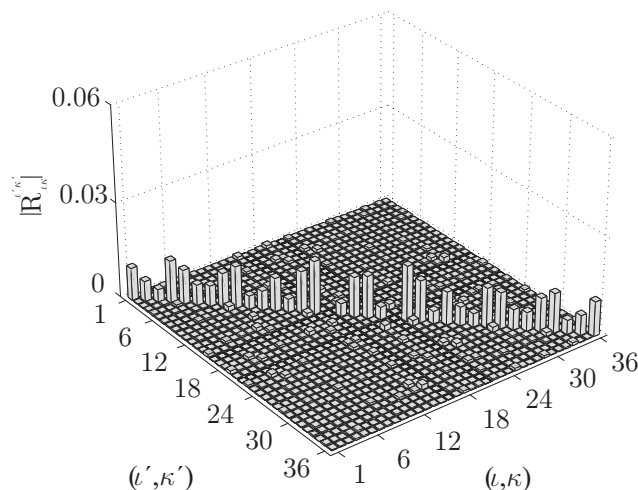
**Figure 5:** Average power mode,  $\langle |M_{l\kappa}|^2 \rangle$ , as function of the multipole indices at the receiver,  $l$ , and the transmitter,  $\kappa$ , for Model-B.

elements of (4.14).

The channel behavior in the multipole modes, i.e., the  $\mathbf{M}$ -matrix, together with the mode behavior of the antennas, given by the transmission and reception matrices  $\mathbf{R}$  and  $\mathbf{T}$ , have direct impact on the behavior of the  $\mathbf{H}$ -matrix. Fig. 8 shows the average power of the elements of the  $2 \times 2$   $\mathbf{H}$ -matrix as a function of the rotation angles at the receiver and the transmitter obtained using Model-A. As we can see, the power is practically the same for all links and it does not depend on the rotation of the antennas. On the other hand, similar results obtained using Model-B show how the XPR and spatial selectivity (and therefore selectivity in modes too) impact the link power. For both Model-A and Model-B the powers of the links  $H_{+-}$  and  $H_{-+}$  are mutually symmetric and symmetric with respect to the rotation angle  $\alpha$ . On the other hand, the powers of the links  $H_{++}$  and  $H_{--}$  are not mutually symmetric, although each of them is symmetric with respect to the rotation angle  $\alpha$ . Clearly, the link is strongest for the  $H_{--}$  link in Model-B since both antennas are collinear and their polarizations coincide with the polarization of the channel (see Figs. 3 and 5 for a comparison).

The correlation coefficients of the  $\mathbf{H}$ -matrix are shown in Figs. 10 and 11, for Model-A and Model-B, respectively. As anticipated by the mode correlation in Fig. 7, the correlation coefficient is low for Model-A as shown in Fig. 10. Here, we also observe a symmetric behavior for the corresponding links. Moreover, we see that for the isotropic channel discussed above the covariance is only determined by the antennas, see (4.15). On the other hand the correlation increases considerably with spatial selectivity as in Model-B, shown in Fig. 11. Also here, the behavior can be anticipated by the covariance of the multimodes observed in Fig. 7. The asymmetrical behavior of the correlation coefficients is explained by the asymmetry in AoA and AoD as well as the power polarization imbalance.

As a final remark we would like to briefly mention spatial-diversity systems,



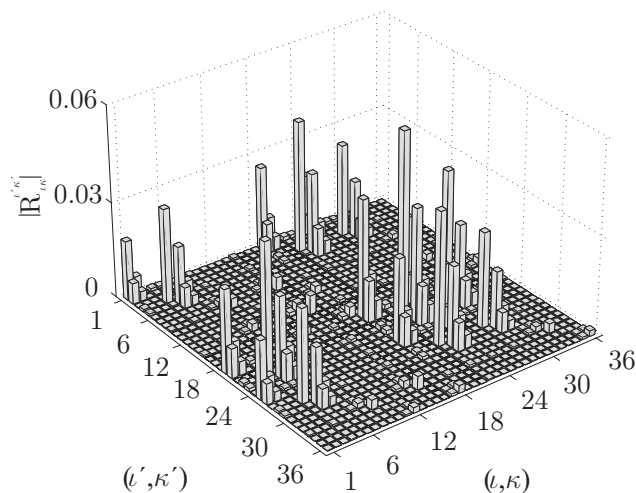
**Figure 6:** Absolute value of the covariance matrix,  $|R_{\iota\kappa}|$ , as function of the multipole indices at the receiver,  $(\iota, \kappa)$ , and the transmitter,  $(\iota', \kappa')$ , for Model-A.

which draw special attention in the design of MIMO wireless systems. The topic of spatial-diversity has been thoroughly analyzed since the seminal work presented in [21]. Spatial-diversity has proven to have a great impact on wireless communications systems both through increased transmission rates and/or increased reliability (lower error probability) [25]. A system of particular interest, due to its “apparent” simplicity, is the  $2 \times 2$  MIMO spatial-diversity system with half-wavelength dipoles placed at some distance  $d_s$  from each other. Here we show a short example when varying  $d_s$ . The squared absolute values of the transmission (or reception) coefficients,  $|R_\iota|^2$  of a vertically polarized half-wavelength dipole antenna as a function of the offset distance from a reference coordinate system,  $2d_s/\lambda$ , is given in Fig. 12<sup>23</sup>(compare with Fig. 3). As can be seen from the plot, the translation of the antenna is equivalent to the excitation of higher order multipoles, e.g., coefficients with multi-pole index  $l = 2$  have to be taken into the analysis of the interaction between the antenna and the channel for a separation as low as  $\lambda/2$ . This “spread” in modes can be seen as the origin of the low correlation between spatially separated antenna branches. For more compact configurations, e.g.,  $d_s \leq \lambda/6$ , the representation of the antenna is mainly concentrated to coefficients corresponding to dipole modes ( $l = 1$ ), more specifically to the “vertical” dipole moment with  $\iota = 4$ . Therefore, the correlation of closely placed dipoles is high unless “intelligent” matching measures are taken [4, 19, 40].

## 6 Summary

In this paper we introduced the concept of mode-to-mode channel matrix, the **M**-matrix, to describe the coupling between the spherical vector wave modes excited

<sup>23</sup>A similar behavior is observed for the second antenna too.



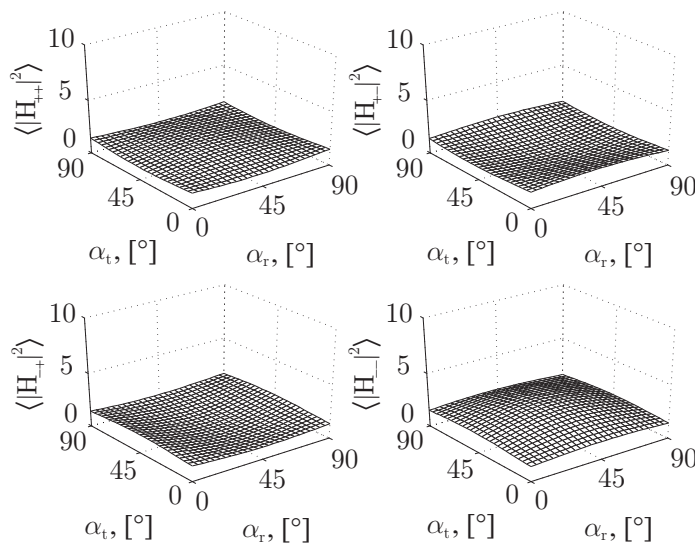
**Figure 7:** Absolute value of the covariance matrix,  $|R_{\iota\kappa}|$ , as function of the multipole indices at the receiver,  $(\iota, \kappa)$ , and the transmitter,  $(\iota', \kappa')$ , for Model-B.

at the transmitter to the modes excited at the receiver of a wireless MIMO system. The  $\mathbf{M}$ -matrix contains all relevant information about the physics involving the excitation of the channel by electromagnetic waves, since it provides a direct coupling among different multi-poles at the receiver and the transmitter. We further discussed the concept of the *channel scattering dyadic*,  $\mathcal{C}$ , which maps the field radiated by the transmitting antenna to the field impinging at the receive antenna obtained by superposition of plane waves. Further, using the correlation model for the amplitudes incident at the receive antenna [10], we developed a more general correlation model for the double-directional channel, i.e., the correlation between the co- and cross-polarized components of the channel scattering dyadic. We then expanded the channel  $\mathbf{H}$ -matrix in spherical vector wave modes using the derived correlation model for the elements of the *channel scattering dyadic*. We also provided results for first and second order statistics of the expansion coefficients based on the assumption that the dyadic elements are independently distributed Gaussian variables. The equations we proved establish a direct relationship between the elements of  $\mathbf{H}$  and  $\mathbf{M}$ , and therefore the spatial, directional and polarization properties of the channel and the antennas for MIMO systems.

Our results can be used to further analyze the interaction between the antennas and the channel and the performance limits of antennas in stochastic channels. Further investigations of the behavior of the mode-to-mode channel matrix should also enable valuable insights into the design of small and efficient MIMO antenna arrays.

## 7 Acknowledgments

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**Figure 8:** Average power of the elements of the  $2 \times 2$   $\mathbf{H}$ -matrix as a function of the rotation angles at the receiver and the transmitter, Model-A.

## Appendix A. Spherical vector waves

The regular spherical vector waves are given by [17]

$$\mathbf{v}_{1ml}(k\mathbf{r}) = j_l(kr)\mathbf{A}_{1ml}(k\hat{\mathbf{r}}), \quad (7.1)$$

$$\begin{aligned} \mathbf{v}_{2ml}(k\mathbf{r}) &= \frac{(krj_l(kr))'}{kr}\mathbf{A}_{2ml}(k\hat{\mathbf{r}}) \\ &+ \sqrt{l(l+1)}\frac{j_l(kr)}{kr}\mathbf{A}_{3ml}(k\hat{\mathbf{r}}), \end{aligned} \quad (7.2)$$

where the time convention  $e^{i\omega t}$  is used,  $\mathbf{r} = r\hat{\mathbf{r}}$ ,  $r = |\hat{\mathbf{r}}|$  and  $j_l(kr)$  denotes the regular spherical Bessel functions [1]. Similarly, the incoming ( $p = 1$ ) and outgoing ( $p = 2$ ) spherical vector waves,  $\mathbf{u}_{\tau ml}^{(p)}(k\mathbf{r})$  are given by

$$\mathbf{u}_{1ml}^{(p)}(k\mathbf{r}) = h_l^{(p)}(kr)\mathbf{A}_{1ml}(k\hat{\mathbf{r}}), \quad (7.3)$$

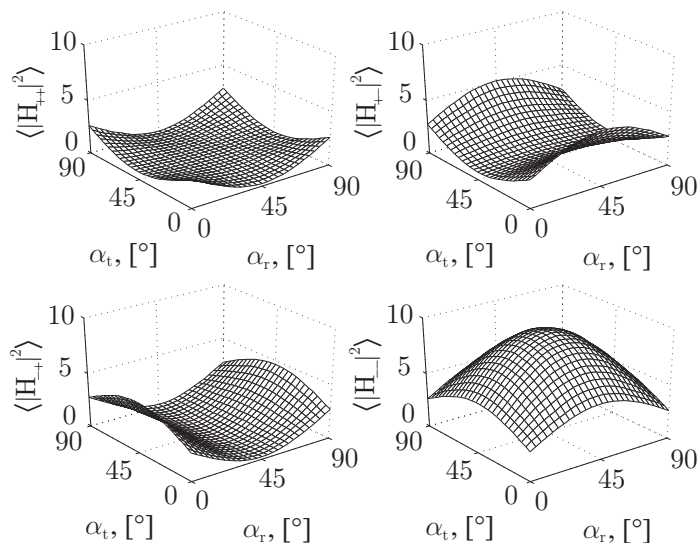
$$\begin{aligned} \mathbf{u}_{2ml}^{(p)}(k\mathbf{r}) &= \frac{(krh_l^{(p)}(kr))'}{kr}\mathbf{A}_{2ml}(k\hat{\mathbf{r}}) \\ &+ \sqrt{l(l+1)}\frac{h_l^{(p)}(kr)}{kr}\mathbf{A}_{3ml}(k\hat{\mathbf{r}}), \end{aligned} \quad (7.4)$$

where  $h_l^{(p)}(kr)$  are the spherical Hankel functions of the  $p$ -th kind.

The functions  $\mathbf{A}_{\tau ml}(k\hat{\mathbf{r}})$  are the spherical vector harmonics that satisfy the complex valued inner product, i.e. orthogonality on the unit sphere [17]

$$\int \mathbf{A}_{\tau ml}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau' m' l'}^*(\hat{\mathbf{r}}) d\Omega = \delta_{\tau\tau'} \delta_{mm'} \delta_{ll'}. \quad (7.5)$$





**Figure 9:** Average power of the elements of the  $2 \times 2$   $\mathbf{H}$ -matrix as a function of the rotation angles at the receiver and the transmitter, Model-B.

The addition theorem for the vector spherical harmonics reads

$$\sum_{m=-l}^l \mathbf{A}_{\tau ml}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau ml}^*(\hat{\mathbf{r}}) = \frac{2l+1}{4\pi} \quad (7.6)$$

## Appendix B. Scattering Matrix

For reciprocal antennas we also note that the following relationship is valid [17]

$$R_{n,\tau ml} = (-1)^m T_{\tau(-m)l,n}, \quad (7.7)$$

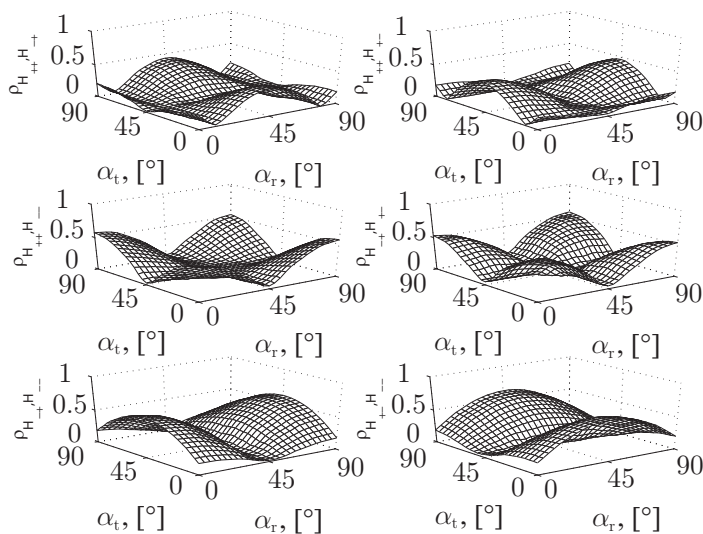
where  $R_{n,\tau ml}$  and  $T_{\tau ml,n}$  are elements of matrix  $\mathbf{R}$  and  $\mathbf{T}$ , respectively.

The transmission matrix is obtained as a projection of the far-field of the antenna on the spherical vector harmonics,  $\mathbf{A}_{\tau ml}$ , in transmitting regime. Hence, the far-field  $\mathbf{F}_n(\hat{\mathbf{r}})$  of port  $n$  is given by

$$\mathbf{F}_n(\hat{\mathbf{r}}) = k\sqrt{2\eta} \sum_{l=1}^{\infty} \sum_{m=-l}^l \sum_{\tau=1}^2 i^{l+2-\tau} T_{\tau ml,n} \mathbf{A}_{\tau ml}(\hat{\mathbf{r}}) v_n, \quad (7.8)$$

where  $\mathbf{A}_{\tau ml}(\hat{\mathbf{r}})$  is defined in Appendix A,  $\hat{\mathbf{r}}$  is the unitary spatial coordinate and  $v_n$  is the signal incident on port  $n$  with corresponding power normalization,  $\|\mathbf{v}\|^2 = 1$ . Further, applying the orthogonality properties of spherical vector harmonics, we obtain the transmission coefficients of the antenna

$$T_{\tau ml,n} v_n = \frac{i^{-l-2+\tau}}{k\sqrt{2\eta}} \int \mathbf{A}_{\tau ml}^*(\hat{\mathbf{r}}) \cdot \mathbf{F}_n(\hat{\mathbf{r}}) d\Omega. \quad (7.9)$$



**Figure 10:** Correlation coefficients between pairs of elements of the  $2 \times 2$   $\mathbf{H}$ -matrix as a function of the rotation angles at the receiver and the transmitter, Model-A.

It is worthwhile to notice that matrix  $\mathbf{T}$  can also be obtained by measurements of the radiated field in anechoic chambers or any other antenna range facility for near- and/or far-field measurements, [17]. Moreover, mutual coupling between antenna elements is implicitly integrated in the scattering matrix (7.7) and if we are interested in calculating the mutual coupling between antennas belong to different antenna systems and spatially separated it can straightforwardly be done as outlined in [22].

## Appendix C. Correlation model

We see from (3.7) and (3.10) that the field incident at the receive antenna labeled with the superscript  $r$  can be expressed through the elements of the "channel scattering dyadic",

$$E_{0j,\alpha}^t = C_{\alpha\theta} F_{j,\theta}^t + C_{\alpha\phi} F_{j,\phi}^t \quad (7.10)$$

$$E_{0j,\beta}^{t*} = C_{\beta\theta}^* F_{j,\theta}^{t*} + C_{\beta\phi}^* F_{j,\phi}^{t*} \quad (7.11)$$

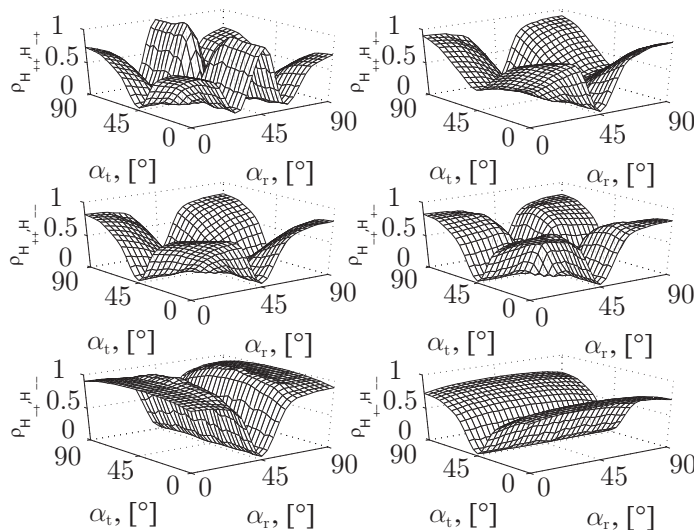
Similarly for the reverse channel

$$E_{0i,\alpha}^r = C_{\alpha\theta} F_{i,\theta}^r + C_{\alpha\phi} F_{i,\phi}^r \quad (7.12)$$

$$E_{0i,\beta}^{r*} = C_{\beta\theta}^* F_{i,\theta}^{r*} + C_{\beta\phi}^* F_{i,\phi}^{r*} \quad (7.13)$$

Similarly to (7.36) (see also Appendix E) we get for the direct link the following relation

$$H_{ij} = -2\pi i \mathbf{F}_i^r \cdot \mathbf{E}_{0j}^t \quad (7.14)$$



**Figure 11:** Correlation coefficients between pairs of elements of the  $2 \times 2$   $\mathbf{H}$ -matrix as a function of the rotation angles at the receiver and the transmitter, Model-B.

and for the reverse channel

$$\tilde{H}_{ji} = -2\pi i \mathbf{F}_j^t \cdot \mathbf{E}_{0i}^r \quad (7.15)$$

where  $\mathbf{F}_i^r = [F_{i,\theta}^t \ F_{i,\phi}^t]^T$  and  $\mathbf{F}_j^t = [F_{j,\theta}^t \ F_{j,\phi}^t]^T$  are the far-field patterns of the antennas at the receiver and transmitter, respectively. The received signal power is proportional to the square of the absolute value of the induced open-circuit voltage (7.14), for the direct channel we obtain the correlation between where observing that  $\langle E_{0j,\theta}^t E_{0j,\phi}^{t*} \rangle = \langle E_{0j,\phi}^t E_{0j,\theta}^{t*} \rangle = 0$

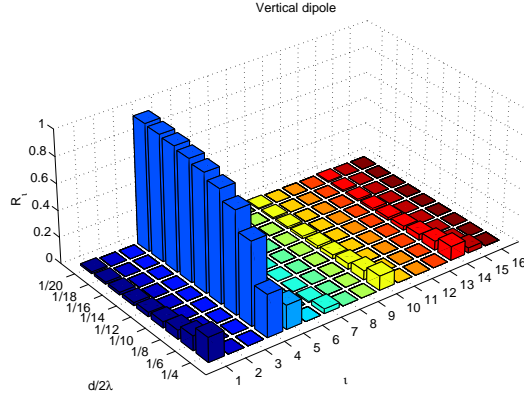
$$\langle H_{ij} H_{ij}^* \rangle \approx F_{i,\theta}^r F_{i,\theta}^{r*} \langle E_{0j,\theta}^t E_{0j,\theta}^{t*} \rangle + F_{i,\phi}^r F_{i,\phi}^{r*} \langle E_{0j,\phi}^t E_{0j,\phi}^{t*} \rangle \quad (7.16)$$

Now from (7.10)-(7.11) we compute the cross-covariance between the orthogonal components

$$\begin{aligned} \langle E_{0j,\alpha}^t E_{0j,\alpha}^{t*} \rangle &= \langle \mathcal{C}_{\alpha\theta} \mathcal{C}_{\alpha\theta}^* \rangle F_{j,\theta}^t F_{j,\theta}^{t*} + \langle \mathcal{C}_{\alpha\phi} \mathcal{C}_{\alpha\phi}^* \rangle F_{j,\theta}^t F_{j,\phi}^{t*} \\ &\quad + \langle \mathcal{C}_{\alpha\phi} \mathcal{C}_{\alpha\theta}^* \rangle F_{j,\phi}^t F_{j,\theta}^{t*} + \langle \mathcal{C}_{\alpha\theta} \mathcal{C}_{\alpha\phi}^* \rangle F_{j,\phi}^t F_{j,\phi}^{t*} \end{aligned} \quad (7.17)$$

which we can substitute in (7.16) and obtain

$$\begin{aligned} \langle H_{ij} H_{ij}^* \rangle &\approx F_{i,\theta}^r F_{i,\theta}^{r*} \langle \mathcal{C}_{\theta\theta} \mathcal{C}_{\theta\theta}^* \rangle F_{j,\theta}^t F_{j,\theta}^{t*} \\ &\quad + F_{i,\theta}^r F_{i,\theta}^{r*} \langle \mathcal{C}_{\theta\theta} \mathcal{C}_{\theta\phi}^* \rangle F_{j,\theta}^t F_{j,\phi}^{t*} \\ &\quad + F_{i,\theta}^r F_{i,\theta}^{r*} \langle \mathcal{C}_{\theta\phi} \mathcal{C}_{\theta\theta}^* \rangle F_{j,\phi}^t F_{j,\theta}^{t*} \\ &\quad + F_{i,\theta}^r F_{i,\theta}^{r*} \langle \mathcal{C}_{\theta\phi} \mathcal{C}_{\theta\phi}^* \rangle F_{j,\phi}^t F_{j,\phi}^{t*} \\ &\quad + F_{i,\phi}^r F_{i,\phi}^{r*} \langle \mathcal{C}_{\phi\theta} \mathcal{C}_{\phi\theta}^* \rangle F_{j,\theta}^t F_{j,\theta}^{t*} \\ &\quad + F_{i,\phi}^r F_{i,\phi}^{r*} \langle \mathcal{C}_{\phi\theta} \mathcal{C}_{\phi\phi}^* \rangle F_{j,\theta}^t F_{j,\phi}^{t*} \\ &\quad + F_{i,\phi}^r F_{i,\phi}^{r*} \langle \mathcal{C}_{\phi\phi} \mathcal{C}_{\phi\theta}^* \rangle F_{j,\phi}^t F_{j,\theta}^{t*} \\ &\quad + F_{i,\phi}^r F_{i,\phi}^{r*} \langle \mathcal{C}_{\phi\phi} \mathcal{C}_{\phi\phi}^* \rangle F_{j,\phi}^t F_{j,\phi}^{t*} \end{aligned} \quad (7.18)$$



**Figure 12:** Squared absolute values of the transmission (or reception) coefficients,  $|R_l|^2$  of a vertically polarized half-wavelength dipole antenna as a function of the offset distance from a reference coordinate system. It is assumed that there is no mutual coupling between the antennas

Similarly, using the reciprocity condition for the channel scattering dyadic (3.9) we obtain for the reverse channel

$$\begin{aligned}
 \langle \tilde{H}_{ji} \tilde{H}_{ji}^{I*} \rangle &\approx F_{j,\theta}^t F_{j,\theta}^{tI*} \langle C_{\theta\theta} C_{\theta\theta}^{I*} \rangle F_{i,\theta}^r F_{i,\theta}^{rI*} \\
 &+ F_{j,\theta}^t F_{j,\theta}^{tI*} \langle C_{\theta\theta} C_{\phi\theta}^{I*} \rangle F_{i,\theta}^r F_{i,\phi}^{rI*} \\
 &+ F_{j,\theta}^t F_{j,\theta}^{tI*} \langle C_{\phi\theta} C_{\theta\theta}^{I*} \rangle F_{i,\phi}^r F_{i,\theta}^{rI*} \\
 &+ F_{j,\theta}^t F_{j,\theta}^{tI*} \langle C_{\phi\theta} C_{\phi\theta}^{I*} \rangle F_{i,\phi}^r F_{i,\phi}^{rI*} \\
 &+ F_{j,\phi}^t F_{j,\phi}^{tI*} \langle C_{\theta\phi} C_{\phi\theta}^{I*} \rangle F_{i,\theta}^r F_{i,\theta}^{rI*} \\
 &+ F_{j,\phi}^t F_{j,\phi}^{tI*} \langle C_{\theta\phi} C_{\phi\phi}^{I*} \rangle F_{i,\theta}^r F_{i,\phi}^{rI*} \\
 &+ F_{j,\phi}^t F_{j,\phi}^{tI*} \langle C_{\phi\phi} C_{\theta\phi}^{I*} \rangle F_{i,\phi}^r F_{i,\theta}^{rI*} \\
 &+ F_{j,\phi}^t F_{j,\phi}^{tI*} \langle C_{\phi\phi} C_{\phi\phi}^{I*} \rangle F_{i,\phi}^r F_{i,\phi}^{rI*}
 \end{aligned} \tag{7.19}$$

by comparing (7.18) and (7.19) we see that for the reciprocity to hold, i.e.,

$$\langle H_{ij} H_{ij}^{I*} \rangle = \langle \tilde{H}_{ji} \tilde{H}_{ji}^{I*} \rangle \tag{7.20}$$

the cross-covariance must satisfy  $\langle C_{\theta\theta} C_{\theta\phi}^{I*} \rangle = \langle C_{\theta\theta} C_{\phi\theta}^{I*} \rangle = \langle C_{\theta\phi} C_{\theta\theta}^{I*} \rangle = \langle C_{\phi\theta} C_{\theta\theta}^{I*} \rangle = \langle C_{\phi\theta} C_{\phi\phi}^{I*} \rangle = \langle C_{\theta\phi} C_{\phi\phi}^{I*} \rangle = \langle C_{\phi\phi} C_{\theta\phi}^{I*} \rangle = \langle C_{\phi\phi} C_{\phi\phi}^{I*} \rangle = 0$ . Furthermore, signals incoming from different directions are uncorrelated at both in the reverse and the direct channels. Hence

$$\begin{aligned}
 \langle H_{ij} H_{ij}^{I*} \rangle &= \langle \tilde{H}_{ji} \tilde{H}_{ji}^{I*} \rangle \\
 &\approx F_{i,\theta}^r F_{i,\theta}^{rI*} \langle C_{\theta\theta} C_{\theta\theta}^{I*} \rangle F_{j,\theta}^t F_{j,\theta}^{tI*} \\
 &+ F_{i,\theta}^r F_{i,\theta}^{rI*} \langle C_{\theta\phi} C_{\theta\phi}^{I*} \rangle F_{j,\phi}^t F_{j,\phi}^{tI*} \\
 &+ F_{i,\phi}^r F_{i,\phi}^{rI*} \langle C_{\phi\theta} C_{\phi\theta}^{I*} \rangle F_{j,\theta}^t F_{j,\theta}^{tI*} \\
 &+ F_{i,\phi}^r F_{i,\phi}^{rI*} \langle C_{\phi\phi} C_{\phi\phi}^{I*} \rangle F_{j,\phi}^t F_{j,\phi}^{tI*}
 \end{aligned} \tag{7.21}$$

Thus, we finally conclude that

$$\langle \mathcal{C}_{\alpha\beta} \mathcal{C}_{\alpha'\beta'} \rangle = \mathcal{P}_{\alpha\beta}(\hat{\mathbf{k}}_r, \hat{\mathbf{k}}_t) \delta^2(\hat{\mathbf{k}}_r - \hat{\mathbf{k}}'_r) \delta^2(\hat{\mathbf{k}}_t - \hat{\mathbf{k}}'_t) \delta_{\alpha\alpha'} \delta_{\beta\beta'} \quad (7.22)$$

## Appendix D. Proof of Proposition 1

Using the scattering matrix (3.10) we can express (4.9) more clearly in terms of the matrix elements

$$M_{\iota\kappa} = 2\pi i^{1+l-\lambda-\tau+t} \int \int A_{\iota,\theta}^* \mathcal{C}_{\theta\theta} A_{\kappa,\theta} + A_{\iota,\theta}^* \mathcal{C}_{\theta\phi} A_{\kappa,\phi} + A_{\iota,\phi}^* \mathcal{C}_{\phi\theta} A_{\kappa,\theta} + A_{\iota,\phi}^* \mathcal{C}_{\phi\phi} A_{\kappa,\phi} d\Omega_r d\Omega_t \quad (7.23)$$

we arrive at

$$\begin{aligned} \mathcal{R}_{\iota\kappa}^{\iota'\kappa'} &= 4\pi^2 i^{l-l'-\lambda+\lambda'-\tau+\tau'+t-t'} \sum_{\alpha} \sum_{\alpha'} \sum_{\beta} \sum_{\beta'} \int \int \int \int \dots \\ &\dots A_{\iota,\alpha}^*(\hat{\mathbf{k}}_r) A_{\iota',\alpha'}(\hat{\mathbf{k}}'_r) \langle \mathcal{C}_{\alpha\beta} \mathcal{C}_{\alpha'\beta'}^* \rangle \dots \\ &\dots A_{\kappa,\beta}(\hat{\mathbf{k}}_t) A_{\kappa',\beta'}^*(\hat{\mathbf{k}}'_t) d\Omega_r d\Omega_t d\Omega'_r d\Omega'_t \end{aligned} \quad (7.24)$$

where the summation is over  $\alpha = \{\theta, \phi\}$ ,  $\alpha' = \{\theta, \phi\}$ ,  $\beta = \{\theta, \phi\}$  and  $\beta' = \{\theta, \phi\}$ . Further substituting (3.12) into (7.24) and using the property of the delta function that  $\int f(\mathbf{x}) \delta^n(\mathbf{x} - \mathbf{a}) d^n \mathbf{x} = f(\mathbf{a})$ , where  $\mathbf{x}$  and  $\mathbf{a}$  are  $n$  dimensional vectors.

## Appendix E.

According (2.7), the output signal at the receiver due to transmitter  $j$  can be written as follows,  $\mathbf{w}_j = \mathbf{R} \mathbf{a}_j$ , then the channel matrix can be expressed as

$$w_{ij} = \sum_{\iota} R_{i\iota} a_{\iota j} \quad (7.25)$$

Now for an incident plane wave at the receive antenna with amplitude  $\mathbf{E}_0$  with  $\hat{\mathbf{k}}_r \cdot \mathbf{E}_0 = 0$  we have that coefficients for the expansion in regular waves is given by

$$f_{\iota j} = \frac{4\pi(-i)^{\lambda-t+1}}{k\sqrt{2\eta}} \mathbf{A}_{\iota}^*(\hat{\mathbf{k}}_r) \cdot \mathbf{E}_0 \quad (7.26)$$

Assuming  $\hat{\mathbf{r}}_r = -\hat{\mathbf{k}}_r$  and observing that

$$\mathbf{A}_{\iota}^*(-\hat{\mathbf{r}}_r) = (-1)^{\lambda-t+1} \mathbf{A}_{\iota}^*(\hat{\mathbf{r}}_r) \quad (7.27)$$

we can rewrite (7.26)

$$f_{\iota j} = \frac{4\pi i^{\lambda-t+1}}{k\sqrt{2\eta}} \mathbf{A}_{\iota}^*(\hat{\mathbf{r}}_r) \cdot \mathbf{E}_0 \quad (7.28)$$

Then using (3.6) and (3.1) and the relationship between expansion coefficients in regular and incoming waves

$$2a_{\iota j} = f_{\iota j} \quad (7.29)$$

we obtain that

$$a_{ij} = \frac{2\pi i^{\lambda-t+1}}{k\sqrt{2\eta}} \mathbf{A}_i^*(\hat{\mathbf{r}}_r) \cdot \mathcal{S}(-\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_t) \cdot \mathbf{F}_j(\hat{\mathbf{r}}_t) \frac{e^{-ik(r_s+r_t)}}{k^2 r_s r_t} \quad (7.30)$$

Inserting (7.30) into (7.25) we get

$$w_{ij} = -2\pi i \sum_{\iota} R_{i,\lambda\mu\iota} i^{\lambda-t+2} \mathbf{A}_{\lambda\mu\iota}^*(\hat{\mathbf{r}}_r) \cdot \mathcal{S}(-\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_t) \cdot \mathbf{F}_j(\hat{\mathbf{r}}_t) \frac{e^{-ik(r_s+r_t)}}{k\sqrt{2\eta}k^2 r_s r_t} \quad (7.31)$$

Further observing that

$$\mathbf{A}_{\lambda(-\mu)t}^*(\hat{\mathbf{k}}_r) = (-1)^\mu \mathbf{A}_{\lambda\mu t}(\hat{\mathbf{k}}_r), \quad (7.32)$$

$$R_{i,\lambda(-\mu)t} = (-1)^\mu T_{\lambda\mu t, i}, \quad (7.33)$$

and that the far-field of the transmitted antenna can be expressed as

$$\mathbf{F}_j(\hat{\mathbf{r}}_t) = k\sqrt{2\eta}v_j \mathbf{F}_j^t(\hat{\mathbf{r}}_t) \quad (7.34)$$

we arrive at

$$\begin{aligned} H_{ij} &= -\frac{2\pi i e^{-ik(r_s+r_t)}}{k^2 r_s r_t} \sum_{\iota} k\sqrt{2\eta} i^{\lambda-t+2} T_{\lambda\mu t, i} \mathbf{A}_{\lambda\mu\iota}^*(\hat{\mathbf{r}}_r) \cdot \mathcal{S}(-\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_t) \cdot \mathbf{F}_j^t(\hat{\mathbf{r}}_t) \\ &= -2\pi i \mathbf{F}_i^r(\hat{\mathbf{r}}_r) \cdot \mathcal{S}(\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_t) \cdot \mathbf{F}_j^t(\hat{\mathbf{r}}_t) \frac{e^{-ik(r_s+r_t)}}{k^2 r_s r_t} \\ &= -2\pi i \mathbf{F}_i^r(\hat{\mathbf{r}}_r) \cdot \mathbf{E}_{0j}^t(\hat{\mathbf{r}}_t) \end{aligned} \quad (7.35)$$

where we have used that  $w_i = \sum_j w_{ij} = \sum_j H_{ij} v_j$  and  $\mathbf{F}_i(\hat{\mathbf{r}}_r) = k\sqrt{2\eta}v_i \mathbf{F}_i^r(\hat{\mathbf{r}}_r)$ . It is worthwhile to notice that both  $\mathbf{F}_i^r(\hat{\mathbf{r}}_r)$ ,  $\mathbf{F}_j^t(\hat{\mathbf{r}}_t)$  and  $\mathbf{E}_{0j}^t(\hat{\mathbf{r}}_t)$  are dimensionless quantities.

Hence, we see that combining (2.7) with (3.4)-(3.6) and using the properties of the spherical harmonics we obtain that the contribution to the signal sensed by antenna  $i$  at the receiver position from antenna  $j$  at the transmitter is then given by

$$H_{ij} = -2\pi i \mathbf{F}_i^r(\hat{\mathbf{r}}_r) \cdot \mathcal{S}(-\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_t) \cdot \mathbf{F}_j^t(\hat{\mathbf{r}}_t) \frac{e^{-ik(r_s+r_t)}}{k^2 r_s r_t}, \quad (7.36)$$

where we have specialized  $\hat{\mathbf{r}}_s = -\hat{\mathbf{r}}_r$ . Equivalently, for the reverse channel, the transmission and reception roles are interchanged between the antennas. Hence

$$\tilde{H}_{ji} = -2\pi i \mathbf{F}_j^t(\hat{\mathbf{r}}_t) \cdot \mathcal{S}(-\hat{\mathbf{r}}_t, \hat{\mathbf{r}}_r) \cdot \mathbf{F}_i^r(\hat{\mathbf{r}}_r) \frac{e^{-ik(r_s+r_t)}}{k^2 r_s r_t}. \quad (7.37)$$

It is worthwhile to notice that in (7.36) and (7.37),  $\mathbf{F}_i^r(\hat{\mathbf{r}}_r)$  and  $\mathbf{F}_j^t(\hat{\mathbf{r}}_t)$  are dimensionless and differ from  $\mathbf{F}_i(\hat{\mathbf{r}}_r)$  and  $\mathbf{F}_j(\hat{\mathbf{r}}_t)$  by a multiplication factor (7.34). For reciprocal channels  $H_{ij} = H_{ji}$ , where the elements are obtained by transposition, the scattering dyadic satisfy the following necessary but not sufficient condition

$$\mathcal{S}(-\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_t) = \mathcal{S}^T(-\hat{\mathbf{r}}_t, \hat{\mathbf{r}}_r). \quad (7.38)$$

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