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Abstract

This thesis considers static and dynamic team decision problems in both stochastic and deterministic settings. The team problem is a cooperative game, where a number of players make up a team that tries to optimize a given cost induced by the uncertainty of nature. The uncertainty is modeled as either stochastic, which gives the stochastic team problem, or modeled as deterministic where the team tries to optimize the worst case scenario. Both the stochastic and deterministic static team problems are stated and solved in a linear quadratic setting. It is shown that linear decisions are optimal in both the stochastic and deterministic framework.

The dynamic team problem is formulated using well known results from graph theory. The dynamic interconnection structure is described by a graph. It appears natural to use a graph theoretical formulation to examine how a decision by a member of the team affects the rest of the members.

Conditions for tractability of the dynamic team problem are given in terms of the graph structure. Tractability of a new class of information constrained team problems is shown, which extends existing results. For the presented tractable classes, necessary and sufficient conditions for stabilizability are given.

The state feedback $\mathcal{H}_2$ and $\mathcal{H}_\infty$ dynamic team problems are solved using a novel approach. The new approach is based on the crucial idea of disturbance feedback, which is used to separate the controller effect from the measured output, to eliminate the controller's dual role.

Finally, a generalized stochastic linear quadratic control problem is considered. A broad class of team problems can be modeled by imposing quadratic constraints of correlation type. Also, power constraints on the control signals are very common. This motivates the development of a generalized control theory for both the finite and infinite horizon case, where power constraints are imposed. It is shown that the solution can be found using finite dimensional convex optimization.

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Optimal Decisions
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Limited Information

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Department of Automatic Control
Lund University
Lund, June 2007

To my parents Awatif and Said

"Any intelligent fool can make things bigger, more complex, and more violent. It takes a touch of genius – and a lot of courage – to move in the opposite direction."

Albert Einstein

Abstract

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Abstract

Preface

The thesis consists of 8 chapters. We will give a brief summary of the content and main contributions of each chapter:

Chapter 1

The first chapter gives a brief introduction to the team problem and its history. Some motivational examples are also given.

Chapter 2

In this chapter, we introduce the notation and mathematical tools used throughout the thesis. The graph theory section is crucial, which introduces the potential use of graph theoretical concepts in the theory of linear dynamical systems.

Chapter 3

This chapter treats static and dynamic estimation problems. The estimation problem is formulated as an optimal decision problem in three kinds of settings: deterministic (or minimax), stochastic, and error-operator minimization. The main contribution is to show that the linear optimal solutions to the three estimation problems mentioned above coincide. We then derive the optimal distributed (or team) estimators in the stochastic and the deterministic setting (known as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ in systems theory).

This chapter is partly based on:

Ather Gattami. *Optimal Distributed Linear Quadratic Control*. Submitted to the 46th IEEE Conference on Decision and Control (CDC), 2007.

Chapter 4

Chapter 4 introduces the team decision problem in the stochastic framework. Old results are reviewed and given new formulations and proofs, which are of more modern character. The results are then used to solve

the stochastic finite horizon linear quadratic team problem, where conditions are given for convexity. The new conditions generalize some previous results.

The chapter is partly based on:

Ather Gattami. *Distributed Stochastic Control:*
A Team Theoretic Approach. In Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, July 2006.

Chapter 5

Chapter 5 introduces the deterministic analog of the stochastic team problem. It is shown that the linear decisions are optimal for the static minimax team problem. Like in Chapter 4, the results are then used to solve the deterministic finite horizon linear quadratic team problem, and conditions on the problem parameters are given for convexity.

The chapter is based on:

Ather Gattami and Bo Bernhardsson. *Minimax Team Decision Problems*. In Proceedings of the American Control Conference 2007, New York, USA.

Chapter 6

This chapter considers team problems in the linear quadratic dynamic setting. Whereas the dynamic team problem was solved for the finite horizon case in chapters 4 and 5, the infinite horizon case is solved here. A general control problem setup is given, where constraints on information of the external signals (such as disturbances) are imposed. Necessary and sufficient conditions are given for stabilizability of distributed control problems with delayed measurements. A novel approach to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problem is developed. The new approach is applied to find the optimal state feedback control law for information constrained control problems. Also, the approach reveals that the optimal state feedback controllers for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems coincide.

The chapter is based on:

Ather Gattami. *Optimal Distributed Linear Quadratic Control*. Submitted to the 46th IEEE Conference on Decision and Control (CDC), 2007.

Chapter 7

Chapter 7 considers the problem of stochastic finite and infinite horizon linear quadratic control under nonconvex power constraints. A broad class of stochastic linear quadratic optimal control problems with information

constraints can be modeled with the help of power constraints. First, the finite horizon state feedback control problem is solved through duality. The computations of the optimal control law can be done off-line as in the classical linear quadratic Gaussian control theory using a combination of dynamic programming and primal-dual methods. Then, a solution to the infinite horizon control problem is presented. Finally, the output feedback problem is solved.

The chapter is based on:

Ather Gattami. *Generalized Linear Quadratic Control Theory*. In Proceedings of the 45th IEEE Conference on Decision and Control (CDC), San Diego, USA, December 2006.

Chapter 8

The main results of the thesis are summarized in this chapter, and some possible future research avenues are discussed.

How to read this Thesis

The dependencies between the chapters are as follows. Notation used throughout the thesis is presented in Chapter 2. A brief introduction to graph theory concepts is also given in Chapter 2. Graph theory is used in chapters 4, 5, and 6. The terminology and results of the game theory section in Chapter 2 are used in Chapter 3. Chapters 1, 3-8 can be read independently.

Other Publications

The following publications are not included in this Thesis, but they are closely related (some of them are cited in the introductory chapter):

Ather Gattami and Richard Murray. *A Frequency Domain Condition for Stability of Interconnected MIMO Systems*. In Proceedings of the American Control Conference, Boston, USA, June 2004.

Ather Gattami and Johannes Berglund. *Stabilization of Vehicle Formations: A Case Study*. In Proceedings of the 3rd Swedish Workshop on Autonomous Robotics, Stockholm, Sweden, September 2005.

Ather Gattami and Anders Rantzer. *Linear Quadratic Performance Criteria for Cascade Control*. In Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference, Seville, Spain, December 2005.

Preface

Acknowledgements

It is a pleasure, privilege, and great honor to be part of the Department of Automatic Control at Lund.

First of all, I would like to thank my supervisors Anders Rantzer and Bo Bernhardsson.

Anders Rantzer has been a source of inspiration for how I pursued my research, and a great influence in forming the way I looked at things in and around research. Anders always gave me valuable comments and suggestions about my work, and helped me to improve on my manuscripts.

After my first year, Anders left for a sabbatical for ten months, and a great time with Bo Bernhardsson started. Bo inspired me in ways that are beyond what I can express in words. His great sense of humor made our almost regular meetings on Fridays the highlight of the week. We got to learn a lot of things together, sometimes out of pure childish curiosity. Bo's creativity has been very inspiring and I must say that he is by far the most creative person I have ever met. It is also worth mentioning Bo's remarkable ability to find errors, from which I benefited greatly.

Teaching has been a big part of my Ph.D. education. It has been great working with Professor Tore Hägglund, who has always been well organized and open to new ideas and suggestions.

I have enjoyed sharing office with Brad Schofield for three years. From day one, we started our struggle in teaching the hardest course at the department, which none of us really mastered (so to all of our students, you are not alone). But we helped each other out. He helped me once again by reading this Thesis and giving linguistic corrections. I also owe Peter Alriksson, Toivo Henningsson, and Erik Johannesson a debt of gratitude for reading this Thesis and giving valuable comments and suggestions. Many thanks to Anton Cervin, Per Hagander, Bo Lincoln, Anders Robertsson, Henrik Sandberg, Björn Wittenmark, and Karl Johan Åström. They never hesitated to take their time to help and answer many questions, in particular at the beginning of my Ph.D. studies.

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Introduction

In the past half century, optimal decision problems have received a large amount of interest by mathematicians, information theorists, and control theorists. Decision theory dates back to the 19th century when statistical decision theory was developed. The main applications concerned the art of war, or economic or financial speculations. In 1921, Émile Borel [10] was the first author to evolve the concept of pure and mixed strategies, in the framework of skew symmetric matrix games. A revolutionary result was given by John von Neumann in 1928, where he presented his celebrated "minimax" solution to the two-person zero-sum matrix games. In the 1950's, a new type of game problems emerged, so-called "Team Problems", introduced by Marshak in 1955 [38]. The team problem is an optimization problem, where a number of decision makers (or players) make up a team, optimizing a common cost function with respect to some uncertainty representing *nature*. Each member of the team has limited information about the global state of nature. Furthermore, the team members could have different pieces of information, which makes the problem different from the one considered in classical optimization, where there is only one decision function that has access to the entire information available about the state of nature.

Team problems seemed to possess certain properties that were considerably different from standard optimization, even for specific problem structures such as the optimization of a quadratic cost in the state of nature and the decisions of the team members. In stochastic linear quadratic decision theory, it was believed for a while that separation holds between estimation and optimal decision with complete information, even for team problems. The separation principle can be briefly explained as follows. First assume that every team member has access to the information about the entire state of nature, and find the corresponding optimal decision for each member. Then, each member makes an estimate of the state of nature, which is in turn combined with the optimal decision obtained from

the full information assumption. It turns out that this strategy does *not* yield an optimal solution (see [42]), as will be shown by the numerical example below:

EXAMPLE 1.1
Consider a team of 2 players with decisions u_1 and u_2, and let the state of nature be a stochastic variable x, with[1] $x \sim \mathcal{N}(0,1)$. Let $v_1, v_2 \sim \mathcal{N}(0,1)$, and assume that x, v_1, v_2 are uncorrelated. The information about x available to decision maker 1 is $y_1 = x + v_1$, and the information available to decision maker 2 is $y_2 = x + v_2$. The team problem is to find optimal decision functions $u_1 = \mu_1(y_1)$ and $u_2 = \mu_2(y_2)$ that minimize the cost

$$\mathbf{E} \begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix}.$$

Standard completion of squares gives:

$$\begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix} = x^T \left[1 - \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] x +$$

$$+ \begin{pmatrix} u_1 + \frac{1}{3}x \\ u_2 + \frac{1}{3}x \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 + \frac{1}{3}x \\ u_2 + \frac{1}{3}x \end{pmatrix}.$$

Since there is nothing to do about the first term in the right hand side of the equation above, we deduce that the optimal decisions with full information are given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x \\ -\frac{1}{3}x \end{pmatrix}.$$

The optimal estimate of x of decision maker 1 is

$$\hat{x}_1 = \mathbf{E}\{x|y_1\} = \frac{1}{2}y_1,$$

and of decision maker 2

$$\hat{x}_2 = \mathbf{E}\{x|y_2\} = \frac{1}{2}y_2.$$

---

[1] $x \sim \mathcal{N}(m, X)$ means that x is a Gaussian variable with mean $\mathbf{E}\,x = m$ and covariance $\mathbf{E}\,xx^T = X$.

Hence, the decision where decision maker i combines the best deterministic decision with her best estimate of x is given by

$$u_i = -\frac{1}{3}\hat{x}_i$$
$$= -\frac{1}{3} \cdot \frac{1}{2}y_i$$
$$= -\frac{1}{6}y_i,$$

for $i = 1, 2$. This decision gives a cost equal to 0.611. However, the team decision given by

$$u_i = -\frac{1}{5}y_i,$$

yields a cost equal to 0.600. Clearly, separation does not hold in team decision problems. $\square$

A general solution to similar stochastic quadratic team problems was presented by Radner [42]. Radner's result gave hope that some related problems could be solved using similar arguments. But in 1968, Witsenhausen [53] showed in his well known paper that finding the optimal decision can be complex if the decision makers affect each other's information. Witsenhausen considered a dynamic decision problem over two time steps to illustrate that difficulty. The dynamic problem can actually be written as a static team problem:

$$\text{minimize } \mathbf{E} \left\{ k_0 u_0^2 + (x + u_0 - u_1)^2 \right\}$$
$$\text{subject to } u_0 = \mu_0(x), \quad u_1 = \mu_1(x + u_0 + w)$$
$$x \sim \mathcal{N}(0, X), w \sim \mathcal{N}(0, W)$$

Here, we have two decision makers, one corresponding to u_0, and the other to u_1. Witsenhausen showed that the optimal decisions μ_0 and μ_1 are not linear because of the *coding incentive* of u_0. Decision maker 1 measures $x + u_0 + w$, and hence, its measurement is affected by u_0. Decision maker 0 tries to *encode* information about x in its decision, which makes the optimal strategy complex. The problem above is actually an information theoretic problem. To see this, consider the slightly modified problem

$$\text{minimize } \mathbf{E} (x - u_1)^2$$
$$\text{subject to } u_0 = \mu_0(x), \quad \mathbf{E} \, u_0^T u_0 \leq 1, \quad u_1 = \mu_1(u_0 + w)$$
$$x \sim \mathcal{N}(0, X), w \sim \mathcal{N}(0, W)$$

The modified problem is exactly the Gaussian channel coding/decoding problem (see Figure 1.1)!
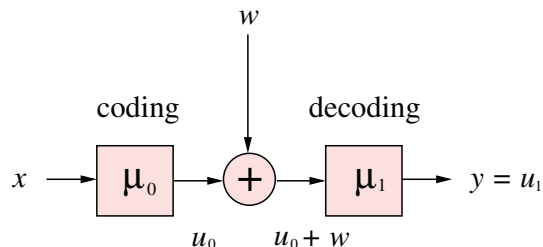
Figure 1.1 Coding-decoding diagram over a Gaussian channel.

The property that the decision made by each team member can affect the information of the other members of the team led to an observation made by Ho and Chu in 1972 [29]:

"If a decision makers action affects our information, then knowing what he knows will yield linear optimal solutions."

This information structure is then called *partially nested*. The problem of the incentive of encoding information in the decisions appears naturally in control of dynamical systems. The most fundamental and well known open problem in linear systems theory, namely the static output feedback problem, is hard because of the coding incentive that arises as a result of the lack of complete output history.

Team decision problems can be found in our everyday life. There are many examples of such systems, and here we give only a sample of different problems that have the issue of team decision in common. The Internet is a very large network where issues of optimal distribution of information flow are of great interest. Although every subsystem in the network tries to maximize its information flow, there is a common interest of stabilizing the entire network. The information flow transported along different links is subject to delays. This makes it hard to stabilize the entire network if the delays are not taken into account.

The power network is probably one of the most complex networks in engineering. We can find stability problems not only when trying to robustly stabilize the physical power network (which is hard enough), but also stabilize the market that is embedded into it. A power company has group of generators in some geographical region, which are dynamically interconnected. There are two important issues in a power network. The first is to stabilize the entire network. The second one is to minimize the power losses along the interconnection links. This can be modeled as a dynamic team problem, where the generators make up a team that minimizes the power losses along the interconnection links, subject to stability of the network. An example of a power network problem arising from a
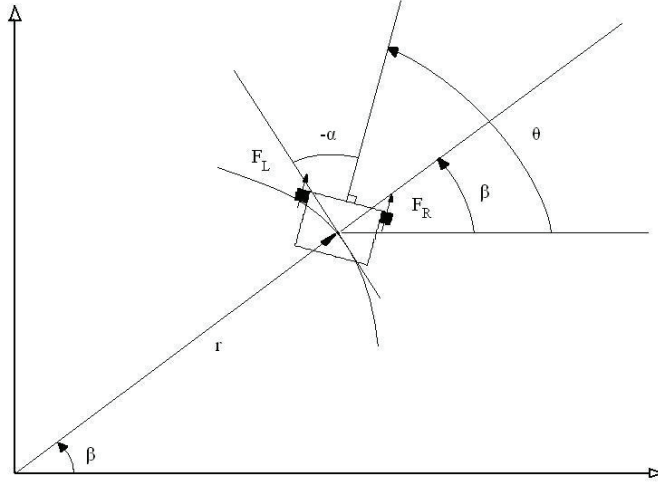
Figure 1.2 The Multi-Vehicle Wireless Testbed vehicle.

combination of the economics and technology of the electricity market is the California power crisis of 2000.

In recent years, stability of vehicle formations has been of great interest. Formation of unmanned air vehicles (UAV), robots, and satellites are a few examples. we will give a toy example of a vehicle formation problem (see [21] and the references therein):

EXAMPLE 1.2
Consider a practical example of stabilization of vehicle formations, namely six vehicles from the Multi-Vehicle Wireless Testbed (MVWT) used at Caltech [14]. The task is to stabilize the six vehicles in a prespecified formation. The dynamics of each vehicle are given by

$$m(\ddot{r} - r\dot{\beta}^2) = -\mu\dot{r} + (F_R + F_L)\cos(\theta - \beta)$$
$$m(r\ddot{\beta} + 2\dot{r}\dot{\beta}) = -\mu r\dot{\beta} + (F_R + F_L)\sin(\theta - \beta) \qquad (1.1)$$
$$J\ddot{\theta} = -\mu r^2\dot{\theta} + (F_R - F_L)r_f$$

Each vehicle has a rectangular shape seen from above, with two fans to control its motion, see Figure 1.2. The nonlinear dynamics are linearized and we obtain a linear system for the error dynamics which has two inputs, the fan forces F_R and F_L and two outputs, the polar coordinates r and β. The task is to stabilize all vehicles in a formation of two groups, with three vehicles in each group, rotating around an agreed coordinate. There is no common coordinate-system. Each vehicle can only measure the relative

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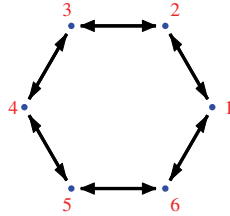
Figure 1.3   The interconnection graph.
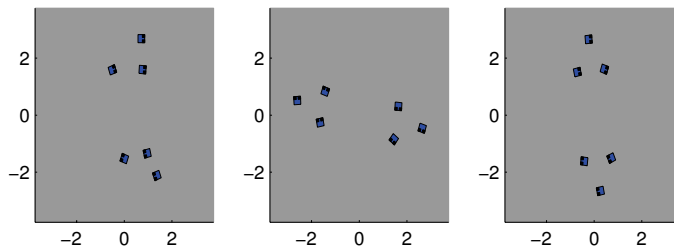
Figure 1.4   The vehicles rotate in two groups around a center that is agreed on on-line. The simulation shows the stable formation, where they rotate in the desired grouping counter-clockwise.

distance to a limited number of other vehicles. Using the fact that the system is homogeneous (that is, the subsystems have identical dynamics), existing results from [20] can be used for *separately* finding a decentralized controller for every vehicle. The graph in Figure 1.3 shows which vehicles that can see each other. Every node denotes a vehicle, and for instance, the graph shows that vehicle 1 can sense the distance to vehicle 2 and 6, vehicle 2 can sense the distance to vehicle 1 and 3, and so on. Other interconnections can also be used using the same methods for analysis and controller design. A simulation is presented in Figure 1.4.   □

The problem presented in the example above has two features. The first feature is that the vehicles have identical dynamics (homogeneous system). The other feature is that there is no interconnection in the dynamics, only in the cost represented by the distances between adjacent vehicles. The analysis and synthesis become more difficult in the case of a heterogeneous system with dynamical coupling. This motivates the study of team problems of a more general structure, where the subsystems are not identical and could be dynamically interconnected.

2

Mathematical Background

2.1 Notation

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. The set of natural numbers $\{0, 1, 2, 3, ...\}$ is denoted by $\mathbb{N}$. The set of integers modulo n, $\{0, 1, ..., n-1\}$, is denoted by $\mathbb{Z}_n$. We denote the open unit disk in $\mathbb{C}$ by $\mathbb{D}$:

$$\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\},$$

and let $\overline{\mathbb{D}}$ and $\partial\mathbb{D}$ be its closure and boundary, respectively:

$$\overline{\mathbb{D}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\},$$
$$\partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

For a stochastic variable x, $x \sim \mathcal{N}(m, X)$ means that x is a Gaussian variable with $\mathbf{E}\, x = m$ and $\mathbf{E}\, (x - m)(x - m)^T = X$. The Dirac delta function $\delta(t)$ is defined as

$$\delta(t) = \left\{ \begin{array}{ll} 1 & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{array} \right.$$

We denote the set of $n \times n$ symmetric, positive semi-definite, and positive definite matrices by $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$, respectively.

A^T denotes the transpose of the the matrix A. The Hermitian adjoint of a matrix A is denoted by A^*. $A^\dagger$ denotes the pseudo-inverse of the matrix A. We write $Q \succ 0$ ($Q \succeq 0$) to denote that Q is positive definite (semi-definite). $Q \succ P$ ($Q \succeq P$) means that $Q - P \succ 0$ ($Q - P \succeq 0$). $\mathbf{Tr}\, A$ denotes the trace of the quadratic matrix A. M_i, or $[M]_i$, denotes either block column i or block row i of a matrix M, which should follow from the

context. For a matrix A partitioned into blocks, $[A]_{ij}$ denotes the block of A in block position (i, j). Given a matrix $M \in \mathbb{R}^{m \times n}$, we let vec$\{M\}$ denote

$$\text{vec}\{M\} = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix},$$

where M_i denotes the ith column of M. Also, let diag(D_i) denote a block diagonal matrix with the matrices D_i on its diagonal. The matrix $\mathbf{1}_{m \times n}$ denotes an $m \times n$ matrix with entries equal to 1. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. The *Kronecker product* $A \otimes B \in \mathbb{R}^{mp \times nq}$ is defined as:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}.$$

2.2 Linear Systems Theory

For a matrix valued function $G(q)$ defined on the unit circle $\partial \mathbb{D}$ we define the norm

$$\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \mathbf{Tr}\{G^*(e^{j\theta})G(e^{j\theta})\}d\theta}.$$

$\mathcal{L}_2$ is a Hilbert space of matrix valued functions G defined on $\partial \mathbb{D}$ with $\|G\|_2 < \infty$. The real rational subspace of $\mathcal{L}_2$, which consists of all strictly proper real rational transfer matrices is denoted by $\mathcal{RL}_2$. $\mathcal{H}_2$ is a (closed) subspace of $\mathcal{L}_2$ with matrix valued functions $G(q)$ analytic in $\mathbb{C}\backslash\overline{\mathbb{D}}$. The real rational subspace of $\mathcal{H}_2$, which consists of all strictly proper real rational transfer matrices analytic in $\mathbb{C}\backslash\overline{\mathbb{D}}$, is denoted by $\mathcal{RH}_2$.

$\mathcal{L}_\infty$ is a Banach space of measurable matrix valued functions G defined on $\partial \mathbb{D}$, with norm

$$\|G\|_\infty = \sup_{f \in \mathcal{H}_2} \{\|Gf\|_2 : \|f\|_2 = 1\}.$$

The real rational subspace of $\mathcal{L}_\infty$ is denoted by $\mathcal{RL}_\infty$ which consists of all proper and real rational transfer matrices. $\mathcal{H}_\infty$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in $\mathbb{C}\backslash\overline{\mathbb{D}}$. The real rational

subspace of $\mathcal{H}_\infty$ is denoted by $\mathcal{R}\mathcal{H}_\infty$ which consists of all proper and real rational transfer matrices analytic and bounded in $\mathbb{C}\backslash\overline{\mathbb{D}}$.

The forward shift operator is denoted by $\mathbf{q}$. That is $x_{k+1} = \mathbf{q}x_k$, where $\{x_k\}$ is a given process. A causal linear time-invariant operator $T(\mathbf{q})$ is given by its generating function $T(\mathbf{q}) = \sum_{k=0}^{\infty} T_k \mathbf{q}^{-k}$, $T_k \in \mathbb{R}^{p\times m}$. A transfer matrix in terms of state-space data is denoted

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := C(\mathbf{q}I - A)^{-1}B + D.$$

For matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, we say that (A, B) is stabilizable if $\left(\begin{array}{cc} \lambda I - A & B \end{array}\right)$ has full row rank for all $\lambda \in \mathbb{C}\backslash\mathbb{D}$, and that (C, A) is detectable if $\left(\begin{array}{c} \lambda I - A \\ C \end{array}\right)$ has full column rank for all $\lambda \in \mathbb{C}\backslash\mathbb{D}$.

2.3 Graph Theory

We will present in brief some graph theoretical definitions and results that could be found in the graph theory or combinatorics literature (see for example [17]). A (simple) graph $\mathcal{G}$ is an ordered pair $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set, whose elements are called *vertices* or *nodes*, $\mathcal{E}$ is a set of pairs (unordered) of distinct vertices, called *edges* or *lines*. The vertices belonging to an edge are called the ends, endpoints, or end vertices of the edge. The set $\mathcal{V}$ (and hence $\mathcal{E}$) is taken to be finite in this thesis. The order of a graph is $|\mathcal{V}|$ (the number of vertices). A graph's size is $|\mathcal{E}|$, the number of edges. The degree of a vertex is the number of other vertices it is connected to by edges. A *loop* is an edge with both ends the same.

A directed graph or digraph $\mathcal{G}$ is a graph where $\mathcal{E}$ is a set of ordered pairs of vertices, called *directed* edges, arcs, or arrows. An edge $e = (v_i, v_j)$ is considered to be directed from v_i to v_j; v_j is called the head and v_i is called the tail of the edge.

A *path* or *walk* Π in a graph of length m from vertex u to v is a sequence $e_1 e_2 \cdots e_m$ of m edges such that the head of e_m is v and the tail of e_1 is u, and the head of e_i is the tail of e_{i+1}, for $i = 1, ..., m - 1$. The first vertex is called the start vertex and the last vertex is called the end vertex. Both of them are called end or terminal vertices of the walk. If also $u = v$, then we say that Π is a *closed walk* based at u. A directed graph is *strongly connected* if for every pair of vertices (v_i, v_j) there is a walk from v_i to v_j.

The *adjacency matrix* of a finite directed graph $\mathcal{G}$ on n vertices is the $n \times n$ matrix where the nondiagonal entry a_{ij} is the number of edges from
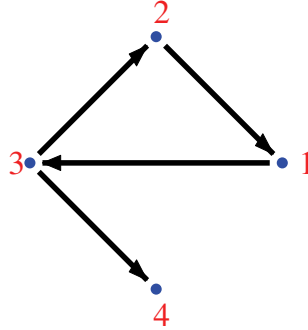
Figure 2.1   An example of a graph $\mathcal{G}$.

vertex i to vertex j, and the diagonal entry a_{ii} is the number of loops at vertex i (the number of loops at every node is defined to be one, unless another number is given on the graph). For instance, the adjacency matrix of the graph in Figure 2.1 is

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A graph $\mathcal{G}$ with adjacency matrix A is *isomorphic* to another graph $\mathcal{G}'$ with adjacency matrix A' if there exists a permutation matrix P such that

$$PAP^T = A'$$

The matrix A is said to be *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix} \tag{2.1}$$

where E and G are square matrices. If A is not reducible, then it is said to be *irreducible*.

PROPOSITION 2.1
A matrix $A \in \mathbb{Z}_2^{n \times n}$ is irreducible if and only if its corresponding graph $\mathcal{G}$ is strongly connected. $\qquad\square$

Proof If A is reducible, then from (2.1) we see that the vertices can be divided in two subsets; one subset belongs to the rows of E and the other belongs to the rows of G. The latter subset is closed, because there is no walk from the second subset to the first one. Hence, the graph is not strongly connected.

Now, suppose that $\mathcal{G}$ is not strongly connected. Then there exists an isolated subset of vertices. Permute the vertices of $\mathcal{G}$ such that the vertices in the isolated subset comes last in the enumeration of $\mathcal{G}$. Then we see that the same permutation with A gives a block triangular form as in (2.1). □

PROPOSITION 2.2
Consider an arbitrary finite graph $\mathcal{G}$ with adjacency matrix A. Then there is a permutation matrix P and a positive integer r such that

$$PAP^T = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1j} \\ 0 & A_2 & \cdots & A_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix}, \tag{2.2}$$

where $A_1, ..., A_r$ are adjacency matrices of strongly connected graphs. □

Proof If A is strongly connected, then it is irreducible according to Proposition 2.1, and there is nothing to do. Suppose now that A is reducible. Then Proposition 2.1 gives that there is a permutation matrix P_1 such that $A_1 = P_1 A P_1^T$ with

$$A_1 = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix},$$

and E and G are square matrices. Now if E and G are irreducible, then we are done. Otherwise we repeat the same argument with E and/or G. Since the graph is finite, we can only repeat this procedure a finite number of times, and hence there is some positive integer r where this procedure stops. Then we arrive at a sequence of permutation matrices $P_1, ..., P_r$, such that $(P_r \cdots P_1)A(P_r \cdots P_1)^T$ has a block triangular structure given by (2.2) with $A_1, ..., A_r$ irreducible, and hence adjacency matrices of strongly connected graphs. Taking $P = P_r \cdots P_1$ completes the proof. □

Now let $\omega : \mathcal{E} \to R$ be a *weight function* on $\mathcal{E}$ with values in some commutative ring R (we can take $R = \mathbb{C}$ or a polynomial ring over $\mathbb{C}$).

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If $\Pi = e_1 e_2 \cdots e_m$ is a walk, then the *weight* of Π is defined by $w(\Pi) = w(e_1)w(e_2)\cdots w(e_m)$.

Let $\mathcal{G}$ be a finite directed graph, with $|\mathcal{V}| = p$. In this case, letting $i, j \in \{1, ..., p\}$ and $n \in \mathbb{N}$, define

$$A_{ij}(n) = \sum_{\Pi} \omega(\Pi),$$

where the sum is over all walks Π in $\mathcal{G}$ of length n from v_i to v_j. In particular $A_{ij}(0) = \delta(i - j)$. Define a $p \times p$ matrix A by

$$A_{ij} = \sum_{e} \omega(e),$$

where the sum is over all edges e with v_i and v_j as the head and tail of e, respectively. In other words, $A_{ij} = A_{ij}(1)$. The matrix A is called the adjacency matrix of $\mathcal{G}$, with respect to the weight function ω.

The following proposition can be found in [50]:

PROPOSITION 2.3
Let $n \in \mathbb{N}$. Then the (i, j)-entry of A^n is equal to $A_{ij}(n)$. □

Proof   This is immediate from the definition of matrix multiplication. Specifically, we have

$$[A^n]_{ij} = \sum A_{ii_1} A_{i_1 i_2} \cdots A_{i_{n-1} j},$$

where the sum is over all sequences $(i_1, ..., i_{n-1}) \in \{1, ..., p\}^{n-1}$. The summand is 0 unless there is a walk $e_1 e_2 \cdots e_n$ from v_i to v_j with v_{i_k} as the tail of e_k $(1 < k \le n)$ and $v_{i_{k-1}}$ as the head of e_k $(1 \le k < n)$. If such a walk exists, then the summand is equal to the sum of the weights of all such walks, and the proof follows. □

COROLLARY 2.1
Let $\mathcal{G}$ be a graph with adjacency matrix $A \in \mathbb{Z}_2^{n \times n}$. Then there is a walk of length k from node v_i to node v_j if and only if $[A^k]_{ij} \ne 0$. In particular, if $[A^{n-1}]_{ij} = 0$, then $[A^k]_{ij} = 0$ for all $k \in \mathbb{N}$. □

A particularly elegant result for the matrices $A_{ij}(n)$ is that the generating function $g_{ij}(\lambda) = \sum_n A_{ij}(n)\lambda^n$ is

$$
\begin{aligned}
g_{ij}(\lambda) &= \sum_n A_{ij}(n)\lambda^n \\
&= \sum_n [A^n]_{ij}\lambda^n
\end{aligned}
$$
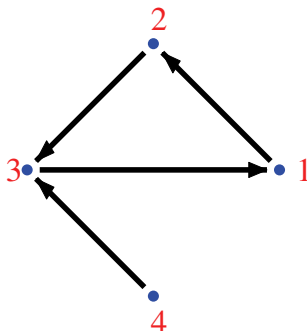
Figure 2.2 The adjoint of the graph G in Figure 2.1.

Thus, we can see that the generating matrix function

$$G(\lambda) = (I - \lambda A)^{-1},$$

is such that $[G(\lambda)]_{ij} = g_{ij}(\lambda)$.

The *adjoint* graph of a finite directed graph G is denoted by G^*, and it is the graph with the orientation of all arrows in G reversed. If the adjacency matrix of G with respect to the weight function ω is A then the adjacency matrix of G^* is A^*.

EXAMPLE 2.1
Consider the graph G in Figure 2.1. The adjacency matrix (in $\mathbb{Z}_2^{4\times4}$) of this graph is

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The adjoint graph G^* is given by Figure 2.2. It is easy to verify that the adjacency matrix (in $\mathbb{Z}_2^{4\times4}$) of G^* is

$$A^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

To see if there is a walk of length 2 or 3 between any two nodes in G, we

calculate A^2 and A^3:

$$A^2 = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 3 & 3 & 3 \\ 3 & 2 & 3 & 1 \\ 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using Corollary 2.1, we can see that there is no walk of length 2 from node 2 to node 4 since $[A^2]_{24} = 0$. On the other hand, there is a walk of length 3 since $[A^3]_{24} = 1 \neq 0$. There is also a walk of length 3 from node 2 to node 3, since we have assumed that every node has a loop. An example of such a walk is node 2 $\rightarrow$ node 1 $\rightarrow$ node 1 $\rightarrow$ node 3. Note also that since $[A^3]_{4i} = 0$ for $i = 1, 2, 3$, there is *no walk* that leads from 4 to any of the nodes 1, 2, or 3. $\qquad\square$

EXAMPLE 2.2
Consider the matrix

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}. \tag{2.3}$$

The sparsity structure of A can be represented by the graph given in Figure 2.1. Hence, if there is an edge from node i to node j, then $A_{ij} \neq 0$. Mainly, the block structure of A is the same as the adjacency matrix of the graph in Figure 2.1. Let $\{B(l)\}$ be a set of k matrices having the same sparsity structure. Consider the product $A(k) = \prod_{l=1}^{k} B(l)$. If the block $A_{ij}(k) \neq 0$, then there is a walk of length less than or equal to k from node i to j. In particular, if each $B(l)$ consists of compatible $n \times n$ block matrices, then Corollary (2.1) gives that $A(k)$ has the same block structure for all $k \geq n - 1$. $\qquad\square$

Systems over Graphs
Consider linear systems $\{G_i(\mathbf{q})\}$ with state space realization

$$G_i(\mathbf{q}) := \begin{cases} x_i(k+1) = A_{ii}x_i(k) + \sum_{j \neq i}^{N} A_{ij}u_{ij}(k) + B_i u_i(k) \\ y_i(k) = C_i x_i(k), \end{cases} \tag{2.4}$$

for $i = 1, ..., N$. Here, $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ for $j \neq i$, $B_i \in \mathbb{R}^{n_i \times m_i}$, and $C_i \in \mathbb{R}^{p_i \times n_i}$. The systems are interconnected as follows. We set $u_{ij} = x_j$ for all i and $j \neq i$. If system $G_j(\mathbf{q})$ affects the dynamics of $G_i(\mathbf{q})$, then $A_{ij} \neq 0$, otherwise $A_{ij} = 0$. This interconnection can then be written as

$$
\underbrace{\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_N(k+1) \end{pmatrix}}_{x(k+1)} = \underbrace{\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_N(k) \end{pmatrix}}_{x(k)} +
$$

$$
+ \underbrace{\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & B_N \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_N(k) \end{pmatrix}}_{u(k)}, \tag{2.5}
$$

$$
\underbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_N(k) \end{pmatrix}}_{y(k)} = \underbrace{\begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & C_N \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_N(k) \end{pmatrix}}_{}.
$$

This block structure can be described by a graph $\mathcal{G}$ of order N, whose adjacency matrix is A, with respect to some weighting function ω. The graph $\mathcal{G}$ has an arrow from node i to j if and only if $A_{ij} \neq 0$. The transfer function of the interconnected systems is given by $G(\mathbf{q}) = C(\mathbf{q}I - A)^{-1}B$. Then, the system $G^T(\mathbf{q})$ is equal to $B^T(\mathbf{q}I - A^T)^{-1}C^T$, and it can be represented by a graph $\mathcal{G}^*$ which is the adjoint of $\mathcal{G}$, since the adjacency matrix of $\mathcal{G}^*$ is $A^* = A^T$. The block diagram for the transposed interconnection is simply obtained by reversing the orientation of the interconnection arrows. This property was observed in [7].

EXAMPLE 2.3
Consider four interconnected systems with state space realization as in (2.5) with $N = 4$ and A given by (2.3). The interconnection can be represented by the graph $\mathcal{G}$ in Figure 2.3. System 2 is affected directly by the dynamics of system 1, and this is reflected in the graph by an arrow from node 2 to node 1. It is also reflected in the A matrix, where $A_{21} \neq 0$. On the other hand, the system 1 is not affected by the dynamics of system 2,
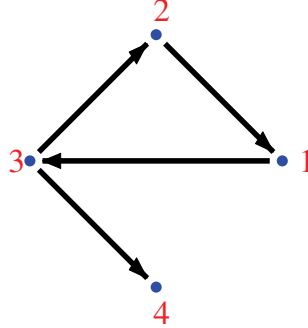
Figure 2.3   The graph reflects the interconnection structure of the dynamics be-
tween four systems. The arrow from node 2 to node 1 indicates that system 2 is
affected directly by the dynamics of system 1.

and therefore there is no arrow from node 1 to node 2, and $A_{12} = 0$. The
state space realization of the transpose of this system is

$$
\left(
\begin{array}{cccc|cccc}
A_{11}^T & A_{21}^T & 0 & 0 & C_1^T & 0 & 0 & 0 \\
0 & A_{22}^T & A_{32}^T & 0 & 0 & C_2^T & 0 & 0 \\
A_{13}^T & 0 & A_{33}^T & 0 & 0 & 0 & C_3^T & 0 \\
0 & 0 & A_{34}^T & A_{44}^T & 0 & 0 & 0 & C_4^T \\
\hline
B_1^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B_2^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_3^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_4^T & 0 & 0 & 0 & 0
\end{array}
\right).
$$

Then interconnection structure for the transposed system can be described
by the adjoint of G in Figure 2.2. ☐

2.4 Game Theory

In this section, we will review some definitions and results from classical
game theory. These can be found in for example [5] and [6].

Let $J = J(u,w)$ be a functional defined on a product vector space
$\mathbb{U} \times \mathbb{W}$, to be minimized by $u \in U \subset \mathbb{U}$ and maximized by $w \in W \subset \mathbb{W}$,
where U and W are the constrained sets. This defines a zero-sum game,
with kernel J, in connection with which we can introduce two values, the

upper value

$$\bar{J} := \inf_{u \in U} \sup_{w \in W} J(u,w),$$

and the *lower value*

$$\underline{J} := \sup_{w \in W} \inf_{u \in U} J(u,w).$$

Obviously, we have the inequality $\bar{J} \geq \underline{J}$. If $\bar{J} = \underline{J} = J^*$, then J^* is called the *value* of the zero-sum game. Furthermore, if there exists a pair $(u^* \in U, w^* \in W)$ such that

$$J(u^*, w^*) = J^*,$$

then the pair (u^*, w^*) is called a (pure-strategy) *saddle-point solution*. In this case, we say that the game admits a *saddle-point* (in pure strategies). Such a saddle-point solution will equivalently satisfy the so-called *pair of saddle-point inequalities*:

$$J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*), \quad \forall u \in \mathbb{U}, \forall w \in \mathbb{W}.$$

We end the chapter with the following proposition:

PROPOSITION 2.4
Consider a two-person zero-sum game on convex finite dimensional action sets $U_1 \times U_2$, defined by the continuous kernel $J(u_1, u_2)$. Suppose that $J(u_1, u_2)$ is strictly convex in u_1 and strictly concave in u_2. Suppose that either

  (*i*) U_1 and U_2 are closed and bounded, or

  (*ii*) $U_i \in \mathbb{R}^{m_i}$, $i = 1, 2$, and $J(u_1, u_2) \to \infty$ as $\|u_1\| \to \infty$, and $J(u_1, u_2) \to -\infty$ as $\|u_2\| \to \infty$.

Then, the game admits a unique pure-strategy saddle-point equilibrium.
□

Proof   Consult [6], pp. 177.   □

3

Static and Dynamic Estimation

In this chapter, static and dynamic estimation problems will be discussed. The main point is to show that some well known formulations of estimation problems have the same solution, although different measures of the size of the error are used. We first consider estimation problems that have been discussed earlier in the literature. These problems can be seen as a an optimal decision problem of *one* decision maker, whose object is to find the best estimate of some given variable, subject to limited information about that variable. Different estimation error measures are considered, and it is shown they all have the same optimal estimator. We then introduce the team estimation problem, where we now have *multi* decision makers. The team members have access to different pieces of information about a given variable to be estimated by the team. Analogous to the estimation problem of one decision maker, the team decision solution is shown to be the same for different measures of the estimation error.

3.1 Static Estimation Problems

Let x, y, z be finite dimensional vectors with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^m$, and

$$z = Mx,$$
$$y = Cx, \tag{3.1}$$

where $M \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{p \times n}$. Without loss of generality, we assume that C has full row rank. It then follows that $p \leq n$. This assumption guarantees that C has a right inverse and that CC^T is invertible.

The static estimation problem is to find a map $\mu : \mathbb{R}^p \to \mathbb{R}^m$ that minimizes a given functional $J(x, \mu(y))$, $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. We will consider

three types of estimation problems: Minimax estimation, stochastic estimation, and error-operator minimization. We will show that the optimal decisions for the three estimation problems coincide.

Minimax Estimation

The cost used here is the induced norm of the error $Mx - \mu(y)$ by x:

$$J(x, \mu(y)) = \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(y)\|^2}{\|x\|^2}.$$

So, the decision problem is to minimize the cost J above:

$$\inf_{\mu} \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(y)\|^2}{\|x\|^2}.$$

Toward this end, we first introduce a related quadratic game with a kernel parametrized by γ:

$$L_\gamma(x, \mu(y)) = \|Mx - \mu(y)\|^2 - \gamma\|x\|^2,$$

for which we seek the *upper value* with μ being the minimizer, and x the maximizer. More precisely, we seek

$$\bar{L}_\gamma = \inf_{\mu} \sup_{x} L_\gamma(x, \mu(y)).$$

The following theorem has appeared in a similar form in [5], but we give a different formulation and proof technique which we will use later for team estimation problems:

THEOREM 3.1
For a quadratic game defined by the kernel $L_\gamma(x, \mu(y)) = \|Mx - \mu(y)\|^2 - \gamma\|x\|^2$, and $y = Cx$:

(*i*) There exists a γ^* such that for $\gamma \geq \gamma^*$ the upper value $\bar{L}_\gamma$ is finite, whereas for $\gamma < \gamma^*$, it is infinite.

(*ii*) For all $\gamma \geq \gamma^*$, the game admits a minimax decision given by $\mu(y) = MC^\dagger y$.

(*iii*) γ^* is the largest eigenvalue of the matrix

$$(I - C^\dagger C)^T M^T M (I - C^\dagger C).$$

$\square$

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Proof   Let $\tilde{x} = (I - C^\dagger C)x$. Then we can write

$$x = (I - C^\dagger C)x + C^\dagger C x = \tilde{x} + C^\dagger y.$$

$\tilde{x}$ is the unobservable part from y, since $C\tilde{x} = C(I - C^\dagger C)x = 0$. Introduce $Q = M(I - C^\dagger C)$, and let $\mu(y) = MC^\dagger y - \nu(y)$, for some policy $\nu(y)$. Then

$$
\begin{aligned}
\bar{L}_\gamma &= \inf_\mu \sup_x L_\gamma(x, \mu(y)) \\
&= \inf_\mu \sup_x \|Mx - \mu(y)\|^2 - \gamma \|x\|^2 \\
&= \inf_\mu \sup_{\tilde{x},y} \|M\tilde{x} + MC^\dagger y - \mu(y)\|^2 - \gamma \|C^\dagger y + \tilde{x}\|^2 \\
&= \inf_\nu \sup_{\tilde{x},y} \|M\tilde{x} + \nu(y)\|^2 - \gamma \|C^\dagger y + \tilde{x}\|^2 \\
&= \sup_y \inf_\nu \sup_{\tilde{x}} \|M\tilde{x} + \nu\|^2 - \gamma \|C^\dagger y + \tilde{x}\|^2
\end{aligned}
$$

Now for each fixed vector y, we will study the *inner* "inf sup" game given by

$$\inf_\nu \sup_{\tilde{x}} \|M\tilde{x} + \nu\|^2 - \gamma \|C^\dagger y + \tilde{x}\|^2.$$

Now we have that

$$\inf_\nu \sup_{\tilde{x}} \|M\tilde{x} + \nu\|^2 - \gamma \|C^\dagger y + \tilde{x}\|^2 = \inf_\nu \sup_x \|Qx + \nu\|^2 - \gamma \|x\|^2$$

and

$$\|Qx + \nu\|^2 - \gamma \|x\|^2 = \begin{pmatrix} x \\ \nu \end{pmatrix}^T \begin{pmatrix} Q^T Q - \gamma I & Q^T \\ Q & I \end{pmatrix} \begin{pmatrix} x \\ \nu \end{pmatrix}.$$

This kernel is strictly convex in ν, but not necessarily strictly concave in x, with the latter condition holding if and only if

$$Q^T Q - \gamma I = (I - C^\dagger C)^T M^T M (I - C^\dagger C) - \gamma I \prec 0, \qquad (3.2)$$

or equivalently, $\gamma > \gamma^*$, where γ^* is the largest eigenvalue of the matrix $(I - C^\dagger C)^T M^T M (I - C^\dagger C)$. If the concavity condition (3.2) is satisfied, then Proposition 2.4 gives that the game admits a unique saddle-point solution. On the other hand, if the matrix in (3.2) has at least one positive eigenvalue, which occurs when $\gamma < \gamma^*$, then x can be chosen in the direction of the eigenvector corresponding to the positive eigenvalue of the matrix in (3.2), implying that the upper value is unbounded. Thus, for $\gamma < \gamma^*$, we have

$$\sup_x \|Mx - \mu(y)\|^2 - \gamma \|x\|^2 = \infty.$$

Now suppose that $\gamma > \gamma^*$, and let $R = (Q^T Q - \gamma I)^{-1} Q^T$. Then, standard completion of squares gives

$$\begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} Q^T Q - \gamma I & Q^T \\ Q & I \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = v^T \left(I - Q \left(Q^T Q - \gamma I \right)^{-1} Q^T \right) v + \\ + (x + Rv)^T \left(Q^T Q - \gamma I \right) (x + Rv).$$

Since

$$Q^T Q - \gamma I \prec 0,$$

and

$$I - Q \left(Q^T Q - \gamma I \right)^{-1} Q^T \succ 0,$$

we get

$$\begin{aligned}
\bar{L}_\gamma &= \inf_v \sup_x \begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} Q^T Q - \gamma I & Q^T \\ Q & I \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \\
&= \sup_x x^T \left(Q^T Q - \gamma I \right) x \\
&= \sup_x x^T \left((I - C^\dagger C)^T M^T M (I - C^\dagger C) - \gamma I \right) x \\
&= 0,
\end{aligned}$$

where the minimizing v is $v = 0$, for all y. Thus, for any $\gamma > \gamma^*$ the minimax policy is $\mu(y) = M C^\dagger y$, which is independent of γ. Furthermore,

$$\frac{\|Mx - \mu(y)\|^2}{\|x\|^2} \leq \gamma, \quad \forall x \neq 0.$$

Since γ can be chosen arbitrarily close to γ^*, we conclude that

$$\frac{\|Mx - \mu(y)\|^2}{\|x\|^2} \leq \gamma^*, \quad \forall x \neq 0,$$

which in turn gives that

$$\|Mx - \mu(y)\|^2 - \gamma^* \|x\|^2 \leq 0, \quad \forall x,$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

THEOREM 3.2
The minimax policy $\mu(Cx)$ for the game

$$\inf_{\mu} \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2},$$

is given by $\mu(Cx) = MC^\dagger Cx$, and the value of the game is γ^*, where γ^* is the largest eigenvalue of the matrix

$$(I - C^\dagger C)^T M^T M (I - C^\dagger C).$$

$\square$

Proof Let γ^* be the largest eigenvalue of the matrix

$$(I - C^\dagger C)^T M^T M (I - C^\dagger C).$$

In the proof of Theorem 3.1, we showed that for $\gamma < \gamma^*$

$$\sup_{x} \|Mx - \mu(Cx)\|^2 - \gamma\|x\|^2 = \infty,$$

which implies that for every decision μ, there is a vector $x \neq 0$ such that

$$\gamma < \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2},$$

and thus

$$\gamma < \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2}.$$

We also showed that for $\mu(Cx) = MC^\dagger Cx$, we have

$$\sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2} \leq \gamma^*,$$

Since γ can be chosen arbitrarily close to γ^*, we conclude that

$$\gamma^* = \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2},$$

and the proof is complete. $\square$

Stochastic Estimation

Let $x \sim \mathcal{N}(0, I)$ and let y, z be given by (3.1). Define the cost $J(x, \mu(y))$ as the variance of the error $Mx - \mu(y)$:

$$J(x, \mu(y)) = \mathbf{E} \, \|Mx - \mu(y)\|^2.$$

We will first consider linear decisions $\mu(y) = Ky = KCx$. Now

$$
\begin{aligned}
J(x, KCx) &= \mathbf{E} \, \|Mx - KCx\|^2 \\
&= \mathbf{E} \, x^T (M - KC)^T (M - KC)x \\
&= \mathbf{E} \, \left\{ \mathbf{Tr} \, (M - KC)xx^T (M - KC)^T \right\} \\
&= \mathbf{Tr} \, (M - KC)(M - KC)^T
\end{aligned}
$$

where the last equality follows from the assumption that $\mathbf{E} \, xx^T = I$. The optimal value of K is obtained by solving $\frac{\partial J}{\partial K} = 0$:

$$
\begin{aligned}
0 &= \frac{\partial J}{\partial K} \\
&= \frac{\partial}{\partial K} \left\{ \mathbf{Tr} \, (M - KC)(M - KC)^T \right\} \\
&= 2KCC^T - 2MC^T.
\end{aligned}
$$

Since CC^T is assumed to be invertible, the optimal K is given by $K = MC^T(CC^T)^{-1} = MC^\dagger$. It is also the optimal policy over all policies, linear and nonlinear, because of the Gaussian assumption of the stochastic variable x, see [49].

Error-Operator Minimization

Let x be any given vector, and let y, z be given by (3.1). Consider linear decisions $\mu(y) = Ky = KCx$, and introduce the *error $e = z - \mu(y) = (M - KC)x$*. The linear operator from the vector x to the error e is given by the matrix $M - KC$. Our estimation problem is to minimize the *Frobenius norm* of the matrix $M - KC$, that is, minimizing the cost

$$J(x, KCx) = \mathbf{Tr} \, \left\{ (M - KC)^T (M - KC) \right\}.$$

Note that J is independent of x here. Just as in the stochastic estimation problem, the optimal K is given by $K = MC^\dagger$.

3.2 Optimal Filtering

Now we will consider filtering problems of linear dynamical systems. Consider the linear system

$$G := \begin{cases} x_{k+1} = Ax_k + Bw_k \\ \quad y_k = Cx_k + Dw_k. \end{cases} \tag{3.3}$$

Assume that (C, A) is detectable. Introduce

$$\mathbf{x} = \begin{pmatrix} w_{k-1} \\ w_{k-1} \\ \vdots \\ w_0 \\ x_0 \end{pmatrix},$$

and

$$Y_{k-1} = \begin{pmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_0 \end{pmatrix}.$$

Since,

$$x_k = A^k x_0 + \sum_{t=1}^{k} A^{k-t} B w_{t-1},$$

$$y_k = CA^k x_0 + \sum_{t=1}^{k} CA^{k-t} B w_{t-1},$$

we can write

$$x_k = M\mathbf{x},$$
$$Y_{k-1} = U\mathbf{x},$$

for some real matrices M and U. We want to find an optimal filter $\mu(Y_{k-1})$ that minimizes a cost $J(\mathbf{x}, \mu)$. Here, we let the cost $J(\mathbf{x}, \mu)$ be any of the costs introduced in Section 3.1, that is

$$J(\mathbf{x}, \mathbf{K}U\mathbf{x}) = \mathbf{Tr} \ (M - \mathbf{K}U)^T (M - \mathbf{K}U),$$

and

$$J(\mathbf{x}, \mu(Y^{k-1})) = \sup_{\|\mathbf{x}\| \neq 0} \frac{\|M\mathbf{x} - \mu(Y^{k-1})\|^2}{\|\mathbf{x}\|^2},$$

or

$$J(\mathbf{x}, \mu(Y_{k-1})) = \mathbf{E} \, \|M\mathbf{x} - \mu(Y_{k-1})\|^2.$$

Hence, on a finite horizon, the filtering problem is converted to a static estimation problem, and the solution is the same, no matter which of the above costs we use. To find an optimal filter, we choose the stochastic formulation, that is:

$$J(\mathbf{x}, \mu(Y_{k-1})) = \mathbf{E} \, \|M\mathbf{x} - \mu(Y_{k-1})\|^2,$$

with $\mathbf{x} \sim \mathcal{N}(0, I)$. The problem above is to find the best estimate of $x_k = Mx$ based on the information Y_{k-1}. Let $\hat{x}_k = \mu(Y_{k-1})$ be the optimal estimate. Introduce the matrix

$$\begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}^T.$$

Then, our estimation problem over a finite horizon can be solved using the standard Kalman filter (see [1]):

$$\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + K(k)(y(k) - C\hat{x}(k)), & (3.4) \\
K(k) &= (AP(k)C^T + R_{12})(CP(k)C^T + R_2)^{-1}, & (3.5) \\
P(k+1) &= AP(k)A^T + R_1 - K(k)(CP(k)C^T + R_2)K^T(k), & (3.6) \\
P(0) &= \mathbf{E} \, x(0)x^T(0) = I, & (3.7)
\end{aligned}$$

where we assumed that $CP(k)C^T + R_2$ is invertible for each k. The filter above is optimal over an arbitrarily long finite horizon, so it is also optimal over the infinite horizon. Since we assumed that (C, A) is detectable, a stationary solution exists $(k \to \infty)$ and it is given by

$$\hat{x}(k+1) = A\hat{x}(k) + K(y(k) - C\hat{x}(k)), \tag{3.8}$$

$$K = (APA^T + R_{12})(CPC^T + R_2)^{-1}, \tag{3.9}$$

where P is the symmetric and positive definite solution to the algebraic Riccati equation

$$P = APA^T + R_1 - (APC^T + R_{12})(CPC^T + R_2)^{-1}(APC^T + R_{12})^T, \tag{3.10}$$

and we assumed that $CPC^T + R_2$ is invertible.

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3.3 Static Team Estimation Problems

The previous section showed that the stochastic and deterministic estimation problems have identical solutions. In this section, we will show that the same property holds for static team estimation problems, which we will describe next.

As in the previous section, let x, y, z be finite dimensional vectors with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^m$, and

$$
\begin{aligned}
z &= Mx, \\
y &= Cx,
\end{aligned}
\tag{3.11}
$$

where $M \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{p \times n}$. Let μ be a map $\mu : \mathbb{R}^p \to \mathbb{R}^m$. Partition z, y, and μ in N compatible blocks:

$$
\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix}, \quad z = \begin{pmatrix} M_1 x \\ M_2 x \\ \vdots \\ M_N x \end{pmatrix},
$$

$$
y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} C_1 x \\ C_2 x \\ \vdots \\ C_N x \end{pmatrix},
$$

where $\mu_i : \mathbb{R}^{p_i} \to \mathbb{R}^{m_i}$, $M_i \in \mathbb{R}^{m_i \times n}$, $C_i \in \mathbb{R}^{p_i \times n}$, $\sum_{i=1}^N m_i = m$, $\sum_{i=1}^N p_i = p$. Without loss of generality, we will assume that C_i has full row rank (hence $p_i \le n$, for all i). This assumption guarantees that C_i has a right inverse and that $C_i C_i^T$ is invertible for all i. The static team estimation problem is to find a map $\mu : \mathbb{R}^p \to \mathbb{R}^m$ that minimizes a given functional $J(x, \mu(y))$, $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, where $\mu(y)$ is constrained to be of the form

$$
\mu(y) = \begin{pmatrix} \mu_1(y_1) \\ \mu_2(y_2) \\ \vdots \\ \mu_N(y_N) \end{pmatrix} = \begin{pmatrix} \mu_1(C_1 x) \\ \mu_2(C_2 x) \\ \vdots \\ \mu_N(C_N x) \end{pmatrix}.
\tag{3.12}
$$

Again, we will consider three types of team estimation problems: Minimax team estimation, stochastic team estimation, and error-operator team minimization. We will show that the linear optimal decisions for the three estimation problems coincide.

Minimax Team Estimation

The cost used here is the induced norm of the error $Mx - \mu(y)$ by x:

$$J(x, \mu(y)) = \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(y)\|^2}{\|x\|^2}.$$

Hence, the decision problem is to minimize the cost J above:

$$\inf_{\mu} \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(y)\|^2}{\|x\|^2}.$$

Toward this end, we first introduce a related quadratic game with a kernel parametrized by γ:

$$L_\gamma(x, \mu(y)) = \|Mx - \mu(y)\|^2 - \gamma \|x\|^2$$

for which we seek the *upper value* with μ being the minimizer, and x the maximizer. More precisely, we seek

$$\bar{L}_\gamma = \inf_{\mu} \sup_{x} L_\gamma(x, \mu(y)).$$

We are now ready to state the main result of this chapter:

THEOREM 3.3
For a quadratic game defined by the kernel $L_\gamma(x, \mu(y)) = \|Mx - \mu(y)\|^2 - \gamma \|x\|^2$ with $y = Cx$ and constrained policies $\mu(y)$ given by (3.12):

(*i*) There exists a γ^* such that for $\gamma \geq \gamma^*$ the upper value $\bar{L}_\gamma$ is finite, whereas for $\gamma < \gamma^*$, it is infinite.

(*ii*) For all $\gamma \geq \gamma^*$, the game admits a minimax decision given by $\mu_i(y_i) = M_i C_i^\dagger y_i$.

(*iii*) γ^* is the largest eigenvalue of the matrix

$$\sum_{i=1}^{N} (I - C_i^\dagger C_i)^T M_i^T M_i (I - C_i^\dagger C_i).$$

$\square$

Proof   Let $\tilde{x}_i = (I - C_i^\dagger C_i)x$. We can then write

$$x = (I - C_i^\dagger C_i)x + C_i^\dagger C_i x$$
$$= \tilde{x}_i + C_i^\dagger y_i, \quad i = 1, 2, ..., N.$$

The vector $\tilde{x}_i$ is the unobservable part from the vector y_i, since $C_i \tilde{x}_i = C_i(I - C_i^\dagger C_i)x = 0$. Introduce $Q_i = M_i(I - C_i^\dagger C_i)$, and let $\mu_i(y_i) = M_i C_i^\dagger y_i - v_i(y_i)$, for some policy $v_i(y_i)$. This gives

$$\bar{L}_\gamma = \inf_\mu \sup_x L_\gamma(x, \mu(y))$$
$$= \inf_\mu \sup_{x,y} \|Mx - \mu(y)\|^2 - \gamma\|x\|^2$$
$$= \inf_\mu \sup_{x,y} \sum_{i=1}^N \|M_i x - \mu_i(y_i)\|^2 - \gamma\|x\|^2$$
$$= \inf_\mu \sup_{x,\tilde{x},y} \sum_{i=1}^N \|M_i \tilde{x}_i + M_i C_i^\dagger y_i - \mu_i(y_i)\|^2 - \gamma\|x\|^2$$
$$= \inf_v \sup_{x,\tilde{x},y} \sum_{i=1}^N \|M_i \tilde{x}_i + v_i(y_i)\|^2 - \gamma\|x\|^2$$
$$= \sup_y \inf_v \sup_{x,\tilde{x}} \sum_{i=1}^N \|M_i \tilde{x}_i + v_i\|^2 - \gamma\|x\|^2$$
$$= \sup_y \inf_v \sup_x \sum_{i=1}^N \|M_i(I - C_i^\dagger C_i)x + v_i\|^2 - \gamma\|x\|^2$$
$$= \sup_y \inf_v \sup_x \sum_{i=1}^N \|Q_i x + v_i\|^2 - \gamma\|x\|^2$$

Now for every vector y, we will study the *inner* "inf sup" game given by

$$\inf_v \sup_x \sum_{i=1}^N \|Q_i x + v_i\|^2 - \gamma\|x\|^2.$$

Introduce

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{pmatrix}.$$

We can now write

$$\sum_{i=1}^{N} \|Q_i x + v_i\|^2 - \gamma \|x\|^2 = \begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} Q^T Q - \gamma I & Q^T \\ Q & I \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}.$$

This kernel is strictly convex in v, but not necessarily strictly concave in x, with the latter condition holding if and only if

$$Q^T Q - \gamma I = \sum_{i=1}^{N} (I - C_i^\dagger C_i)^T M_i^T M_i (I - C_i^\dagger C_i) - \gamma I \prec 0, \qquad (3.13)$$

or equivalently, $\gamma > \gamma^*$, where γ^* is the largest eigenvalue of the matrix

$$\sum_{i=1}^{N} (I - C_i^\dagger C_i)^T M_i^T M_i (I - C_i^\dagger C_i).$$

If the concavity condition (3.13) is satisfied, then Proposition 2.4 gives that the game admits a unique saddle-point solution. On the other hand, if the matrix in (3.13) has at least one positive eigenvalue, which occurs when $\gamma < \gamma^*$, then x can be chosen in the direction of the eigenvector corresponding to the positive eigenvalue of the matrix in (3.13), implying that the upper value is unbounded. Thus, for $\gamma < \gamma^*$, we have

$$\sup_{x} \|M x - \mu(y)\|^2 - \gamma \|x\|^2 = \infty.$$

Now suppose that $\gamma > \gamma^*$, and let $R = (Q^T Q - \gamma I)^{-1} Q^T$. Then, standard completion of squares gives

$$\begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} Q^T Q - \gamma I & Q^T \\ Q & I \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = v^T \left(I - Q \left(Q^T Q - \gamma I \right)^{-1} Q^T \right) v +$$
$$+ (x + Rv)^T \left(Q^T Q - \gamma I \right) (x + Rv).$$

Since

$$Q^T Q - \gamma I \prec 0,$$

and

$$I - Q \left(Q^T Q - \gamma I \right)^{-1} Q^T \succ 0,$$

we get

$$\inf_{v} \sup_{x} \sum_{i=1}^{N} \|Q_i x + v_i\|^2 - \gamma \|x\|^2 =$$

$$= \inf_{v} \sup_{x} \begin{pmatrix} x \\ v \end{pmatrix}^T \begin{pmatrix} Q^T Q - \gamma I & Q^T \\ Q & I \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

$$= \sup_{x} x^T \left(Q^T Q - \gamma I \right) x$$

$$= \sup_{x} x^T \left((I - C^\dagger C)^T M^T M (I - C^\dagger C) - \gamma I \right) x$$

$$= 0,$$

where the minimizing v is $v = 0$, for all y. Thus, for any $\gamma > \gamma^*$ the minimax policy is given by $\mu_i(y_i) = M_i C_i^\dagger y_i$, which is clearly independent of γ. Furthermore,

$$\frac{\|Mx - \mu(y)\|^2}{\|x\|^2} \le \gamma, \quad \forall x \ne 0.$$

Since γ can be chosen arbitrarily close to γ^*, we conclude that

$$\frac{\|Mx - \mu(y)\|^2}{\|x\|^2} \le \gamma^*, \quad \forall x \ne 0,$$

which in turn gives that

$$\|Mx - \mu(y)\|^2 - \gamma^* \|x\|^2 \le 0, \quad \forall x,$$

and the proof is complete. $\qquad\square$

THEOREM 3.4
For the game

$$\inf_{\mu} \sup_{\|x\| \ne 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2},$$

the minimax policy $\mu(Cx)$ with the constraint (3.12) is given by $\mu_i(C_i x) = M_i C_i^\dagger C_i x$. The value of the game is γ^*, where γ^* is the largest eigenvalue of the matrix

$$\sum_{i=1}^{N} (I - C_i^\dagger C_i)^T M_i^T M_i (I - C_i^\dagger C_i). \tag{3.14}$$

$\qquad\square$

Proof   Let γ^* be the largest eigenvalue of the matrix in (3.14). In the proof of Theorem 3.3, we showed that for $\gamma < \gamma^*$

$$\sup_x \|Mx - \mu(Cx)\|^2 - \gamma \|x\|^2 = \infty,$$

which implies that for every decision μ, there is a vector $x \neq 0$ such that

$$\gamma < \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2},$$

and thus

$$\gamma < \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2}.$$

We also showed that for $\mu_i(C_i x) = M C_i^\dagger C_i x$, we have

$$\sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2} \leq \gamma^*,$$

Since γ can be chosen arbitrarily close to γ^*, we conclude that

$$\gamma^* = \sup_{\|x\| \neq 0} \frac{\|Mx - \mu(Cx)\|^2}{\|x\|^2},$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Stochastic Team Estimation

Let $x \sim \mathcal{N}(0, I)$ and let y, z be given by (3.1). Define the cost $J(x, \mu(y))$ as the variance of the error $Mx - \mu(y)$:

$$J(x, \mu(y)) = \mathbf{E} \, \|Mx - \mu(y)\|^2.$$

We will first consider linear decisions $\mu_i(y_i) = K_i y_i = K_i C_i x$. Now

$$
\begin{aligned}
J(x, \mu(y)) &= \mathbf{E} \, \|Mx - \mu(y)\|^2 \\
&= \sum_{i=1}^{N} \mathbf{E} \, \|M_i x - \mu_i(y_i)\|^2 \\
&= \sum_{i=1}^{N} \mathbf{E} \, x^T (M_i - K_i C_i)^T (M_i - K_i C_i) x \\
&= \sum_{i=1}^{N} \mathbf{E} \, \left\{ \mathbf{Tr} \, (M_i - K_i C_i) x x^T (M_i - K_i C_i)^T \right\} \\
&= \sum_{i=1}^{N} \mathbf{Tr} \, (M_i - K_i C_i)(M_i - K_i C_i)^T
\end{aligned}
$$

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where the last equality follows from the assumption that $\mathbf{E}\,xx^T = I$. Note that there is no coupling between the matrices K_1, K_2, ..., and K_N. Hence, the optimal K_i can be found by solving $\frac{\partial J}{\partial K_i} = 0$:

$$
\begin{aligned}
0 &= \frac{\partial J}{\partial K_i} \\
&= \frac{\partial}{\partial K_i}\left\{\mathbf{Tr}\,(M_i - K_iC_i)(M_i - K_iC_i)^T\right\} \\
&= 2K_iC_iC_i^T - 2M_iC_i^T.
\end{aligned}
$$

Since $C_iC_i^T$ is assumed to be invertible for all i, the optimal K_i is given by $K_i = M_iC_i^T(C_iC_i^T)^{-1} = M_iC_i^\dagger$. It is also the optimal policy over all policies, linear and nonlinear, because of the Gaussian assumption of the stochastic variables, see [49].

Error-Operator Team Minimization

Let x be any given vector, and let y, z be given by (3.1). Consider linear decisions $\mu_i(y_i) = K_iy_i = K_iC_ix$, and introduce the *error* $e = z - \mu(y) = (M - KC)x$, where

$$
K = \begin{pmatrix}
K_1 & 0 & \cdots & 0 \\
0 & K_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & K_N
\end{pmatrix}.
$$

The linear operator from the vector x to the error e is given by the matrix $M - KC$. Our estimation problem is to minimize the *Frobenius norm* of the matrix $M - KC$, that is, minimizing the cost

$$
\begin{aligned}
J(x, KCx) &= \mathbf{Tr}\,\left\{(M - KC)^T(M - KC)\right\} \\
&= \sum_{i=1}^{N} \mathbf{Tr}\,\left\{(M_i - K_iC_i)^T(M_i - K_iC_i)\right\}
\end{aligned}
$$

Note that J is independent of x. Just as in the stochastic team estimation problem, the optimal K_i is given by $K_i = M_iC_i^\dagger$.

3.4 Optimal Distributed Filtering

Consider the linear systems $x = H(\mathbf{q})w$ and $y = G(\mathbf{q})w$, where

$$
H = \left(\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right), \tag{3.15}
$$
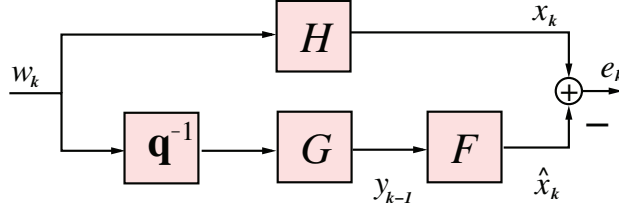
Figure 3.1 A standard state estimation problem with one step delayed measurements.

and

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right). \tag{3.16}$$

The problem we consider is to find an optimal filter $F(\mathbf{q})$, with some structure that will be specified later, such that $\|H - FG\mathbf{q}^{-1}\|_\alpha$ is minimized. This is simply the problem of finding the optimal estimate of the state of G with respect to some constraints on the filter (see Figure 3.1). We let the constraints be delays on some entries of the transfer matrix of the filter. That is, $F_{ij}(\mathbf{q}) = f_{ij}(\mathbf{q})\mathbf{q}^{-\tau_{ij}}$, where $f_{ij}(\mathbf{q})$ is a transfer matrix to be optimized. Let $\tau = \max_{i,j} \tau_{ij}$, and let F_i and H_i be the ith block row of F and H, respectively. Then, the estimation problem is to minimize

$$\left\| \begin{array}{c} H_1 - F_1 G \mathbf{q}^{-1} \\ \vdots \\ H_N - F_N G \mathbf{q}^{-1} \end{array} \right\|_\alpha.$$

Introduce the extended state z:

$$z(k) = \begin{pmatrix} x(k) \\ y(k-1) \\ \vdots \\ y(k-\tau) \end{pmatrix}.$$

Let $G_e(\mathbf{q})$ be the extended system of $G(\mathbf{q})$, with state z and output

$$\bar{y}(k) = \begin{pmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-\tau) \end{pmatrix}.$$

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Then, a state space realization of $G_e(\mathbf{q})$ is given by:

$$
G_e =
\left(
\begin{array}{ccccc|c}
A & 0 & \cdots & 0 & 0 & B \\
C & 0 & \cdots & 0 & 0 & D \\
0 & I & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0 & 0 \\
\hline
C & 0 & \cdots & 0 & 0 & D \\
0 & I & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0 & 0 \\
0 & 0 & \cdots & 0 & I & 0
\end{array}
\right) .
$$

Now the delayed information constraints on every filter F_i can be seen as if the input to F_i is given by the system $G_e^i(\mathbf{q})\mathbf{q}^{-1}$, where

$$
\begin{aligned}
G_e^i &=
\left(
\begin{array}{c|c}
A_e & B_e \\
\hline
E_i & D_i
\end{array}
\right) \\[2mm]
&=
\left(
\begin{array}{ccccc|c}
A & 0 & \cdots & 0 & 0 & B \\
C & 0 & \cdots & 0 & 0 & D \\
0 & I & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0 & 0 \\
\hline
E_{i0} & 0 & \cdots & 0 & 0 & D_{i0} \\
0 & E_{i1} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & E_{i(\tau-1)} & 0 & 0 \\
0 & 0 & \cdots & 0 & E_{i\tau} & 0
\end{array}
\right) ,
\end{aligned}
\tag{3.17}
$$

where E_{it} is a block diagonal matrix partitioned in N^2 blocks such that the jth diagonal block is the identity matrix I (C_i) for $t > 0$ ($t = 0$) if the filter F_i has access to $y_j(t)$, and zero otherwise. Mathematically, for $r, s = 1, ..., N$, we can define E_{it} as

$$
[E_{it}]_{rs} =
\begin{cases}
C_i & \text{if } r = s \text{ and } \tau_{ir} = t = 0 \\
I & \text{if } r = s \text{ and } \tau_{ir} \le t \ne 0 \\
0 & \text{otherwise .}
\end{cases}
\tag{3.18}
$$

In a similar way, for $r = 1, ..., N$, D_{i0} is defined as

$$[D_{i0}]_r = \begin{cases} [D]_r & \text{if } \tau_{ir} = 0 \\ 0 & \text{otherwise} . \end{cases} \qquad (3.19)$$

THEOREM 3.5
Consider the filtering problem

$$\min_{F \in \mathbf{S}} \|H - FG\mathbf{q}^{-1}\|_\alpha,$$

where $S = \{F : f_{ij} \in \mathcal{RL}_\alpha, F_{ij} = f_{ij}\mathbf{q}^{-\tau_{ij}}\}$, $\tau = \max \tau_{ij}$, and H, G, G_e^i are given by (3.15)-(3.18). Assume that (A_e, E_i) is detectable for all i. Then, the optimal filter for $\alpha = 2$ or $\alpha = \infty$ is given by

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix},$$

where F_i has the state space realization for $i = 1, ..., N$:

$$\left(\begin{array}{c|c} A_e - K_i E_i & K_i \\ \hline \Gamma_i & 0 \end{array} \right), \qquad (3.20)$$

with

$$\Gamma_i = \begin{pmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{pmatrix}, \qquad (3.21)$$

where the identity matrix I in Γ_i is in block position i,

$$K_i = (A_e P_i E_i^T + B_e^T D_i)(E_i P_i E_i^T + D_i^T D_i)^{-1}, \qquad (3.22)$$

and P_i is the symmetric positive definite solution to the Riccati equation

$$\begin{aligned} P_i = A_e P_i A_e^T + B_e B_e^T - \\ - (A_e P_i E_i^T + B_e^T D_i)(E_i P_i E_i^T + D_i^T D_i)^{-1}(A_e P_i E_i^T + B_e^T D_i)^T. \end{aligned} \qquad (3.23)$$

$\square$

Proof The estimation problem

$$\min_{F \in S} \| H - FG\mathbf{q}^{-1} \|_{\alpha}$$

can be written as

$$\min_{F \in S} \left\| \begin{matrix} H_1 - F_1 G \mathbf{q}^{-1} \\ \vdots \\ H_N - F_N G \mathbf{q}^{-1} \end{matrix} \right\|_{\alpha}.$$

Introduce the system $\bar{G}_e$

$$\bar{G}_e := \left\{ \begin{matrix} z(k+1) = A_e z(k) + B_e w(k) \\ \begin{pmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_N(k) \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{pmatrix} z(k). \end{matrix} \right. \qquad (3.24)$$

Let

$$\mathbf{x} = \begin{pmatrix} w(k-1) \\ w(k-2) \\ \vdots \\ w(0) \\ z(0) \end{pmatrix},$$

$$Y_i^{k-1} = \begin{pmatrix} y_i(k-1) \\ y_i(k-2) \\ \vdots \\ y_i(0) \end{pmatrix}, \qquad Y^{k-1} = \begin{pmatrix} Y_1^{k-1} \\ Y_2^{k-1} \\ \vdots \\ Y_N^{k-1} \end{pmatrix}.$$

Since,

$$z(k) = A_e^k z(0) + \sum_{t=1}^{k} A_e^{k-t} B_e w(t-1),$$

$$y_i(k) = E_i A_e^k z(0) + \sum_{t=1}^{k} E_i A_e^{k-t} B_e w(t-1),$$

we can write

$$
\begin{pmatrix} z_1(k) \\ z_2(k) \\ \vdots \\ z_N(k) \end{pmatrix} = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_N \end{pmatrix} \mathbf{x},
$$

$$
\begin{pmatrix} Y_1^{k-1} \\ Y_2^{k-1} \\ \vdots \\ Y_N^{k-1} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{pmatrix} \mathbf{x},
$$

for some real matrices M_i and U_i, $i = 1, ..., N$. Let

$$
M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_N \end{pmatrix},
$$

and

$$
U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{pmatrix}.
$$

We want to find a constrained optimal filter $\mu(Y^{k-1})$ with

$$
\mu(Y^{k-1}) = \begin{pmatrix} \mu_1(Y_1^{k-1}) \\ \mu_2(Y_2^{k-1}) \\ \vdots \\ \mu_N(Y_N^{k-1}) \end{pmatrix} = \begin{pmatrix} \mu_1(U_1\mathbf{x}) \\ \mu_2(U_2\mathbf{x}) \\ \vdots \\ \mu_N(U_N\mathbf{x}) \end{pmatrix},
$$

that minimizes a cost $J(x, \mu)$. I follows from the definition of the norms $\| \cdot \|_2$ and $\| \cdot \|_\infty$ that minimization of the error norms $\|H - FG\mathbf{q}^{-1}\|_2^2$ and $\|H - FG\mathbf{q}^{-1}\|_\infty^2$ correspond to minimizing the costs

$$
J(\mathbf{x}, \mathbf{K}U\mathbf{x}) = \mathbf{Tr}\ (M - \mathbf{K}U)^T(M - \mathbf{K}U),
$$

and

$$
J(\mathbf{x}, \mu(Y^{k-1})) = \sup_{\|\mathbf{x}\| \neq 0} \frac{\|M\mathbf{x} - \mu(Y^{k-1})\|^2}{\|\mathbf{x}\|^2},
$$

respectively, where

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{K}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{K}_N \end{pmatrix}.$$

It was shown in Section 3.3 that minimization of the two costs above have the same solution $\mu(Y^{k-1}) = \mathbf{K} Y^{k-1} = \mathbf{K} U \mathbf{x}$ as that of the the stochastic problem with cost

$$J(\mathbf{x}, \mu(Y^{k-1})) = \mathbf{E} \, \|M\mathbf{x} - \mu(Y^{k-1})\|^2.$$

To find an optimal filter, we choose the stochastic formulation, that is:

$$J(\mathbf{x}, \mu(Y^{k-1})) = \mathbf{E} \, \|M\mathbf{x} - \mu(Y^{k-1})\|^2 = \sum_{i=1}^{N} \mathbf{E} \, \|M_i \mathbf{x} - \mu_i(Y_i^{k-1})\|^2,$$

where $\mathbf{x} \sim \mathcal{N}(0, I)$. Hence, the solution can be found by separately finding an estimator $\mu_i(Y_i^{k-1})$ that minimizes $\mathbf{E} \, \|M_i \mathbf{x} - \mu_i(y_i)\|^2$, for $i = 1, ..., N$. Let $\hat{z}_i$ be the optimal Kalman filter estimate of z with respect to Y_i^{k-1}, which is given by

$$\begin{aligned}
\hat{z}_i &= \bar{F}_i y_i, \\
\bar{F}_i &= \left(\begin{array}{c|c} A_e - K_i E_i & K_i \\ \hline I & 0 \end{array} \right), \\
K_i &= (A_e P_i E_i^T + B_e^T D_i)(E_i P_i E_i^T + D_i^T D_i)^{-1},
\end{aligned}$$

where P_i is the symmetric positive definite solution to the Riccati equation

$$\begin{aligned}
P_i = {}& A_e P_i A_e^T + B_e B_e^T - \\
& - (A_e P_i E_i^T + B_e^T D_i)(E_i P_i E_i^T + D_i^T D_i)^{-1}(A_e P_i E_i^T + B_e^T D_i)^T.
\end{aligned}$$

Let

$$\Gamma_i = \left(\begin{array}{ccccccc} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{array} \right),$$

where the identity matrix I in Γ_i is in block position i. Then,

$$M_i \mathbf{x} = x_i(k) = \Gamma_i z.$$

So, we obtain

$$\mathbf{E} \, \|M_i \mathbf{x} - \mu_i(Y_i^{k-1})\|^2 = \mathbf{E} \, \|\Gamma_i z - \mu_i(Y_i^{k-1})\|^2,$$

and the minimizing $\mu_i(Y_i^{k-1})$ is given by

$$\mu_i(Y_i^{k-1}) = \Gamma_i \hat{z}_i.$$

To conclude, the minimizing $\mu_i(Y_i^{k-1})$ is given by $\mu_i(Y_i^{k-1}) = F_i y_i$, where $F_i = \Gamma_i \bar{F}_i$. That is,

$$F_i = \left(\begin{array}{c|c} A_e - K_i E_i & K_i \\ \hline \Gamma_i & 0 \end{array} \right),$$

and the proof is complete. $\qquad\qquad\square$

4

Stochastic Team Decision Problems

In the previous chapter, we considered team estimation problems with different error measures. In this chapter, we will consider a more general class of quadratic team decision problems in a stochastic framework. The goal of this chapter is twofold. The first goal is to introduce the reader to static team decision theory, and reproduce some of the earlier results, where we give modern and easy presentation, including the proof techniques. The second goal is to show how dynamic team problems can be formulated as static team problems, and then derive conditions under which the problems become tractable, with respect to the existing mathematical tools.

4.1 Introduction

The problem of distributed control with information constraints is considered in this chapter. For instance, information constraints appear naturally when making decisions over networks. These problems can be formulated as team problems. Early results considering team theory in [26], [42], [48], [53], and [54] showed the possibilities and difficulties of the linear quadratic Gaussian control problem with non-classical information structure. Recently, Bamieh *et al* [4] and Rotkowitz *et al* [46] showed that the distributed linear optimal control problem is convex if the rate of information propagation is faster than the dynamics.

In this chapter, we consider the distributed linear quadratic Gaussian control problem and give a solution using statistical decision theory. We give a mathematical definition of *signaling* in team decision problems, and give necessary and sufficient conditions, determined by the system parameters (A, B, C), for elimination of the signaling incentive in optimal

control. Under the conditions for eliminating the signaling incentive, the optimal distributed controller is shown to be linear. Furthermore, it can be found by solving a linear system of equations.

4.2 Team Decision Theory

In this section we will review some classical results in team theory.

The Static Team Decision Problem

In the static team decision problem, one would like to solve

$$\text{minimize } \mathbf{E} \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

$$\text{subject to } y_i = C_i x + v_i \qquad (4.1)$$

$$u_i = \mu_i(y_i)$$

$$\text{for } i = 1, ..., N.$$

Here, x and v are independent Gaussian variables taking values in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, with $x \sim \mathcal{N}(0, V_{xx})$ and $v \sim \mathcal{N}(0, V_{vv})$. Also, y_i and u_i will be stochastic variables taking values in $\mathbb{R}^{p_i}$, $\mathbb{R}^{m_i}$, respectively, and $p_1 + ... + p_N = p$. We assume that

$$\begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \in \mathbb{S}^{m+n}, \qquad (4.2)$$

and $Q_{uu} \in \mathbb{S}^m_{++}$, $m = m_1 + \cdots + m_N$.

If full state information about x is available to each *decision maker* u_i, the minimizing u can be found easily by completion of squares. It is given by $u = Lx$, where L is the solution to

$$Q_{uu} L = -Q_{ux}.$$

Then, the cost function in (4.1) can be rewritten as

$$J(x, u) = \mathbf{E} \, x^T (Q_{xx} - L^T Q_{uu} L) x + \mathbf{E} \, (u - Lx)^T Q_{uu} (u - Lx). \qquad (4.3)$$

Minimizing the cost function $J(x, u)$, is equivalent to minimizing

$$\mathbf{E} \, (u - Lx)^T Q_{uu} (u - Lx),$$

since nothing can be done about $\mathbf{E} \, \{x^T (Q_{xx} - L^T Q_{uu} L) x\}$ (the cost when u has full information).

Now we will give the first team theoretic result of this chapter. The result was in principle shown by Radner [42], but we give a different formulation and proof:

THEOREM 4.1

Let x and v_i be Gaussian variables with zero mean, taking values in $\mathbb{R}^n$ and $\mathbb{R}^{p_i}$, respectively, with $p_1 + ... + p_N = p$. Also, let u_i be a stochastic variable taking values in $\mathbb{R}^{m_i}$, $Q_{uu} \in \mathbb{S}^m_{++}$, $m = m_1 + \cdots + m_N$, $L \in \mathbb{R}^{m \times n}$, $C_i \in \mathbb{R}^{p_i \times n}$, for $i = 1, ..., N$. Set $y_i = C_i x + v_i$, and assume that $\mathbf{E}\, y_i y_i^T \succ 0$. Then, the optimal decision μ to the optimization problem

$$\min_{\mu} \quad \mathbf{E}\, (u - Lx)^T Q_{uu}(u - Lx)$$
$$\text{subject to} \quad u_i = \mu_i(y_i) \tag{4.4}$$
$$\text{for } i = 1, ..., N.$$

is unique and linear in y.

$\square$

Proof   Let $\mathbb{Y}_i$ be a linear space of stochastic Gaussian variables taking values in $\mathbb{R}^{p_i}$, such that $y_i \in \mathbb{Y}_i$ if and only if $\mathbf{E}\, y_i y_i^T \succ 0$, for $i = 1, ..., N$. Also, let $\mathbb{Y}$ be a linear space of stochastic Gaussian variables y such that $y = (y_1^T, ..., y_N^T)^T$, with $y_i \in \mathbb{Y}_i$. Denote $\mathbb{H}$ as the space of all measurable functions $g(y)$ from $\mathbb{Y}$ to $\mathbb{R}^p$ for which

$$g(y) = \begin{pmatrix} g_1(y_1) \\ g_2(y_2) \\ \vdots \\ g_N(y_n) \end{pmatrix}, \tag{4.5}$$

and $\mathbf{E}\,\{g^T(y)Q_{uu}g(y)\} < \infty$. Since $Q_{uu} \succ 0$ and $\mathbf{E}\, y_i y_i^T \succ 0$, $\mathbb{H}$ is a Hilbert space under the inner product

$$\langle g, h \rangle = \mathbf{E}\,\{g^T(y)Q_{uu}h(y)\},$$

and norm

$$\|g(y)\|^2 = \mathbf{E}\,\{g^T(y)Q_{uu}g(y)\}.$$

For a fixed stochastic variable $y \in \mathbb{Y}$, let $\mathcal{Z}$ be a linear space such that $z \in \mathcal{Z}$ if z_i is a linear transformation of y_i, that is $z_i = A_i y_i$ for some real matrix $A_i \in \mathbb{R}^{m_i \times n_i}$. Clearly, $\mathcal{Z} \subset \mathbb{H}$. Now the optimization problem in equation (4.4) where we search for the *linear* optimal decision can be written as

$$\min_{u \in \mathcal{Z}} \quad \|u - Lx\|^2 \tag{4.6}$$

Finding the best linear optimal decision u^* to the above problem is equivalent to finding the shortest distance from the subspace $\mathcal{Z}$ to the element

Lx, where the minimizing u^* is the projection of Lx on $\mathcal{Z}$, and hence unique. Also, since u^* is the projection, we have

$$0 = \langle u^* - Lx, u \rangle = \mathbf{E} \, (u^* - Lx)^T Q_{uu} u,$$

for all u. In particular, for $f_i = (0, 0, ..., z_i, 0, ..., 0)$ with $z_i \in \mathcal{Z}$, we have

$$\mathbf{E} \, (u^* - Lx)^T Q_{uu} f_i = \mathbf{E} \, \{ [(u^* - Lx)^T Q_{uu}]_i z_i \} = 0.$$

The Gaussian assumption implies that $[(u^* - Lx)^T Q_{uu}]_i$ is independent of $z_i = A_i y_i$, for all linear transformations A_i. This gives in turn that $[(u^* - Lx)^T Q_{uu}]_i$ is independent of y_i. Hence, for any decision u, *linear or nonlinear*, we have that

$$
\begin{aligned}
\langle u^* - Lx, u \rangle &= \mathbf{E} \, (u^* - Lx)^T Q_{uu} u \\
&= \sum_i \mathbf{E} \, \{ [(u^* - Lx)^T Q_{uu}]_i u_i \} \\
&= \sum_i \mathbf{E} \, \{ [(u^* - Lx)^T Q_{uu}]_i \mu_i(y_i) \} = 0.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
||u - Lx||^2 &= \langle u - Lx, u - Lx \rangle \\
&= \langle u^* - Lx + u - u^*, u^* - Lx + u - u^* \rangle \\
&= \langle u^* - Lx, u^* - Lx \rangle + \langle u - u^*, u - u^* \rangle + \\
&\quad + 2\langle u^* - Lx, u^* - u \rangle \\
&= \langle u^* - Lx, u^* - Lx \rangle + \langle u - u^*, u - u^* \rangle \\
&\geq \langle u^* - Lx, u^* - Lx \rangle
\end{aligned}
$$

with equality if and only if $u = u^*$. This concludes the proof.

$\square$

The next theorem shows how to find the linear optimal control law $u = Ky$, where K is block diagonal:

$$
K = \begin{pmatrix}
K_1 & 0 & \cdots & 0 \\
0 & K_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & K_N
\end{pmatrix}.
$$

THEOREM 4.2

Let x and v_i be independent Gaussian variables taking values in $\mathbb{R}^n$ and $\mathbb{R}^{p_i}$, respectively with $x \sim \mathcal{N}(0, V_{xx})$, $v \sim \mathcal{N}(0, V_{vv})$. Also, let u_i be a stochastic variable taking values in $\mathbb{R}^{m_i}$, $m = m_1 + \cdots + m_N$, $Q_{xu} \in \mathbb{R}^{n \times m}$, $Q_{uu} \in \mathbb{S}^m_{++}$, $C_i \in \mathbb{R}^{p_i \times n}$, and $L = -Q_{uu}^{-1} Q_{ux}$. Set $y_i = C_i x + v_i$, with $\mathbf{E} \, y_i y_i^T \succ 0$. Then, the linear optimal solution $u_i = K_i y_i$ to the optimization problem

$$\min_{K_i} \quad \mathbf{E} \, (u - Lx)^T Q_{uu} (u - Lx)$$
$$\text{subject to} \quad u_i = K_i y_i, \quad i = 1, ..., N. \tag{4.7}$$

is the solution to the linear system of equations

$$\sum_{j=1}^N [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) = -[Q_{ux}]_i V_{xx} C_i^T,$$
$$K_i \in \mathbb{R}^{p_i \times n_i}, \text{ for } i = 1, ..., N.$$

$\square$

Proof Let $K = \text{diag}(K_i)$ and $C = \begin{pmatrix} C_1^T & \cdots & C_N^T \end{pmatrix}^T$. The problem of finding the optimal linear feedback law $u_i = K_i y_i$ can be written as

$$\text{minimize } \mathbf{Tr} \, \{\mathbf{E} \, Q_{uu} (u - Lx)(u - Lx)^T\}$$
$$\text{subject to } u = K(Cx + v) \tag{4.8}$$

Now

$$
\begin{aligned}
f(K) &= \mathbf{Tr} \, \left\{\mathbf{E} \, Q_{uu} (u - Lx)(u - Lx)^T\right\} \\
&= \mathbf{Tr} \, \left\{\mathbf{E} \, Q_{uu} (KCx + Kv - Lx)(KCx + Kv - Lx)^T\right\} \\
&= \mathbf{Tr} \, \left\{\mathbf{E} \, Q_{uu} (K(Cxx^T C^T + vv^T)K^T - 2Lxx^T C^T K^T + Lxx^T L^T)\right\} + \\
&\quad + \mathbf{Tr} \, \left\{\mathbf{E} \, 2Q_{uu}(KC - L)xv^T K^T\right\} \\
&= \mathbf{Tr} \, \left\{Q_{uu}(K(CV_{xx}C^T + V_{vv})K^T - 2LV_{xx}C^T K^T + LV_{xx}L^T)\right\} \\
&= \mathbf{Tr} \, \left\{\sum_{i,j=1}^N [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) K_i^T - \right. \\
&\quad \left. -2\sum_{i,j=1}^N [Q_{uu}]_{ij} L_j V_{xx} C_i^T K_i^T\right\} + \mathbf{Tr} \, \{Q_{uu} L V_{xx} L^T\}.
\end{aligned}
$$
$$\tag{4.9}$$

The minimizing K is obtained by solving $\nabla_{K_i} f(K) = 0$:

$$
\begin{aligned}
0 &= \nabla_{K_i} f(K) \\
&= 2 \sum_{j=1}^{N} [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) - 2 \sum_{j=1}^{N} [Q_{uu}]_{ij} L_j V_{xx} C_i^T.
\end{aligned}
\tag{4.10}
$$

Since $Q_{uu} L = -Q_{ux}$, we get that

$$
\sum_{j=1}^{N} [Q_{uu}]_{ij} L_j V_{xx} C_i^T = -[Q_{ux}]_i V_{xx} C_i^T.
$$

Hence, the equality in (4.10) is equivalent to

$$
\sum_{j=1}^{N} [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) = -[Q_{ux}]_i V_{xx} C_i^T,
\tag{4.11}
$$

and the proof is complete. $\qquad\square$

To verify existence and uniqueness of the linear solution obtained from Theorem 4.2, we proceed as follows. The system of equations in (4.11) is easy to pose as a standard linear system of equations $Hz = g$, in a simple structural way as follows. For matrices V and U of compatible sizes, we have the relation

$$
\text{vec}\{UXV\} = (V^T \otimes U)\text{vec}\{X\}.
\tag{4.12}
$$

Taking the transpose of the equations in (4.11), we obtain the equivalent linear system:

$$
\sum_{j=1}^{N} [CV_{xx}C^T + V_{vv}]_{ij} K_j^T [Q_{uu}]_{ij}^T = -C_i V_{xx} [Q_{xu}]_i,
\tag{4.13}
$$

where we have used that

$$
\begin{aligned}
(C_j V_{xx} C_i^T + [V_{vv}]_{ji})^T &= C_i V_{xx} C_j^T + [V_{vv}]_{ij} \\
&= [CV_{xx}C^T + V_{vv}]_{ij},
\end{aligned}
$$

since V_{xx} and V_{vv} are symmetric. Using the relation in (4.12), we can write (4.13) as

$$
\sum_{j=1}^{N} ([Q_{uu}]_{ij} \otimes [CV_{xx}C^T + V_{vv}]_{ij}) \text{vec}\{K_j^T\} = -\text{vec}\{C_i V_{xx} [Q_{xu}]_i\},
$$

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or equivalently as

$$Hz = g,$$

where H consists of blocks H_{ij} given by

$$H_{ij} = [Q_{uu}]_{ij} \otimes [CV_{xx}C^T + V_{vv}]_{ij},$$

$$z = \begin{pmatrix} \text{vec}\{K_1^T\} \\ \text{vec}\{K_2^T\} \\ \vdots \\ \text{vec}\{K_N^T\} \end{pmatrix},$$

and

$$g = - \begin{pmatrix} \text{vec}\{C_1 V_{xx}[Q_{xu}]_1\} \\ \text{vec}\{C_2 V_{xx}[Q_{xu}]_2\} \\ \vdots \\ \text{vec}\{C_N V_{xx}[Q_{xu}]_N\} \end{pmatrix}.$$

Now we will show that H is positive definite, and hence invertible, which proves existence and uniqueness of the solution of K_i. But first we need the following lemma from [42]:

LEMMA 4.1
If D is a $d \times d$ symmetric positive semi-definite matrix, partitioned symmetrically into m^2 blocks D_{ij} of size $d_i \times d_j$ $(i, j = 1, ..., m)$, such that D_{ii} is positive definite for every i; and if Q is an $m \times m$ positive definite matrix with elements q_{ij}; then the matrix H composed of blocks $H_{ij} = q_{ij}D_{ij}$ is positive definite. $\qquad \square$

Proof   Since D is symmetric positive semidefinite, it can be expressed as

$$D = \sum_k r(k)r^T(k),$$

where for each k, $r(k) \in \mathbb{R}^d$. For any vector $v \in \mathbb{R}^d$, let $\{v_i\}$ be a partitioning of v into subvectors, corresponding to the partitioning of D; then for every i and j

$$D_{ij} = \sum_k r_i(k)r_j^T(k).$$

For any v

$$
\begin{aligned}
v^T H v &= \sum_{i,j} q_{ij} v_i^T D_{ij} v_j \\
&= \sum_k \sum_{i,j} q_{ij} v_i^T r_i(k) r_j^T(k) v_j \\
&= \sum_k \sum_{i,j} q_{ij} w_i(k) w_j(k),
\end{aligned}
\tag{4.14}
$$

where $w_i(k) = v_i^T r_i(k)$. Hence $v^T H v \geq 0$ for all v. Now let $v \neq 0$; then for some i, $v_i \neq 0$. For that i, and for some k, $v_i^T r_i(k) \neq 0$, because D_{ii} is positive definite. Hence, from (4.14), $v^T H v > 0$ if $v \neq 0$, which completes the proof. $\qquad\square$

Now let $Y = CV_{xx}C^T + V_{vv}$. Partition Y into blocks such that

$$
Y_{ij} = [CV_{xx}C^T + V_{vv}]_{ij}.
$$

Note that $Y_{ii} = \mathbf{E}\, y_i y_i^T \succ 0$. Introduce

$$
D = \begin{pmatrix}
D_{11} & D_{12} & \cdots & D_{1m} \\
D_{21} & D_{22} & \cdots & D_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
D_{m1} & D_{m2} & \cdots & D_{mm}
\end{pmatrix},
$$

where $D_{ij} = \mathbf{1}_{m_i \times m_j} \otimes Y_{ij}$, that is:

$$
D = \begin{pmatrix}
Y_{11} & \cdots & Y_{11} & \cdots & Y_{1N} & \cdots & Y_{1N} \\
\vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
Y_{11} & \cdots & Y_{11} & \cdots & Y_{1N} & \cdots & Y_{1N} \\
\vdots & & \vdots & & \vdots & & \vdots \\
Y_{N1} & \cdots & Y_{N1} & \cdots & Y_{NN} & \cdots & Y_{NN} \\
\vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
Y_{N1} & \cdots & Y_{N1} & \cdots & Y_{NN} & \cdots & Y_{NN}
\end{pmatrix},
$$

where every block Y_{ij} is repeated m_i times vertically and m_j times horizontally. Then, it is easy to verify that the matrix H is exactly the matrix with blocks consisting of $q_{ij} D_{ij}$, where $q_{ij} = [Q_{uu}]_{ij}$. Applying Lemma 4.1 gives that H is positive definite, which proves existence and uniqueness of the solution of K.

In general, separation does not hold for the static team problem when constraints on the information available for every decision maker u_i are imposed, as was already demonstrated in the introductory chapter. That is, the optimal decision is *not* given by $u_i = L\hat{x}_i$, where $\hat{x}_i$ is the optimal estimated value of x by decision maker i. We will now revisit the example that was given in the introductory chapter.

EXAMPLE 4.1
Consider the team problem

$$\text{minimize } \mathbf{E} \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

$$\text{subject to } y_i = C_i x + v_i$$
$$u_i = \mu_i(y_i)$$
$$\text{for } i = 1, ..., N$$

The data we will consider is:

$$N = 2$$
$$C_1 = C_2 = 1$$
$$Q_{xx} = 1$$
$$Q_{uu} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$
$$Q_{xu} = Q_{ux}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$
$$x \sim \mathcal{N}(0,1), \ v_1 \sim \mathcal{N}(0,1), \ v_2 \sim \mathcal{N}(0,1)$$

The best decision with full information is given by

$$u = -Q_{uu}^{-1} Q_{ux} x$$
$$= - \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x$$
$$= - \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} x.$$

The optimal estimate of x of decision maker 1 is

$$\hat{x}_1 = \mathbf{E}\{x|y_1\} = \frac{1}{2}y_1,$$

and of decision maker 2

$$\hat{x}_2 = \mathbf{E}\{x|y_2\} = \frac{1}{2}y_2.$$

Hence, the decision where each decision maker combines the best deterministic decision with her best estimate of x is given by

$$u_i = -\frac{1}{3}\hat{x}_i$$
$$= -\frac{1}{3} \cdot \frac{1}{2}y_i$$
$$= -\frac{1}{6}y_i,$$

for $i = 1, 2$. This policy gives a cost equal to 0.611. However, solving the team problem yields $K_1 = K_2 = -\frac{1}{5}$, and hence the optimal team decision is given by

$$u_i = -\frac{1}{5}y_i.$$

The cost obtained from the team problem is 0.600. Clearly, separation does not hold in team decision problems. $\square$

Team Decision Problems and Signaling

Consider a modified version of the static team problem posed in the previous section, where the observation y_i for every decision maker i is affected by the inputs of the other decision makers, that is

$$y_i = C_i x + \sum_j D_{ij}u_j + v_i,$$

where $D_{ij} = 0$ if decision maker j does not affect the observation y_i. The modified optimization problem becomes

$$\begin{aligned}
&\text{minimize } \mathbf{E} \left(u - Lx\right)^T Q_{uu}(u - Lx)\\
&\text{subject to } y_i = C_i x + \sum_j D_{ij}u_j + v_i\\
&\qquad u_i = \mu_i(y_i)\\
&\qquad \text{for } i = 1, ..., N.
\end{aligned} \tag{4.15}$$

The problem above is, in general, very complex if decision maker i does not have information about the decisions u_i that appear in y_i (see [8]). It has been shown by Witsenhausen [53], by means of a counterexample, that for such problems there could be nonlinear decisions in the observations that perform better than any linear decision. This is referred to as the problem of *signaling*, where decision maker j tries to *encode* information in his decision that could be *decoded* by other decision makers whose

observation is affected (see Ho [26]).

If we assume that decision maker i has the value of u_j available for every j such that $D_{ij} \neq 0$, then he/she could form the new output measurement

$$\check{y}_i = y_i - \sum_j D_{ij} u_j = C_i x + v_i,$$

which transforms the problem to a static team problem without signaling, and the optimal solution is linear and can be found according to theorems 4.1 and 4.2. Note that if decision maker i has the information available that every decision maker j has for which $D_{ij} \neq 0$, then the decision u_j is also available to decision maker i. This information structure is closely related to the *partially nested* information structure, which was introduced by Ho and Chu in [29].

Finally, we state a mathematical definition of signaling incentive in static teams:

DEFINITION 4.1—SIGNALING INCENTIVE
Consider the static team problem given by

$$
\begin{aligned}
&\text{minimize } \mathbf{E}\, (u - Lx)^T Q_{uu} (u - Lx) \\
&\text{subject to } y_i = C_i x + \sum_j D_{ij} u_j + v_i \\
&\qquad u_i = \mu_i : \mathbb{I}_i \mapsto \mathbb{R}^{m_i} \\
&\qquad \text{for } i = 1, ..., N,
\end{aligned}
\tag{4.16}
$$

where $\mathbb{I}_i$ denotes the information y_j available to decision maker i, for $j = 1, ..., N$. Then, the problem is said to have a signaling incentive if there exist i, j such that $\mathbb{I}_j \not\subseteq \mathbb{I}_i$ and $D_{ij} \neq 0$. □

4.3 Distributed Linear Quadratic Gaussian Control

In this section, we will treat the distributed linear quadratic Gaussian control problem with information constraints, which can be seen as a dynamic team decision problem.

Consider an example of four dynamically coupled systems according to the graph in Figure 4.1. The equations for the interconnected system are
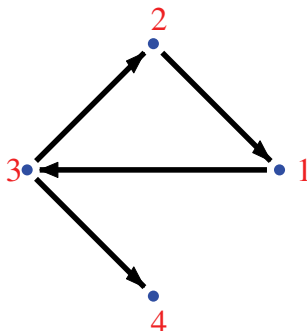
Figure 4.1   The graph reflects the interconnection structure of the dynamics be-
tween four systems. The arrow from node 2 to node 1 indicates that system 1 affects
the dynamics of system 2 directly.

then given by

$$
\underbrace{\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{pmatrix}}_{x(k+1)} = \underbrace{\begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix}}_{x(k)} +
$$
$$
+ \underbrace{\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ u_4(k) \end{pmatrix}}_{u(k)} + \underbrace{\begin{pmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \end{pmatrix}}_{w(k)} .
$$

(4.17)

 For instance, the arrow from node 2 to node 1 in the graph means that the
dynamics of system 2 are directly affected by system 1, which is reflected
in the system matrix A, where the element $A_{21} \neq 0$. On the other hand,
system 2 does not affect system 1 directly, which implies that $A_{12} = 0$.
Because of the "physical" distance between the subsystems, there will be
some constraints on the information available to each node.

   The structure of the matrix A can be described by the *adjacency matrix*
$\mathcal{A}$ of the graph. For instance, the adjacency matrix of the graph in Figure

4.1 is given by

$$
\mathcal{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The observation of system i at time k is given by

$$
y_i(k) = C_i x(k),
$$

where

$$
C_i = \begin{pmatrix} C_{i1} & 0 & 0 & 0 \\ 0 & C_{i2} & 0 & 0 \\ 0 & 0 & C_{i3} & 0 \\ 0 & 0 & 0 & C_{i4} \end{pmatrix}. \tag{4.18}
$$

Here, $C_{ij} = 0$ if system i does not have access to $y_j(k)$. Let $\mathbb{I}_i^k$ denote the set of information $(y_j(n), u_j(n))$ available to node i up to time k, $n \leq k$, $j = 1, ..., N$.

Consider the following (general) dynamic team decision problem:

$$
\text{minimize} \sum_{k=0}^{M} \mathbf{E} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}
$$

$$
\begin{aligned}
\text{subject to } & x(k+1) = Ax(k) + Bu(k) + w(k) \\
& y_i(k) = C_i x(k) + v_i(k) \\
& u_i(k) = \mu_i : \mathbb{I}_i^k \mapsto \mathbb{R}^{m_i} \\
& \text{for } i = 1, ..., N.
\end{aligned} \tag{4.19}
$$

where $x(k) \in \mathbb{R}^n$, $y_i(k) \in \mathbb{R}^{p_i}$, $u_i(k) \in \mathbb{R}^{m_i}$, $x(0) \sim \mathcal{N}(0, R_0)$, $\{v(k)\}$ and $\{w(k)\}$ are sequences of independent Gaussian variables, uncorrelated with $x(0)$, such that

$$
\mathbf{E} \begin{pmatrix} v(k) \\ w(k) \end{pmatrix} \begin{pmatrix} v(l) \\ w(l) \end{pmatrix}^T = \delta(k - l)R,
$$

and the weighting matrix Q_{uu} is positive definite. Now for any $t \in \mathbb{N}$ such that $t \leq k$, we can write $x(k)$ and $y(k)$ as

$$x(k) = A^t x(k-t) + \sum_{n=1}^{t} A^{n-1} B u(k-n) + \sum_{n=1}^{t} A^{n-1} w(k-n),$$

$$y_i(k) = C_i A^t x(k-t) + \sum_{n=1}^{t} C_i A^{n-1} B u(k-n) + \qquad (4.20)$$

$$+ \sum_{n=1}^{t} C_i A^{n-1} w(k-n) + v_i(k).$$

Note that the summation over n is defined to be zero when $t = 0$.

THEOREM 4.3
Consider the optimization problem given by (4.19). The problem has no signaling incentive if

$$\mathbb{I}_j^n \subseteq \mathbb{I}_i^k \text{ for } [C_i A^n B]_j \neq 0 \qquad (4.21)$$

for all n such that $0 \leq n < k$, and $k = 0, ..., M-1$.
In addition, the optimal solution to the optimization problem given by (4.19) is linear in the observations $\mathbb{I}_i^k$ if condition (4.21) is satisfied, and has an analytical solution that can be found by solving a linear system of equations.

$\square$

Proof   Introduce

$$\bar{x} = \begin{pmatrix} w(M-1) \\ w(M-2) \\ \vdots \\ w(0) \\ x(0) \end{pmatrix}, \quad \bar{u}_i = \begin{pmatrix} u_i(M-1) \\ u_i(M-2) \\ \vdots \\ u_i(0) \end{pmatrix}.$$

Then, we can write the cost function

$$\sum_{k=0}^{M-1} \mathbf{E} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}$$

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as

$$\mathbf{E} \left(\begin{array}{c} \bar{x} \\ \bar{u} \end{array} \right)^T \bar{Q} \left(\begin{array}{c} \bar{x} \\ \bar{u} \end{array} \right), \qquad (4.22)$$

for some symmetric matrix $\bar{Q}$ with

$$\bar{Q} = \left(\begin{array}{cc} \bar{Q}_{xx} & \bar{Q}_{xu} \\ \bar{Q}_{ux} & \bar{Q}_{uu} \end{array} \right),$$

and $\bar{Q}_{uu} \succ 0$. Consider the expansion given by (4.20). The problem here is that $y_i(k)$ depends on previous values of the control signals $u(n)$ for $n = 0, ..., k-1$. The components $u_j(n)$ that $y_i(k)$ depends on are completely determined by the structure of the matrix $[C_i A^n B]_j$. This means that, if for every node i we have $\mathbb{I}_j^n \subseteq \mathbb{I}_i^k$ for $[C_i A^n B]_j \neq 0$, then there is no signaling incentive. Thus, we have proved the first statement of the theorem.

Now if condition (4.21) is satisfied, we can form the new output measurement

$$\begin{aligned} \check{y}_i(k) &= y_i(k) - \sum_{n=1}^{k} C_i A^{n-1} B u(k - n) \\ &= C_i A^k x(0) + \sum_{n=1}^{k} C_i A^{n-1} w(k - n) + v_i(k). \end{aligned} \qquad (4.23)$$

Let

$$\bar{y}_i(k) = \left(\begin{array}{c} \check{y}_i(k) \\ \check{y}_i(k - 1) \\ \vdots \\ \check{y}_i(0) \end{array} \right).$$

With these new variables introduced, the optimization problem given by equation (4.19) reduces to the following static team decision problem:

$$\begin{aligned} \min_{\mu} \mathbf{E} \left(\begin{array}{c} \bar{x} \\ \bar{u} \end{array} \right)^T \bar{Q} \left(\begin{array}{c} \bar{x} \\ \bar{u} \end{array} \right) & \\ \text{subject to } u_i(k) = \bar{\mu}_i(\bar{y}_i(k)), \quad k = 0, ..., M - 1 & \\ \text{for } i = 1, ..., N. & \end{aligned} \qquad (4.24)$$

and the optimal solution $\bar{u}$ is unique and linear according to Theorem 4.1, and can be obtained using Theorem 4.2, *QED*.

$\square$

5

Minimax Team Decision Problems

We considered the problem of static and dynamic stochastic team decision in the previous chapter. This chapter treats an analogous version for the deterministic (or worst case) problem. Although the problem formulation is very similar, the ideas of the solution are considerably different, and in a sense more difficult.

The deterministic problem considered is a quadratic game between a team of players and nature. Each player has limited information that could be different from the other players in the team. This game is formulated as a minimax problem, where the team is the minimizer and nature is the maximizer. We show that if there is a solution to the static minimax team problem, then linear decisions are optimal, and we show how to find a linear optimal solution by solving a linear matrix inequality. The result is used to solve the distributed finite horizon $\mathcal{H}_\infty$ control problem. It shows that information exchange with neighbours on the graph only, is enough to obtain a linear optimal policy.

5.1 Introduction

We consider the problem of static minimax team decision. A team of players is to optimize a worst case scenario given limited information of nature's decision for each player. The problem can be considered as the *deterministic* analog of the stochastic team decision problems that were solved by Radner [42].

An initial step for solving the static deterministic problem was made in [5], where a team of two players is considered using a stochastic framework. The solution given in [5] cannot easily be extended to more than two players, since it uses *common* information for the two players, a concept

that does not necessarily exist for more than two players. Also, the one step delay $\mathcal{H}_\infty$ control problem is solved in [5].

In this chapter, we solve the static minimax (or deterministic) team decision problem completely for an arbitrary number of players, and show that the optimal solution is linear and can be found by solving a linear matrix inequality. Also, we show how to solve the dynamic finite horizon $\mathcal{H}_\infty$ team problem, under some conditions that prevent *signaling*. The dynamic finite horizon $\mathcal{H}_\infty$ team problem is identical to the distributed finite horizon stochastic linear quadratic control problem treated in [26] and its generalization in Chapter 4, see also [22]. For the infinite horizon problem, similar conditions were obtained in [4] and [46]. We show that the information structure where subsystems on a graph are restricted to exchange information with neighbours only, is enough to obtain an optimal feedback law which turns out to be linear. This reveals a broader class of information structures that lead to tractable problems.

5.2 The Static Minimax Team Decision Problem

Consider the following team decision problem

$$
\begin{aligned}
&\inf_\mu \sup_{x \neq 0} \frac{J(x,u)}{||x||^2} \\
&\text{subject to} \quad y_i = C_i x \\
&\qquad\qquad\quad u_i = \mu_i(y_i) \\
&\qquad\qquad\quad \text{for } i = 1, ..., N
\end{aligned}
\tag{5.1}
$$

where $u_i \in \mathbb{R}^{m_i}$, $m = m_1 + \cdots + m_N$, $C_i \in \mathbb{R}^{p_i \times n}$.
$J(x,u)$ is a quadratic cost given by

$$
J(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix},
$$

where

$$
\begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \in \mathbb{S}^{m+n}.
$$

We will be interested in the case $Q_{uu} \succ 0$ (this can be generalized to $Q_{uu} \succeq 0$, but the presentation of the chapter becomes more technical). The players $u_1, ..., u_N$ make up a *team*, which plays against *nature* represented

by the vector x, using

$$\mu(x) = \begin{pmatrix} \mu_1(C_1x) \\ \vdots \\ \mu_N(C_Nx) \end{pmatrix}.$$

PROPOSITION 5.1
The value of the game in (5.1) is

$$\gamma^* = \inf_{\mu} \sup_{x \neq 0} \frac{J(x,u)}{||x||^2},$$

if and only if for every $\epsilon > 0$ there is a decision μ_ϵ such that

$$\gamma^* \leq \sup_{x \neq 0} \frac{J(x,\mu_\epsilon(x))}{||x||^2} < \gamma^* + \epsilon,$$

and γ^* is the smallest such number. □

Proof If the value of the game (5.1) is γ^*, then clearly

$$\gamma^* \leq \sup_{x \neq 0} \frac{J(x,\mu_\epsilon(x))}{||x||^2}.$$

for every policy μ_ϵ. Also, it follows from the definition of the infimum that for every $x \neq 0$ there is a vector $u_x \in \mathbb{R}^m$ such that

$$\frac{J(x,u_x)}{||x||^2} < \gamma^* + \epsilon. \tag{5.2}$$

For every pair (x, u_x) satisfying (5.2), define the decision $\mu_\epsilon(x) = u_x$. Then,

$$\sup_{x \neq 0} \frac{J(x,\mu_\epsilon(x))}{||x||^2} < \gamma^* + \epsilon.$$

On the other hand, if γ^* is the infimal value such that for every $\epsilon > 0$ there is a decision $\mu_\epsilon(x)$ with

$$\gamma^* \leq \sup_{x \neq 0} \frac{J(x,\mu_\epsilon(x))}{||x||^2} < \gamma^* + \epsilon,$$

then it follows that

$$\gamma^* \leq \inf_{\mu_\epsilon} \sup_{x} \frac{J(x,\mu_\epsilon(x))}{||x||^2} < \gamma^* + \epsilon,$$

Since ϵ can be chosen arbitrarily small, we conclude that the value of the game must be γ^*, and the proof is complete. $\qquad\square$

Proposition 5.1 shows that if γ^* is the value of the game in (5.1), then for any given real number $\gamma > \gamma^*$, there exists a policy μ such that $J(x, \mu(x)) - \gamma\|x\|^2 \leq 0$ for all x. Hence, we can formulate the alternative team decision problem:

$$
\begin{aligned}
\inf_{\mu} \sup_{x \neq 0} \ & J(x, u) - \gamma\|x\|^2 \\
\text{subject to} \ & y_i = C_i x \\
& u_i = \mu_i(y_i) \\
& \text{for } i = 1, ..., N
\end{aligned}
\tag{5.3}
$$

The formulation above can be seen as the problem of looking for suboptimal solutions to the game given by (5.1). Clearly, Proposition 5.1 shows that the value of the game resulting from the decision obtained in (5.3) approaches the optimal value in (5.1) as γ approaches γ^* (or as $\epsilon \to 0$). From now on we will consider the equivalent game given by (5.3). Introduce the matrix

$$
C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{pmatrix}.
$$

C is a $p \times n$ matrix, where $p = p_1 + p_2 + \cdots + p_N$. Let rank $C = r$. For any given vector y, a vector x with $y = Cx$ can be written as $x = C^\dagger y + \tilde{x}$, where $\tilde{x}$ is the unobservable part of x from the vector y, that is $C\tilde{x} = 0$. Let $F \in \mathbb{R}^{n \times (n-r)}$ be a nullspace generator of the matrix C, that is, $CF = 0$ (F can be taken as the matrix with column vectors orthogonal to the column vectors of $C^\dagger$). Then, any vector $\tilde{x}$ such that $C\tilde{x} = 0$ can be written as $\tilde{x} = F\tilde{y}$ for some vector $\tilde{y} \in \mathbb{R}^{n-r}$. We will now show how to eliminate the unobservable part of x from our problem. Define

$$
Q_\gamma = \begin{pmatrix} Q_{xx} - \gamma I & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix},
\tag{5.4}
$$

and let V be given by

$$
V = \begin{pmatrix} F & C^\dagger & 0 \\ 0 & 0 & I \end{pmatrix}.
\tag{5.5}
$$

Then,

$$J(x,u) - \gamma\|x\|^2 = \begin{pmatrix} x \\ u \end{pmatrix}^T Q_\gamma \begin{pmatrix} x \\ u \end{pmatrix}$$

$$= \begin{pmatrix} C^\dagger y + F\tilde{y} \\ u \end{pmatrix}^T Q_\gamma \begin{pmatrix} C^\dagger y + F\tilde{y} \\ u \end{pmatrix} \quad (5.6)$$

$$= \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}^T V^T Q_\gamma V \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}.$$

Let $V^T Q_\gamma V$ be partitioned as

$$V^T Q_\gamma V = Z = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix}, \quad (5.7)$$

$$Z_{11} \in \mathbb{R}^{(n-p)\times(n-p)}, \quad Z_{22} \in \mathbb{R}^{p\times p}, \quad Z_{33} \in \mathbb{R}^{m\times m}.$$

Thus, we have

$$\begin{pmatrix} x \\ u \end{pmatrix}^T Q_\gamma \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}^T Z \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}.$$

Then, the game (5.3) can be equivalently formulated as

$$\inf_{\mu} \sup_{y_i = C_i x, x \neq 0} \sup_{\tilde{y}} \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}^T Z \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}$$

$$\text{subject to } y_i = C_i x$$
$$u_i = \mu_i(y_i)$$
$$\text{for } i = 1, ..., N \quad (5.8)$$

PROPOSITION 5.2
Let Z be the matrix given by (5.7). Then, the value of the game

$$\inf_{\mu} \sup_{y_i = C_i x, x \neq 0} \sup_{\tilde{y}} \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}^T Z \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}$$

$$\text{subject to } y_i = C_i x, \ u_i = \mu_i(y_i), \text{ for } i = 1, ..., N,$$

can be zero only if $Z_{11} \preceq 0$.

$\square$

Proof If $Z_{11} \not\preceq 0$, then $\tilde{y}$ can be chosen in the direction of the eigenvector corresponding to the positive eigenvalue of Z_{11}, which makes the value of the game arbitrarily large. Hence, a necessary condition for the game to have value zero is that $Z_{11} \preceq 0$. $\qquad\square$

To ease the exposition of the chapter, we will consider the case where $Z_{11} \prec 0$. The case where Z_{11} is semi-definite can be treated similarly, but is more technical, and therefore omitted here.

PROPOSITION 5.3
If $Z_{11} \prec 0$, then

$$
\sup_{\tilde{y}} \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}^T Z \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix} =
$$

$$
\begin{pmatrix} y \\ u \end{pmatrix} \left(\begin{pmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{pmatrix} - \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix} Z_{11}^{-1} \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix}^T \right) \begin{pmatrix} y \\ u \end{pmatrix}. \tag{5.9}
$$

$\qquad\square$

Proof Completion of squares gives

$$
\begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix}^T Z \begin{pmatrix} \tilde{y} \\ y \\ u \end{pmatrix} = \left(\tilde{y} + F \begin{pmatrix} y \\ u \end{pmatrix} \right)^T Z_{11} \left(\tilde{y} + F \begin{pmatrix} y \\ u \end{pmatrix} \right) +
$$

$$
+ \begin{pmatrix} y \\ u \end{pmatrix} \left(\begin{pmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{pmatrix} - \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix} Z_{11}^{-1} \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix}^T \right) \begin{pmatrix} y \\ u \end{pmatrix} \tag{5.10}
$$

where F is given by

$$
F = Z_{11}^{-1} \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix}^T. \tag{5.11}
$$

Since $Z_{11} \prec 0$, the quadratic form in (5.10) is maximized for

$$
\tilde{y} = -F \begin{pmatrix} y \\ u \end{pmatrix},
$$

which proves our proposition. $\qquad\square$

Introduce the matrix

$$
Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{pmatrix} - \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix} Z_{11}^{-1} \begin{pmatrix} Z_{21} \\ Z_{31} \end{pmatrix}^T. \tag{5.12}
$$

Recall that $Z_{33} = Q_{uu} \succ 0$, and $Z_{11} \prec 0$, which implies that $Q_{22} \succ 0$. Now using Proposition (5.3), the game described by (5.8) reduces to

$$\inf_{\mu} \; \sup_{y_i = C_i x, x \neq 0} \begin{pmatrix} y \\ u \end{pmatrix}^T Q \begin{pmatrix} y \\ u \end{pmatrix}$$
$$\text{subject to} \;\; y_i = C_i x \tag{5.13}$$
$$u_i = \mu_i(y_i)$$
$$\text{for } i = 1, ..., N$$

Hence, we consider the problem of finding policies $\mu_i(y_i)$ such that $u_i = \mu_i(y_i)$ and

$$\begin{pmatrix} Cx \\ u \end{pmatrix}^T Q \begin{pmatrix} Cx \\ u \end{pmatrix} \leq 0$$

for all x. Now we are ready to state the main result of the chapter where we show linearity of the optimal decisions:

THEOREM 5.1
Let $Q_{22} \succ 0$ and $y_i = C_i x$, $i = 1, ..., N$. If there exist policies $\mu_i(y_i)$ such that

$$\sup_{x \neq 0} \begin{pmatrix} Cx \\ \mu(y) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} Cx \\ \mu(y) \end{pmatrix} \leq 0, \tag{5.14}$$

then there exist *linear* policies $\mu_i(y_i) = K_i y_i$ that satisfy (5.14).

$\square$

Proof   Assume existence of a policy μ that satisfies (5.14). If $y_i = C_i x = 0$ for some i, then the optimal decision for player i is to set $\mu_i(0) = 0$. To see this, take $y = 0$. Then

$$\begin{pmatrix} y \\ \mu(y) \end{pmatrix}^T Q \begin{pmatrix} y \\ \mu(y) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(0) \end{pmatrix}^T Q \begin{pmatrix} 0 \\ \mu(0) \end{pmatrix} = \mu^T(0) Q_{22} \mu(0).$$

Since $Q_{22} \succ 0$, we see that $\mu(0) = 0$ is the optimal decision. In particular, $\mu_i(0) = 0$ is the optimal decision for decision maker i.

Now suppose that $y_i \neq 0$ for $i = 1, 2, ..., N$. Define $K_i(y_i)$ as

$$K_i(y_i) = \frac{\mu_i(y_i) \cdot y_i^T}{\|y_i\|^2}, \quad y_i \neq 0, \tag{5.15}$$

for $i = 1, ..., N$. Also, define $K(x)$ as

$$K(x) = \begin{pmatrix} K_1(C_1x) & 0 & \cdots & 0 \\ 0 & K_2(C_2x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N(C_Nx) \end{pmatrix}. \qquad (5.16)$$

It is easy to see that (5.14) is equivalent to

$$x^T C^T \begin{pmatrix} I \\ K(x) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} I \\ K(x) \end{pmatrix} Cx \leq 0, \quad \forall x \neq 0. \qquad (5.17)$$

Hence, we have obtained an equivalent problem for which existence of policies μ_i is the same as existence of matrix functions $K_1(y_1), ..., K_N(y_N)$, and $K(x)$ satisfying (5.16) and (5.17). Note that the problem of searching for linear policies corresponds to that of searching for constant matrices $K_i(C_ix) = K_i$. Furthermore, (5.17) is equivalent to the problem of finding a matrix function $X \mapsto K$ such that for every $X = xx^T \neq 0$,

$$\mathbf{Tr}\, C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX \leq 0. \qquad (5.18)$$

To see this, take a matrix K satisfying (5.18), for $X = xx^T \neq 0$. Then, $K(x) = K$ satisfies (5.17). Conversely, given $K(x)$ satisfying (5.17), we can take $K = K(x)$ and (5.18) is satisfied.

Now if for a given matrix $X \neq 0$, a matrix K is such that the inequality in (5.18) is satisfied, then the same matrix K satisfies (5.18) with the matrix $X / \mathbf{Tr}\, X$ instead of X. Thus, since we are considering matrices $X = xx^T \neq 0$, it is enough to consider matrices X with $\mathbf{Tr}\, X = 1$. Define the set

$$S_1 = \{X : x \in \mathbb{R}^n, X = xx^T, \mathbf{Tr}\, X = 1\}$$

Then (5.18) implies that

$$\max_{X \in S_1} \min_K \mathbf{Tr}\, C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX \leq 0. \qquad (5.19)$$

We will now extend the set of matrices X from S_1 to the set

$$S = \{X : X \succeq 0, \mathbf{Tr}\, X = 1\}.$$

That is, we will consider the extended problem

$$\max_{X \in S} \min_K \mathbf{Tr}\, C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX. \qquad (5.20)$$

Clearly, we have that

$$
\max_{X \in S_1} \min_{K} \mathbf{Tr} \; C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX \leq
$$
$$
\max_{X \in S} \min_{K} \mathbf{Tr} \; C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX. \tag{5.21}
$$

Now consider the extended minimax problem (5.20), and suppose that

$$
\max_{X \in S} \min_{K} \mathbf{Tr} \; C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX = \alpha
$$

for some real number α. This is equivalent to

$$
\max_{X \in S} \min_{K} \mathbf{Tr} \left\{ C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX - \alpha X \right\} = 0 \tag{5.22}
$$

Note that

$$
\max_{X \in S} \min_{K} \mathbf{Tr} \left\{ C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX - \alpha X \right\}
$$

is the dual to the following convex optimization problem (see [11]):

$$
\min_{K,s} \; s
$$
$$
\text{subject to} \;\; C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} C - \alpha I \preceq sI. \tag{5.23}
$$

Strong duality holds since the primal problem (5.23) is convex ($Q_{22} \succ 0$) and Slater's condition is satisfied, see [11]. Thus, existence of a decision matrix $K(x)$ fulfilling (5.22) implies existence of a constant matrix K that fulfills

$$
\max_{X} \mathbf{Tr} \left\{ C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX - \alpha X \right\} = 0. \tag{5.24}
$$

Now take any positive semi-definite matrix X of rank $k \leq n$ and $\mathbf{Tr}X = 1$. Then, we can write X as

$$
X = \sum_{i=1}^{k} \lambda_i X_i,
$$

where $X_i = x_i x_i^T$, $\|x_i\| = 1$, $x_i^T x_j = 0$ for $i \neq j$, $\lambda_i > 0$, and $\sum_{i=1}^{k} \lambda_i = 1$ (see [30], pp.457). Let $X_* = \sum_{i=1}^{k} \lambda_i X_i$ be

$$X_* = \arg\max_{X \in S} \mathbf{Tr} \left\{ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX - \alpha X \right\}.$$

This gives together with equation (5.24):

$$\mathbf{Tr}\ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX_* = \mathbf{Tr}\ \alpha X_* = \alpha.$$

Let X_j be the matrix for which

$$\mathbf{Tr}\ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX_i$$

is maximized among $X_1, ..., X_k$. Then

$$\alpha = \mathbf{Tr}\ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX_*$$

$$= \sum_{i=1}^{k} \lambda_i \mathbf{Tr}\ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX_i$$

$$\leq \sum_{i=1}^{k} \lambda_i \mathbf{Tr}\ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX_j$$

$$= \mathbf{Tr}\ C^T \begin{pmatrix} I \\ K \end{pmatrix}^T Q \begin{pmatrix} I \\ K \end{pmatrix} CX_j \leq 0.$$

Hence, we have proved that the worst case is attained for a matrix X with rank 1, and the extension of the set S_1 to the set S does *not* increase the cost. We conclude that the optimal decision can be taken to be a linear decision with $\mu(y) = Ky$, and the proof is complete. $\square$

5.3 Computation of Optimal Team Decisions

In the previous section we showed that for the minimax team problem given by (5.3), the linear policy $u = KCx$ is optimal, where K is given by

$$K = \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N \end{pmatrix}. \tag{5.25}$$

The problem of finding linear policies satisfying (5.17) can be written as the following convex feasibility problem:

$$\text{Find} \quad K$$

$$\text{such that} \quad \begin{pmatrix} C \\ KC \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} C \\ KC \end{pmatrix} \preceq 0. \qquad (5.26)$$

where $Q_{22} \succ 0$. The inequality in (5.26) can be written as

$$R - (KC - LC)^T Q_{22}(KC - LC) \succeq 0, \qquad (5.27)$$

where $L = Q_{22}^{-1} Q_{21}$, and $R = -C^T Q_{11} C + C^T Q_{12} Q_{22}^{-1} Q_{21} C$. First note that a necessary condition for (5.27) to be satisfied is that $R \succeq 0$. If $R \succeq 0$, then using the Schur complement gives that the inequality in (5.27) can be written as a linear matrix inequality (LMI):

$$\begin{pmatrix} R & (KC - LC)^T \\ KC - LC & Q_{22}^{-1} \end{pmatrix} \succeq 0,$$

which can be solved efficiently.

5.4 Relation to the Stochastic Team Decision Problem

In this section we consider the stochastic minimax team decision problem

$$\min_K \max_{\mathbf{E}\|x\|^2 = 1} \mathbf{E} \left\{ x^T \begin{pmatrix} C \\ KC \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} C \\ KC \end{pmatrix} x \right\}.$$

Taking the expectation of the cost in the stochastic problem above yields the equivalent problem

$$\min_K \max_{\mathbf{Tr}X = 1} \mathbf{Tr} \begin{pmatrix} C \\ KC \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} C \\ KC \end{pmatrix} X$$

where X is a positive semi-definite matrix, and is the covariance matrix of x, i. e. $X = \mathbf{E}\, xx^T$. Hence, we see that the stochastic minimax team problem is equivalent to the deterministic minimax team problem, where nature maximizes with respect to all covariance matrices X of the stochastic variable x with variance $\mathbf{E}\, \|x\|^2 = \mathbf{E}\, x^T x = \mathbf{Tr}\, X = 1$.

5.5 Team Decision Problems and Signaling

Consider a modified version of the static team problem posed in the previous section, where the observation y_i for every decision maker i is affected by the inputs of the other decision makers, that is

$$y_i = C_i x + \sum_j D_{ij} u_j,$$

where $D_{ij} = 0$ if decision maker j does not affect the observation y_i. The modified optimization problem becomes

$$\inf_\mu \sup_x \begin{pmatrix} x \\ u \end{pmatrix}^T Q \begin{pmatrix} x \\ u \end{pmatrix}$$
$$\text{subject to } y_i = C_i x + \sum_j D_{ij} u_j \qquad (5.28)$$
$$u_i = \mu_i(y_i)$$
$$\text{for } i = 1, ..., N.$$

The problem above is in general very complex if decision maker i does not have access to the information about the decisions u_j that appear in y_i. We say that the problem give rise to a signaling incentive for decision maker j, which is the same definition as in Chapter 4. If we assume that decision maker i has the value of u_j available for every j such that $D_{ij} \neq 0$, then she can form the new output measurement

$$\check{y}_i = y_i - \sum_j D_{ij} u_j = C_i x,$$

which transforms the problem to a static team problem without signaling, and the optimal solution is linear and can be found according to Theorem 5.1 and Section 5.3. Note that if decision maker i has the information available that decision maker j has, then the decision u_j is also available to decision maker i.

5.6 Distributed $\mathcal{H}_\infty$ Control

In this section, we will treat the distributed linear quadratic $\mathcal{H}_\infty$ control problem with information constraints, which can be seen as a dynamic team decision problem. The idea is to transform the dynamic team problem to a static one, and then exploit information structures for every time step.
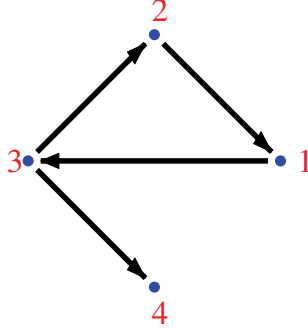
Figure 5.1   The graph reflects the interconnection structure of the dynamics between four systems. The arrow from node 2 to node 1 indicates that system 1 affects the dynamics of system 2 directly.

Consider an example of four dynamically coupled systems according to the graph in Figure 5.1. The equations for the interconnected system are given by

$$
\underbrace{\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{pmatrix}}_{x(k+1)} = \underbrace{\begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix}}_{x(k)} +
$$

$$
+ \underbrace{\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ u_4(k) \end{pmatrix}}_{u(k)} + \underbrace{\begin{pmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \end{pmatrix}}_{w(k)} .
$$

(5.29)

For instance, the arrow from node 2 to node 1 in the graph means that the dynamics of system 2 are directly affected by system 1, which is reflected in the system matrix A, where the block $A_{21} \neq 0$. On the other hand, system 2 does not affect system 1 directly, which implies that $A_{12} = 0$. Because of the "physical" distance between the subsystems, there will be some constraints on the information available to each node.

The observation of system i at time k is given by

$$
y_i(k) = C_i x_i(k),
$$

where

$$C_i = \begin{pmatrix} C_{i1} & 0 & 0 & 0 \\ 0 & C_{i2} & 0 & 0 \\ 0 & 0 & C_{i3} & 0 \\ 0 & 0 & 0 & C_{i4} \end{pmatrix}. \tag{5.30}$$

Here, $C_{ij} = 0$ if system i does not have access to $y_j(k)$. The subsystems could exchange information about their outputs. Every subsystem recieves the information with some time delay, which is reflected by the interconnection structure. As in the previous chapter, let $\mathbb{I}_i^k$ denote the set of observations $y_j(n)$ and control signals $u_j(n)$ available to node i up to time k, $n \le k$, $j = 1, ..., N$.

Consider the following (general) dynamic team decision problem:

$$\inf_{\mu} \sup_{w} J(u, w)$$
$$\text{subject to } x(k+1) = Ax(k) + Bu(k) + w(k)$$
$$y_i(k) = C_i x(k) \tag{5.31}$$
$$u_i(k) = \mu_i : \mathbb{I}_i^k \mapsto \mathbb{R}^{p_i}$$
$$\text{for } i = 1, ..., N,$$

where

$$J(u, w) = x^T(M) Q_f x^T(M) + \sum_{k=0}^{M-1} \left\{ \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} - \gamma \|w(k)\|^2 \right\}, \tag{5.32}$$

$$Q = \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \in \mathbb{S}_+^{m+n},$$

$Q_f \succeq 0$, $Q_{uu} \succ 0$, $x(k) \in \mathbb{R}^n$, $y_i(k) \in \mathbb{R}^{m_i}$, $u_i(k) \in \mathbb{R}^{p_i}$. Now write $x(k)$ and $y(k)$ as

$$x(k) = A^t x(k-t) + \sum_{n=1}^{t} A^{n-1} Bu(k-n) + \sum_{n=1}^{t} A^{n-1} w(k-n),$$

$$y_i(k) = C_i A^t x(k-t) + \sum_{n=1}^{t} C_i A^{n-1} Bu(k-n) + \sum_{n=1}^{t} C_i A^{n-1} w(k-n).$$
$$\tag{5.33}$$

Note that the summation over n is defined to be zero when $t = 0$. The next theorem gives conditions where the signaling incentive is eliminated, and states that under these considions, an optimal decision is linear in the observations:

Theorem 5.2
Consider the optimization problem given by (5.31). The problem has no signaling incentive if

$$\mathbb{I}_j^n \subseteq \mathbb{I}_i^k \text{ for } [C_i A^n B]_j \neq 0 \tag{5.34}$$

for all n such that $0 \leq n < k$, and $k = 0, ..., M-1$.

In addition, an optimal solution to the optimization problem given by (5.31) is linear in the observations $\mathbb{I}_i^k$ if condition (5.34) is satisfied, and has a solution that can be found by solving a linear matrix inequality.

$\square$

Proof Introduce

$$\bar{x} = \begin{pmatrix} w(M-1) \\ w(M-2) \\ \vdots \\ w(0) \\ x(0) \end{pmatrix}, \quad \bar{u}_i = \begin{pmatrix} u_i(M-1) \\ u_i(M-2) \\ \vdots \\ u_i(0) \end{pmatrix}.$$

Then, we can write the cost function $J(x, u)$ as

$$\begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}^T \bar{Q} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix},$$

for some symmetric matrix $\bar{Q}$ with

$$\bar{Q} = \begin{pmatrix} \bar{Q}_{xx} & \bar{Q}_{xu} \\ \bar{Q}_{ux} & \bar{Q}_{uu} \end{pmatrix},$$

and $\bar{Q}_{uu} \succ 0$. Consider the expansion given by (5.33). The problem here is that $y_i(k)$ depends on previous values of the control signals $u(n)$ for $n = 0, ..., k-1$. The components $u_j(n)$ that $y_i(k)$ depends on are completely determined by the structure of the matrix $[C_i A^n B]_j$. This means that, if for every node i we have $\mathbb{I}_j^n \subseteq \mathbb{I}_i^k$ for $[C_i A^n B]_j \neq 0$, then there is no signaling incentive. Thus, we have proved the first statement of the theorem.

Now if condition (5.34) is satisfied, we can form the new output measurement

$$\begin{aligned} \check{y}_i(k) &= y_i(k) - \sum_{n=1}^{k} C_i A^{n-1} B u(k-n) \\ &= A^k x(0) + \sum_{n=1}^{k} C_i A^{n-1} w(k-n). \end{aligned} \tag{5.35}$$

Let

$$\bar{y}_i(k) = \begin{pmatrix} \check{y}_i(k) \\ \check{y}_i(k-1) \\ \vdots \\ \check{y}_i(0) \end{pmatrix}.$$

With these new variables introduced, the optimization problem given by equation (5.31) reduces to the following static team decision problem:

$$\begin{aligned} \inf_{\mu} \sup_{\bar{x}} \quad & \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}^T \bar{Q} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \\ \text{subject to } & u_i(k) = \mu_i(\bar{y}_i(k)), \quad k = 0, ..., M-1 \\ & \text{for } i = 1, ..., N. \end{aligned} \tag{5.36}$$

and the optimal solution $\bar{u}$ is linear according to Theorem 5.1, and can be obtained by solving a linear matrix inequality as described in Section 5.3. □

In fact, using the static team formulation reveals a broad class of information structures that lead to convex problems. It turns out to be *enough to exchange information with the neighbours on the graph*. We illustrate this by an example:

EXAMPLE 5.1
Consider the example presented at the beginning of this section. The dynamics of the second subsystem is given by

$$x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2u_2(k) + w_2(k).$$

If at time $k+1$, subsystem 2 has information about the state of its neighbour $x_1(k)$, then it has knowledge about the value of $w_2(k)$:

$$w_2(k) = x_2(k+1) - A_{21}x_1(k) - A_{22}x_2(k) - B_2u_2(k).$$

Hence, if we restrict the control law $u_2(k+1)$ to be a function of $x_1(k)$, $x_2(k)$, $u_2(k)$ (information about the state of its neighbour and its own state and control input at time step k), and restrict it to be based only on the information about $w_2(k)$, then we can set $u_2(k+1) = \mu_2(w_2(k))$. The same information restriction can be similarly imposed on the other subsystems. Just as before, the dynamic $\mathcal{H}_\infty$ team problem can be reduced to a static team problem (5.36), where $u_i(k+1) = \mu_i(w_i(k))$. This problem has an optimal solution that is linear and can be found by solving a linear matrix inequality. □

6

Optimal Distributed Linear Quadratic Control

In the previous chapters, we considered distributed stochastic and deterministic linear quadratic dynamic team problems over a finite horizon. In this chapter, the problem of optimal distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control is considered over an infinite horizon (the steady state problem). A general control problem setup is given, where constraints on information of the external signals (such as disturbances) are imposed. Necessary and sufficient conditions are given for stabilizability of distributed control problems with delayed measurements. A novel approach to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problem is developed. The approach is based on the crucial idea of disturbance feedback, which transforms a state feedback problem to a feedforward problem. The feedforward problem is transformed to a filtering problem, which is then solved using the methods of Chapter 3. The new approach is applied to find the optimal state feedback control law for information constrained control problems.

6.1 Introduction

Control of dynamical systems with information structures imposed on the decision maker(s) has been very challenging for decision theory researchers. Even in the simple linear quadratic static decision problem, it has been shown that complex nonlinear decisions could outperform any given linear decision (see [53]). Important progress was made for the stochastic static team decision problems in [38] and [42]. These results were later used to solve the one step delay control problem in [48], and a special two player deterministic version was developed in [16] to solve the one step delay $\mathcal{H}_\infty$ control problem. New information structures were explored in [26] for the stochastic linear quadratic finite horizon. These

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were recently explored in [4], [46], and [47] to show that the constrained linear optimal decision problem, for infinite horizon linear quadratic control, can be posed as an infinite dimensional convex optimization problem, given that the system considered is stable. The approach is very similar to the problem of symmetric controllers discussed earlier in [56]. The distributed stochastic linear quadratic team problem was revisited in [22], which generalizes previous results for tractability of the dynamic team decision problem with information constraints. An analog deterministic version of the stochastic team decision problem was solved in the previous section, which showed that for the finite horizon linear quadratic $\mathcal{H}_\infty$ control problem with bounds on the information propagation, the optimal decision is linear and can be found by solving a linear matrix inequality. In [43], the stationary state feedback stochastic linear quadratic control problem was solved using state space formulation and covariance constraints, under the condition that all of the subsystems have a *common past*. With a common past, we mean that all subsystems have information about the *global* state from some time step in the past. The problem was posed as a finite dimensional convex optimization problem. The stationary output feedback version was solved in [44] and [23]. Also, [23] generalizes the result to the finite horizon case with general non-convex quadratic constraints. Other approaches explore homogeneous systems on graphs ([3], [15], [20]). Heterogeneous systems over graphs were considered using approximate methods in [19], [35], [36], and [32].

6.2 Structured Linear Optimal Control

Consider a linear operator $P = P(\mathbf{q}) \in \mathcal{RL}_\alpha$ with state space realization

$$P := \begin{cases} x_{k+1} = Ax_k + Bu_k + w_k \\ \quad z_k = C_1 x_k + D_1 u_k \\ \quad y_k = C_2 x_k \end{cases} \tag{6.1}$$

where $\alpha = 2$ or $\alpha = \infty$. The inputs are u_k, w_k and the outputs are y_k, z_k. Here, y_k is the measured output and z_k is the controlled output. The classical (or centralized) linear quadratic optimal control problem is to find a stabilizing linear optimal control law $u_k = K(\mathbf{q})y_k$ such that a quadratic performance index is minimized with respect to $\{w_k\}$, where $\{w_k\}$ is the disturbance injected in the system. There are many ways of solving the problem above. We choose to approach the problem by using disturbance feedback. The disturbance feedback approach is reminiscent of the Youla parametrization (or Q-parametrization) [57], although, no
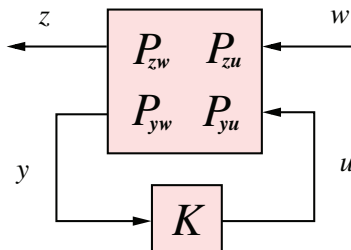
Figure 6.1 Standard feedback setting.

parametrization is used. Define the following linear systems:

$$
P_{zw} := \begin{cases} x^w_{k+1} = Ax^w_k + w_k \\ z^w_k = C_1 x^w_k \end{cases}
$$

$$
P_{zu} := \begin{cases} x^u_{k+1} = Ax^u_k + Bu_k \\ z^u_k = C_1 x^u_k + D_1 u_k \end{cases}
$$

$$
P_{yw} := \begin{cases} x^w_{k+1} = Ax^w_k + w_k \\ y^w_k = C_2 x^w_k \end{cases}
$$

$$
P_{yu} := \begin{cases} x^u_{k+1} = Ax^u_k + Bu_k \\ y^u_k = C_2 x^u_k \end{cases}
$$

(6.2)

Note that $x_k = x^w_k + x^u_k$, $z_k = z^w_k + z^u_k$ and $y_k = y^w_k + y^u_k$. Hence, we have separated the signals x_k, z_k and y_k into two modes; one corresponding to the disturbance w and the other to the controller u. Note that the controller is restricted to be a linear function of the outputs $y_k, y_{k-1}, \dots$. In turn, the outputs will be a linear combination of the disturbance w. To avoid the controller's *dual effect* (the signaling effect), we will let the controller be a function of the output sequence y^w_k of the disturbance process given by P_{yw}. This is done in practice by taking the difference $y_k - y^u_k$, which is possible since we have access to both the process output y_k and the *effect* of the controller output y^u_k (the controller saves the value of x^u_k at every time step, then we obtain $y^u_k = Cx^u_k$). Thus, we will restrict the control law to be $u_k = Q(\mathbf{q})y^w_k$, where $Q \in \mathcal{RL}_\alpha$. Hence, we get that $z = (P_{zw} + P_{zu}QP_{yw})w$, and the optimization problem becomes

$$
\min_Q \ \|P_{zw} + P_{zu}QP_{yw}\|_\alpha
$$

where $\alpha = 2$ or ∞. Compare with figures 6.1 and 6.2. We can put different constraints on the structure of the Q parameter. For instance, we can require that $Q \in \mathbf{S}$, where $\mathbf{S}$ is the set of elements Q which have the
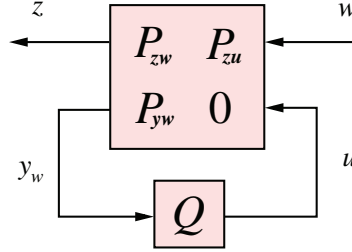
Figure 6.2   Feedback with respect to the mode of the output corresponding to the disturbance.

structure $Q_{ij}(\mathbf{q}) = q_{ij}(\mathbf{q}) f_{ij}(\mathbf{q})$, where $f_{ij}(\mathbf{q})$ is a given operator, and $q_{ij}(\mathbf{q})$ is to be optimized.

6.3 Distributed Control with Delayed Measurements

Consider an example of four dynamically coupled systems according to the graph in Figure 2.1. The equation for the interconnected systems is then given by

$$
\underbrace{\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{pmatrix}}_{x(k+1)} = \underbrace{\begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix}}_{x(k)} +
$$
$$
+ \underbrace{\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ u_4(k) \end{pmatrix}}_{u(k)} + \underbrace{\begin{pmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \end{pmatrix}}_{w(k)} \tag{6.3}
$$

For instance, the arrow from node 2 to node 1 in the graph means that the dynamics of system 2 are directly affected by system 1, which is reflected in the system matrix A where the element $A_{21} \neq 0$. On the other hand, system 2 does not affect system 1 directly, which means that $A_{12} = 0$. Because of the "physical" distance between the subsystems, there will be some constraints on the information available to each node.

Every subsystem i measures at time k its own output

$$y_i(k) = C_i x_i(k).$$

The nodes are allowed to exchange information about their own outputs, that may be subject to some transmission delays. Let $\mathbb{I}_i^k$ denote the set of observations $y_j(n)$ and control signals $u_j(n)$ available to node i up to time k, $n \leq k$, $j = 1, ..., N$. We will start by considering the the problem of full state measurement, that is, $C_i = I$ for all i.

Using a graph theoretic formulation makes it easy to describe how every subsystem is affected by the disturbance and the control signals of the other subsystems. The following result is a special case of Theorem 4.3 and Theorem 5.2, but is formulated and proved differently:

THEOREM 6.1
Consider a linear system given by $x(k+1) = Ax(k) + Bu(k) + w(k)$, with A and B partitioned into blocks according to equation (2.5). The disturbance $w_i(k - t)$ and control signal $B_i u_i(k - t)$ affect the state $x_j(k + 1)$ if and only if $[A^t]_{ji} \neq 0$. In particular, if A is partitioned symmetrically in $N \times N$ blocks, and the block $[A^t]_{ji} = 0$ for $t = 1, ..., N - 1$, then $w_i(k - t)$ and $u_i(k - t)$ never affect $x_j(k + 1)$, for every $t \in \mathbb{N}$ and $t \leq k$.   □

Proof   We can write the state of the whole system as

$$
\begin{aligned}
x(k + 1) &= Ax(k) + Bu(k) + w(k) \\
&= A^{t+1}x(k - t) + (I + A\mathbf{q}^{-1} + \cdots + A^t\mathbf{q}^{-t})(Bu(k) + w(k))
\end{aligned}
\tag{6.4}
$$

Then we see that $(B_i u_i(k - t) + w_i(k - t))$ affects $x_j(k + 1)$ if and only if $[A^t]_{ji} \neq 0$. Also, if $[A^t]_{ji} = 0$ for $t = 1, ..., n - 1$, then $[A^t]_{ji} = 0$ for all $t \in \mathbb{N}$ according to Corollary 2.1. Hence, $w_i(k - t)$ and $u_i(k - t)$ never affect $x_j(k + 1)$, for all $t \in \mathbb{N}$.   □

We will introduce an information structure that will be the *basis* for all information structures that will be treated in this chapter. The main idea is to *put constraints on the information available about the disturbance entering the system, rather than on the state of the system*. The basic information structure that will be required is that, at time step k, every system i has access to information about the disturbance $w_i(t)$, for all $t < k$. This requires that system i has access to its own state up to time k, and to the states of its neighbours on the graph up to time $k - 1$. To show the idea, take the example given by equation (6.3) in the beginning of the section. We can see that if system 1 has access to $x_1(k)$, $x_1(k - 1)$, and $x_3(k - 1)$, then it can build $x_1(k) - A_{11}x_1(k - 1) - A_{13}x_3(k - 1) = w_1(k - 1)$. Then,

the information that every system i transmits is $w_i(k-1)$, which will be received by other nodes with some delay. An information structure that has been studied first in [26], and recently in [4], [47], and [43], is when information propagates at least as fast as the dynamics on the graph, and is called a *partially nested* information structure.

DEFINITION 6.1
We say that a given delayed information structure is partially nested if $\mathbb{I}_j^t \subset \mathbb{I}_i^k$ when $u_j(t)$ affects the information set $\mathbb{I}_i^k$.   □

The definition above of partially nested information structure states that if the decision of system j at time t affects the dynamics of system i at time k, then system i has access to the information of system j up to time step t.

6.4 Stabilizability

We will consider the partially nested information structure (see Definition 6.1) to explain how to analyze stabilizability with respect to disturbance feedback. Consider again the example given by (6.3). It can be written as

$$x(k+1) = \begin{pmatrix} \bar{A} & \star \\ 0 & A_{44} \end{pmatrix} x(k) + Bu(k) + w(k)$$

where $\bar{A}$ is the upper left block of A, that is

$$\bar{A} = \begin{pmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}.$$

Then we can see (using Theorem 6.1 for instance) that the disturbances $w_1(t), w_2(t), w_3(t)$ and control signals $u_1(t), u_2(t), u_3(t)$ never affect $x_4(k)$, while they *all* affect $x_1(k), x_2(k), x_3(k)$ for $t \le k-3$. Also, $w_4(t)$ and $u_4(t)$ affect system 1, 2 and 3 for all $t \le k-4$. Then, we require that system 1, 2, and 3 has access to $x(t)$ and $u(t-1)$ for $t \le k-3$. Let $\mathbb{I}_{k-3}$ denote the set of information containing $(x(t), u(t-1))$ for $t \le k-3$. Then, $\mathbb{I}_{k-3} \subset \mathbb{I}_1^{k-3}, \mathbb{I}_2^{k-3}, \mathbb{I}_3^{k-3}$, that is, system 1, 2, and 3 have a *common past*. Also, system 4 requires that $u_4(t)$ and $w_4(t)$ are available at time k for all $t \le k-1$, that is $u_4(t-1), x_4(t) \in \mathbb{I}_4^k$ for $t \le k-1$. We will give conditions on the system parameters in order for a controller with the delayed information structure above to exist. First we need:

THEOREM 6.2

Consider a linear system with N-step delayed output measurement

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ \quad y_k = Cx_{k-N}. \end{cases} \tag{6.5}$$

The following two statements are equivalent:

(i) (A, B) is stabilizable and (C, A) is detectable.

(ii) System (6.5) can be stabilized by output feedback.

$\square$

Proof $(i) \Rightarrow (ii)$

Introduce

$$z_k = \begin{pmatrix} x_k \\ y_{k-1} \\ \vdots \\ y_{k-N} \end{pmatrix}.$$

We can write system (6.5) as

$$\begin{cases} z_{k+1} = \underbrace{\begin{pmatrix} A & 0 & \cdots & 0 & 0 \\ C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}}_{A_z} z_k + \underbrace{\begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{B_z} u_k \\[1em] y_k = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & I \end{pmatrix}}_{N+1 \text{ blocks}} z_k = C_z z_k \end{cases} \tag{6.6}$$

The matrix $\begin{pmatrix} \lambda I - A_z & B_z \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C} \backslash \mathbb{D}$ since (A, B) is stabilizable, and

$$\begin{pmatrix} \lambda I - A_z \\ C_z \end{pmatrix}$$

has full column rank for all $\lambda \in \mathbb{C} \backslash \mathbb{D}$, since (C, A) is detectable. We deduce that system (6.6) is stabilizable and detectable, which implies that a stabilizing output feedback exists.

$(ii) \Rightarrow (i)$

The row rank of $\begin{pmatrix} \lambda I - A_z & B_z \end{pmatrix}$ decreases if and only if the row rank of $\begin{pmatrix} \lambda I - A & B \end{pmatrix}$ decreases, for $\lambda \in \mathbb{C} \backslash \mathbb{D}$. Hence stabilizability of (A_z, B_z) implies stabilizability of (A, B). A similar argument shows that detectability of (C_z, A_z) implies detectability of (C, A), and we are done. $\quad\square$

According to Proposition 2.2, we can write the A matrix for a system in a block triangular form as in (2.2) after a suitable permutation of its blocks. The graphs of the diagonal blocks are then strongly connected (the adjacency matrices of the diagonal blocks are irreducible). Now we are ready to state:

THEOREM 6.3
Let P be a linear system with state space realization

$$P := \begin{cases} x(k+1) = Ax(k) + Bu(k) + w(k) \\ \qquad y(k) = Cx(k) \end{cases}$$

where

$$A = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1j} \\ 0 & A_2 & \cdots & A_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{pmatrix},$$

and

$$C = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_r \end{pmatrix}.$$

Then, under a partially nested information structure, the system P is stabilizable by linear controllers if and only if (A_j, B_j) is stabilizable and (C_j, A_j) is detectable, for $j = 1, ..., r$. $\quad\square$

Proof Since we are considering linear controllers, we will let every sub-controller $u_i(k)$ be a sum of linear functions of the disturbance corresponding to each block of the system, that is $u_i(k) = \sum_{j=1}^{r} u_i^j(k)$, where

$$u_i^j(k) = \mu_i^j(\bar{w}_j(k - d_{ij})),$$
$$\bar{w}_j(k - d_{ij}) = (w_j(k - d_{ij}), w_j(k - d_{ij} - 1), \ldots),$$

and d_{ij} is the time it takes for w_j to affect x_i. Then, we can write $x(k)$ as the sum of r modes, $x(k) = \sum_{j=1}^{r} x^j(k)$, where x^j is the state of system $G_j(\mathbf{q})$ with output $y^j = Cx^j$, and the dynamics of x^j are given by

$$
x^j(k+1) = \underbrace{\begin{pmatrix} A_1 & A_{12} & \cdots & A_{1j} & \cdots & 0 \\ 0 & A_2 & \cdots & A_{2j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}}_{F_j} x^j(k) +
$$

$$
\quad + \underbrace{\begin{pmatrix} B_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}}_{G_j} u^j(k) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ w_j(k) \\ \vdots \\ 0 \end{pmatrix}
\tag{6.7}
$$

The partially nested information structure implies that the controllers $u_i(k)$, $i = 1, \ldots, j$, have common information about $\bar{w}_j(k - d_j - 1)$, for some d_j. Equivalently, $u^j(k)$ has information about $x^j(k - d_j)$, $x^j(k - d_j - 1)$, $\ldots$ But then, controlling the system G_j is a d_j-step delay problem. Now we will prove the statement of the theorem by induction over j. For $j = 1$, G_1 is stabilizable if and only if (A_1, B_1) is stabilizable and (C_1, A_1) is detectable. Assume that (A_i, B_i) is stabilizable and (C_i, A_i) is detectable for $i = 1, \ldots, j - 1$. Then, G_j is stabilizable only if (A_j, B_j) is stabilizable and (C_j, A_j) is detectable. On the other hand, if (A_j, B_j) is stabilizable and (C_j, A_j) is detectable, then G_j is stabilizable, since (A_i, B_i) is stabilizable and (C_i, A_i) is detectable for $i = 1, \ldots, j - 1$ by the induction hypothesis, and the proof is complete. $\qquad\square$

6.5 Stabilizing Controllers

We will now show how to find a stabilizing controller for the control problem with N-step delayed measurements. Consider the system

$$P := \begin{cases} x_{k+1} = Ax_k + Bu_k + w_k \\ \quad z_k = C_1 x_k + D_1 u_k \\ \quad y_k = C_2 x_k, \end{cases}$$

where (A, B) is stabilizable and (C_2, A) is detectable. Writing the system above as a transfer function, we get

$$\begin{pmatrix} z_k \\ y_k \end{pmatrix} = P \begin{pmatrix} w_k \\ u_k \end{pmatrix} = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} \begin{pmatrix} w_k \\ u_k \end{pmatrix}.$$

We will now construct a stabilizing controller $K(\mathbf{q})$ based on the delayed measurement $y_{k-N} = C_2 x_{k-N}$, that is, $u_k = K(\mathbf{q})\mathbf{q}^{-N} y_k$. To design a controller based on the delayed measurement y_{k-N}, we need to include the delayed states, and then write a modified version of P with output y_{k-N} instead of y_k:

$$\begin{cases} \bar{x}_{k+1} = \underbrace{\begin{pmatrix} A & 0 & \cdots & 0 & 0 \\ C_2 & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}}_{A_d} \bar{x}_k + \underbrace{\begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{B_d} u_k + \begin{pmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} w_k \\ y_{k-N} = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & I \end{pmatrix}}_{N+1 \text{ blocks}} \bar{x}_k \end{cases}$$

Obviously, this is just a standard linear quadratic control problem that can be readily solved using standard tools. Closing the loop with the stabilizing controller yields a closed-loop system

$$\begin{pmatrix} z_k \\ y_k \end{pmatrix} = T \begin{pmatrix} w_k \\ u_k \end{pmatrix} = \begin{pmatrix} T_{zw} & T_{zu} \\ T_{yw} & T_{yu} \end{pmatrix} \begin{pmatrix} w_k \\ u_k \end{pmatrix},$$

with T stable. Then, all stabilizing controllers can be parametrized by $Q(\mathbf{q}) \in \mathbf{S}$: $T_{zw} + T_{zu}QT_{yw} \in \mathcal{RH}_\alpha$.

6.6 A New Approach to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ State Feedback

Consider the problem of state feedback control of the system

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix}$$

where

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} = \left(\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array} \right).$$

The aim is to minimize the $\mathcal{H}_\alpha$ norm of the linear operator from the disturbance w to the performance index z, for $\alpha = 2$ or $\alpha = \infty$. As discussed in section 6.3, we will consider the problem of feedback with respect to the disturbance. Since we consider constraints on the disturbance, we will also restrict the controller to be a function of the disturbance, that is, $u_k = -R(\mathbf{q})\mathbf{q}^{-1}w_k$. This can be compared to the control with respect to the disturbance driven output discussed in section 6.2. Since we also have that $u_k = QP_{yw}w_k = Q(\mathbf{q}I - A)^{-1}w_k = Q(I - A\mathbf{q}^{-1})^{-1}\mathbf{q}^{-1}w_k$, the relation between the filters Q and R is obviously given by $R = -Q(I - A\mathbf{q}^{-1})^{-1}$. The optimization problem we will be considering is then:

$$\min_{R \in \mathbf{S}} \; \|P_{zw} - P_{zu}R\mathbf{q}^{-1}\|_\alpha$$

which is a feedforward problem. When we obtain the optimal R, we can recover Q by setting $Q = -R(I - A\mathbf{q}^{-1})$. Note that minimizing $\|P_{zw} - P_{zu}R\mathbf{q}^{-1}\|_\alpha$ with respect to R is equivalent to minimizing its transpose $\|P_{zw}^T - R^T P_{zu}^T \mathbf{q}^{-1}\|_\alpha$, with respect to R^T. But minimizing $\|P_{zw}^T - R^T P_{zu}^T \mathbf{q}^{-1}\|_\alpha$ with respect to R^T is a filtering problem, see Figure 6.3. Now

$$P_{zw}^T = \left(\begin{array}{c|c} A^T & C^T \\ \hline I & 0 \end{array} \right),$$

and

$$P_{zu}^T = \left(\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right).$$

Clearly, the filtering problem is to find an optimal filter R^T that finds the optimal state estimate of P_{zu}^T, with respect to its output delayed with one time step. An optimal filter is the Kalman filter, for both $\alpha = 2$ and $\alpha = \infty$, see Section 3.2. Introduce
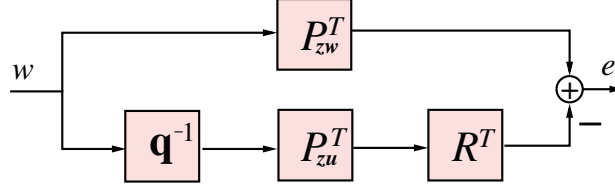
Figure 6.3   The filtering problem obtained from the feedforward problem.

$$
\begin{pmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{pmatrix} = \begin{pmatrix} C^T \\ D^T \end{pmatrix} \begin{pmatrix} C^T \\ D^T \end{pmatrix}^T
$$

$$
= \begin{pmatrix} C^T C & C^T D \\ D^T C & D^T D \end{pmatrix}.
$$

Then, the Kalman filter R^T has the state space realization

$$
R^T = \left(\begin{array}{c|c} A^T - K^T B^T & K^T \\ \hline I & 0 \end{array} \right),
$$

where

$$
K^T = (A^T S A + W_{12})(B^T S B + W_2)^{-1},
$$

and S is the symmetric and positive definite solution to the Riccati equation

$$
S = A^T S A + W_1 - (A^T S B + W_{12})(B^T S B + W_2)^{-1}(A^T S B + W_{12})^T.
$$

(6.8)

This gives the optimal filter R:

$$
R = \left(\begin{array}{c|c} A - BK & I \\ \hline K & 0 \end{array} \right).
$$

Then, the optimal control law is given by

$$
u = -R\mathbf{q}^{-1}w
$$

$$
= -R(I - A\mathbf{q}^{-1})y_w
$$

$$
= -R(I - A\mathbf{q}^{-1})(y - P_{yu}u).
$$

Noting that the state space realization of the transfer function P_{ywu} from $\begin{pmatrix} w \\ u \end{pmatrix}$ to the output y is given by

$$
P_{ywu} := \left\{ \begin{array}{l} x_{k+1} = Ax_k + Bu_k + w_k \\ \quad\; y_k = x_k, \end{array} \right.
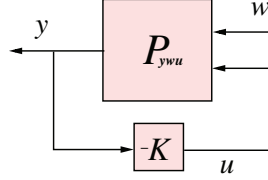$$

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Figure 6.4 The closed loop of P_{yuw} with the control law $u = R\mathbf{q}^{-1}w$.

we can see that $u = -Kx$ yields a feedback with respect to the disturbance which is exactly $u = -R\mathbf{q}^{-1}w$. Hence, we conclude that $u = -R\mathbf{q}^{-1}w$ is equivalent to the state feedback law $u = -Kx$ (see Figure 6.4). We have arrived at:

THEOREM 6.4
Consider the system

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix}.$$

where

$$\begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} = \left(\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array} \right).$$

with state x. Assume that (A, B) is stabilizable. Then, the optimal controller $u = Q(y - P_{yu}u)$ that minimizes the norm $\|P_{zw} + P_{zu}QP_{yw}\|_\alpha$ for $\alpha = 2$ or $\alpha = \infty$, is obtained with $Q = -R(I - A\mathbf{q}^{-1})$, where

$$R = \left(\begin{array}{c|c} A - BK & I \\ \hline K & 0 \end{array} \right),$$

and K^T is given by

$$K^T = (A^T SA + C^T D)(C^T SC + D^T D)^{-1},$$

with S as the symmetric positive definite solution to the Riccati equation

$$S = A^T SA + C^T C - (A^T SB + C^T D)(B^T SB + D^T D)^{-1}(A^T SB + C^T D)^T. \tag{6.9}$$

Furthermore, the optimal controller $u = QP_{yw}w$ is equivalent to the state feedback law $u = Kx$. $\qquad\square$

It is striking that the optimal control law coincide for both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ state feedback control problem. The solution shows also that duality between state feedback control and filtering holds under the $\mathcal{H}_\infty$ setting, just like the duality in $\mathcal{H}_2$ control and filtering.

6.7 Optimal Distributed State Feedback Control

Consider again the system

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix},$$

where

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} = \left(\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array} \right),$$

and

$$B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_N \end{pmatrix}.$$

Let $\mathbf{S}$ be the set of all transfer matrices $F \in \mathcal{RL}_\alpha$ that have the same *delay structure* as $P_{yw}(\mathbf{q})$. That is, $F_{ij}(\mathbf{q}) = F_{ij}(\mathbf{q})\mathbf{q}^{-\tau_{ij}}$ if $[P_{yw}(\mathbf{q})]_{ij}$ has a factor equal to $\mathbf{q}^{-\tau_{ij}}$. The aim is to find a controller $u = Qy_w = Q(y - y_u) = Q(y - P_{yu}u) = -R\mathbf{q}^{-1}w$ that minimizes $\|P_{zw} + P_{zu}QP_{yw}\|_\alpha^2$, $\alpha = 2$ or $\alpha = \infty$, $Q(\mathbf{q}) \in \mathbf{S}$. As in the previous section, the control problem can be written as a feedforward problem:

$$\min_{R \in \mathbf{S}} \|P_{zw} - P_{zu}R\mathbf{q}^{-1}\|_\alpha$$

where

$$\mathbf{S} = \left\{ F : f_{ij} \in \mathcal{RL}_\alpha, F_{ij} = f_{ij}\mathbf{q}^{-\tau_{ij}} \right\}.$$

By taking the transpose, the problem above can be transformed to a distributed filtering problem that can be solved as in Section 3.4. Then, the optimal Q can be recovered by setting $Q = -R(I - A\mathbf{q}^{-1})$. We summarize the discussion above:

THEOREM 6.5
Consider the linear system

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix},$$

where

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} = \left(\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array} \right),$$

$$B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_N \end{pmatrix}.$$

Assume that P is stabilizable under partially nested information structure, and let $P_{yw} \in \mathbf{S}$, where

$$\mathbf{S} = \{ F : f_{ij} \in \mathcal{RL}_\alpha, F_{ij} = f_{ij}\mathbf{q}^{-\tau_{ij}} \}.$$

Then, the optimal control law is given by $u = -R(I - A\mathbf{q}^{-1})(y - P_{yu}u)$, where R^T is the optimal distributed filter, obtained using Theorem 3.5, that solves

$$\min_{R \in \mathbf{S}} \| P_{zw}^T - R^T P_{zu}^T \mathbf{q}^{-1} \|_\alpha,$$

for $\alpha = 2$ or $\alpha = \infty$. $\qquad\qquad\square$

Discussion on the Optimal Distributed Controller Structure

We will now present the equations for the optimal distributed controllers obtained in Theorem 6.5.

Recall that the optimal distributed state feedback control problem is transformed to an optimal distributed filtering problem

$$\min_{R \in \mathbf{S}} \| P_{zw}^T - R^T P_{zu}^T \mathbf{q}^{-1} \|_\alpha,$$

for $\alpha = 2$ or $\alpha = \infty$. Set $H = P_{zw}^T$, $G = P_{zu}^T$, and $F = R^T$. Then, Theorem 3.5 gives that the optimal filter F is given by

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix},$$

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where F_i has the state space realization for $i = 1, ..., N$:

$$\left(\begin{array}{c|c} A_e - K_i E_i & K_i \\ \hline \Gamma_i & 0 \end{array} \right) , \qquad (6.10)$$

with

$$\Gamma_i = \left(\begin{array}{ccccccc} 0 & \cdots & 0 & I & 0 & \cdots & 0, \end{array} \right)$$

where the identity matrix I in Γ_i is in block position i,

$$K_i = (A_e S_i E_i^T + B_e^T D_i)(E_i S_i E_i^T + D_i^T D_i)^{-1},$$

and S_i is the symmetric positive definite solution to the Riccati equation

$$S_i = A_e S_i A_e^T + B_e B_e^T - \\ - (A_e S_i E_i^T + B_e^T D_i)(E_i S_i E_i^T + D_i^T D_i)^{-1}(A_e S_i E_i^T + B_e^T D_i)^T.$$

Since $R^T = F$, we get

$$R = \left(\begin{array}{cccc} F_1^T & F_2^T & \cdots & F_N^T \end{array} \right) .$$

Let

$$w = \left(\begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_N \end{array} \right) .$$

Then

$$u = -R\mathbf{q}^{-1}w$$
$$= -\sum_{i=1}^{N} F_i^T \mathbf{q}^{-1} w_i.$$

We can see that the controller can be written as the sum of N controllers, $u = \sum_{i=1}^{N} u_i$, with $u_i = -F_i^T \mathbf{q}^{-1} w_i$ as the the feedback law with respect to the disturbance w_i entering system i. Taking the transpose of (6.10) gives the state space realization of F_i^T:

$$\left(\begin{array}{c|c} A_e^T - E_i^T K_i^T & \Gamma_i^T \\ \hline K_i^T & 0 \end{array} \right) . \qquad (6.11)$$

Let

$$\Sigma_i := \quad z_i(k+1) = A_e^T z_i(k) + E_i^T u_i(k) + \Gamma_i w_i(k).$$

It is easy to verify that $u_i(k) = -K_i^T z_i(k)$ and $u_i(k) = -F_i^T \mathbf{q}^{-1} w_i(k)$ are equivalent. Hence, the optimal distributed controller $u_i = -F_i^T \mathbf{q}^{-1} w_i$ is equivalent to the state feedback controller, with respect to the mode generated by w_i, for $i = 1, 2, ..., N$.

7

Generalized Linear
Quadratic Control

This chapter considers the problem of stochastic finite and infinite horizon linear quadratic control under power constraints. Problems such as linear quadratic optimal control with information constraints are special cases of the problem considered. The calculations of the optimal control law can be done off-line as in the classical linear quadratic Gaussian control theory using dynamic programming, which turns out to be a special case of the new theory developed in this chapter. A numerical example is solved using the new methods.

7.1 Introduction

In this chapter we consider the problem of linear quadratic control with power constraints. Power constraints are very common in control problems. For instance, we often have some limitations on the control signal, which we can express as $\mathbf{E}\, u^T u \leq \gamma$. Also, Gaussian channel capacity limitation can be modeled through power constraints. There has been much work on control under power constraints, see [39], [12], [45], [55]. In [43], it was shown how to use power constraints for distributed state feedback control. What is common to previously published papers is that they solve the stationary state feedback infinite horizon case using convex optimization. Output feedback is only discussed in [45], where the quadratic (power) constraints are restricted to be convex.

The aim of this chapter is to give a complete solution to the *non-stationary* and *finite horizon* problem for linear systems, including *time varying*, with power constraints. The solution is obtained using dynamic programming. A solution of the infinite horizon linear quadratic control problem is derived from the finite horizon results. Also, the output feed-

back problem with *non-convex* quadratic constraints is solved.

The outline of the chapter is as follows. In Section 7.2, we introduce a novel approach for solving the state-feedback linear quadratic control problem. Relations to the classical approach is discussed in Section 7.3. The new approach is then used in Section 7.4 to give the main result of the chapter, the finite horizon state-feedback linear quadratic control with *power constraints*. We show how a solution to the constrained infinite horizon control problem can be derived in Section 7.5. The constrained output feedback control problem is solved in Section 7.6. A numerical example is given in Section 7.7.

7.2 Optimal State Feedback Control through Duality

In this section, we will derive a state feedback solution to the classical linear quadratic control problem using duality. This method will be used to solve the problem of linear quadratic optimal control with power constraints. Consider the linear quadratic stochastic control problem

$$
\begin{aligned}
\min_{\mu_k} \ & \mathbf{E} \, x^T(N) Q_{xx} x(N) + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\
\text{subject to} \ & x(k+1) = Ax(k) + Bu(k) + w(k) \\
& \mathbf{E} \, w(k) x^T(l) = 0, \forall l \le k \\
& \mathbf{E} \, w(k) w^T(k) = V_{ww}(k) \\
& u(k) = \mu_k(x(0), ..., x(k))
\end{aligned}
\tag{7.1}
$$

where $Q \succ 0$ and it is partitioned according to the dimensions of x and u as

$$
Q = \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{xu}^T & Q_{uu} \end{pmatrix} .
$$

Without loss of generality, we assume that

$$
\begin{aligned}
\mathbf{E} \, x(0) &= \mathbf{E} \, w(k) = 0 \\
\mathbf{E} \, x(0) x^T(0) &= V_{ww}(k) = I
\end{aligned}
$$

The quadratic cost in (7.1) can be written as

$$\mathbf{E}\left\{\mathbf{Tr}\ Q_{xx}x(N)x^T(N)\right\} + \sum_{k=0}^{N-1}\mathbf{E}\left\{\mathbf{Tr}\ Q\begin{pmatrix}x(k)\\u(k)\end{pmatrix}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}^T\right\} =$$

$$= \mathbf{Tr}\ Q_{xx}\{\mathbf{E}\ x(N)x^T(N)\} + \sum_{k=0}^{N-1}\mathbf{Tr}\ Q\left\{\mathbf{E}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}^T\right\}$$

$$= \mathbf{Tr}\ Q_{xx}V_{xx}(N) + \sum_{k=0}^{N-1}\mathbf{Tr}\ QV(k),$$

where

$$V(k) = \begin{pmatrix}V_{xx}(k) & V_{xu}(k)\\V_{xu}^T(k) & V_{uu}(k)\end{pmatrix} = \mathbf{E}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}^T.$$

Let $F \in \mathbb{R}^{n\times(m+n)}$ be

$$F = \begin{pmatrix}I & 0\end{pmatrix}.$$

Then,

$$V_{xx}(k) = FV(k)F^T.$$

The system dynamics implicate the following recursive equation for the covariance matrices $V(k)$

$$FV(k+1)F^T = V_{xx}(k+1) = \mathbf{E}\ x(k+1)x^T(k+1) =$$

$$= \mathbf{E}\ (Ax(k) + Bu(k) + w(k))(Ax(k) + Bu(k) + w(k))^T$$

$$= \mathbf{E}\left\{\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}\begin{pmatrix}x(k)\\u(k)\end{pmatrix}^T\begin{pmatrix}A & B\end{pmatrix}^T + w(k)w^T(k)\right\}$$

$$= \begin{pmatrix}A & B\end{pmatrix}V(k)\begin{pmatrix}A & B\end{pmatrix}^T + I.$$

The initial condition $V_{xx}(0) = \mathbf{E}\ x(0)x^T(0) = I$ can be written as

$$FV(0)F^T = I.$$

We summarize the discussion above:

PROPOSITION 7.1
The linear quadratic problem (7.1) is equivalent to the covariance selection
problem

$$
\min_{V(0),...,V(N)} \mathbf{Tr}\, Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \mathbf{Tr}\, QV(k)
$$

$$
\text{subject to } V(k) \succeq 0, \quad FV(0)F^T = I
$$

$$
\begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T + I = FV(k+1)F^T
$$

(7.2)

In particular, it is convex in $V(0), ..., V(N)$. $\square$

The dual problem of (7.2) is given by

$$
\max_{S(0),...,S(N)} \min_{V(0),...,V(N)} J(V(0), ..., V(N), S(0), ..., S(N))
$$

where

$$
J(V(0), ..., V(N), S(0), ..., S(N)) =
$$

$$
\mathbf{Tr}\, Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \mathbf{Tr}\, QV(k) + \mathbf{Tr}\, \left\{ S(0)(I - FV(0)F^T) \right\} +
$$

$$
\sum_{k=0}^{N-1} \mathbf{Tr}\, \left\{ S(k+1) \left(\begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T + I - FV(k+1)F^T \right) \right\}
$$

$$
= \mathbf{Tr}\, Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \mathbf{Tr}\, QV(k) + \mathbf{Tr}\, S(0) - \mathbf{Tr}\, \left\{ S(0)FV(0)F^T \right\}
$$

$$
+ \sum_{k=0}^{N-1} \mathbf{Tr}\, S(k+1) + \sum_{k=0}^{N-1} \mathbf{Tr}\, \left\{ S(k+1) \begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T \right\}
$$

$$
- \sum_{k=0}^{N-1} \mathbf{Tr}\, \left\{ S(k+1)FV(k+1)F^T \right\}
$$

$$
= \mathbf{Tr}\, Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \mathbf{Tr}\, QV(k) - \mathbf{Tr}\, \left\{ S(N)FV(N)F^T \right\} + \mathbf{Tr}\, S(0) +
$$

$$
+ \sum_{k=0}^{N-1} \mathbf{Tr}\, S(k+1) + \sum_{k=0}^{N-1} \mathbf{Tr}\, \left\{ S(k+1) \begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T \right\} -
$$

$$
- \sum_{k=0}^{N-1} \mathbf{Tr}\, \left\{ S(k)FV(k)F^T \right\},
$$

(7.3)

and $S(0), ..., S(N) \in \mathbb{S}^n$ are the Lagrange multipliers. Thus, the dual problem can be written as

$$\max_{S(0),...,S(N)} \min_{V(0),...,V(N)} H(N) + \sum_{k=0}^{N-1} \{H(k) + \mathbf{Tr}\ S(k+1)\} + \mathbf{Tr}\ S(0) \quad (7.4)$$

where

$$H(N) = \mathbf{Tr}\ \{Q_{xx} V_{xx}(N) - S(N)V_{xx}(N)\}$$
$$= \mathbf{Tr}\ \{[Q_{xx} - S(N)]V_{xx}(N)\}$$

and

$$H(k) =$$

$$= \mathbf{Tr}\ \left\{ QV(k) + S(k+1) \begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T - S(k)FV(k)F^T \right\}$$

$$= \mathbf{Tr}\ \left\{ \left(Q + \begin{pmatrix} A & B \end{pmatrix}^T S(k+1) \begin{pmatrix} A & B \end{pmatrix} - F^T S(k)F \right) V(k) \right\}$$

$$\quad (7.5)$$

for $k = 0, ..., N-1$. Here, $H(k)$ plays the role of the *Hamiltonian* of the system. The duality gap between (7.2) and (7.3) is zero, since Slater's condition is satisfied for the primal (and dual) problem (see [11] for a reference on Slater's condition). Now for the optimal selection of the dual variables $S(k)$, we must have

$$Q_{xx} - S(N) \succeq 0,$$

and

$$Q + \begin{pmatrix} A & B \end{pmatrix}^T S(k+1) \begin{pmatrix} A & B \end{pmatrix} - F^T S(k)F \succeq 0, \quad (7.6)$$

because otherwise, the value of the cost function in (7.3) becomes infinite. In order for the dual variables to maximize the cost in (7.3), $S(N)$ is chosen equal to Q_{xx}

$$S(N) = Q_{xx}, \quad (7.7)$$

and $S(k)$ is chosen to maximize $\mathbf{Tr}\ S(k)$ subject to the constraint (7.6). Now

$$F^T S(k)F = \begin{pmatrix} I \\ 0 \end{pmatrix} S(k) \begin{pmatrix} I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} S(k) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$Q + \begin{pmatrix} A & B \end{pmatrix}^T S(k+1) \begin{pmatrix} A & B \end{pmatrix} =$$

$$= \begin{pmatrix} A^T S(k+1)A + Q_{xx} & A^T S(k+1)B + Q_{xu} \\ B^T S(k+1)A + Q_{xu}^T & B^T S(k+1)B + Q_{uu} \end{pmatrix}.$$

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For every k, let the matrix $L(k)$ be a solution to the equation

$$(B^T S(k+1)B + Q_{uu})L(k) = B^T S(k+1)A + Q_{xu}^T. \qquad (7.8)$$

Also, let $R(k)$ be

$$R(k) = L(k)^T (B^T S(k+1)B + Q_{uu})L(k). \qquad (7.9)$$

Then, the matrix $S(k)$ with

$$S(k) = A^T S(k+1)A + Q_{xx} - R(k) \qquad (7.10)$$

fulfills (7.6) or equivalently fulfills the inequality

$$Q + \begin{pmatrix} A^T S(k+1)A - S(k) & A^T S(k+1)B \\ B^T S(k+1)A & B^T S(k+1)B \end{pmatrix} \succeq 0, \qquad (7.11)$$

and any other matrix P with $\mathbf{Tr}\,P > \mathbf{Tr}\,S(k)$ violates the inequality in (7.11). This is seen by taking the Schur complement of the matrix in the right hand side of (7.11) to obtain an inequality equivalent to (7.11):

$$\begin{pmatrix} A^T S(k+1)A + Q_{xx} - R(k) - S(k) & 0 \\ 0 & B^T S(k+1)B + Q_{uu} \end{pmatrix} \succeq 0 \;\; (7.12)$$

Since

$$A^T S(k+1)A + Q_{xx} - R(k) - S(k) \succeq 0,$$

we must have

$$\mathbf{Tr}\,\left\{ A^T S(k+1)A + Q_{xx} - R(k) - S(k) \right\} \geq 0,$$

which is equivalent to

$$\mathbf{Tr}\,\left\{ A^T S(k+1)A + Q_{xx} - R(k) \right\} \geq \mathbf{Tr}\,S(k).$$

If we take $S(k) = P$ with P such that

$$\mathbf{Tr}\,\left\{ A^T S(k+1)A + Q_{xx} - R(k) \right\} < \mathbf{Tr}\,P,$$

then the inequality (7.12) is not satisfied. Hence, the choice of $S(k)$ given by (7.10) is optimal.

THEOREM 7.1
The dual problem of (7.1) and (7.2) is given by

$$\max_{S(k)\in\mathbb{S}^n} \sum_{k=0}^{N-1} \mathbf{Tr}\, S(k)$$

$$\text{subject to}\quad Q + \begin{pmatrix} A^T S(k+1)A - S(k) & A^T S(k+1)B \\ B^T S(k+1)A & B^T S(k+1)B \end{pmatrix} \succeq 0 \tag{7.13}$$

$$k = 0, ..., N-1.$$

The problem (7.13) can be solved dynamically by sequentially solving

$$\max_{S(k)\in\mathbb{S}^n} \mathbf{Tr}\, S(k)$$

$$\text{subject to}\quad Q + \begin{pmatrix} A^T S(k+1)A - S(k) & A^T S(k+1)B \\ B^T S(k+1)A & B^T S(k+1)B \end{pmatrix} \succeq 0 \tag{7.14}$$

for $k = N-1, ..., 0$, with $S(N+1) = 0$. The optimal solution is given by equations (7.7)-(7.10). $\qquad\square$

With this optimal choice of the multipliers $S(0), ..., S(N)$, the dual problem (7.3) becomes

$$\min_{V(0),...,V(N-1)} \sum_{k=0}^{N-1} \mathbf{Tr}\, Z(k)V(k) + \sum_{k=0}^{N} \mathbf{Tr}\, S(k)$$

$$\text{subject to}\quad FV(0)F^T - I = 0$$

$$V(k) \succeq 0$$

where

$$Z(k) = \begin{pmatrix} R(k) & A^T S(k+1)B + Q_{xu} \\ B^T S(k+1)A + Q_{xu}^T & B^T S(k+1)B + Q_{uu} \end{pmatrix},$$

and $R(k)$ is given by (7.9). The matrix $Z(k)$ is of the form

$$Z = \begin{pmatrix} XY^{-1}X^T & X \\ X^T & Y \end{pmatrix},$$

where

$$X = A^T S(k+1)B + Q_{xu},$$

and

$$Y = B^T S(k+1)B + Q_{uu}.$$

Note that the matrix Y is invertible since $Q_{uu} \succ 0$ and $S(k+1) \succeq 0$. The matrix V given by

$$V = \begin{pmatrix} I & -XY^{-1} \\ -Y^{-1}X^T & Y^{-1}X^TXY^{-1} \end{pmatrix}$$

is such that $\mathbf{Tr}\, ZV = 0$. In general, for any given matrix $V_{xx} \succeq 0$, we can choose V as

$$V = \begin{pmatrix} V_{xx} & -V_{xx}XY^{-1} \\ -Y^{-1}X^TV_{xx} & Y^{-1}X^TV_{xx}XY^{-1} \end{pmatrix}$$

and we get $\mathbf{Tr}\, ZV = 0$. Therefore, we see that the minimizing covariances are given by

$$
\begin{aligned}
V_{xx}(0) &= I \\
V_{ux}(k) &= -L(k)V_{xx}(k) \\
V_{uu}(k) &= V_{ux}(k)V_{xx}^{-1}(k)V_{xu}(k) \\
V_{xx}(k+1) &= \begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T + I,
\end{aligned}
\tag{7.15}
$$

and the optimal cost is given by

$$\sum_{k=0}^{N} \mathbf{Tr}\, S(k).$$

Now that we have found the covariances, it is easy to see that the optimal control law is

$$
\begin{aligned}
u(k) &= V_{ux}(k)V_{xx}^{-1}(k)x(k) \\
&= -L(k)x(k)
\end{aligned}
\tag{7.16}
$$

where $L(k)$ is, as before, given by (7.8).

THEOREM 7.2
The optimal solution of the covariance selection problem (7.2) is given by the equations in (7.15). The corresponding optimal control law is given by (7.16). ☐

Note that we could have assigned covariance matrices other than the identity matrix for the initial value of the state $x(0)$ and the disturbances $w(k)$, and the solution would be similar to the case treated.

We could also have treated a time-varying system with time-varying quadratic cost functions. The only change is that we replace Q by $Q(k)$, A by $A(k)$, etc.

7.3 Relations to Classical Linear Quadratic Control

The covariance selection method developed in the previous Section is very closely related to the classical method of calculating the optimal state-feedback control law. Consider the dual variable $S(k)$. At each time-step k, $S(k)$ was chosen to be

$$S(k) = A^T S(k+1)A + Q_{xx} - L^T(k)(B^T S(k+1)B + Q_{uu})L(k)$$

with $L(k)$ such that

$$(B^T S(k+1)B + Q_{uu})L(k) = B^T S(k+1)A + Q_{xu}^T.$$

This value of $S(k)$ is exactly the quadratic matrix for the cost to go function from time-step k to N, given by $x^T(k)S(k)x(k)$ (see [1]).

Now we will take a closer look at the optimal cost. In the previous section, we obtained the cost

$$\sum_{k=0}^{N} \mathbf{Tr}\, S(k).$$

In general, when $\mathbf{E}\, x(0)x^T(0) = V_{xx}(0)$ and $\mathbf{E}\, w(k)w^T(k) = V_{ww}(k)$, it turns out that the cost becomes

$$\mathbf{Tr}\, S(0)V_{xx}(0) + \sum_{k=1}^{N} \mathbf{Tr}\, S(k)V_{ww}(k).$$

For $\mathbf{E}\, x(0)x^T(0) = I$ and $\mathbf{E}\, w(k)w^T(k) = I$, we get the cost obtained in the previous Section. Since

$$\mathbf{Tr}\, S(0)V_{xx}(0) = \mathbf{Tr}\, \mathbf{E}\, S(0)x(0)x^T(0)$$
$$= \mathbf{E}\, x^T(0)S(0)x(0),$$

and

$$\mathbf{Tr}\, S(k+1)V_{ww}(k) = \mathbf{Tr}\, \{\mathbf{E}\, S(k+1)w(k)w^T(k)\}$$
$$= \mathbf{E}\, w^T(k)S(k+1)w(k),$$

the optimal cost can be written as

$$\sum_{k=0}^{N} \mathbf{Tr}\, S(k) = \mathbf{E}\, x^T(0)S(0)x(0) + \sum_{k=0}^{N-1} \mathbf{E}\, w^T(k)S(k+1)w(k).$$

We see that the cost $\mathbf{E}\, x^T(0)S(0)x(0)$ is due to the initial value $x(0)$, and

$$\sum_{k=0}^{N-1} \mathbf{E}\, w^T(k)S(k+1)w(k)$$

is the cost caused by the disturbance $\{w(k)\}_{k=0}^{N-1}$.

Having realized that the cost can be expressed as a quadratic function of the uncertainty represented by $x(0)$ and $\{w(k)\}_{k=1}^{N-1}$, the dual (maximin) problem can be seen as a *game* between the controller and *nature's* choice of uncertainty.

7.4 Optimal State Feedback with Power Constraints

In this Section we consider a linear quadratic problem given by (7.1), with additional constraints of the form

$$\mathbf{E}\, \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q_i \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \leq \gamma_i(k)$$

or equivalently

$$\mathbf{Tr}\, Q_i V(k) \leq \gamma_i(k), \tag{7.17}$$

for $k = 0, ..., N-1$, $i = 1, ..., m$. Note that we do *not* make any other assumptions about Q_i except that it is symmetric, $Q_i \in \mathbb{S}^{m+n}$. Note also that the covariance constraints in (7.17) are linear in the elements of the covariance matrices $V(k)$, and hence convex. The dual problem, including the covariance constraints above, becomes

$$\max_{S(k),\tau_i(k)\geq 0} \min_{V(k)} H(N) + \sum_{k=0}^{N-1} \{H(k) + \mathbf{Tr}\, S(k+1)\} +$$
$$+ \mathbf{Tr}\, S(0) - \sum_{k=0}^{N-1}\sum_{i=1}^{m} \tau_i(k)\gamma_i(k) \tag{7.18}$$

where $\tau_i(k) \geq 0$ and $H(k)$, the Hamiltonian of the system, is given by

$$H(N) = \mathbf{Tr}\, \{[Q_{xx} - S(N)]\, V_{xx}(N)\},$$

and

$$H(k) = \mathbf{Tr}\, \left\{ \left(Q + \begin{pmatrix} A & B \end{pmatrix}^T S(k+1) \begin{pmatrix} A & B \end{pmatrix} - \right.\right.$$
$$\left.\left. -F^T S(k)F + \sum_{i=1}^{m} \tau_i(k)Q_i \right) V(k) \right\}, \tag{7.19}$$

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for $k = N - 1, ..., 0$. The dual problem (7.18) is finite if and only if

$$Q + \begin{pmatrix} A & B \end{pmatrix}^T S(k+1) \begin{pmatrix} A & B \end{pmatrix} - F^T S(k)F + \sum_{i=1}^{m} \tau_i(k)Q_i \succeq 0$$

The duality gap is zero, since Slater's condition is satisfied for the dual problem (7.18) (see [11]). Just like in the previous Section, for every time step k and for fixed values of $\tau_i(k)$, we solve

$$\max_{S(k) \in \mathbb{S}^n} \mathbf{Tr}\, S(k) - \sum_{i=1}^{m} \tau_i(k)\gamma_i(k)$$

subject to

$$Q(k) + \begin{pmatrix} A^T S(k+1)A - S(k) & A^T S(k+1)B \\ B^T S(k+1)A & B^T S(k+1)B \end{pmatrix} \succeq 0$$

(7.20)

where

$$Q(k) = Q + \sum_{i=1}^{m} \tau_i(k)Q_i.$$

Now for any fixed values of $\tau_i(k)$, $\mathbf{Tr}\, S(k)$ is maximized by

$$S(k) = A^T S(k+1)A + Q_{xx}(k) - R(k) \tag{7.21}$$

with

$$R(k) = L(k)^T (B^T S(k+1)B + Q_{uu}(k))L(k) \succ 0, \tag{7.22}$$

and $L(k)$ such that

$$(B^T S(k+1)B + Q_{uu}(k))L(k) = B^T S(k+1)A + Q_{xu}^T(k), \tag{7.23}$$

and any other matrix P with $\mathbf{Tr}\, P > \mathbf{Tr}\, S(k)$ violates the inequality in (7.20). Hence, the choice of $S(k)$ given by (7.21) is the optimal that is obtained through the eigenvalue problem (7.20).

With the optimal values of $S(k)$ and $\tau_i(k)$, the dual problem (7.18) becomes

$$\min_{V(0),...,V(N-1)} \sum_{k=0}^{N-1} \mathbf{Tr}\, Z(k)V(k) + \sum_{k=0}^{N} \mathbf{Tr}\, S(k) - \sum_{k=0}^{N-1} \sum_{i=1}^{m} \tau_i(k)\gamma_i(k)$$

subject to $\quad FV(0)F^T - I = 0$

$\qquad\qquad V(k) \succeq 0$

where the matrix $Z(k)$ is given by

$$Z = \begin{pmatrix} XY^{-1}X^T & X \\ X^T & Y \end{pmatrix},$$

with

$$X = A^T S(k+1)B + Q_{xu}(k)$$

and

$$Y = B^T S(k+1)B + Q_{uu}(k).$$

The optimal covariances $V(k)$ are obtained just like in the previous Section, by taking

$$
\begin{aligned}
V_{xx}(0) &= I \\
V_{ux}(k) &= -L(k)V_{xx}(k) \\
V_{uu}(k) &= V_{ux}(k)V_{xx}^{-1}(k)V_{xu}(k) \\
V_{xx}(k+1) &= \begin{pmatrix} A & B \end{pmatrix} V(k) \begin{pmatrix} A & B \end{pmatrix}^T + I \\
u(k) &= V_{ux}(k)V_{xx}^{-1}(k)x(k) = -L(k)x(k),
\end{aligned}
\tag{7.24}
$$

where $L(k)$ is the solution of (7.23). The problem above can be solved efficiently using primal-dual interior point methods (see [11], pp. 609), where iteration is performed with respect to the dual variables $\tau_i(0), \dots \tau_i(N-1)$, $i = 1, \dots, m$.

7.5 Stationary State Feedback with Power Constraints

Consider the infinite horizon linear quadratic control problem with power constraints:

$$
\begin{aligned}
\min_{\mu_k} \quad & \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q_0 \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \\
\text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) + w(k) \\
& u(k) = \mu_k(x(0), \dots, x(k)) \\
& \mathbf{E} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q_i \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \leq \gamma_i \\
& \text{for } i = 1, \dots, N.
\end{aligned}
\tag{7.25}
$$

The solution to this problem is easily obtained using the results for the finite horizon problem in the previous section. We have seen that the optimal choice of $S(k)$ is given by equation (7.21). When the control system is stationary, we have $\lim_{k \to \infty} S(k) = S$. Thus, when $k \to \infty$, the convex

optimization problem of the cost in (7.20) becomes

$$\max_{S \in \mathbb{S}^n, \tau_i \geq 0} \mathbf{Tr}\, S - \sum_{i=1}^{m} \tau_i \gamma_i$$

$$\text{subject to}\quad Q + \begin{pmatrix} A^T S A - S & A^T S B \\ B^T S A & B^T S B \end{pmatrix} \succeq 0 \qquad (7.26)$$

$$Q = Q_0 + \sum_{i=1}^{m} \tau_i Q_i$$

The optimal control law is then given by

$$u(k) = -Lx(k),$$

where L is the solution to

$$(B^T S B + Q_{uu})L = B^T S A + Q_{xu}^T.$$

We will now show that the controller $u(k) = -Lx(k)$ is stabilizing. Indeed, equation (7.24) in the previous chapter gives that the optimal stationary covariance V is obtained according to:

$$V_{ux} = -LV_{xx}$$
$$V_{uu} = V_{ux}V_{xx}^{-1}V_{xu} = LV_{xx}L^T$$
$$V_{xx} = \begin{pmatrix} A & B \end{pmatrix} V \begin{pmatrix} A & B \end{pmatrix}^T + I$$

Then,

$$\begin{aligned}
V_{xx} &= \begin{pmatrix} A & B \end{pmatrix} V \begin{pmatrix} A & B \end{pmatrix}^T + I \\
&= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} V_{xx} & -V_{xx}L^T \\ -LV_{xx} & LV_{xx}L^T \end{pmatrix} \begin{pmatrix} A & B \end{pmatrix}^T + I \\
&= AV_{xx}A^T - AV_{xx}L^T B^T - BLV_{xx}A^T + BLV_{xx}L^T B^T + I \\
&= (A - BL)V_{xx}(A - BL)^T + I.
\end{aligned}$$

The equality above can be equivalently written as

$$(A - BL)V_{xx}(A - BL)^T - V_{xx} = -I,$$

which is a Lyapunov matrix equation. Since $V_{xx} \succeq 0$, we conclude that $A - BL$ is stable according to Lyapunov's theorem (see [31], pp. 95-100).

THEOREM 7.3
The dual of the infinite horizon control problem (7.25) is given by (7.26). The optimal value of (7.26) is equal to the optimal value of the primal problem (7.25). The optimal control law is given by

$$u(k) = -Lx(k),$$

where L solves

$$(B^T S B + Q_{uu})L = B^T S A + Q_{xu}^T.$$

$\square$

Remark.   The result above is similar to previous results obtained for the continuous time infinite horizon control problem with power constraints. The main contribution of the result above is that it gives the optimal controller, not only the optimal value (see for instance [39], [12]).

7.6 Optimal Output Feedback Control

The problem of optimal output feedback control will be treated in this Section. The solution will be observer-based using the optimal Kalman filter.

The optimization problem to be considered is given by

$$\min_{\mu_k} \mathbf{E}\, x^T(N)Q_{xx}x(N) + \sum_{k=0}^{N-1} \mathbf{E} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}$$

$$\begin{aligned}
\text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) + w(k) \\
& y(k) = Cx(k) + v(k) \\
& u(k) = \mu_k(y(0), ..., y(k)) \\
& \mathbf{E} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q_i \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \leq \gamma_i(k) \\
& \text{for } i = 1, ..., N
\end{aligned} \tag{7.27}$$

We make the following assumptions:

$$\mathbf{E}\, x(0) = \mathbf{E}\, w(k) = \mathbf{E}\, v(k) = 0$$

$$\mathbf{E}\, x(0)x^T(0) = I$$

$$\mathbf{E} \begin{pmatrix} w(k) \\ v(k) \end{pmatrix} \begin{pmatrix} w(l) \\ v(l) \end{pmatrix}^T = \delta(k-l) \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix}.$$

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Consider the standard Kalman filter [1]:

$$\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K(k)(y(k) - C\hat{x}(k)), \\
K(k) &= (AP(k)C^T + R_{12})(CP(k)C^T + R_2)^{-1}, \\
P(k+1) &= AP(k)A^T + R_1 - K(k)(CP(k)C^T + R_2)K^T(k), \\
P(0) &= \mathbf{E}\, x(0)x^T(0) = I.
\end{aligned} \tag{7.28}$$

$P(k)$ is the covariance matrix of the error

$$\tilde{x}(k) = x(k) - \hat{x}(k).$$

Now define the innovations

$$e(k) = y(k) - C\hat{x}(k) = C\tilde{x}(k) + v(k).$$

The covariance matrix of the innovations is given by

$$\begin{aligned}
V_{ee}(k) &= \mathbf{E}\, e(k)e^T(k) \\
&= C\mathbf{E}\, [\tilde{x}(k)\tilde{x}^T(k)]C^T + \mathbf{E}\, v(k)v^T(k) \tag{7.29} \\
&= CP(k)C^T + R_2.
\end{aligned}$$

Define

$$\hat{w}(k) = K(k)e(k). \tag{7.30}$$

Then

$$\begin{aligned}
V_{\hat{w}\hat{w}}(k) &= \mathbf{E}\, \hat{w}(k)\hat{w}^T(k) \\
&= \mathbf{E}\, K(k)e(k)e^T(k)K^T(k) \tag{7.31} \\
&= K(k)V_{ee}(k)K^T(k).
\end{aligned}$$

Since $\tilde{x}(k)$ is the error obtained from the Kalman filter, we have that $\mathbf{E}\, y(t)\tilde{x}^T(k) = 0$ and $\mathbf{E}\, \hat{x}(t)\tilde{x}^T(k) = 0$ for $t \le k$. Also, $\mathbf{E}\, u(k)\tilde{x}^T(k) = 0$, since $u(k) = \mu(y(0), ..., y(k))$. Hence,

$$\begin{aligned}
J(x,u) &= \mathbf{E}\, x^T(N)Q_{xx}x(N) + \sum_{k=0}^{N-1} \mathbf{E}\, \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \\
&= \mathbf{E}\, \hat{x}^T(N)Q_{xx}\hat{x}(N) + \sum_{k=0}^{N-1} \mathbf{E}\, \begin{pmatrix} \hat{x}(k) \\ u(k) \end{pmatrix}^T Q \begin{pmatrix} \hat{x}(k) \\ u(k) \end{pmatrix} + \tag{7.32} \\
&\qquad + \sum_{k=0}^{N} \mathbf{E}\, \tilde{x}^T(k)Q_{xx}\tilde{x}(k).
\end{aligned}$$

Therefore, minimization of $J(x, u)$ is the same as minimizing

$$\mathbf{E}\,\hat{x}^T(N)Q_{xx}\hat{x}(N) + \sum_{k=0}^{N-1}\mathbf{E}\,\begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}^T Q \begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix},$$

since nothing can be done about the sum

$$\sum_{k=0}^{N}\mathbf{E}\,\tilde{x}^T(k)Q_{xx}\tilde{x}(k) = \sum_{k=0}^{N}\mathbf{Tr}\,Q_{xx}P(k),$$

which is already a constant. We also have the inequality

$$\gamma_i(k) \geq \mathbf{E}\,\begin{pmatrix}x(k) \\ u(k)\end{pmatrix}^T Q_i \begin{pmatrix}x(k) \\ u(k)\end{pmatrix}$$

$$= \mathbf{E}\,\begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}^T Q_i \begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix} + \mathbf{E}\,\tilde{x}^T(k)Q_{i_{xx}}\tilde{x}(k)$$

$$= \mathbf{E}\,\begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}^T Q_i \begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix} + \mathbf{Tr}\,Q_{i_{xx}}P(k)$$

Since the value of $\mathbf{Tr}\,Q_{i_{xx}}P(k)$ is known, we can define the new constant $\hat{\gamma}_i(k) = \gamma_i(k) - \mathbf{Tr}\,Q_{xx}P(k)$ to obtain the equivalent inequality

$$\mathbf{E}\,\begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}^T Q_i \begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix} \leq \hat{\gamma}_i(k).$$

Thus, our output feedback problem is equivalent to the following problem:

$$\min_{\mu_k}\ \mathbf{E}\,\hat{x}^T(N)Q_{xx}\hat{x}(N) + \sum_{k=0}^{N-1}\mathbf{E}\,\begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}^T Q \begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}$$

$$\text{subject to}\ \ \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + \hat{w}(k)$$

$$\mathbf{E}\,\hat{w}(k)\hat{x}^T(0) = 0$$

$$\mathbf{E}\,\hat{w}(k)\hat{w}^T(l) = V_{\hat{w}\hat{w}}(k) \tag{7.33}$$

$$u(k) = \mu_k(\hat{x}(0), ..., \hat{x}(k))$$

$$\mathbf{E}\,\begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix}^T Q_i \begin{pmatrix}\hat{x}(k) \\ u(k)\end{pmatrix} \leq \hat{\gamma}_i(k)$$

$$\text{for}\ i = 1, ..., N.$$

We see that we have transformed the output feedback problem to a state feedback problem, which can be solved as in the previous Sections. We have obtained:

The optimal output feedback problem (7.27) is equivalent to the static feedback problem (7.33), where $\hat{x}(k)$ is the optimal estimate of $\hat{x}(k)$ obtained from the Kalman filter given by (7.28) with $P(k)$ as the covariance matrix of the estimation error $x(k) - \hat{x}(k)$. The covariance matrix $V_{\hat{w}\hat{w}}(k)$ is calculated according to (7.29)-(7.31), and $\hat{\gamma}_i(k) = \gamma_i(k) - \mathbf{Tr}\ Q_{xx}P(k)$.

$\square$

7.7 Example

Consider the following scalar stochastic linear quadratic control problem:

$$\min_{\mu_k}\ \mathbf{E}\ x_2^2 + \sum_{k=0}^{1}\mathbf{E}\ \{x_k^2 + u_k^2\}$$

$$\text{subject to}\quad x_{k+1} = x_k + u_k + w_k$$

$$x_0, w_0, w_1 \in \mathcal{N}(0,1) \tag{7.34}$$

$$\mathbf{E}\ u_0^2 \le \frac{1}{10},\ \mathbf{E}\ u_1^2 \le \frac{1}{4}\mathbf{E}\ x_1^2$$

$$u_0 = \mu_0(x_0)$$

$$u_1 = \mu_1(x_0, x_1)$$

Note first that

$$x_k^2 + u_k^2 = \begin{pmatrix} x_k \\ u_k \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix},\quad u_0^2 = \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}.$$

Also, $\mathbf{E}\ u_1^2 \le \frac{1}{4}\mathbf{E}\ x_1^2$ can be written as

$$0 \ge \mathbf{E}\ \{4u_1^2 - x_1^2\} = \mathbf{E}\ \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}.$$

We now have the weighting matrices Q, $Q_1(0)$, and $Q_1(1)$. The optimal control law of the optimization (7.34) can be calculated efficiently using the methods presented in Section 7.4. It is given by $u_0 = -0.3162x_0$ and $u_1 = -0.5x_1$. The minimal cost is 4.3013, compared to the cost of the unconstrained problem (that is, without the third constraint in (7.34)) which is 4.1. We can also check that the quadratic constraints are satisfied; $\mathbf{E}\ u_0^2 = \frac{1}{10}\mathbf{E}\ x_0^2 = \frac{1}{10}$, and $\mathbf{E}\ u_1^2 = \frac{1}{4}\mathbf{E}\ x_1^2$.

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Conclusions

The thesis considers optimal decision problems where a team of decision makers optimize a given cost induced by the uncertainty of the state of nature. The crucial property for the team problems considered in the thesis is that the team members have different pieces of information about the state of nature.

A brief introduction was given, and in particular a combination of graph theory and linear systems theory was presented.

Optimal static team estimation was developed for different measures of the estimation error. Then, the dynamic team estimation problem was solved with the use of the results obtained for the static team estimation.

Linear quadratic static and dynamic team decision problems were considered in both the stochastic and deterministic setting. It was shown that when the information structure is such that coding incentives in the decisions are eliminated, linear decisions are optimal and can be easily found using convex optimization. The results showed a broader class of information structures that lead to convex team decision problems.

The finite horizon stochastic and deterministic team decision problems, or as they are known in systems theory, the distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems, were solved for the state feedback case under limitations on the rate of information propagation. A novel approach to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problem was developed, by using the crucial idea of disturbance feedback. It was shown that by using disturbance feedback, the team control problem can be transformed to a team estimation problem. This problem can be readily solved using the theory developed in the thesis for team estimation.

Necessary and sufficient conditions were given for stabilizability of systems over graphs under distributed delayed measurements, by using the idea of disturbance feedback.

The thesis treats a generalized stochastic linear quadratic control setting, for both the finite and infinite horizon case. The main contribution

is that non-convex quadratic constraints can be added in the optimization problem, and the problem remains convex, by a simple restatement of the problem. A broad class of stochastic linear quadratic optimal control problems with information constraints can be modeled with the help of quadratic constraints. Also, many distributed control problems can be modeled through quadratic constraints of correlation type. An example is the distributed stochastic control problem where non-convex quadratic constraints are forced to be zero. The advantage with this setting is that other constraints such as limitations on the power of the control signal can be easily added in the optimization problem. First, the finite horizon state feedback control problem is solved through duality. The calculations of the optimal control law can be done off-line as in the classical linear quadratic Gaussian control theory using a combination of dynamic programming and primal-dual methods. Then, a solution to the infinite horizon control problem is presented. Finally, the output feedback problem is solved.

Future Challenges

There are two main research directions that are of interest. The first is to solve the output feedback deterministic (or $\mathcal{H}_\infty$) team problem for the infinite horizon. The other direction, which is most likely much harder, is to examine the team decision problem where the coding incentive is not eliminated. This is a very challenging problem, and it is anticipated that its solution will show new avenues in systems and information theory.

Another issue that has a great importance is the implementation of distributed controllers on a graph. In this thesis, the obtained state feedback controller for each subsystem is of an order equal to the sum of the orders of all subsystems on the graph times the number of subsystems, (that is, $O(n \times N)$, where n is the order of the global state and N is the number of subsystems on the graph). When, the number of subsystems of the graph is very large, the implementation might be difficult. One way to approach this problem is to try model reduction. Another approach is to consider the distributed control problem from a perspective similar to that of static output feedback, with the great disadvantage that the static output is still an open problem. The output feedback problem is in turn of the same nature as that of team problems with coding incentives, which gives yet another motivation for the importance of studying decisions with coding incentives.

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