Articles on Potential Theory, Functional Analysis and Hankel Forms

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Abstracts

The thesis consists of three research articles. For convencience the respective references and abstracts are given here.

Paper I: Perfekt, K.-M. and Putinar, M., *Spectral bounds for the Neumann-Poincaré operator on planar domains with corners*, to appear in J. Anal. Math., (2012).

The boundary double layer potential, or the Neumann-Poincaré operator, is studied on the Sobolev space of order $1/2$ along the boundary, coinciding with the space of charges giving rise to double layer potentials with finite energy in the whole space. Poincaré's program of studying the spectrum of the boundary double layer potential is developed in complete generality, on closed Lipschitz hypersurfaces in Euclidean space. Furthermore, the Neumann-Poincaré operator is realized as a singular integral transform bearing similarities to the Beurling-Ahlfors transform in 2D. As an application, bounds for the spectrum of the Neumann-Poincaré operator are derived from recent results in quasi-conformal mapping theory, in the case of planar curves with corners.

Paper II: Perfekt, K.-M., *Duality and distance formulas in spaces defined by means of oscillation*, Arkiv för Matematik, (2012), pp. 1–17 (Online First).

For the classical space of functions with bounded mean oscillation, it is well known that $\mathrm{VMO}^{**} = \mathrm{BMO}$ and there are many characterizations of the distance from a function f in BMO to VMO. When considering the Bloch space, results in the same vein are available with respect to the little Bloch space. In this paper such duality results and distance formulas are obtained by pure functional analysis. Applications include general Möbius invariant spaces such as Q_K-spaces, weighted spaces, Lipschitz-Hölder spaces and rectangular BMO of several variables.

Paper III: Aleman, A. and Perfekt, K.-M., *Hankel forms and embedding theorems in weighted Dirichlet spaces*, Int. Math. Res. Not., 2012 (2012), pp. 4435–4448.

We show that for a fixed operator-valued analytic function g, the boundedness of the bilinear (Hankel-type) form

$$(f,h) \to \int_{\mathbb{D}} \operatorname{tr}\left(g'^{*} f h'\right) (1-|z|^2)^{\alpha} \, \mathrm{d}A,$$

defined on appropriate cartesian products of dual weighted Dirichlet spaces of Schatten class-valued functions, is equivalent to corresponding Carleson embedding estimates.

Populärvetenskaplig Sammanfattning

Avhandlingen består av tre matematiska forskningsartiklar.

Den första artikeln, Paper I, undersöker spektrala egenskaper hos Neumann-Poincaré-operatorer, även kallade dubbelskiktspotentialoperatorer, för områden med låg regularitet. Operatorernas spektra har kopplingar till elektrostatiska problem, och författarens intresse inleddes i samband med en tidigare artikel som behandlade elektromagnetiska egenskaper hos en dielektrisk kub inbäddad i ett annat dielektriskt material. Mycket fascinerande aspekter av ämnet är de dramatiska förändringar som sker i spektralbilden när ett i övrigt glatt område tillfogas hörn och de stora teoretiska och beräkningsmässiga svårigheter som orsakas av sådana irregulariteter. I Paper I behandlas teorin för Lipschitzområden, vilka grovt kan beskrivas som områden som tillåter förekomst av hörn. För varje sådant område placeras motsvarande operator i ett fysikaliskt naturligt ramverk som tydliggör spektrets uppträdande och belyser kontrasten som uppstår när man jämför operatorns egenskaper för glatta områden och Lipschitzområden. Klassiska idéer härrörande från Poincaré och Schiffer generaliseras till Lipschitz-fallet, vilket leder till principer som kan nyttjas till att bestämma spektret. Teorin tillämpas för att uppskatta storleken på spektret och det essentiella spektret för områden med hörn i två dimensioner. Uppskattningarna som erhålls stämmer överens med de få resultat som funnits tillgängliga tidigare.

Paper II handlar om bidualrum, dualrummens dualrum, till vissa Banachrum. Givet en samling storheter, av godtycklig natur, betraktas i artikeln Banachrum M som består av funktioner med egenskapen att storheterna är likformigt begränsade, tillsammans med motsvarande "små rum" M_0 som innehåller de funktioner för vilka storheterna går mot noll under vissa, också godtyckligt valda, villkor. I många konkreta exempel av ovan beskrivna konstruktion är det känt att bidualen M_0^{**} kan representeras som M på ett naturligt sätt, och att avståndet från en funktion $f \in M$ till underrummet M_0 kan beskrivas i termer av storheterna. Artikelns syfte är att visa att giltigheten av dessa typer av resultat följer allmänt ur abstrakta resonemang från funktionalanalys och vektorvärd måtteori.

I den sista artikeln, Paper III, ger huvudresultatet en karakterisering av kontinuerliga Hankelformer med vektorvärda symbolfunktioner. En klassisk Hankelform svarar mot en matris med oändligt många rader och kolumner där elementen

är lika längs skevdiagonalerna (de uppåt lutande diagonalerna, sedda från vänster till höger). Varje ändlig matris ger alltid upphov till en kontinuerlig operator, vilket i vardagligt språk innebär att en liten förändring i indata inte kan orsaka drastiska förändringar i utdata. I motsats till detta är det för en given oändlig matris inte ett trivialt problem att avgöra om motsvarande operator är kontinuerlig. Resultaten av många matematikers skilda insatser behövde förenas för att slutligen kunna visa att en klassisk Hankelform är kontinuerlig om och endast om symbolfunktionen som genererar dess matriselement uppfyller Carlesons inbäddningsvillkor. I artikeln betraktas en variant av Hankelformer, motsvarande matriser vars element är oändliga vektorer och vars struktur påminner om den för klassiska Hankelformer. Det visas att kontinuitet gäller om och endast om motsvarande vektorvärt inbäddningsvillkor är uppfyllt. En intressant följd av resultatet är att samma inbäddningsvillkor omöjligt kan karakterisera kontinuitet för klassiska Hankelformer med vektorelement.

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With these words I would like to express my appreciation of those who have supported me in the development of this thesis.

Above all, I am deeply grateful to my advisor Alexandru Aleman. Without his steadfast support, advice, collaboration, and belief in me, the work presented in this thesis and my time as a Ph.D. student could not have been the same.

I am also indebted to all the people who I have had the opportunity to collaborate with. They have provided me with inspiration and success in research, and therefore I want to acknowledge Johan Helsing, Yurii Lyubarskii, Eugenia Malinnikova, Sandra Pott, Mihai Putinar, Stefan Richter, Carl Sundberg, and Paco Villarroya.

The moral support of my closest family has been invaluable, and I give my thanks and love to Ellen, Axel, Erika, Sarah, Margareta, Roland, Elsie, Eskil, and Sten.

Finally, I thank my friends for the many good memories that have been created during my Ph.D. studies.

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Preface

Preface

This thesis consists of three independent research articles, ordered by date of publication:

Paper I: Perfekt, K.-M. and Putinar, M., *Spectral bounds for the Neumann-Poincaré operator on planar domains with corners*, to appear in J. Anal. Math., (2012). [11].
Paper II: Perfekt, K.-M., *Duality and distance formulas in spaces defined by means of oscillation*, Arkiv för Matematik, (2012), pp. 1–17 (Online First). [10].
Paper III: Aleman, A. and Perfekt, K.-M., *Hankel forms and embedding theorems in weighted Dirichlet spaces*, Int. Math. Res. Not., 2012 (2012), pp. 4435–4448. [1].

Only slight modifications have been made to the papers from their printed versions, most changes being typographical in nature.

The purpose of the present chapter is to provide an introduction and further comments to each of the articles. To facilitate reading accessibility, the discourse will partially take place on a more informal and less precise level than in the respective papers.

1 Spectral Bounds for the Neumann-Poincaré Operator

Paper I is concerned with examining the spectral features of layer potential operators associated with the Laplacian. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with connected boundary, $n \geq 2$, the *double layer potential operator* on the boundary, or the *Neumann-Poincaré operator*, K, is given by

$$Kf(x) = -2 \text{ p.v.} \int_{\partial\Omega} \partial_{n_y} G(x,y) f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

where $f\colon \partial\Omega \to \mathbb{C}$ is a function on the boundary, ∂_{n_y} denotes the outward normal derivative at $y \in \partial\Omega$, σ denotes surface measure on $\partial\Omega$, and $G(x,y)$ is the Newtonian kernel

$$G(x,y) = \begin{cases} -\omega_n^{-1} \log|x-y|, & n = 2, \\ \omega_n^{-1} |x-y|^{2-n}, & n \geq 3, \end{cases}$$

normalized with a constant ω_n so that $\Delta_x G(x,0) = -\delta$.

The author's initial interest in the operator K arose from the study of an electrostatic problem, where $\mathbb{R}^n$ is interpreted as a composite with two constituents, the exterior domain $\Omega_e = \overline{\Omega}^c$ with permittivity constant $\varepsilon_1 \in \mathbb{C}$ and the interior domain Ω with a different permittivity constant $\varepsilon_2 \in \mathbb{C}$. For a given applied unit field $e \in \mathbb{R}^n$, the problem seeks a potential $U\colon \Omega \cup \Omega_e \to \mathbb{C}$ such that

$$\begin{cases} \Delta U(x) = 0, & x \in \Omega \cup \Omega_e, \\ \operatorname{Tr}_{\text{ext}} U(x) = \operatorname{Tr}_{\text{int}} U(x), & x \in \partial\Omega, \\ \varepsilon_1 \partial_n^{\text{ext}} U(x) = \varepsilon_2 \partial_n^{\text{int}} U(x), & x \in \partial\Omega, \\ \lim_{x\to\infty} \nabla U(x) = e. \end{cases} \tag{1.1}$$

Here $\operatorname{Tr}_{\text{ext}} U$ and $\operatorname{Tr}_{\text{int}} U$ denote the traces (boundary values) of U from the exterior and interior domains Ω_e and Ω, respectively. Similarly, $\partial_n^{\text{ext}} U$ and $\partial_n^{\text{int}} U$ denote exterior and interior trace normal derivatives with respect to $\partial\Omega$.

In many settings, the harmonic potentials V with equal boundary values from the interior and exterior and decay at infinity correspond to the family of single layer potentials, $V = S\rho$. Recall that for a charge $\rho\colon \partial\Omega \to \mathbb{C}$, the corresponding *single layer potential* is defined by

$$S\rho(x) = \int_{\partial\Omega} G(x,y)\rho(y)\, d\sigma(y), \quad x \in \mathbb{R}^n.$$

Inserting the ansatz $U(x) = S\rho(x) + e\cdot x$ into the equation of normal derivatives in (1.1), having already cared for the other conditions, leads to the problem

$$(K^* - z)\rho(x) = g(x), \quad x \in \partial\Omega \tag{1.2}$$

where $z = \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_2-\varepsilon_1}$ and $g(x) = 2(e\cdot n_x)$, n_x denoting the exterior unit normal of $\partial\Omega$ at x.

We have hence related the electrostatic problem (1.1) to the equation (1.2), motivating the study of the spectrum of K and K^*. In fact, the entire spectral measure of K, and in particular knowing its support, is of importance to applications in physics where the system (1.1) appears (for example in computing the polarizability or effective thermal conductivity of a composite). For further details and applications of the theory of Paper I, see Helsing and Perfekt [7].

One of the main points made in [7] is that in order to study the physically natural situation of finite energy single layer potentials $S\rho$, one has to consider

charges $\rho \in H^{-1/2}(\partial\Omega)$ in the fractional Sobolev space of order $-1/2$ along the boundary $\partial\Omega$. Hence, the spectral properties of the double layer potential K should be studied when $K\colon H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is considered as an operator on the Sobolev space $H^{1/2}$.

It is a striking fact that the spectral properties of K acting on said space $H^{1/2}$ turn out to be completely different from the properties seen in the more familiar setting of K acting on $L^2(\partial\Omega) = H^0$. In the case that $\partial\Omega$ is a smooth surface the distinction is not important to make, and the spectrum of the compact operator $K\colon H^s \to H^s$ consists of the same eigenvalues regardless of the choice of parameter s, $0 \leq s \leq 1$. However, when $\partial\Omega$ is imposed with Lipschitz behavior, such as giving it a corner, the picture changes. When Ω is a curvilinear polygon in two dimensions, I. Mitrea [9] has determined the spectrum of $K\colon L^2 \to L^2$ acting on L^2, showing that it contains closed lemniscate domains extending into the complex plane, one for each corner of $\partial\Omega$. This is in stark contrast to the situation on $H^{1/2}$, where the spectral picture is closer to what may be physically expected; a first indication of this is that the spectrum is contained in the real line $\mathbb{R}$, owing to the symmetry that K exhibits on $H^{1/2}$.

Paper I sets out to study the spectrum of K on $H^{1/2}(\partial\Omega)$ for Lipschitz domains Ω, a problem for which little has been known. Generalizing the results of Khavinson, Putinar, and Shapiro [8] to the non-smooth case, a framework is developed for studying finite energy potentials in the Lipschitz setting. The framework, of determining the spectrum of K either via a balance of energies or through a Beurling-Ahlfors type transform, incorporates many classical ideas, tracing back to Poincaré and M. Schiffer.

The framework is then utilized to give sharp bounds for the spectral radius and essential spectral radius of $K\colon H^{1/2} \to H^{1/2}$ in the case of curvilinear polygonal domains $\Omega \subset \mathbb{R}^2$. The main difficulty in characterizing the spectrum for such domains lies in obtaining a suitable localization principle in order to reduce to the case of studying only one corner, a technique which has proven successful for studying layer potentials on (weighted) L^p-spaces. See for example [9] and Qiao and Nistor [12]. In lieu of an available method of localization, we combine in Paper I recent results in quasiconformal mapping theory with the explicit construction of corner-preserving conformal maps to obtain the desired spectral bounds, under a hypothesis on the angles related to convexity.

2 Duality and Distance Formulas

Consider the following two examples.

Example 2.1. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$, let $X = L^2(\mathbb{T})/\mathbb{C}$ be the space of square integrable functions on $\mathbb{T}$ modulo constants, and let $Y = L^1(\mathbb{T})$. For each non-empty arc $I \subset \mathbb{T}$, define the linear operator $L_I \colon X \to Y$ by

$$L_I f = \chi_I \frac{1}{|I|}(f - f_I),$$

where χ_I is the characteristic function of I, $|I|$ is the length of the arc, and $f_I = \frac{1}{|I|}\int_I f\, ds$ is the average of f on I, ds denoting the Lebesgue measure on $\mathbb{T}$. With this notation, the space $\mathrm{BMO}(\mathbb{T})$ of functions on $\mathbb{T}$ of bounded mean oscillation may be defined as

$$\mathrm{BMO}(\mathbb{T}) = \{f \in X \,:\, \sup_I \|L_I f\|_Y < \infty\}.$$

Similarly, the space $\mathrm{VMO}(\mathbb{T})$ of vanishing mean oscillation is given by

$$\mathrm{VMO}(\mathbb{T}) = \{f \in \mathrm{BMO} \,:\, \overline{\lim_{|I|\to 0}} \|L_I f\|_Y = 0\}.$$

Central to our discussion will be the two facts that the bidual of VMO can be identified with BMO under the $L^2(\mathbb{T})$-pairing, $\mathrm{VMO}(\mathbb{T})^{**} \simeq \mathrm{BMO}(\mathbb{T})$ (Garnett [6]) and for $f \in \mathrm{BMO}(\mathbb{T})$ that

$$\mathrm{dist}(f, \mathrm{VMO})_{\mathrm{BMO}} \sim \overline{\lim_{|I|\to 0}} \|L_I f\|_Y = \overline{\lim_{|I|\to 0}} \frac{1}{|I|}\int_I |f - f_I|\, ds$$

(Stegenga and Stephenson [14]).

Example 2.2. Next consider the subspace $\mathrm{BMOA}(\mathbb{T}) \subset \mathrm{BMO}(\mathbb{T})$ consisting of analytic functions. That is, $\mathrm{BMOA} = \mathrm{BMO} \cap H^2/\mathbb{C}$, where $H^2/\mathbb{C}$ is the usual Hardy space of the disc, modulo constants. $\mathrm{BMOA}(\mathbb{T})$ has an alternate construction as a *Möbius invariant* Banach space. For $a \in \mathbb{D}$ and $\lambda \in \mathbb{T}$, let $\phi_{a,\lambda} \colon \mathbb{D} \to \mathbb{D}$ denote the conformal automorphism given by

$$\phi_{a,\lambda}(z) = \lambda \frac{a - z}{1 - \bar{a}z},$$

and associate with it the linear map $L_{\phi_{a,\lambda}} : H^2/\mathbb{C} \to H^2/\mathbb{C}$,

$$L_{\phi_{a,\lambda}} f(z) = f \circ \phi_{a,\lambda}.$$

To elucidate notation that will appear shortly, put $X = Y = H^2/\mathbb{C}$. We then have that

$$\mathrm{BMOA}(\mathbb{T}) = \{f \in X \,:\, \sup_{a,\lambda} \|L_{\phi_{a,\lambda}} f\|_Y < \infty\},$$

see for example [6]. Similarly, for analytic VMO,

$$\mathrm{VMOA}(\mathbb{T}) = \{f \in \mathrm{BMOA} \,:\, \overline{\lim_{|a|\to 1}} \|L_{\phi_{a,\lambda}} f\|_Y = 0\}.$$

Just as in the case of real BMO, it holds that $VMOA(\mathbb{T})^{**} \simeq \mathrm{BMOA}(\mathbb{T})$ under the H^2-pairing, and for $f \in \mathrm{BMOA}(\mathbb{T})$ that

$$\mathrm{dist}(f, \mathrm{VMOA})_{\mathrm{BMOA}} \sim \overline{\lim_{|a|\to 1}} \|L_{\phi_{a,\lambda}} f\|_Y = \overline{\lim_{|a|\to 1}} \|f \circ \phi_{a,\lambda}\|_{H^2/\mathbb{C}}.$$

See for example Carmona and Cufí [4].

The theme of the previous examples is that of a "large space" M defined by a big-O condition and a "small space" M_0 given by the corresponding little-O condition. In both cases it holds that $M_0^{**} \simeq M$ and that the distance from an element $x \in M$ to M_0 can be computed in terms of the defining conditions of M and M_0. The purpose of Paper II is to demonstrate that such results can be obtained in general, relying only on tools of functional analysis.

To explain the main result of Paper II, we require the following notation. X and Y will be two Banach spaces, with X separable and reflexive. $\mathcal{L}$ will be a given collection of bounded operators $L\colon X \to Y$ that is accompanied by a σ-compact locally compact Hausdorff topology τ such that for every $x \in X$, the map $T_x\colon \mathcal{L} \to Y$ given by $T_x L = Lx$ is continuous from $(\mathcal{L}, \tau)$ to $(Y, \|\cdot\|_Y)$. In accordance with the two examples given, the spaces M and M_0 are defined by

$$M(X, \mathcal{L}) = \left\{ x \in X \,:\, \sup_{L\in\mathcal{L}} \|Lx\|_Y < \infty \right\}$$

and

$$M_0(X, \mathcal{L}) = \left\{ x \in M(X, \mathcal{L}) \,:\, \overline{\lim_{\mathcal{L}\ni L\to\infty}} \|Lx\|_Y = 0 \right\},$$

where the limit $L \to \infty$ is taken in the sense of the one-point compactification of $(\mathcal{L}, \tau)$.

We work under the hypothesis that $M(X, \mathcal{L})$ is a Banach space when normed naturally, and that it is continuously contained and dense in X. Then the biduality relation $M_0(X, \mathcal{L})^{**} \simeq M(X, \mathcal{L})$ (under the X-pairing) implies that for every $x \in M(X, \mathcal{L})$ there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset M_0(X, \mathcal{L})$ such that x_n converges to x (weakly) in X with $\sup_n \|x_n\|_{M(X,\mathcal{L})} < \infty$. To obtain a result it is therefore necessary to *assume* that this approximation property holds. In the two examples given, the property can easily be directly verified to hold by convolving $f \in \mathrm{BMO}$ with Poisson kernels and applying standard results.

Under the assumption of the above approximation property, the two main theorems of Paper II state that the biduality relation

$$M_0(X, \mathcal{L})^{**} \simeq M(X, \mathcal{L}) \tag{2.1}$$

holds in a canonical way, and furthermore that the desired distance formula holds with equality,

$$\operatorname{dist}(x, M_0(X, \mathcal{L}))_{M(X,\mathcal{L})} = \overline{\lim_{\mathcal{L} \ni L \to \infty}} \|Lx\|_Y \,. \tag{2.2}$$

The theorems not only apply to the examples of BMO already given, but also to general Möbius invariant spaces of analytic functions including a large class of Q_K-spaces, weighted spaces, rectangular BMO of several variables and Lipschitz-Hölder spaces.

The approach of Paper II is to consider the isometric embedding $x \hookrightarrow T_x$ of $M(X, \mathcal{L})$ into the space $C_b(\mathcal{L}, Y)$ of bounded continuous Y-valued functions on $(\mathcal{L}, \tau)$. Note that $M_0(X, \mathcal{L})$ embeds into the space $C_0(\mathcal{L}, Y)$ of continuous functions vanishing at infinity. Duality is then studied with help of the Riesz-Zinger theorem, which identifies the dual $C_0(\mathcal{L}, Y)^*$ with a space of measures.

The validity of (2.2) turns out to be intimately connected with the question of recognizing which elements of M_0^{***} actually belong to M_0^*. This question is in turn related to whether M_0^* is the unique (isometric) predual of $M_0^{**} \simeq M$. As a corollary of the techniques involved in Paper II, it will be obtained that this is indeed the case.

3 Hankel Forms and Embedding Theorems

The topic of Paper III is that of Hankel forms on weighted Dirichlet spaces in the vector-valued setting. To explain the results' vantage point, let us begin by combining several aspects of the classical theory of Hankel forms on the scalar-valued Hardy space H^2. Given a holomorphic symbol $g\colon \mathbb{D} \to \mathbb{C}$ with $g(0) = 0$ we will by its Hankel form on H^2 mean the sesqui-linear form on $H^2 \times H^2$ defined at least for polynomials f and h by

$$(f, h)_g = \lim_{r \to 1^-} \int_{\mathbb{T}} f(z)\overline{h(\bar{z})g(rz)}\frac{ds(z)}{2\pi},$$

where ds is the Lebesgue measure on $\mathbb{T}$. The form induces a corresponding Hankel operator Γ_g,

$$\langle \Gamma_g f, h\rangle_{H^2} = (f, h)_g,$$

named such because when written as a matrix (A_{ij}) in the standard basis, A_{ij} depends only on $i + j$.

C. Fefferman proved that $(H^1)^* = \mathrm{BMOA}$. In combination with the factorization $H^1 = H^2 \cdot H^2$ this shows that $(f, h)_g$ is a bounded form if and only if $g \in \mathrm{BMOA}$, an argument due to Nehari. Furthermore, it is a well known fact that $g \in \mathrm{BMOA}$ if and only if $|g'(z)|^2(1 - |z|^2)\, dA(z)$ is a *Carleson measure* for H^2, where dA is the area measure on $\mathbb{D}$. That is, if and only if the embedding of H^2 into $L^2(|g'|^2(1 - |z|^2)\, dA)$ is bounded. From the corresponding norm equivalences we get that

$$\sup_{\|f\|_2 = \|h\|_2 = 1} |(f, h)_g|^2 \sim \|g\|^2_{\mathrm{BMOA}} \sim \sup_{\|f\|_2 = 1} \int_{\mathbb{D}} |f(z)g'(z)|^2(1 - |z|^2)\, dA(z).$$

For $p > 1$ and $\beta > -1$, the weighted Dirichlet space $D^{p,\beta}$ consists of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{p,\beta} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^\beta\, dA(z)\right)^{1/p} < \infty.$$

To connect our discussion of H^2 with Dirichlet spaces, we recall that the identity of Littlewood-Paley says that $H^2 = D^{2,1}$, with equivalent norms. The previous

discussion can hence be summarized as

$$\sup_{\|f\|_{2,1}=\|h\|_{2,1}=1} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} (fh)'(z)\overline{g'(z)}(1-|z|^2)\,dA(z) \right|^2 \sim \sup_{\|f\|_{2,1}=1} \int_{\mathbb{D}} |f(z)g'(z)|^2(1-|z|^2)\,dA(z).$$

It is natural to ask if the boundedness of a (small) Hankel form on the Dirichlet space $D^{2,\beta}$, $0 \leq \beta < 1$, also is equivalent to a corresponding Carleson embedding condition. This question turns out to be very hard to answer, an obvious obstacle being that the Carleson measures for $D^{2,\beta}$ are characterized in terms of capacities, and therefore difficult to deal with. Nonetheless, for $\beta = 0$ it has been answered in the positive direction recently by Arcozzi, Rochberg, Sawyer and Wick [2], by a method of replacing $D^{2,0}$ with a discretized version imbued with a certain tree structure. That is, for the standard Dirichlet space $D^{2,0}$ we have

$$\sup_{\|f\|_{2,0}=\|h\|_{2,0}=1} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} (fh)'(z)\overline{g'(z)}\,dA(z) \right|^2 \sim \sup_{\|f\|_{2,0}=1} \int_{\mathbb{D}} |f(z)g'(z)|^2\,dA(z).$$

Comparing with the argument of Nehari for the Hardy space, we see that the natural analogue of H^1 is the weak product space $D^{2,0} \odot D^{2,0}$, a space which does not appear to have a more direct definition. With respect to the duality $(H^1)^* = \mathrm{BMOA}$, the natural analogue of analytic BMO is thus the space of holomorphic functions g such that $|g'|^2\,dA$ is a Carleson measure for $D^{2,0}$. For further discussion on this topic, see the excellent Dirichlet space survey [3]. Recall that $h \in D^{2,0} \odot D^{2,0}$ if and only if $h = \sum f_n h_n$ with $\sum \|f_n\|_{2,0}\|h_n\|_{2,0} < \infty$.

One may also consider the weak product space $\partial^{-1}(D^{2,\beta} \odot \partial D^{2,\beta})$, consisting of functions h such that $h' = \sum f_n h_n'$. Characterizing the dual of this space corresponds to characterizing the boundedness of the Hankel-type form in which we replace $(fh)' = f'h + fh'$ with fh', taking only one "half" of the original Hankel form. It turns out that these half-forms are considerably less resilient to analysis; that their boundedness on $D^{2,\beta}$ is equivalent to the Carleson embedding condition was proven by Rochberg and Wu [13] almost 20 years before the

corresponding result for the full Hankel form. For $\beta = 0$, their result reads

$$\sup_{\|f\|_{2,0}=\|h\|_{2,0}=1} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} f(z)h'(z)\overline{g'(z)}\, dA(z) \right|^2 \sim \sup_{\|f\|_{2,0}=1} \int_{\mathbb{D}} |f(z)g'(z)|^2\, dA(z).$$

The two types of Hankel forms therefore are bounded simultaneously. Equivalently,

$$\partial^{-1}(D^{2,0} \odot \partial D^{2,0}) = D^{2,0} \odot D^{2,0}.$$

Note that in the case of the scalar-valued Hardy space, $\beta = 1$, it is straightforward to directly check, using the square function, that

$$\partial^{-1}(H^2 \odot \partial H^2) = H^2 \odot H^2 = H^2 \cdot H^2 = H^1. \tag{3.1}$$

The main result of Paper III states that the boundedness of the half-form is equivalent to the Carleson embedding condition for general parameters in the *vector-valued* case. Let $1 < p, q < \infty$, $\alpha \geq 0$, and $\beta, \gamma > -1$ satisfy the duality relations $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{\beta}{p} + \frac{\gamma}{q} = \alpha$. Then Theorem 3.1 of Paper III, stated in the scalar-valued case for simplicity, says that

$$\sup_{\|f\|_{p,\beta}=\|h\|_{q,\gamma}=1} \left| \lim_{r\to 1} \int_{\mathbb{D}_r} f(z)h'(z)\overline{g'(z)}(1-|z|^2)^\alpha \, \mathrm{d}A(z) \right| \sim \sup_{\|f\|_{p,\beta}=1} \left(\int_{\mathbb{D}} |\overline{g'(z)}f(z)|^p (1-|z|^2)^\beta \, \mathrm{d}A(z) \right)^{1/p}. \tag{3.2}$$

As before, this result can be recast as a Carleson embedding characterization of the dual of $\partial^{-1}(D^{p,\beta} \odot \partial D^{q,\gamma})$.

The vector-valued case is obtained by letting g be a holomorphic operator-valued function and substituting products with scalar products. We postpone making a precise statement to Paper III, but remark here that the Carleson condition is quite curious in the vector-valued setting. For example, we show that the anti-analytic factor $\overline{g'(z)}$ in the embedding condition may not be replaced with the analytic factor $g'(z)$. Furthermore, equation (3.1) is known to be false in the vector-valued setting (Davidson and Paulsen [5]). Therefore, the boundedness of the full Hankel form on the vector-valued Hardy space must have a different characterization.

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