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# A General Power-Series Expansion Method
# for Scalar Analysis of
# the Guided Modes in an Optical Fiber

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A power-series expansion method for approximate analysis of the guided modes in a cylindrical, radially inhomogeneous, dielectric waveguide is presented. The method is developed for an arbitrary piece-wise polynomial permittivity profile. The solution of the scalar wave equation is constructed by a sequence of power-series expansions. Convergence is ensured and accuracy maintained by regulating the distance ( = the steplength) between successive expansion points. It is demonstrated, by analytical reasoning and numerical examples, that a short steplength is inefficient. The characteristic equation and the cutoff condition are expressed in a unified manner. The method presented is competitive in terms of accuracy achieved and computing time required.

# INTRODUCTION

An optical fiber can be idealized to a cylindrical dielectric waveguide (see Fig. 1 and Fig. 2). In order to investigate the propagation characteristics of the guided modes, Maxwell's equations must be solved. A general (i.e. treating an arbitrary piece-wise polynomial permittivity profile) and exact (i.e. no approximation in the process of solving Maxwell's equations) power-series expansion method (abbreviated PSEM) is presented in Ref. 1. In this paper a corresponding general PSEM is presented which solves the well-known scalar equation exactly. Since scalar analysis involves an approximation the analysis is approximate.

The formulas are less complex in the scalar case. As a consequence the computer programming is easier. If multi-variable optimization is to be performed then the shorter computer time required by scalar analysis may prove advantageous. The result obtained by scalar analysis may be adjusted and verified by exact analysis. Further, a basis for comparative study is provided by the corresponding methods of exact and scalar analysis.

## MAXWELL'S EQUATIONS

Maxwell's field equations are

$$\nabla \times \mathbf{E} = - j k_0 Z_0 \mathbf{H}, \tag{1}$$

$$\nabla \times \mathbf{H} = j \, \varepsilon(r) \, \frac{k_0}{Z_0} \, \mathbf{E}, \tag{2}$$

where $\varepsilon(r)$ is the relative permittivity profile, $Z_0$ is the wave impedance of vacuum and $k_0$ is the vacuum wavenumber,

$$k_0 = 2\pi/\lambda_0 \, , \tag{3}$$

where $\lambda_0$ is the vacuum wavelength. If terms involving $\varepsilon'(r)$ are neglected then

$$\nabla^2 \mathbf{E} + \varepsilon(r) \, k_0^2 \, \mathbf{E} = 0, \tag{4}$$

$$\nabla^2 \mathbf{H} + \varepsilon(r) \, k_0^2 \, \mathbf{H} = 0. \tag{5}$$

The electric and magnetic field vectors of a guided mode can be expressed as

$$\mathbf{E} = e_r(r) \, \mathbf{u}_r + j \, e_\varphi(r) \, \mathbf{u}_\varphi + j \, e_z(r) \, \mathbf{u}_z \, , \tag{6}$$

$$\mathbf{H} = \frac{1}{Z_0} \, g_r(r) \, \mathbf{u}_r + \frac{1}{Z_0} \, g_\varphi(r) \, \mathbf{u}_\varphi + \frac{1}{Z_0} \, g_z(r) \, \mathbf{u}_z \, , \tag{7}$$

where the common factor

$$\exp(j(\omega t - \beta z - \nu\varphi)) \tag{8}$$

has been omitted.

## SCALAR ANALYSIS

The transverse field function R(s) is a bounded solution
of the scalar wave equation

$$R''(s) + \frac{1}{s} R'(s) + [\,\varepsilon - \lambda^2 - \frac{m^2}{s^2}\,]\, R(s) = 0 \;, \tag{9}$$

where the normalized radial coordinate s is defined as

$$s = k_0 r = 2\pi r/\lambda_0 \;, \tag{10}$$

and the normalized propagation constant $\lambda$ is defined as

$$\lambda = \beta/k_0 = n_{effective} \;. \tag{11}$$

The normalized propagation constant $\lambda$ can be interpreted as
an "effective refractive-index" and the quantity $\lambda^2$ as an
"effective relative permittivity". If the maximum value of
the relative permittivity is $\varepsilon_1$ and the cladding value is $\varepsilon_2$
then

$$\varepsilon_2 \leq \lambda^2 \leq \varepsilon_1 \;. \tag{12}$$

R and R' are, as a consequence of Eq. (9), continuous
functions of the normalized radial coordinate s. A
discontinuity in the permittivity profile $\varepsilon$ induces a
discontinuity in R''.

When $v = 0$ the scalar solution is

TE-modes: $\quad\quad\quad\quad\quad m = 1,$ $\quad\quad\quad\quad\quad\quad$ (13a)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_\varphi = R,$ $\quad\quad\quad\quad\quad\quad$ (13b)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad g_r = -\lambda R,$ $\quad\quad\quad\quad\quad\quad$ (13c)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad g_z = -[R' + \frac{1}{s} R],$ $\quad\quad\quad\quad$ (13d)

and

TM-modes: $\quad\quad\quad\quad\quad m = 1,$ $\quad\quad\quad\quad\quad\quad$ (14a)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_r = R,$ $\quad\quad\quad\quad\quad\quad$ (14b)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad g_\varphi = \lambda R,$ $\quad\quad\quad\quad\quad\quad$ (14c)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_z = -\frac{1}{\lambda} [R' + \frac{1}{s} R].$ $\quad\quad\quad$ (14d)

When $v \neq 0$ the scalar solution is

HE-/EH modes: $\quad\quad\quad\quad m = |v| + q,$ $\quad\quad\quad\quad\quad$ (15a)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_r = R,$ $\quad\quad\quad\quad\quad\quad$ (15b)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_\varphi = q \frac{v}{|v|} R,$ $\quad\quad\quad\quad\quad$ (15c)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_z = -\frac{1}{\lambda} [R' + q \frac{m}{s} R],$ $\quad\quad\quad$ (15d)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad g_r = -q \frac{v}{|v|} \lambda R,$ $\quad\quad\quad\quad$ (15e)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad g_\varphi = \lambda R,$ $\quad\quad\quad\quad\quad\quad$ (15f)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad g_z = -q \frac{v}{|v|} [R' + q \frac{m}{s} R],$ $\quad\quad$ (15g)

where

$$q = -1 \quad \Leftrightarrow \quad \text{HE-mode,} \qquad (16a)$$

$$q = +1 \quad \Leftrightarrow \quad \text{EH-mode.} \qquad (16b)$$

The scalar modes are designated LP$_{ml}$ where

$$m = 0,1, \cdots , \qquad (17a)$$

and

$$l = 1,2, \cdots . \qquad (17b)$$

The scalar expressions for the transverse amplitude functions $e_r$ , $e_\varphi$ , $g_r$ and $g_\varphi$ are exact solutions of the first and second components of Eqs. (4) and (5). For example, the first component of Eq. (4) is

$$e_r'' + \frac{1}{s} e_r' + [\, \varepsilon - \lambda^2 - \frac{1+v^2}{s^2} \,] \; e_r - \frac{2v}{s^2} \, e_\varphi = 0. \qquad (18)$$

The scalar expressions for the axial amplitude functions $e_z$ and $g_z$ are subsequently obtained from the third components of Eqs. (1) and (2). In this process the approximation

$$\varepsilon \approx \lambda^2 \qquad (19)$$

is made. Substitution of the scalar expressions into Eqs. (1) and (2) demonstrates that scalar and exact analyses coincide only in the case of TE-modes.

## A POWER-SERIES EXPANSION METHOD

The method of undetermined coefficients is now employed to solve Eq. (9). This mathematical technique is described in Ref. 2. The fiber profile $\varepsilon(s)$ is assumed given as in Ref. 1. The transverse field function R is expressed at an arbitrary expansion point p as

$$R = \sum_{n=0}^{\infty} a_n(p)(s-p)^n. \qquad (20)$$

At the origin $p = 0$ and substitution of Eq. (20) into Eq. (9) yields

$$a_n = 0, \qquad n < m, \qquad (21a)$$

$$a_m = C_1, \qquad (21b)$$

$$a_n = \frac{1}{m^2-n^2}\left[-\lambda^2 a_{n-2} + \sum_{l=0}^{L_0} e_l a_{n-2-l}\right], \quad n > m, \qquad (21c)$$

where $C_1$ is an arbitrary constant and

$$L_0 = \min(n-2-m, N_0), \qquad (21d)$$

$$a_{-1} = 0. \qquad (21e)$$

If $L_0$ is less than zero then the sum in Eq. (21c) is zero.

At an expansion point $p \neq 0$ the coefficients $a_0$ and $a_1$ are obtained through proper summation of the previous power-series (R and R' are continuous). Again, substitution of Eq. (20) into Eq. (9) yields

$$a_{n+2} = \frac{1}{(n+1)\,(n+2)\,p^2} \left[\, \lambda^2 a_{n-2} + 2p\lambda^2 a_{n-1} \right.$$

$$+ (p^2\lambda^2 + m^2 - n^2)\,a_n - p\,(n+1)\,(2n+1)\,a_{n+1} - \sum_{l=0}^{L_1} p^2 e_l\, a_{n-l}$$

$$\left. - \sum_{l=1}^{L_2} 2p\,e_{l-1} a_{n-l} - \sum_{l=2}^{L_3} e_{l-2} a_{n-l} \,\right], \qquad n = 0,1,\cdots, \qquad (22a)$$

where

$$L_k = \min(n, N_m + k - 1), \qquad k = 1,2,3 . \qquad (22b)$$

The solution in the core is

$$R(s) = C_1\, R_1(s), \qquad\qquad s \le k_0 a, \qquad (23)$$

and the solution in the cladding is

$$R(s) = C_2\, K_m(\,(\lambda^2 - \varepsilon_2)^{1/2}\, s\,), \qquad s \ge k_0 a, \qquad (24)$$

where $K_m$ are modified Bessel functions[3]. The boundary condition at the core-cladding interface is (R and R' are continuous)

$$C_1\, R_1(s_0) = C_2\, K_m(\,(\lambda^2 - \varepsilon_2)^{1/2}\, s_0\,), \qquad (25)$$

$$C_1\, R_1'(s_0) = C_2\, (\lambda^2 - \varepsilon_2)^{1/2}\, K_m'(\,(\lambda^2 - \varepsilon_2)^{1/2}\, s_0\,), \qquad (26)$$

where

$$s_0 = k_0 a. \qquad (27)$$

The condition for a non-trivial coefficient-vector $(C_1, C_2)$ to exist is

$$\det \mathbf{D} = - R_1 \, (\lambda^2 - \varepsilon_2)^{1/2} \, K_m{'} + R_1{'} \, K_m = 0. \tag{28}$$

The normalized propagation constants corresponding to guided modes appear as zeroes of the characteristic equation (28). At cutoff the solution in the cladding is

$$R(s) = C_2 \, s^{-m}, \qquad\qquad s \geq k_0 a, \tag{29}$$

and the boundary condition is

$$C_1 \, R_1(s_0) = C_2 \, s_0^{-m}, \tag{30}$$

$$C_1 \, R_1{'}(s_0) = C_2 \, (-m) \, s_0^{-m-1}. \tag{31}$$

Consequently the cutoff condition is

$$\det \mathbf{D} = R_1 \, m \, s_0^{-m-1} + R_1{'} \, s_0^{-m} = 0. \tag{32}$$

The characteristic equation (28) and the cutoff condition (32) can be unified into

$$L_m = [m + h_m(w)] \, R_1 / s_0 + R_1{'} = 0, \tag{33}$$

where

$$w = (\lambda^2 - \varepsilon_2)^{1/2} \, s_0, \tag{34}$$

and

$$h_m(w) = w\, K_{m-1}(w)/K_m(w), \qquad w > 0, \tag{35a}$$

$$h_m(0) = 0. \qquad \cdot \tag{35b}$$

It is evident from (9) that the origin is a singular point. If the power-series representing the relative permittivity profile $\varepsilon(s)$ is infinite, then there may be singular points of $\varepsilon$ in the complex s-plane. Normally the profile is given as piece-wise polynomial and then the origin is the only singular point.

The steplength $\delta$ must be less than the radius of convergence. If the profile is piece-wise polynomial then the radius of convergence is infinite when expanding at the origin and equal to the distance to the origin, i.e. equal to p, when expanding at $p \neq 0$. The steplength must also, of course, be less than or equal to the distance to the next profile expansion point $p_m$. The steplength may also, especially in multi-mode fibers, be limited by loss of accuracy due to the appearance of terms of great magnitude and alternating signs.

A graded profile may be approximated as piece-wise constant. Then Eq. (9) can be solved in each layer in terms of Bessel functions. Arnold et al.[4] compare the efficiency of this approach with the efficiency of a fourth order Runge-Kutta method. It is found that the Runge-Kutta method

is superior to the stratification method in its computing time by a substantial factor ($\approx 20$). Arnold points out that the stratification procedure itself introduces an error of $O(\delta)$ in each layer (= step) and that redundant information is generated when evaluating the Bessel functions. When employing a fourth order Runge-Kutta method the error in each step is $O(\delta^5)$ and the total error is $O(\delta^4)$.

Consider now the power-series expansion method. We want to investigate, qualitatively, how accuracy and computing time depend on steplength $\delta$ and expansion order N. The investigation is based on the assumption that the behavior of the coefficients of the oscillating function R(s) is comparable with the behavior of the elementary functions sine and cosine. Consequently the absolute value of term number n is estimated as

$$\delta^n/n! \ . \tag{36}$$

If the terms are summed up to and including term number N then the error E can be estimated as

$$E = \delta^{N+1}/(N+1)! \ , \tag{37}$$

and consequently

$$\delta = [\, E \cdot (N+1)! \,]^{1/(N+1)} \ . \tag{38}$$

The accuracy in each step is estimated by E and the computing time in each step is proportional to N. The total computing time is thus proportional to $N/\delta$. The absolute value of the greatest term in expression (36) can be estimated as

$$t_{max} = \max(1, \delta^{\delta}/\delta!).\tag{39}$$

If the sum is assumed to be approximately unity then the number of lost significant figures can be estimated as

$$^{10}\log(t_{max}).\tag{40}$$

Numerical values are given in Table 1 and 2. Stirling's formula is

$$N! \approx N^N \exp(-N)(2\pi N)^{1/2}, \qquad N = 1, 2, \cdots,\tag{41}$$

and yields

$$N/\delta \approx e \approx 2.71, \qquad N \gg 1,\tag{42}$$

$$^{10}\log(\delta^{\delta}/\delta!) \approx {}^{10}\log(e) \cdot \delta \approx 0.43\,\delta, \qquad \delta \gg 1.\tag{43}$$

The total error is the accumulated effect of the errors in each step. Thus, for a certain fixed degree of accuracy in each step, a shorter steplength will produce a larger total error. It is evident from Table 1 and 2 that a small

steplength is inefficient in terms of computing time. A
fourth order Runge-Kutta method corresponds to expansion
order N equal to four. If a high degree of accuracy is
required then the reduction in computing time when changing
from a fourth order method to a high order method may be
very substantial. A very long steplength is prohibited by
severe loss of accuracy. For example, if the machine
precision is eight decimal digits then all accuracy is lost
if a steplength $\delta \approx 20$ is employed.


## NUMERICAL EXAMPLES


The efficiency of the power-series expansion method is
compared with the efficiency of a fourth order Runge-Kutta
method. The computing time for a single evaluation of the
function $L_0$ was measured for different expansion
orders (PSEM) and different steplengths (R.-K.). This was
done for a single-mode step-index fiber (see Fig. 3) and
for a single-mode triangular-ring fiber (see Fig. 4).
Realistic fiber parameters were chosen and a fixed trial
value of the normalized propagation $\lambda$ constant was used.
The step-index core-profile was, of course, represented by
a single constant. The triangular-ring core-profile was
represented by a second degree polynomial (linear in
refractive index) and two constants. When employing the
PSEM the quantities $R_1(s_0)$ and $R_1'(s_0)$ were computed in a

single-step (step-index fiber) and three steps (triangular-ring fiber). The machine precision was 18 decimal digits.

These numerical examples indicate that if a high degree of accuracy is required then the PSEM is superior by a very substantial factor.

**CONCLUSION**

A power-series expansion method, which solves the well-known scalar wave equation for an arbitrary piece-wise polynomial permittivity profile, has been presented. This PSEM is, at least if a high degree of accuracy is required and for certain permittivity profiles, superior in terms of computing time when compared with short steplength methods such as stratification methods and Runge-Kutta methods.

# REFERENCES

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Table titles:

**Table 1.** **Six-figure accuracy in each step ( E = $10^{-6}$ )**

**Table 2.** **Twelve-figure accuracy in each step ( E = $10^{-12}$ )**

Figure captions:

Fig. 1. A cylindrical structure.

Fig. 2. The relative permittivity profile.

Fig. 3. Computing time as a function of accuracy achieved in a single-mode step-index fiber. The dashed line represents a fourth-order Runge-Kutta method and the solid line represents the power-series expansion method.
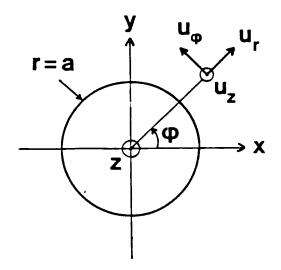
Fig. 4. Computing time as a function of accuracy achieved in a single-mode triangular-ring fiber. The dashed line represents a fourth-order Runge-Kutta method and the solid line represents the power-series expansion method.

| expansion order N | steplength $\delta$ | computing time $N/\delta$ | number of lost figures $^{10}\log(t_{max})$ |
|---|---|---|---|
| 4 | 0.12 | 33.6 | 0.0 |
| 10 | 1.12 | 8.9 | 0.0 |
| 20 | 3.89 | 5.1 | 1.0 |
| 40 | 10.53 | 3.8 | 3.7 |
| 80 | 24.73 | 3.2 | 9.6 |

Table 1

| expansion order N | steplength $\delta$ | computing time $N/\delta$ | number of lost figures $^{10}\log(t_{max})$ |
|---|---|---|---|
| 4 | 0.008 | 532.1 | 0.0 |
| 10 | 0.32 | 31.2 | 0.0 |
| 20 | 2.01 | 9.9 | 0.3 |
| 40 | 7.51 | 5.3 | 2.4 |
| 80 | 20.85 | 3.8 | 8.0 |

Table 2

Fig. 1



Fig. 2

Fig. 3

Fig. 4