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**Abstract**

The modeling of non-reflecting one-dimensional dispersive media is discussed. The media are temporal dispersive with a spatially varying impedance. It is shown that the effects from the variation of the impedance can be matched by the temporal dispersive effects so that the media do not reflect any field regardless of the shape of the incident transient field. The problem of finding reflectionless media is formulated as an inverse problem where the constitutive relation is to be determined as a function of depth given a reflection kernel which is zero. A time domain Green functions technique is used to solve the inverse problem.

# 1 Introduction

Problems concerning non-reflecting layers have been paid much attention during the last decade. Often the aim is to find layers for which the reflection coefficient is minimized in certain frequency bands. Especially this is the case in many radar applications where there is a perfectly conducting wall behind the layer. These frequency selective layers are often constructed using different patches distributed on the surface, *cf.* [2, 11, 13]. There are also attempts to use chiral media as low-reflecting materials, *cf.* [5]. The frequency selective layers give small reflections in one frequency band but may give large reflections at other frequencies. Thus a transient wave containing a wide spectrum of frequencies can be highly reflected from such a layer. In quantum mechanical scattering, inverse scattering methods have been used to construct non-reflecting potentials, *cf.* [8] and [12]. The idea to use inverse scattering methods to design media with certain scattering properties will be adopted in this paper.

This paper concerns the design of a dispersive half-space which is reflectionless for all frequencies, or equivalently, reflectionless for any incident pulse. The problem is formulated as the inverse problem of finding the constitutive relation as a function of depth, given that the reflection data are known to be zero for all times. In general this problem has non-unique solutions. However, it may be reduced to a problem which is uniquely solvable, by assuming some of the parameters in the constitutive relation to be known. In the case of a non-dispersive medium it is well-known that it is sufficient to keep the impedance constant throughout the medium in order to get a non reflecting medium. This is true even if the medium is of finite length, as long as the medium is impedance matched to the surrounding media. For media with temporal dispersion it is no longer possible to obtain a reflectionless medium by keeping the impedance constant. The reason for this is that the polarization of a dispersive medium depends on the history of the waves that have propagated through the medium. As the polarization diminish with time the medium send out energy and even if the wave front experiences a constant impedance the medium will scatter energy.

The technique which is used in the search for reflectionless media is based upon an inverse scattering method in the time domain, referred to as the Green functions technique. The Green functions technique was originally developed for problems concerning one-dimensional direct and inverse scattering from non-dispersive media, *cf.* [10]. Recently it has also been applied to different types of dispersive media [9]. In the Green functions technique Green operators are introduced which map the incident wave to the internal fields in the medium. The operators are represented in terms of kernels, which satisfy a system of first order hyperbolic equations. Most inverse scattering problems are ill-posed and hence scattering data that are contaminated with noise cause large errors in the solution. In the present case the ill-posedness never shows up in the numerical solution, since the scattering data are prescribed and exactly known rather than obtained from measurements. The errors in the solution of the inverse problem are then only due to discretization errors and round-off effects due to the finite precision used by the computer. In practice these errors can be made very small.

In the next section the constitutive relation for a dispersive medium is given in the time domain. In the third section a wave splitting is introduced and the wave equation is rewritten in terms of the split fields. The fourth section contains a presentation of the Green functions technique for dispersive media. In section 5 the inverse problem of finding non-reflecting media is stated and the solution is discussed in some specific cases. Two necessary conditions for a solution to exist are also given in this section. The numerical section, section 6, contains two examples of non-reflecting media. The dispersive models in the examples are based upon the Debye model, which is appropriate for liquids, and the Lorentz model, which is appropriate for solid materials.

# 2 Formulation of the problem

A general linear isotropic dispersive media is described by the following constitutive relation, *cf.* [4], [1] and [6],

$$\left\{ \begin{array}{l} \mathbf{D}(z,t) = \varepsilon_0(\varepsilon_r(z)\mathbf{E}(z,t) + \int_{-\infty}^{t} \chi_e(z,t-t')\mathbf{E}(z,t')\,dt') \\ \mathbf{B}(z,t) = \mu_0(\mu_r(z)\mathbf{H}(z,t) + \int_{-\infty}^{t} \chi_m(z,t-t')\mathbf{H}(z,t')\,dt'), \end{array} \right. \tag{2.1}$$

where $\mathbf{D}(z,t)$ is the displacement field, $\mathbf{E}(z,t)$ is the electric field, $\mathbf{B}(z,t)$ is the magnetic induction and $\mathbf{H}(z,t)$ is the magnetic field. The medium thus has a non-constant relative permittivity, $\varepsilon_r(z)$, and a non-constant relative permeability, $\mu_r(z)$. The medium is furthermore dispersive and the dispersion is in the time domain modelled by the electric and magnetic susceptibility kernels, $\chi_e(z,t)$ and $\chi_m(z,t)$. A more general model for linear dispersive isotropic media would include biisotropic effects, but that will not be considered in this paper.

The dispersive medium occupies the semi-infinite region $z > 0$. The other half-space, $z < 0$, is a non-dispersive medium with a constant permittivity $\varepsilon_r$ and permeability $\mu_r$ such that $\varepsilon_r/\mu_r = \varepsilon_r(0)/\mu_r(0)$, i.e. the two half-spaces are impedance matched at $z = 0$. A transient wave propagating in the positive $z-$direction impinges at $z = 0$ on the dispersive half-space at time $t = 0$. The aim of the present paper is to find different $z-$dependent constitutive relations for the medium such that the reflected field from the medium vanishes for any incident field. In order to clarify the basic idea of the technique used in the paper, only the constitutive relation with $\chi_m \equiv 0$ will be considered in the analysis. From now on the index of the electric susceptibility kernel will be omitted, i.e. $\chi(z,t) \equiv \chi_e(z,t)$.

# 3 The wave equation and wave splitting

In this section the wave equation for the medium described by the constitutive relation in Eq. (2.1) will be presented and written as a system of first order hyperbolic equations. A wave splitting technique will be introduced, where the total electric field is split into two parts corresponding to waves traveling in the positive and negative $z$-direction. This wave splitting technique is the basis for the Green functions technique presented in the next section.

Since the incident wave is a plane wave at normal incidence it can, without loss of generality, be considered to be linearly polarized in the $x-$direction i.e. $\mathbf{E}(z,t) = E(z,t)\hat{x}$. The corresponding magnetic field is then directed in the $y-$direction, i.e. $\mathbf{H}(z,t) = H(z,t)\hat{y}$. In the region $z > 0$ the constitutive relation in Eq. (2.1), with $\chi_m(z,t) \equiv 0$, and the Maxwell equations lead to

$$\begin{aligned} \partial_z E(z,t) &= -\mu_0\mu_r(z)\partial_t H(z,t) \\ \partial_z H(z,t) &= -\varepsilon_0\left(\varepsilon_r(z)\partial_t E(z,t) + [\chi(z,\cdot) * \partial_t E(z,\cdot)](t)\right). \end{aligned} \tag{3.1}$$

The conventional short hand notation for convolution

$$[f * g](t) = \int_0^t f(t-t')g(t')\,dt'$$

introduced in Eq. (3.1) will be used throughout the rest of the paper. Notice that the lower integration limit in Eq. (3.1) is zero, since there is no electric field in the half space for negative times. The Maxwell equations can be written in a matrix notation as

$$\partial_z \begin{pmatrix} E \\ H \end{pmatrix} = -\begin{pmatrix} 0 & \mu_0\mu_r \\ \varepsilon_0\varepsilon_r + \varepsilon_0\chi* & 0 \end{pmatrix} \partial_t \begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{A}\begin{pmatrix} E \\ H \end{pmatrix}. \tag{3.2}$$

This system of first order partial differential equations (PDE) are equivalent to the wave equation for the electric field

$$\left(\partial_z^2 - \frac{1}{c(z)^2}\partial_t^2\right) E(z,t) - \frac{\mu_r(z)}{c_0^2}\left[\chi(z,\cdot) * \partial_t^2 E(z,\cdot)\right](t) = 0,$$

where $c_0 = (\varepsilon_0\mu_0)^{-\frac{1}{2}}$ is the speed of light in vacuum and $c(z) = (\varepsilon_0\varepsilon_r(z)\mu_0\mu_r(z))^{-\frac{1}{2}}$ is the wavefront speed in the medium.

A wave splitting is now done according to the principal part, $\partial_z^2 - c(z)^{-2}\partial_t^2$, of the wave equation. The following change of basis is introduced:

$$E^{\pm}(z,t) = \frac{1}{2}\left\{E(z,t) \pm Z(z)H(z,t)\right\},$$

where $Z(z)$ is the wave impedance

$$Z(z) = \sqrt{\frac{\mu_0 \mu_r(z)}{\varepsilon_0 \varepsilon_r(z)}}.$$

In a non-dispersive region with constant permittivity, the split fields $E^+$ and $E^-$ are left (negative $z$ direction) and right (positive $z$ direction) moving waves, respectively. In a matrix form the change of basis reads

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & Z(z) \\ 1 & -Z(z) \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{P} \begin{pmatrix} E \\ H \end{pmatrix}.$$

Thus the split fields $E^+$ and $E^-$ satisfy the following relations

$$\begin{aligned} E^+ + E^- &= E \\ E^- - E^+ &= Z(z)H. \end{aligned}$$

It is also seen that in a non-dispersive medium the matrix $\mathbf{P}$ diagonalizes the matrix $\mathbf{A}$ in Eq. (3.2). The Maxwell equations may now be written in terms of the new basis $E^{\pm}$ as a system of first order hyperbolic equations

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = ((\partial_z \mathbf{P})\mathbf{P}^{-1} + \mathbf{P}\mathbf{A}\mathbf{P}^{-1}) \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}. \tag{3.3}$$

The elements $\alpha$, $\beta$, $\gamma$ and $\delta$ read

$$\begin{cases} \alpha = -\dfrac{1}{c(z)}\partial_t + \dfrac{1}{2}\partial_z \ln Z(z) - \dfrac{c(z)}{2c_0^2}\chi * \partial_t \\[2ex] \beta = -\dfrac{1}{2}\partial_z \ln Z(z) - \dfrac{c(z)}{2c_0^2}\chi * \partial_t \\[2ex] \gamma = -\dfrac{1}{2}\partial_z \ln Z(z) + \dfrac{c(z)}{2c_0^2}\chi * \partial_t \\[2ex] \delta = \dfrac{1}{2}\partial_z \ln Z(z) + \dfrac{1}{c(z)}\partial_t + \dfrac{c(z)}{2c_0^2}\chi * \partial_t. \end{cases}$$

# 4 The Green operators

From arguments based upon invariance under time translation and causality the following representation of the internal split fields are seen to hold, *cf.* [10] and [9],

$$\begin{aligned} E^+(z, t + \tau(z)) &= \mathcal{G}^+ E^+(0, t) = a(z)E^+(0, t) + [G^+(z, \cdot) * E^+(0, \cdot)](t) \\ E^-(z, t + \tau(z)) &= \mathcal{G}^- E^+(0, t) = [G^-(z, \cdot) * E^+(0, \cdot)](t). \end{aligned} \tag{4.1}$$

Here $\tau(z) = \int_0^z 1/c(z')\, dz'$ is the travel-time for the wave-front and $a(z)$ is the attenuation of the wave-front. In these representations wave-front time is used, *i.e.*,

$t = 0$ at a point $z$ is the time the wave front reaches that point. The operators $\mathcal{G}^{\pm}$ are referred to as the Green operators and the kernels $G^{\pm}(z,t)$ as the Green kernels. The representations in Eq. (4.1) imply the boundary values of the Green kernels

$$\begin{aligned} G^{+}(0,t) &= 0 \\ G^{-}(0,t) &= R(t), \end{aligned}$$

where $R(t)$ is the reflection kernel of the medium. Thus $G^{-}(0,t) \equiv 0$ for a medium which is non-reflecting for any incident field $E^{+}(0,t)$. A set of first order hyperbolic equations can be obtained for the Green kernels, *cf.* [10] and [7]. Differentiation with respect to $z$ of the representations in Eq. (4.1) gives

$$\begin{aligned} &\begin{pmatrix} \partial_z E^{+} \\ \partial_z E^{-} \end{pmatrix} (z, t+\tau(z)) + \partial_z \tau(z) \begin{pmatrix} \partial_t E^{+} \\ \partial_t E^{-} \end{pmatrix} (z, t+\tau(z)) = \\ &\qquad = \begin{pmatrix} \partial_z a(z) \\ 0 \end{pmatrix} E^{+}(0,t) + \left[ \begin{pmatrix} \partial_z G^{+}(z,\cdot) \\ \partial_z G^{-}(z,\cdot) \end{pmatrix} * E^{+}(0,\cdot) \right](t). \end{aligned} \tag{4.2}$$

The $z-$derivatives of $E^{\pm}$ are eliminated by utilizing the dynamics in Eq. (3.3). Furthermore, $E^{\pm}(z, t+\tau)$ and $\partial_t E^{\pm}(z, t+\tau)$ are expressed in terms of $E^{+}(0,t)$ using Eq. (4.1). The only field appearing in the resulting equations is $E^{+}(0,t)$ and since this is an arbitrary incident field the equations and initial condition for the Green kernels follow

$$\begin{aligned} \partial_z G^{+}(z,\ t) = \frac{1}{2} \left( \partial_z \ln Z(z) \right) \left( G^{+}(z,t) - G^{-}(z,t) \right) - \frac{c(z)}{2c_0^2} \left( a(z) \partial_t \chi(z,t) + \right. \\ \left. + \partial_t \left[ \chi(z,\cdot) * \left( G^{+}(z,\cdot) + G^{-}(z,\cdot) \right) \right](t) \right) \end{aligned} \tag{4.3}$$

$$\begin{aligned} \partial_z G^{-}(z,\ t) - \frac{2}{c(z)} \partial_t G^{-}(z,t) = -\frac{1}{2} \left( \partial_z \ln Z(z) \right) \left( G^{+}(z,t) - G^{-}(z,t) \right) + \\ + \frac{c(z)}{2c_0^2} \left( a(z) \partial_t \chi(z,t) + \partial_t \left[ \chi(z,\cdot) * \left( G^{+}(z,\cdot) + G^{-}(z,\cdot) \right) \right](t) \right) \end{aligned} \tag{4.4}$$

$$G^{-}(z,0) = \frac{1}{4} a(z) \left( c(z) \partial_z \ln Z(z) - \left( \frac{c(z)}{c_0} \right)^2 \chi(z,0) \right). \tag{4.5}$$

Also the equation for the attenuation $a(z)$ follows from Eq. (4.2)

$$\partial_z a(z) + a(z) \left( \frac{c}{2c_0^2} \chi(z,0) - \frac{1}{2} \partial_z \ln Z(z) \right) = 0.$$

Since $a(0) = 1$ it is seen that

$$a(z) = \sqrt{\frac{Z(z)}{Z(0)}} \exp\left\{ -\frac{1}{2c_0^2} \int_0^z c(z') \chi(z',0)\, dz' \right\}.$$

# 5 Non-reflecting media

In order to find non-reflecting media, Eqs. (4.3) and (4.4) are to be solved for $\varepsilon_r(z)$, $\mu_r(z)$ and $\chi(z,t)$ using the initial condition in Eq. (4.5) and the boundary conditions

$$\begin{cases} G^+(0,t) = 0 \\ G^-(0,t) = 0. \end{cases} \tag{5.1}$$

This problem has non-unique solutions, since a function of both $z$ and $t$ and two functions of $z$ are to be determined from one function $G^-(0,t)$.

## 5.1 Non-dispersive media

First consider the simple case of a non-dispersive medium, i.e. when $\chi(z,t) \equiv 0$. All media which have constant impedance, $Z(z) = const.$, for $-\infty < z < \infty$ are then reflectionless. This well-known fact follows immediately from Eqs. (4.3) and (4.4), since the source term in these equations as well as the initial condition, Eq. (4.5), vanish.

An interesting question is if there are any solutions to the non-dispersive problem for which the impedance is non-constant. This question is a special case of the uniqueness problem for the inverse problem. It is plausible that the inverse problem of finding the impedance profile from the reflection kernel is unique and the numerical scheme indicates that this is the case. However, the authors have not seen any general uniqueness proof.

## 5.2 Dispersive media

In the case of dispersive media the inverse problem is non-unique. In order to formulate an inverse problem which has the potential of being uniquely solvable, the functions to be determined have to be reduced to a single function of one argument. The price to be paid for this is that a corresponding physical solution may not exist. There are four natural choices of function to be determined, the impedance $Z(z)$, the phase velocity $c(z)$, the susceptibility kernel $\chi(z,t)$ with a given time dependence, and the susceptibility kernel with a given $z-$dependence. The last choice is not a good one since $\chi(z,t)$ has to satisfy certain conditions in order for the medium to satisfy energy conservation, *cf.* [6]. Moreover, even a $\chi(z,t)$ that satisfies these conditions might be unphysical. The first three choices are assumed to be applicable. Of course one may use the permittivity, $\varepsilon_r(z)$, and the permeability, $\mu_r(z)$, as independent functions instead of the impedance, $Z(z)$, and phase velocity, $c(z)$.

It is hard to find sufficient conditions for a solution to exist for the inverse problem of finding a reflectionless half-space. However, necessary conditions are easier to find and below two such conditions are discussed. The first condition says that the impedance has to be an increasing function of $z$ at $z = 0$ if the susceptibility kernel is non-zero at $z = 0$, i.e. if $\chi(0,t) \not\equiv 0$. The second condition is that the initial condition of the Green kernel $G^-$ has to be a $C^\infty$ function, i.e. infinitely differentiable, with respect to the $z-$coordinate.

The condition that the impedance is an increasing function of $z$ at $z = 0$ follows from energy conservation. In [6] it was shown that the susceptibility kernel $\chi(z,t)$ for a passive medium has to satisfy

$$\chi(z,0) \geq 0, \tag{5.2}$$

and if $\chi(z,0) = 0$ then

$$\chi_t(z,0) \geq |\chi_t(z,t)|, \qquad t > 0. \tag{5.3}$$

Assume first that $\chi(0,0) \neq 0$. From Eqs. (4.5) and (5.2) it follows that

$$\partial_z \ln Z(z)|_{z=0} = \frac{c(0)}{c_0^2}\chi(0,0) > 0.$$

Thus the $z-$derivative of the impedance is a positive quantity at $z = 0$. Next consider the case $\chi(0,0) = 0$. From the initial condition, Eq. (4.5), it is seen that $\partial_z \ln Z(z)|_{z=0} = 0$ By letting $t = 0$ and $z = 0$ in Eq. (4.4) it follows that

$$G_t^-(0,0) = \frac{c(0)}{2}G_z^-(0,0) - \left(\frac{c(0)}{2c_0}\right)^2 \chi_t(0,0).$$

It has then been used that $G^+(0,0) = G^-(0,0) = 0$ and $a(0) = 1$. From Eq. (4.5)

$$G_z^-(0,0) = \frac{1}{4}\left(c(0)\partial_{zz} \ln Z(z)|_{z=0} - \left(\frac{c(0)}{c_0}\right)^2 \chi_z(0,0)\right).$$

Since $G_t^-(0,0) = 0$ for a non-reflecting medium it then follows that

$$\partial_{zz} \ln Z(z)|_{z=0} = \frac{1}{c_0^2}\left(c(0)\chi_z(0,0) + 2\chi_t(0,0)\right).$$

But when $\chi(0,0) = 0$ then $\chi_z(0,0) \geq 0$ according to Eq. (5.2) and since $\chi(0,t)$ is assumed not to be identically zero it follows that $\chi_t(0,0) > 0$ according to Eq. (5.3). The result is that $\partial_{zz} \ln Z(z)|_{z=0} > 0$ and since $\partial_z \ln Z(z)|_{z=0} = 0$ it follows that $\partial_{zz} Z(z)|_{z=0} > 0$. Hence also in this case the impedance must be an increasing function of $z$ at $z = 0$.

Another necessary condition for a solution to exist is that $G^-(z,0)$ has to be a $C^\infty$ function on the interval $z > 0$. This condition results from the initial condition, Eq. (4.5), and propagation of singularity arguments, which say that a discontinuity in the $p$:th $z$-derivative of $G^-(z,0)$ will propagate along the characteristic of Eq. (4.4) and give rise to a discontinuity in the $p$ :th derivative of $G^-(0,t)$. Thus for a non-reflecting medium $G^-(z,0)$ has to be a $C^\infty$ function on the interval $z > 0$, since $G(0,t) \equiv 0$ is a $C^\infty$ function of time. This condition is satisfied if $\varepsilon_r(z)$, $\mu_r(z)$ and $\chi(z,0)$ are $C^\infty$ functions of $z$.

### 5.3 Non-magnetic dispersive media

Most materials are non-magnetic, i.e. they have $\mu_r(z) = 1$. In the previous subsection it was seen that the impedance has to be an increasing function of $z$ at the boundary of the half-space in order for the medium to be non-reflecting. In the case of a non-magnetic medium it follows that the permittivity has to be a decreasing function of $z$ at $z = 0$. Thus the following fundamental necessary condition can be stated.

> *A non-magnetic dispersive medium with $\chi(z,t) \not\equiv 0$ and/or $\varepsilon_r(z) \not\equiv 0$ can not be reflectionless for a transient wave incident from vacuum.*

In other words, a non-reflecting non-magnetic medium has to have $\varepsilon_r > 1$ for $z < 0$.

## 6 Numerical examples

In this section two different types of non-reflecting media are presented. Both media are non-magnetic, which implies that the half-space $z < 0$ is a dielectric medium with $\varepsilon_r > 1$. In both examples $\varepsilon_r = 2$ for $z < 0$. A given susceptibility kernel for the medium is assumed and the permittivity as a function of $z$ is to be determined. The permittivity profile can be constructed down to any depth, but in the examples only the profile for the first 10 centimeters of the medium are shown. The equations for the Green kernels were discretized by the trapezoidal rule, *cf.* [10]. The convergence of the numerical algorithm is then quadratic. In the first example the dispersive medium is a Debye medium and the susceptibility kernel in Eq. (2.1) is given by

$$\chi(t) = \alpha \exp(-\frac{t}{\tau}).$$

This model is relevant for polar liquids, *cf.* [1]. The parameters $\alpha$ and $\tau$ are in the example independent of $z$ and their values are $\alpha = 5 \cdot 10^8\,\mathrm{s}^{-1}$ and $\tau = 10^{-9}\,\mathrm{s}$. Given these values of $\alpha$ and $\tau$ the Green functions equations (4.3) and (4.4) were solved for $\varepsilon_r(z)$ using the initial condition in Eq. (4.5) and the homogeneous boundary conditions Eq. (5.1). The resulting permittivity is given in figure 1. In this example 500 timesteps were used in the discretization.

In the second example a Lorentz medium is considered. Normally this model is used when frequencies above the microwave region are involved. The dispersion then is a result of resonances of the atoms, *cf.* [1]. However there are media which are of Lorentz type even at low frequencies, e.g. gyrotropic media. The simplest case of a Lorentz medium has the following susceptibilty kernel:

$$\chi(t) = \omega_p^2 \frac{\sin \nu t}{\nu} e^{-\gamma t}. \tag{6.1}$$

Here $\omega_p$ is the plasma frequency and the frequency $\nu$ can be expressed in terms of the internal resonant frequency $\omega_0$ and the damping constant $\gamma$ as $\nu = \sqrt{\omega_0^2 - \gamma^2}$.

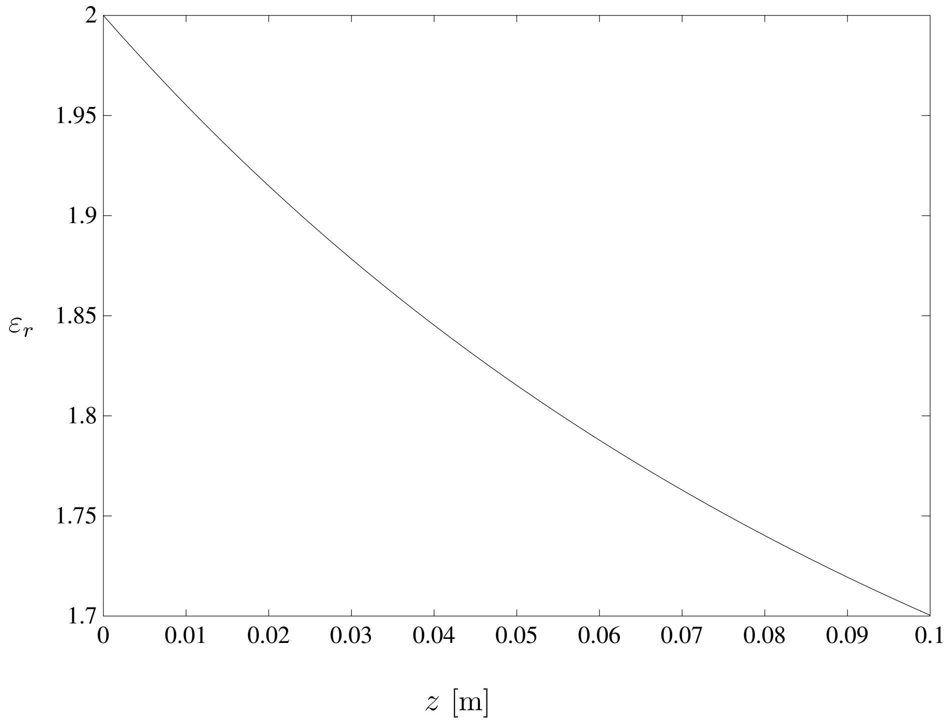


**Figure 1**: The relative permittivity as a function of depth for a reflectionless Debye medium with $\alpha = 5 \cdot 10^8\,\mathrm{s}^{-1}$ and $\tau = 10^{-9}\,\mathrm{s}$

Also in the second example the susceptibilty kernel is independent of $z$. The values of the parameters in the susceptibility kernel are in this example $\omega_p = 5 \cdot 10^9\,\mathrm{s}^{-1}$, $\omega_0 = 2 \cdot 10^{10}\,\mathrm{s}^{-1}$ and $\gamma = 5 \cdot 10^9\,\mathrm{s}^{-1}$.

The numerical solution of Eqs. (4.3) and (4.4) gives the permittivity in figure 3 for a non-reflecting medium. The oscillating behavior of the permittivity is expected, since the variation of the permittivity is to compensate the oscillatory behavior of the susceptibility kernel. In the discretization of the equations 500 gridpoints were used. To see that the permittivity profile in figure 2 really results in a reflectionless medium the constitutive relation for the constructed material was used in the solution of the direct scattering problem. The resulting reflection kernel was compared to the reflection kernel for a non-dispersive medium with the permittivity profile in figure 2. As seen from figure 3 the reflection kernel for the reflectionless medium is negligible, and thus enough number of gridpoints were used in the construction of the permittivity profile.

# 7 Conclusions

The present paper gives an example of media-design using inverse scattering methods. A nice feature is that clean data can be used in the solution of the inverse problem. The inverse problem is then well posed and well conditioned. The intention with the paper is to present a method for designing media with prescribed

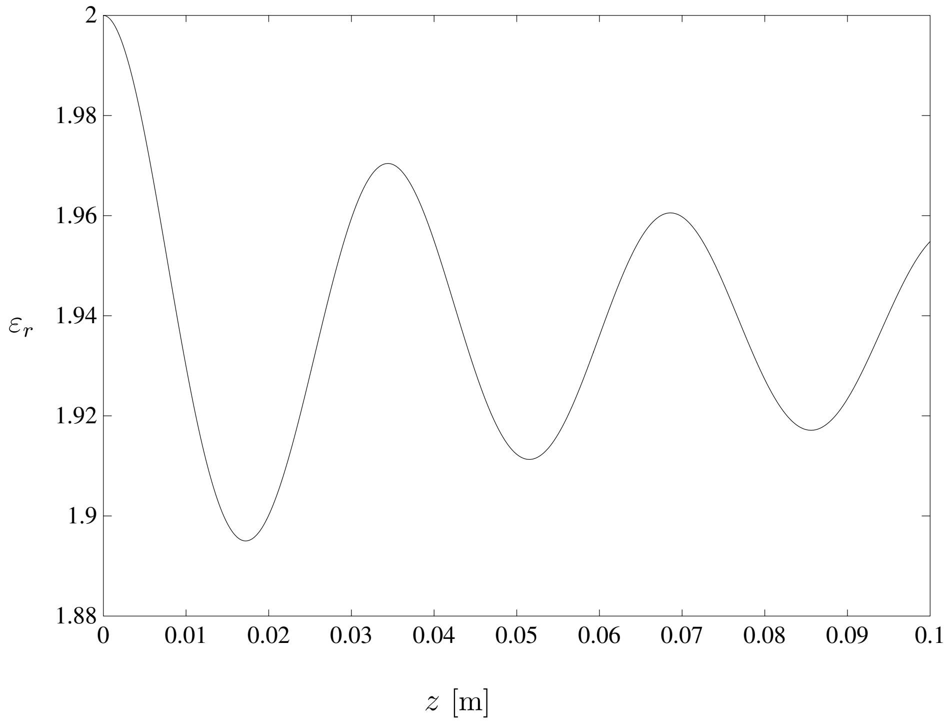


**Figure 2**: The relative permittivity as a function of depth for a reflectionless Lorentz medium with $\omega_p = 5 \cdot 10^9\,\mathrm{s}^{-1}$, $\omega_0 = 2 \cdot 10^9\,\mathrm{s}^{-1}$ and $\gamma = 5 \cdot 10^9\,\mathrm{s}^{-1}$

reflection properties. At this stage there have been no considerations of applications of the results. There are a number of similar problems that can be looked upon with the present technique. One obvious extension of the present project is to consider reflectionless media with high losses. This is a classic problem with obvious applications that has been extensively studied. The new ingredients that the present technique can add is a time domain treatment and the use of dispersive media. Another area where the present technique might be of interest is the construction of damping materials in elastodynamics, *cf.* [3].

## Acknowledgment

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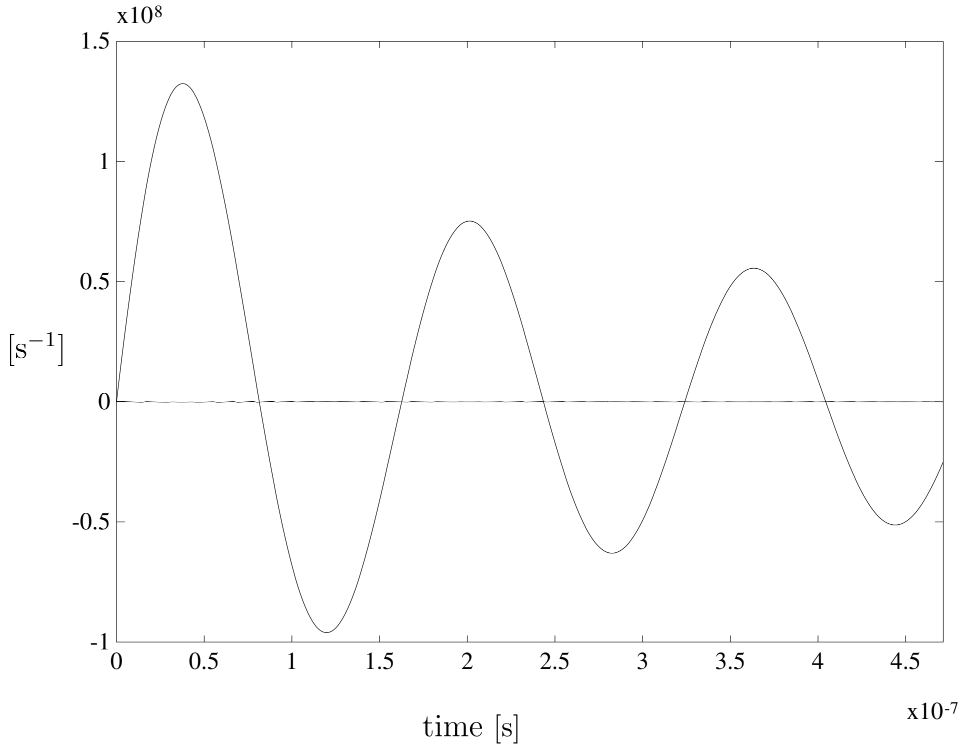


**Figure 3**: The reflection kernel for a medium with zero susceptibility kernel (oscillating curve) and for the constructed reflectionless Lorentz medium with $\omega_p = 5 \cdot 10^9$ s$^{-1}$, $\omega_0 = 2 \cdot 10^9$ s$^{-1}$ and $\gamma = 5 \cdot 10^9$ s$^{-1}$ (almost straight line). Both media have a permittivity given by figure 2.


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