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Abstract

A general mixed initial-boundary value problem for a non-local hyperbolic equation, relevant for the study of the propagation of transient electro-magnetic waves in flat slabs, consisting of dispersive stratified complex media, and excited by incident plane waves, is treated. Theorems regarding unique solvability and causality are presented and proved by a time domain method similar to the one used in the scalar (isotropic) case. The Green functions equations for the problem are derived and proved to be uniquely solvable. The theorems are applicable to wave propagation in, e.g., large classes of dispersive bi-isotropic, an-isotropic, and bi-an-isotropic media.

1 Introduction

Propagation of transient waves in stratified dispersive media has been studied extensively during the last ten years, see, e.g., Refs. [3, 4, 8–10, 12, 14, 15, 17]. Due to a large number of applications, special interest has been paid to electro-magnetic wave propagation in linear dispersive complex media, e.g., chiral or, more generally, bi-isotropic media [15, 17], and an-isotropic media [8]. The dispersive effects in these media are modelled with time convolution in the constitutive relations. The corresponding inverse scattering problems have also been addressed [3, 4, 7, 9, 14, 16]. In the analysis of the propagation of transient waves in these media, two different, but related, methods are available, namely the invariant imbedding technique and the Green functions approach. These methods rely on the assumption, that the propagation problem has a unique well-behaved solution in every bounded time interval, which is sufficient from the numerical point of view, and that strict causality holds for dispersive media, i.e., the speed of the wave front is lower than or equal to the speed given by the non-dispersive properties of the medium. In Ref. [19], it is proved that this is true for stratified, dispersive, isotropic slabs, subject to transient normal plane wave incidence, provided the permittivity and permeability at the boundaries are continuous, i.e., in the case without wave impedance mismatch. Furthermore, it is proved that the speed of the wave front is independent of the dispersion of the medium as well as the incident wave. The fundamental theorem in Ref. [19] asserts the existence of a unique well-behaved solution to the weak canonical problem in every bounded time interval. In the present paper, the theorems in Ref. [19] are generalized, not only to a large class of dispersive complex media, but also to various wave impedance mismatch cases, by employing the time domain method in Ref. [19], i.e., with classical means.

The investigations in, e.g., [7, 8, 15–17] suggest the study of the following mixed initial-boundary value problem, defined in the product set $(x, s) \in (0, 1) \times \mathbb{R}$:

$$\begin{cases} \begin{pmatrix} (\partial_x + \partial_s)\boldsymbol{e}^+(x,s) \\ (\partial_x - \partial_s)\boldsymbol{e}^-(x,s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \boldsymbol{e}^+(x,s) \\ \boldsymbol{e}^-(x,s) \end{pmatrix} + \int_{-\infty}^{s} \mathbf{a}(x, s-s') \begin{pmatrix} \boldsymbol{e}^+(x,s') \\ \boldsymbol{e}^-(x,s') \end{pmatrix} ds', \\ \boldsymbol{e}^{\pm}(x,s) = \mathbf{0}, \quad s \le 0, \\ 2\boldsymbol{e}^i(s) = t_0 \boldsymbol{e}^+(+0,s) + r_0 \boldsymbol{e}^-(+0,s), \\ \boldsymbol{e}^-(1-0,s) = r_1 \boldsymbol{e}^+(1-0,s). \end{cases} \tag{1.1}$$

Here, s and x are the travel-time coordinates for time and slab depth, respectively, and $\boldsymbol{e}^i(s)$, $\boldsymbol{e}^{\pm}(x,s) \in M_{2\times 1}(\mathbb{R})$, while $\mathbf{a}(x,s)$, $\mathbf{b}(x) \in M_{4\times 4}(\mathbb{R})$, for each ordered pair (x,s), where $M_{m\times n}(\mathbb{R})$ is the linear space over $\mathbb{R}$ consisting of the $m \times n$-matrices with real entries. The real numbers r_0, $t_0 \neq 0$, and r_1 are due to wave impedance mismatch at the slab walls, $x = 0$ and $x = 1$, respectively. The relevant material properties of the complex medium are contained in the functions $\mathbf{a}$ and $\mathbf{b}$. Moreover, the incident electric field $\boldsymbol{e}^i$ at the front wall is causal, i.e., $\boldsymbol{e}^i(s) = \mathbf{0}$ for all $s < 0$. The vector fields $\boldsymbol{e}^{\pm}$ have been obtained by a wave splitting, i.e., a change of the dependent variables, such that $\boldsymbol{e}^{\pm}$ represent the right- and left going waves in the isotropic media outside the slab, respectively, see, e.g., Refs. [7, 8, 15–17]. Further details on the physical interpretation of these vector fields and the functions $\mathbf{a}$ and $\mathbf{b}$ can also be found in Section 5. The second equation above shows that the slab is initially quiescent; therefore, $\int_{-\infty}^{s}$ can be substituted for $\int_0^s$ in the first one.

The matrix notation in Eq. (1.1) is appropriate for the wave propagation and scattering problems referred to above, and it is employed throughout this paper. Every vector (in the plane) is identified with its column vector representation in the usual basis, i.e., as a 2×1-matrix, and is typed in italic boldface. Quadratic matrices are typed in Roman boldface.

In Section 2 of this paper, the canonical problem corresponding to Eq. (1.1) is examined. In Section 3, it is proved, that the problem (1.1) is indeed uniquely solvable in each bounded time interval, that strict causality holds in this general vector case, and, furthermore, that the speed of the wave front is precisely one. In Section 4, the Green functions equations are derived and proved uniquely solvable. Finally, in Section 5, Eq. (1.1) is derived for a large class of bi-isotropic cases.

2 The canonical problem

In this section, the canonical problem corresponding to the problem (1.1) is studied. In Theorem 2.1 below, it is proved that this problem is uniquely solvable in a weak sense in each bounded time interval. The proof is similar to the one given in the scalar, isotropic case by T. M. Roberts in Ref. [19], generalizing theorems of S. Aoubi [1] and Courant-Hilbert [6]. See also Ref. [2], where unique solvability of partial integro-differential equations is discussed. The basic idea in the proof of Theorem 2.1 is the repeated use of the the Banach fixed-point theorem, see, e.g., Ref. [20]. Theorem 2.2 shows that the regularity of the solution in the previous theorem is increased with the regularity of the memory function $\mathbf{a}$, which admits the definition of the Green functions employed in, e.g., [7, 8, 15–17].

In the theorems and the proofs below, the following definitions and facts are employed: If $A \subset \mathbb{R}^d$ is an open set, the real linear space consisting of all functions $\mathbf{f} : A \to M_{m\times n}(\mathbb{R})$ with **bounded and continuous** derivatives up to order k in A is denoted by $\mathcal{C}^k_{m\times n}(A)$. This function space is complete furnished with the norm

$$\|\mathbf{f}\| = \max_{i,j} \|f_{i,j}\|_\infty = \max_{i,j}(\sup_{\boldsymbol{x}\in A} |f_{i,j}(\boldsymbol{x})|), \tag{2.1}$$

where $f_{i,j}$ are the components of $\mathbf{f}$. The class $\mathcal{C}_{m\times n}(\overline{A})$ is defined analogously. The product space $\mathcal{C}_{m\times n}(A) \times \mathcal{C}_{m\times n}(A)$ over the real numbers, equipped with the norm $\|(\cdot, \cdot\cdot)\| = \max(\|\cdot\|, \|\cdot\cdot\|)$, where the norm $\|\cdot\|$ is defined in Eq. (2.1), is also a Banach space. Convergence in these norm-topologies is called uniform. By straightforward generalization of a theorem in real analysis, one can prove, that, if the sequence $(\mathbf{f}_j)_{j=1}^{\infty} \in \mathcal{C}^1_{m\times n}(A) \times \mathcal{C}^1_{m\times n}(A)$ converges pointwise to $\mathbf{f}$ in A, and if $(\partial_i \mathbf{f}_j)_{j=1}^{\infty} \in \mathcal{C}_{m\times n}(A) \times \mathcal{C}_{m\times n}(A)$ converges uniformly to $\mathbf{g}$ in A, then $\mathbf{g} \in \mathcal{C}_{m\times n}(A) \times \mathcal{C}_{m\times n}(A)$, $\mathbf{f}$ is differentiable in A with respect to the i-th coordinate, and $\partial_i \mathbf{f} = \mathbf{g}$ in A. Furthermore, recall that a function f on a Banach space $(B, \|\cdot\|)$ is called a contraction, if there exists a non-negative number $r < 1$ such that $\|f(x) - f(y)\| \le r\|x - y\|$ for all points x and y in B, and that the Banach fixed-point theorem under these circumstances guarantees that f has a unique fixed point in B, i.e., there exists precisely one point $x \in B$ such that $f(x) = x$. Finally, the Heaviside step function is denoted by H, and $\mathbf{I}$ is the 2×2 identity matrix.

In the light of Duhamel's principle, it is appropriate to use the matrix setting in Theorem 2.1, i.e., to treat the two canonical problems, due to the different polarizations of the incident wave, together. Geometrical quantities defined in the theorem or the proof are illustrated in Figure 1. The main theorem of this section is

Theorem 2.1. *(Weak canonical problem) Let the given functions* $\mathbf{a} \in \mathcal{C}_{4\times4}(\mathbb{I} \times \mathbb{R}_+)$ *and* $\mathbf{b} \in \mathcal{C}_{4\times4}(\mathbb{I})$*, where* $\mathbb{I} = (0, 1)$ *and* $\mathbb{R}_+ = (0, \infty)$*, be decomposed into* $\mathcal{C}_{2\times2}$*-blocks according to*

$$\mathbf{a}(x, s) = \begin{pmatrix} \mathbf{a}_{11}(x, s) & \mathbf{a}_{12}(x, s) \\ \mathbf{a}_{21}(x, s) & \mathbf{a}_{22}(x, s) \end{pmatrix}, \quad \mathbf{b}(x) = \begin{pmatrix} \mathbf{b}_{11}(x) & \mathbf{b}_{12}(x) \\ \mathbf{b}_{21}(x) & \mathbf{b}_{22}(x) \end{pmatrix}, \quad (x, s) \in \mathbb{I} \times \mathbb{R}_+.$$

Define trapezoids by $Q_{2n} = \{(x, s) \in \mathbb{I} \times \mathbb{R} : -\infty < s < x + 2n\}$*, and line segments by* $L^{\pm} = \cup_{k=0}^{\infty} \{(x, \pm x + 2k) \in \mathbb{I} \times \mathbb{R}_+\}$ *and* $L = L^+ \cup L^-$*. Furthermore, let* r_1*,* r_0*, and* $t_0 \neq 0$ *be given real numbers. Then, for every integer* $n \ge 0$*, the initial-boundary value problem defined in the product set* $\mathbb{I} \times \mathbb{R}$ *by*

$$\begin{cases} \begin{pmatrix} (\partial_x + \partial_s)\mathbf{u}^+(x, s) \\ (\partial_x - \partial_s)\mathbf{u}^-(x, s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \mathbf{u}^+(x, s) \\ \mathbf{u}^-(x, s) \end{pmatrix} + \displaystyle\int_{-\infty}^{s} \mathbf{a}(x, s - s') \begin{pmatrix} \mathbf{u}^+(x, s') \\ \mathbf{u}^-(x, s') \end{pmatrix} ds', \\ \mathbf{u}^{\pm}(x, s) = \mathbf{0}, \quad s \le 0, \\ 2\mathbf{I}H(s) = t_0 \mathbf{u}^+(+0, s) + r_0 \mathbf{u}^-(+0, s), \\ \mathbf{u}^-(1 - 0, s) = r_1 \mathbf{u}^+(1 - 0, s), \end{cases} \tag{2.2}$$

has a unique solution $\mathbf{u}^{\pm} \in \mathcal{C}_{2\times2}(Q_{2n} \setminus L)$ *in the weak sense in* Q_{2n}*, i.e., integrated along the characteristics. Thus,* $(\partial_x \pm \partial_s)\mathbf{u}^{\pm}$ *are understood as derivatives with respect to the vectors* $(1, \pm1)$*, respectively. The solution is equal to zero in* Q_0*, and for every* $j \in \{0, 1, \ldots, 2n\}$*, the restrictions of* $\mathbf{u}^{\pm}$ *to* T_j *can be extended continuously to* $\overline{T_j}$*, where the open set* T_j *is defined as* $\{(x, s) \in \mathbb{I} \times \mathbb{R} : x + j - 1 < s < -x + j + 1\}$ *if* j *is an odd integer, and* $T_j = \{(x, s) \in \mathbb{I} \times \mathbb{R} : -x + j < s < x + j\}$ *if* j *is even. Moreover,* $\mathbf{u}^{\pm}$ *have jump discontinuities across* $L^{\pm}$*, respectively. The jump at the point* $(x, s) \in L^{\pm}$*, defined by* $[\mathbf{u}^{\pm}(x, s)] := \mathbf{u}^{\pm}(x, s + 0) - \mathbf{u}^{\pm}(x, s - 0)$*, satisfies the following ordinary differential equation, where* $\boldsymbol{\beta}^+ := \mathbf{b}_{11}$ *and* $\boldsymbol{\beta}^- := \mathbf{b}_{22}$*:*

$$\frac{d}{dx}[\mathbf{u}^{\pm}(x, \pm x + 2k)] = [\frac{d}{dx}\mathbf{u}^{\pm}(x, \pm x + 2k)] = \boldsymbol{\beta}^{\pm}(x)[\mathbf{u}^{\pm}(x, \pm x + 2k)], \quad x \in \mathbb{I}. \tag{2.3}$$

At the boundary, the jumps are coupled to one another as

$$\begin{cases} [\mathbf{u}^+(+0,0)] = \dfrac{2}{t_0}\mathbf{I}, \qquad [\mathbf{u}^+(+0,2k)] = -\dfrac{r_0}{t_0}[\mathbf{u}^-(+0,2k)], \quad k>0, \\ [\mathbf{u}^-(1-0,2k-1)] = r_1[\mathbf{u}^+(1-0,2k-1)], \quad k>0. \end{cases} \tag{2.4}$$

Finally, if $[\mathbf{u}^+(+0,2k)] \neq \mathbf{0}$*, then* $[\mathbf{u}^+(x,x+2k)]$ *is non-singular for each* $x \in \mathbb{I}$*, and, if* $[\mathbf{u}^-(1-0,2k-1)] \neq \mathbf{0}$*, then* $[\mathbf{u}^-(x,-x+2k)]$ *is non-singular for each* $x \in \mathbb{I}$*. In particular,* $\mathbf{u}^+$*,* $\mathbf{u}^+ \pm \mathbf{u}^-$*, but not* $\mathbf{u}^-$*, are discontinuous across the line* $s = x$*.*

Note that if the matrices $(\mathbf{b}_{11}(x))_{0\le x\le 1}$ all commute, a closed form expression for the jumps in $\mathbf{u}^+$ can be obtained: $[\mathbf{u}^+(x,x+2k)] = \exp\left(\int_0^x \mathbf{b}_{11}(x')dx'\right)[\mathbf{u}^+(+0,2k)]$. Analogously, if the matrices $(\mathbf{b}_{22}(x))_{0\le x\le 1}$ all commute, integration of Eq. (2.3) yields $[\mathbf{u}^-(x,-x+2k)] = \exp\left(\int_1^x \mathbf{b}_{22}(x')dx'\right)[\mathbf{u}^-(1-0,2k-1)]$. Note also that if $\mathbf{I}$ is replaced by $\mathbf{0}$ in Theorem 2.1, then the solution $\mathbf{u}_0^\pm$ to Eq. (2.2) is identically zero. This follows from the uniqueness assertion, since $\mathbf{u}^\pm$ and $\mathbf{u}^\pm + \mathbf{u}_0^\pm$ both solve Eq. (2.2), if $\mathbf{u}^\pm$ is the solution in Theorem 2.1. More generally, it follows that the unique weak solution to Eq. (2.2) subject to the input $\sum c_i H_{s_i}$ instead of H, where $c_i \in \mathbb{R}$, $s_i \in \mathbb{R}_+$, and $H_{s_i}(s) = H(s-s_i)$, is given by $\sum c_i \mathbf{u}_{s_i}^\pm$, where $\mathbf{u}_{s_i}^\pm(x,s) = \mathbf{u}^\pm(x,s-s_i)$.

Theorem 2.2. *If, in the foregoing theorem,* $\mathbf{a}$ *is differentiable with respect to* s *in* $\mathbb{I}\times\mathbb{R}_+$*, and if* $\partial_s\mathbf{a} \in \mathcal{C}_{4\times4}(\mathbb{I}\times\mathbb{R}_+)$*, then* $\mathbf{u}^\pm \in \mathcal{C}^1_{2\times2}(Q_{2n}\setminus L)$*, and the restrictions of the partial derivatives of* $\mathbf{u}^\pm$ *to* T_j *can be extended continuously to* $\overline{T_j}$ *for each* j*.*

Before the presentation of the proofs of the theorems above, some consequences concerning the regularity of the solution are discussed. Note that by necessity in Theorem 2.1, $\mathbf{u}^\pm$ become continuous on $L^\mp$, respectively. The existence of $\mathbf{g}^\pm := \partial_s\mathbf{u}^\pm$ in $Q_{2n}\setminus L$, guaranteed by Theorem 2.2, is crucial for the Green functions formulation. By Theorem 2.2, the jumps in $\mathbf{g}^\pm$ across L also exist, and they are easily computed. For instance, if $(x,s)\in L^-$, then $[\partial_x\mathbf{u}^+(x,s)] - [\partial_s\mathbf{u}^+(x,s)] = \mathbf{0}$, since $\mathbf{u}^+$ is continuous across L^-. On the other hand, $[\partial_x\mathbf{u}^+(x,s)] + [\partial_s\mathbf{u}^+(x,s)] = \mathbf{b}_{12}(x)[\mathbf{u}^-(x,s)]$ by Eq. (2.2). Hence,

$$[\mathbf{g}^+(x,s)] = [\partial_x\mathbf{u}^+(x,s)] = \mathbf{b}_{12}(x)[\mathbf{u}^-(x,s)]/2, \quad (x,s)\in L^-, \tag{2.5}$$

which shows that $\mathbf{u}^+$ is in general not differentiable on L^-. Analogously,

$$[\mathbf{g}^-(x,s)] = -[\partial_x\mathbf{u}^-(x,s)] = -\mathbf{b}_{21}(x)[\mathbf{u}^+(x,s)]/2, \quad (x,s)\in L^+. \tag{2.6}$$

Furthermore, note that $\mathbf{u}^\pm$ might not be differentiable on $L^\pm$, even if $[\mathbf{u}^\pm] = \mathbf{0}$, respectively. To see this, integrate Eq. (2.2) along both sides of the characteristics, differentiate with respect to s, and subtract. In the limit, these operations yield

$$\begin{cases} [\mathbf{g}^+(x,x+2k)] = [\mathbf{g}^+(+0,2k)] + \displaystyle\int_0^x \mathbf{b}_{11}(x')[\mathbf{g}^+(x',x'+2k)]\,dx' + \\ \qquad + \displaystyle\int_0^x \left(\mathbf{a}_{11}(x',+0) - \mathbf{b}_{12}(x')\mathbf{b}_{21}(x')/2\right)[\mathbf{u}^+(x',x'+2k)]\,dx', \\ [\mathbf{g}^-(x,-x+2k)] = [\mathbf{g}^-(1-0,2k-1)] + \displaystyle\int_1^x \mathbf{b}_{22}(x')[\mathbf{g}^-(x',-x'+2k))]\,dx' + \\ \qquad + \displaystyle\int_1^x \left(\mathbf{a}_{22}(x',+0) + \mathbf{b}_{21}(x')\mathbf{b}_{12}(x')/2\right)[\mathbf{u}^-(x',-x'+2k)]\,dx', \end{cases}$$

where also Eqs. (2.5) and (2.6) have been employed. This equation can be solved. By a well known theorem in real analysis, $[\mathbf{g}^{\pm}(\cdot, \pm\cdot+2k)] \in \mathcal{C}^1_{2\times 2}(\mathbb{I})$ and

$$\frac{d}{dx}[\mathbf{g}^{\pm}(x, \pm x + 2k)] = \boldsymbol{\beta}^{\pm}(x)[\mathbf{g}^{\pm}(x, \pm x + 2k)] + \boldsymbol{\alpha}^{\pm}(x)[\mathbf{u}^{\pm}(x, \pm x + 2k)], \tag{2.7}$$

where $\boldsymbol{\alpha}^{+}(x) = \mathbf{a}_{11}(x, +0) - \mathbf{b}_{12}(x)\mathbf{b}_{21}(x)/2$, $\boldsymbol{\alpha}^{-}(x) = \mathbf{a}_{22}(x, +0) + \mathbf{b}_{21}(x)\mathbf{b}_{12}(x)/2$, $\boldsymbol{\beta}^{+}(x) = \mathbf{b}_{11}(x)$, and $\boldsymbol{\beta}^{-}(x) = \mathbf{b}_{22}(x)$ for $x \in \mathbb{I}$. If $[\mathbf{u}^{+}(+0, 2k)] \neq \mathbf{0}$, one obtains

$$\begin{aligned}[\mathbf{g}^{+}(x, x + 2k)] = {} & [\mathbf{u}^{+}(x, x + 2k)][\mathbf{u}^{+}(+0, 2k)]^{-1}[\mathbf{g}^{+}(+0, 2k)]+ \\ & + [\mathbf{u}^{+}(x, x + 2k)] \int_0^x [\mathbf{u}^{+}(x', x' + 2k)]^{-1}\boldsymbol{\alpha}^{+}(x')[\mathbf{u}^{+}(x', x' + 2k)]\, dx',\end{aligned} \tag{2.8}$$

where the results in Theorem 2.1 have been used. If $[\mathbf{u}^{-}(1 - 0, 2k - 1)] \neq \mathbf{0}$, then

$$\begin{aligned}[\mathbf{g}^{-}(x, -x + 2k)] = {} & [\mathbf{u}^{-}(x, -x + 2k)][\mathbf{u}^{-}(1 - 0, 2k - 1)]^{-1}[\mathbf{g}^{-}(1 - 0, 2k - 1)]+ \\ & + [\mathbf{u}^{-}(x, -x + 2k)] \int_1^x [\mathbf{u}^{-}(x', -x' + 2k)]^{-1}\boldsymbol{\alpha}^{-}(x')[\mathbf{u}^{-}(x', -x' + 2k)]\, dx'.\end{aligned}$$

At the boundary, the jumps in $\mathbf{g}^{\pm}$ are related to each other as

$$\begin{cases} [\mathbf{g}^{+}(+0, 0)] = -\dfrac{r_0}{t_0}[\mathbf{g}^{-}(+0, 0)] = \dfrac{r_0}{2t_0}\mathbf{b}_{21}(+0)[\mathbf{u}^{+}(+0, 0)] = \dfrac{r_0}{t_0^2}\mathbf{b}_{21}(+0), \\ \displaystyle\sum_{j=+,-} [\mathbf{g}^{-}(1 - 0, 2k - 1)]_j = r_1 \sum_{j=+,-} [\mathbf{g}^{+}(1 - 0, 2k - 1)]_j, \quad k > 0, \\ \displaystyle\sum_{j=+,-} [\mathbf{g}^{+}(+0, 2k)]_j = -\frac{r_0}{t_0} \sum_{j=+,-} [\mathbf{g}^{-}(+0, 2k)]_j, \quad k > 0, \end{cases} \tag{2.9}$$

where the subscript $+(-)$ indicates that the jump across $L^{+}(L^{-})$ is referred to. The jumps in the x-derivatives of $\mathbf{u}^{\pm}$ across $L^{\pm}$, respectively, are of less interest, but can be computed by Eq. (2.3), once the jumps in the s-derivatives have been calculated.

From the above results, it is possible to make statements about the regularity of the canonical solutions on $L^{\pm}$ in the partial mismatch cases (1): $r_1 = 0$, and (2): $r_1 \neq 0$ and $r_0 = 0$, which are of special interest. In both cases, $\mathbf{u}^{+}$ is discontinuous across the line $s = x$, while $\mathbf{u}^{-}$ is continuous, but not differentiable, across this line.
(1): $\mathbf{u}^{-}$ is continuous across the line $s = 2 - x$, but not differentiable, while $\mathbf{u}^{+}$ is differentiable across this line. Across the line $s = 2 + x$, $\mathbf{u}^{+}$ is continuous but, in general, not differentiable, while $\mathbf{u}^{-}$ is differentiable. If also $r_0 = 0$, $\mathbf{u}^{\pm}$ are both differentiable on this line. On the rest of L, $\mathbf{u}^{\pm}$ are both differentiable.
(2): $\mathbf{u}^{-}$ is discontinuous across the line $s = 2 - x$, while $\mathbf{u}^{+}$ is continuous, but not differentiable. Across the line $s = 2 + x$, $\mathbf{u}^{+}$ is continuous, but not differentiable, and $\mathbf{u}^{-}$ is differentiable. Across the line $s = 4 - x$, $\mathbf{u}^{-}$ is continuous, but not differentiable, and $\mathbf{u}^{+}$ is differentiable. On the rest of L, $\mathbf{u}^{\pm}$ are both differentiable.
Proof of Theorem 2.1. A necessary condition for the existence of a weak solution $\mathbf{u}^{\pm} \in \mathcal{C}_{2\times 2}(Q_{2n} \setminus L)$ in Q_{2n} to Eq. (2.2) is that $\mathbf{u}^{+}$ and $\mathbf{u}^{-}$ can be extended to continuous functions in $Q_{2n} \setminus L^{+}$ and $Q_{2n} \setminus L^{-}$, respectively. This fact is used below in the construction of the solution. Choose $T > 0$, such that $Tn_T = 2$ for some even integer n_T, and such that

$$b(T) := 2(1 + |r_1| + |r_0/t_0|)(\|\mathbf{b}\|T + \|\mathbf{a}\|T^2) < 1/2, \tag{2.10}$$

where the different norms $\|\cdot\|$ are defined in (2.1). Assume that the theorem holds in the set $\cup_{j=0}^{k-1} T_j \cup \{(x,s)\in Q_0\colon s\le 0\}$ for some k, where $0\le k\le 2n$, and prove that it holds also in $\cup_{j=0}^{k} T_j \cup \{(x,s)\in Q_0\colon s\le 0\}$, by using the Banach fixed-point theorem n_T $(n_T/2)$ times if $k\neq 0$ $(k=0)$. By Eq. (2.10), it will be clear, that the method works for all k, including $k=0$, and the theorem follows from the induction axiom. Consider the continuous map $\mathbf{f}\equiv(\mathbf{f}^+,\mathbf{f}^-)$ on the Banach space $\mathcal{C}_{2\times 2}(T_k)\times\mathcal{C}_{2\times 2}(T_k)$ defined by

$$
\begin{aligned}
(\mathbf{f}^+(\mathbf{u}^+,\mathbf{u}^-))(x,s) &= (\mathbf{B}^+(\mathbf{u}^+,\mathbf{u}^-))(x,s)+\\
&+\int_{x^+}^{x} \mathbf{b}_{11}(x')\,\mathbf{u}^+(x',s-x+x')\,dx' + \int_{x^+}^{x} \mathbf{b}_{12}(x')\,\mathbf{u}^-(x',s-x+x')\,dx'+\\
&+\int_{x^+}^{x}\left(\int_{-\infty}^{s-x+x'} \mathbf{a}_{11}(x',s-x+x'-s'')\,\mathbf{u}^+(x',s'')\,ds''\right)dx'+\\
&+\int_{x^+}^{x}\left(\int_{-\infty}^{s-x+x'} \mathbf{a}_{12}(x',s-x+x'-s'')\,\mathbf{u}^-(x',s'')\,ds''\right)dx',\\
(\mathbf{f}^-(\mathbf{u}^+,\mathbf{u}^-))(x,s) &= (\mathbf{B}^-(\mathbf{u}^+,\mathbf{u}^-))(x,s)+\\
&+\int_{x^-}^{x} \mathbf{b}_{21}(x')\,\mathbf{u}^+(x',s-x'+x)\,dx' + \int_{x^-}^{x} \mathbf{b}_{22}(x')\,\mathbf{u}^-(x',s-x'+x)\,dx'+\\
&+\int_{x^-}^{x}\left(\int_{-\infty}^{s-x'+x} \mathbf{a}_{21}(x',s-x'+x-s'')\,\mathbf{u}^+(x',s'')\,ds''\right)dx'+\\
&+\int_{x^-}^{x}\left(\int_{-\infty}^{s-x'+x} \mathbf{a}_{22}(x',s-x'+x-s'')\,\mathbf{u}^-(x',s'')\,ds''\right)dx',
\end{aligned}
\tag{2.11}
$$

where it is agreed that $\mathbf{u}^\pm(x,s)$ for points $(x,s)\in\cup_{j=0}^{k-1} T_j \cup \{(x,s)\in Q_0\colon s\le 0\}$ attain the values computed in the previous steps. This map is induced by line-integration along the characteristics of Eq. (2.2), and the points $(x^\pm,s^\pm)$, where $s^\pm<s$, are the points where the straight lines emanating from (x,s) with slopes ±1, respectively, cut the boundary of T_k, ∂T_k, and $\mathbf{B}^\pm(\mathbf{u}^+,\mathbf{u}^-)$ are the corresponding initial-boundary values of $\mathbf{f}^\pm(\mathbf{u}^+,\mathbf{u}^-)$ at these points, see Figure 1. For a fixed element $(\mathbf{u}^+,\mathbf{u}^-)$ in the domain of $\mathbf{f}$, these quantities are functions defined on T_k. If k is odd, they are given by

$$
\begin{cases}
(x^+(x,s),s^+(x,s)) = (0,s-x),\\
(x^-(x,s),s^-(x,s)) = 2^{-1}(x+s-k+1,x+s+k-1)\in\partial T_k\cap\partial T_{k-1},\\
(\mathbf{B}^-(\mathbf{u}^+,\mathbf{u}^-))(x,s) := \mathbf{u}^-(x^-,s^--0),\\
(\mathbf{B}^+(\mathbf{u}^+,\mathbf{u}^-))(x,s) := \dfrac{2}{t_0}\mathbf{I}-\dfrac{r_0}{t_0}(\mathbf{f}^-(\mathbf{u}^+,\mathbf{u}^-))(x^++0,s^+),
\end{cases}
\tag{2.12}
$$

and if k is even, they are

$$
\begin{cases}
(x^+(x,s),s^+(x,s)) = 2^{-1}(k-s+x,k+s-x)\in\partial T_k\cap\partial T_{k-1},\\
(x^-(x,s),s^-(x,s)) = (1,s+x-1),\\
(\mathbf{B}^+(\mathbf{u}^+,\mathbf{u}^-))(x,s) := \mathbf{u}^+(x^+,s^+-0),\\
(\mathbf{B}^-(\mathbf{u}^+,\mathbf{u}^-))(x,s) := r_1(\mathbf{f}^+(\mathbf{u}^+,\mathbf{u}^-))(x^--0,s^-).
\end{cases}
\tag{2.13}
$$

It is obvious, that every element $(\mathbf{f}^+(\mathbf{u}^+,\mathbf{u}^-),\mathbf{f}^-(\mathbf{u}^+,\mathbf{u}^-))$ in the range of $\mathbf{f}$, can be extended to a function in $\mathcal{C}_{2\times 2}(\overline{T_k})\times\mathcal{C}_{2\times 2}(\overline{T_k})$ in a natural way, and by the fourth formula in Eqs. (2.12) and (2.13), it is clear, that this extension satisfies the boundary values in Eq. (2.2). Furthermore, by the third condition in Eqs. (2.13) and (2.12), it follows, that $\mathbf{f}^\pm(\mathbf{u}^+,\mathbf{u}^-)$ are restrictions to T_k of continuous extensions of $\mathbf{u}^\pm$ from $\overline{T_{k-1}}$ to $\overline{T_{k-1}}\cup T_k$, respectively, see the first sentence of the proof. Differentiation of $\mathbf{f}^\pm(\mathbf{u}^+,\mathbf{u}^-)$ with respect to the vectors $(1,\pm 1)$, respectively, yields

$$\begin{pmatrix}(\partial_x+\partial_s)\mathbf{f}^+(\mathbf{u}^+,\mathbf{u}^-)(x,s)\\(\partial_x-\partial_s)\mathbf{f}^-(\mathbf{u}^+,\mathbf{u}^-)(x,s)\end{pmatrix}=\mathbf{b}(x)\begin{pmatrix}\mathbf{u}^+(x,s)\\\mathbf{u}^-(x,s)\end{pmatrix}+\int_{-\infty}^{s}\mathbf{a}(x,s-s')\begin{pmatrix}\mathbf{u}^+(x,s')\\\mathbf{u}^-(x,s')\end{pmatrix}ds'.$$

Note that the derivatives of the boundary-value functions $\mathbf{B}^\pm(\mathbf{u}^+,\mathbf{u}^-)$ are zero. The theorem is essentially proved, if it can be shown, that the map $\mathbf{f}$ has a unique fixed point $(\mathbf{u}^+,\mathbf{u}^-)\in\mathcal{C}_{2\times 2}(T_k)\times\mathcal{C}_{2\times 2}(T_k)$. Unfortunately, this cannot be accomplished in one step only; therefore, the following subdivision of T_k is introduced, see also Figure 1: $P_{k,j}:=T_k\cap(\mathbb{I}\times(k-1+(j-1)T,k-1+jT))$, $j\in\{1,\dots,n_T\}$.

Since $b(T)<2^{-1}$, the Banach fixed-point theorem implies that $\mathbf{f}$ has a unique fixed point in $\mathcal{C}_{2\times 2}(P_{k,1})\times\mathcal{C}_{2\times 2}(P_{k,1})$ if $k\neq 0$ and in $\mathcal{C}_{2\times 2}(P_{0,n_T/2+1})\times\mathcal{C}_{2\times 2}(P_{0,n_T/2+1})$ if $k=0$. In the latter case, the solution is obviously zero. That $\mathbf{f}$ actually is a contraction follows easily from Eqs. (2.10)-(2.13): $\|\mathbf{f}(\mathbf{u}^+,\mathbf{u}^-)-\mathbf{f}(\mathbf{v}^+,\mathbf{v}^-)\|\le b(T)\,\|(\mathbf{u}^+,\mathbf{u}^-)-(\mathbf{v}^+,\mathbf{v}^-)\|$, for all $(\mathbf{u}^+,\mathbf{u}^-),(\mathbf{v}^+,\mathbf{v}^-)\in\mathcal{C}_{2\times 2}(P_{k,j})\times\mathcal{C}_{2\times 2}(P_{k,j})$.

In the next step, the procedure in the previous paragraph is repeated to show that $\mathbf{f}$ has a unique fix point in the Banach space $\mathcal{C}_{2\times 2}(P_{k,2})\times\mathcal{C}_{2\times 2}(P_{k,2})$ if $k\neq 0$ (and in $\mathcal{C}_{2\times 2}(P_{0,n_T/2+2})\times\mathcal{C}_{2\times 2}(P_{0,n_T/2+2})$ if $k=0$), at which the restriction of $(\mathbf{u}^+,\mathbf{u}^-)$ to $P_{k,1}$ ($P_{0,n_T/2+1}$) in Eq. (2.11) is the unique solution obtained in the first step. Clearly, $\mathbf{u}^\pm$ become continuous on the part of the horizontal line $s=k-1+T$ ($s=T$) that is contained in T_k (T_0). It takes n_T-2 ($n_T/2-2$) another steps to show that the map $\mathbf{f}$ has a unique fixed point $(\mathbf{u}^+,\mathbf{u}^-)\in\mathcal{C}_{2\times 2}(T_k)\times\mathcal{C}_{2\times 2}(T_k)$, which is equal to zero if $k=0$.

It remains to verify the statements concerning the jump-discontinuities. The solution to Eq. (2.2) is zero in Q_0, and by Eq. (2.12), $\mathbf{u}^-$ is continuous on the line $s=x$. Eq. (2.11) then gives that $\mathbf{u}^+(x,x+0)=\frac{2}{t_0}\mathbf{I}+\int_0^x\mathbf{b}_{11}(x')\,\mathbf{u}^+(x',x'+0)\,dx'$ for all $x\in\mathbb{I}$, which is the required result for the function $x\to[\mathbf{u}^+(x,x)]$. Finally, if $\mathbf{Q}^+(x):=\mathbf{u}^+(x,x+0)$, $x\in\mathbb{I}$, then $\det\mathbf{Q}^+(0)\neq 0$, and basic matrix theory yields

$$\begin{aligned}\frac{d}{dx}\det\mathbf{Q}^+(x)&=\det\left(\tfrac{d}{dx}\boldsymbol{Q}_1(x)\quad\boldsymbol{Q}_2(x)\right)+\det\left(\boldsymbol{Q}_1(x)\quad\tfrac{d}{dx}\boldsymbol{Q}_2(x)\right)=\\&=\det\left(\mathbf{b}_{11}(x)\boldsymbol{Q}_1(x)\quad\boldsymbol{Q}_2(x)\right)+\det\left(\boldsymbol{Q}_1(x)\quad\mathbf{b}_{11}(x)\boldsymbol{Q}_2(x)\right)=\mathrm{tr}(\mathbf{b}_{11}(x))\det\mathbf{Q}^+(x)\end{aligned}$$

for all $x\in\mathbb{I}$, where $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are the column vectors of $\mathbf{Q}^+$. Thus, $\det\mathbf{Q}^+(x)=\exp\left(\int_0^x\mathrm{tr}(\mathbf{b}_{11}(x'))\,dx'\right)\det\mathbf{Q}^+(0)\neq 0$, $x\in\mathbb{I}$, which proves that $\mathbf{Q}^+(x)$ is non-singular for each $x\in\mathbb{I}$. The other results follow analogously. The proof is finished.

Proof of Theorem 2.2. Assume that for some k, $0<k\le 2n$, the theorem holds in $\cup_{j=1}^{k-1}T_j\cup Q_0$, and prove the validity of the theorem in $\cup_{j=1}^{k}T_j\cup Q_0$. The theorem then follows by induction in k, since it will appear, that no special consideration has to be made in the first step or depending on whether k is odd or even. The map $\mathbf{f}$ in the proof of Theorem 2.1 has a unique fixed point $(\mathbf{u}^+,\mathbf{u}^-)\in\mathcal{C}_{2\times 2}(T_k)\times\mathcal{C}_{2\times 2}(T_k)$. It must be shown that $\mathbf{u}^\pm$ actually belong to $\mathcal{C}^1_{2\times 2}(T_k)$. Since $(\partial_x\pm\partial_s)\mathbf{u}^\pm\in\mathcal{C}_{2\times 2}(T_k)$

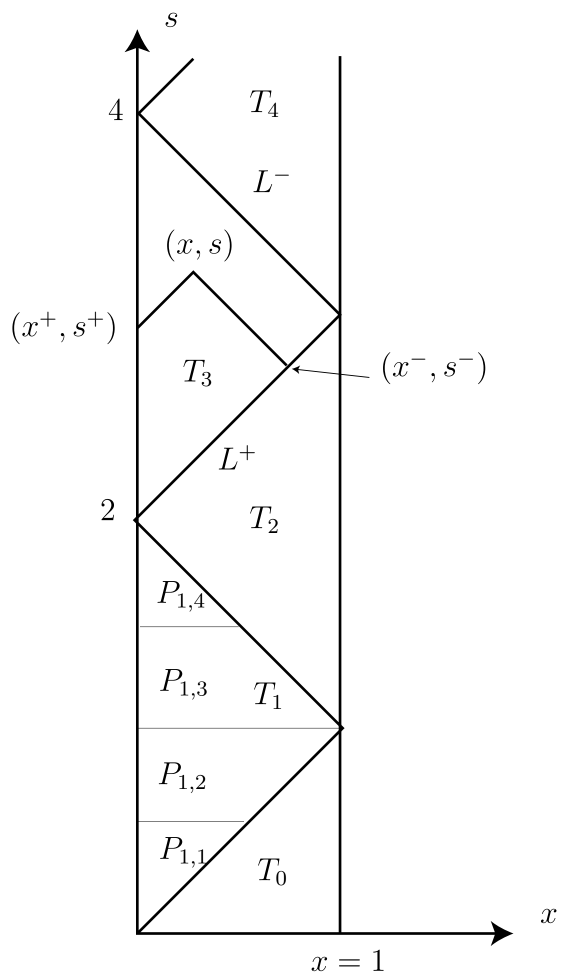

Figure 1: Geometrical quantities defined in Theorem 2.1 and its proof.

by Theorem 2.1, it is sufficient to show that $\partial_s \mathbf{u}^{\pm}$ exist and belong to $\mathcal{C}_{2\times 2}(T_k)$. To this end, define recursively a sequence $(\mathbf{u}_j^+, \mathbf{u}_j^-)_{j=0}^{\infty}$ in $\mathcal{C}^1_{2\times 2}(P_{k,1}) \times \mathcal{C}^1_{2\times 2}(P_{k,1})$ by $(\mathbf{u}_{j+1}^+, \mathbf{u}_{j+1}^-) = (\mathbf{f}^+(\mathbf{u}_j^+, \mathbf{u}_j^-), \mathbf{f}^-(\mathbf{u}_j^+, \mathbf{u}_j^-))$, which is possible since $\partial_s \mathbf{a}$ exists in $\mathbb{I} \times \mathbb{R}_+$ and $\partial_s \mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I} \times \mathbb{R}_+)$. The proof of the Banach fixed-point theorem — this is actually the method of successive approximations, where the first element in the sequence can be chosen arbitrarily — implies that this sequence converges uniformly to $(\mathbf{u}^+, \mathbf{u}^-)$ in $P_{k,1}$, since

$$\left\|(\mathbf{u}_j^+, \mathbf{u}_j^-) - (\mathbf{u}_i^+, \mathbf{u}_i^-)\right\| \leq b(T) \left\|(\mathbf{u}_{j-1}^+, \mathbf{u}_{j-1}^-) - (\mathbf{u}_{i-1}^+, \mathbf{u}_{i-1}^-)\right\| \tag{2.14}$$

for all $i, j > 0$ by Eqs. (2.10)-(2.13). Similarly, these equations yield

$$\begin{aligned} \left\|\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-) - \partial_s(\mathbf{u}_i^+, \mathbf{u}_i^-)\right\| &\leq b(T) \left\|\partial_s(\mathbf{u}_{j-1}^+, \mathbf{u}_{j-1}^-) - \partial_s(\mathbf{u}_{i-1}^+, \mathbf{u}_{i-1}^-)\right\| + \\ &+ a(T) \left\|(\mathbf{u}_{j-1}^+, \mathbf{u}_{j-1}^-) - (\mathbf{u}_{i-1}^+, \mathbf{u}_{i-1}^-)\right\|, \qquad \forall i, j > 0, \end{aligned} \tag{2.15}$$

where $a(T)$ is independent of k and $(\mathbf{u}_j^+, \mathbf{u}_j^-)_{j=0}^{\infty}$. Eqs. (2.14) and (2.15) imply that

$$\begin{aligned} &\left\|\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-) - \partial_s(\mathbf{u}_i^+, \mathbf{u}_i^-)\right\| \leq \\ &\leq \frac{(2b(T))^{i-1}}{(1-2b(T))} \left(\left\|\partial_s(\mathbf{u}_1^+, \mathbf{u}_1^-) - \partial_s(\mathbf{u}_0^+, \mathbf{u}_0^-)\right\| + 2a(T) \left\|(\mathbf{u}_1^+, \mathbf{u}_1^-) - (\mathbf{u}_0^+, \mathbf{u}_0^-)\right\|\right) \end{aligned}$$

if $0 < i < j$, i.e., $(\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-))_{j=0}^{\infty}$ is a Cauchy sequence in $\mathcal{C}_{2\times 2}(P_{k,1}) \times \mathcal{C}_{2\times 2}(P_{k,1})$. Since this function space is complete, the sequence $(\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-))_{j=0}^{\infty}$ converges uniformly in $P_{k,1}$ to a bounded and continuous function $(\mathbf{v}^+, \mathbf{v}^-)$. By the second paragraph of this section, it follows that $\partial_s(\mathbf{u}^+, \mathbf{u}^-)$ exists and equals $(\mathbf{v}^+, \mathbf{v}^-)$. Thus, $\mathbf{f}$ has a unique fixed point in $\mathcal{C}^1_{2\times 2}(P_{k,1}) \times \mathcal{C}^1_{2\times 2}(P_{k,1})$.

In the next step, the procedure in the previous paragraph is repeated to show that $\mathbf{f}$ has a unique fix point in $\mathcal{C}^1_{2\times 2}(P_{k,2}) \times \mathcal{C}^1_{2\times 2}(P_{k,2})$. In this second step, we let the restriction of $(\mathbf{u}^+, \mathbf{u}^-)$ to $P_{k,1}$ be the unique solution obtained in the first step. Since $\partial_s \mathbf{u}^{\pm}$ exist and are continuous on both sides of the part of the horizontal line $s = k - 1 + T$ that is contained in T_k, and since, by construction, $(\partial_x \pm \partial_s)\mathbf{u}^{\pm}$ and $\partial_x \mathbf{u}^{\pm}$ exist and are continuous on this part of the line, $\partial_s \mathbf{u}^{\pm}$ exist and are continuous here also. Just as in the proof of Theorem 2.1, it takes $n_T - 2$ similar steps to show that there is a unique $\mathcal{C}^1_{2\times 2} \times \mathcal{C}^1_{2\times 2}$-solution $(\mathbf{u}^+, \mathbf{u}^-)$ to Eq. (2.2) in $\cup_{j=1}^k T_j \cup Q_0$. From the explicit form of the derivatives of the solution, it is clear that these functions can be extended to bounded and continuous functions in $\overline{T_k}$. The proof is finished.

3 The full propagation problem

In this section, the results in the preceding section are extended to a more general input $\boldsymbol{e}^i$. Theorem 3.1 below asserts unique solvability for the general propagation problem (1.1). As an immediate consequence of this, strict causality holds, that is, the speed of the wave front is ≤ 1, see Section 1. In Theorem 3.2, it is proved that the speed of the wave front is precisely 1, independent of the input.

Theorem 3.1. *Let* $\mathbf{a}$, $\mathbf{b}$, r_1, r_0, t_0, $L^{\pm}$, Q_{2n}, *and* $\mathbf{u}^{\pm}$ *be as in Theorem 2.1. Moreover, let* $\boldsymbol{e}^i : \mathbb{R} \to M_{2\times 1}(\mathbb{R})$ *be causal, i.e.,* $\boldsymbol{e}^i(s) = \mathbf{0}$ *for all* $s < 0$, *and continuously differentiable with bounded derivative, with exception for at most a finite number of points,* $0 \leq s_1 < s_2 < \cdots < s_p$, *where it is undefined. Finally, let* $\Gamma = \Gamma^+ \cup \Gamma^-$, *where* $\Gamma^{\pm} = \cup_{k=0}^p \{(x,s) \in \mathbb{R}^2 : (0, s_k) + L^{\pm}\}$. *Then, for every integer* $n \geq 0$, *the initial-boundary value problem (1.1), defined in* $\mathbb{I} \times \mathbb{R}$, *has a unique solution,* $\boldsymbol{e}^{\pm} \in \mathcal{C}_{2\times 1}(Q_{2n} \setminus \Gamma^{\pm})$, *in the weak sense of integration along the characteristics within* Q_{2n}, *i.e., when the derivatives* $\partial_x \pm \partial_s$ *are interpreted as derivatives with respect to the vectors* $(1, \pm 1)$, *respectively. The solution is given by*

$$\boldsymbol{e}^{\pm}(x,s) = \partial_s \int_{-\infty}^{s-x} \mathbf{u}^{\pm}(x, s-s')\boldsymbol{e}^i(s')\, ds'. \tag{3.1}$$

The vector fields $\boldsymbol{e}^{\pm} \in \mathcal{C}_{2\times 1}(Q_{2n} \setminus \Gamma)$ *are zero in* Q_{s_0}, *if* $\boldsymbol{e}^i(s) = \mathbf{0}$ *for all* $s < s_0$. *If* $\partial_s \mathbf{a}$ *exists in* $\mathbb{I} \times \mathbb{R}_+$ *and* $\partial_s \mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I} \times \mathbb{R}_+)$, *then* $\boldsymbol{e}^{\pm} \in \mathcal{C}^1_{2\times 1}(Q_{2n} \setminus \Gamma)$.

Proof. The Cauchy convergence principle guarantees the existence of $\boldsymbol{e}^i(s_j \pm 0) := \lim \boldsymbol{e}^i(s)$, as $s \to s_j \pm 0$, at each discontinuity point s_j, since, e.g., $\boldsymbol{e}^i(s') -$

$\boldsymbol{e}^i(s'') = \int_{s''}^{s'} \frac{d}{ds}\boldsymbol{e}^i(s)\, ds$, $s_j < s'' < s' < s_{j+1}$, has the limit $\mathbf{0}$, when $s', s'' \searrow s_j$. Thus, $\boldsymbol{e}^i$ has a finite jump discontinuity at the point s_j, and the jump in $\boldsymbol{e}^i$ at s_j is defined as $[\boldsymbol{e}^i(s_j)] := \boldsymbol{e}^i(s_j+0) - \boldsymbol{e}^i(s_j-0)$. A solution to the problem (1.1) is immediately obtained by a straightforward extension of Duhamel's principle, see Ref. [6]: $\boldsymbol{e}^{\pm}(x,s) = \sum_{k=1}^{p} \mathbf{u}^{\pm}(x, s-s_k)[\boldsymbol{e}^i(s_k)] + \int_{-\infty}^{\infty} \mathbf{u}^{\pm}(x, s-s')\{\frac{d}{ds'}\boldsymbol{e}^i\}(s')ds'$, for all $(x,s) \in Q_{2n}\setminus\Gamma$, where $\{\frac{d}{ds}\boldsymbol{e}^i\}(s)$ denotes the classical derivative of $\boldsymbol{e}^i$ at s. Use of the fact that $\mathbf{u}^+(x,s) = \mathbf{0}$ when $x > s$ yields the desired solution (3.1), which has the regularity inherited by $\mathbf{u}^{\pm}$. Moreover, this is the only solution. For let $(\boldsymbol{e}^+, \boldsymbol{e}^-)$ be the difference between two solutions. Clearly, $\begin{pmatrix}\boldsymbol{e}^{\pm} & \mathbf{0}\end{pmatrix}$ solve the canonical problem in Theorem 2.1, with $\mathbf{I}$ replaced by $\mathbf{0}$, and since the solution to this problem is unique, $(\boldsymbol{e}^+, \boldsymbol{e}^-)$ is zero. Finally, the last sentence in the theorem holds by Theorem 2.2, since one can let the derivative on $\boldsymbol{e}^{\pm}$ act upon $\boldsymbol{u}^{\pm}$. The proof is finished.

The vector fields $\boldsymbol{e}^+ \pm \boldsymbol{e}^-$ are essentially the electric and magnetic fields, see Section 5. Trivially, if the conditions in Theorem 3.1 are fulfilled, the vector fields $\boldsymbol{e}^+ \pm \boldsymbol{e}^- \in \mathcal{C}_{2\times 1}(Q_{2n}\setminus\Gamma)$, and they are equal to zero in Q_{s_0}, if $\boldsymbol{e}^i(s) = \mathbf{0}$ for all $s < s_0$. The following theorem shows that the speed of the wavefront in Theorem 3.1 is precisely one.

Theorem 3.2. *(Wave front speed) Let, in the preceding theorem, $\mathbf{e}^i$ have the following additional property: there is a number $\delta > 0$ such that the restriction of $\mathbf{e}^i$ to $(0,\delta)$ is continuously differentiable and $\mathbf{e}^i(s) \neq \mathbf{0}$ for all $0 < s < \delta$. Then for each $x \in (0,1)$, there is a number s, $x < s < x+\delta$, such that $\mathbf{e}^+(x,s) \neq \mathbf{0}$, where $(\mathbf{e}^+, \mathbf{e}^-)$ is the unique solution to the problem (1.1) given by equation (3.1). The same statement is true for the vector fields $\mathbf{e}^+ \pm \mathbf{e}^-$.*

Proof. Put $\boldsymbol{e}^i = (e_1^i, e_2^i)$, and choose a real number δ_0, $0 < \delta_0 < \delta$, such that both e_1^i and e_2^i do not change sign in the interval $0 < s < \delta_0$. In particular, this implies that at least one of the terms $\int_0^{\delta_0} e_1^i(s')\, ds'$, $\int_0^{\delta_0} e_2^i(s')\, ds'$ is non-zero. Assume, on the contrary to the hypothesis of the theorem, that there is a point $x \in (0,1)$ such that $\boldsymbol{e}^+(x, s+x) = \mathbf{0}$ for all $s \in (0,\delta)$. By Theorem 2.1 in the previous section, $\det(\mathbf{u}^+(x, x+0)) \neq 0$, and since $\mathbf{u}^+(x, x+0)$ is an continuous extension, there is a number $\delta_1 > 0$ such that

$$\det\begin{pmatrix} u_{11}^+(x, x+s_1) & u_{12}^+(x, x+s_2) \\ u_{21}^+(x, x+s_3) & u_{22}^+(x, x+s_4) \end{pmatrix} \neq 0 \tag{3.2}$$

for all s_1, s_2, s_3, s_4 such that $0 < s_1, s_2, s_3, s_4 < \delta_1$. It is not a restriction to assume that $\delta_0 < \delta_1$. Eq. (3.1) implies that $\mathbf{0} = \partial_s \int_0^s \mathbf{u}^+(x, x+s-s')\boldsymbol{e}^i(s')\, ds'$, $0 < s < \delta_0$, so that $\mathbf{0} = \int_0^s \mathbf{u}^+(x, x+s-s')\boldsymbol{e}^i(s')\, ds'$, $0 < s < \delta_0$. The mean value theorem of integral calculus asserts that there are positive real numbers $\delta_2, \delta_3, \delta_4, \delta_5$, such that $\delta_2, \delta_3, \delta_4, \delta_5 < \delta_0$ and

$$\begin{cases} u_{11}^+(x, x+\delta_0-\delta_2) \displaystyle\int_0^{\delta_0} e_1^i(s')\, ds' + u_{12}^+(x, x+\delta_0-\delta_3) \displaystyle\int_0^{\delta_0} e_2^i(s')\, ds' = 0, \\ u_{21}^+(x, x+\delta_0-\delta_4) \displaystyle\int_0^{\delta_0} e_1^i(s')\, ds' + u_{22}^+(x, x+\delta_0-\delta_5) \displaystyle\int_0^{\delta_0} e_2^i(s')\, ds' = 0. \end{cases}$$

Eq. (3.2) implies that this system of equations has the trivial solution only, i.e., $\int_0^{\delta_0} e_1^i(s')\, ds' = 0$ and $\int_0^{\delta_0} e_2^i(s')\, ds' = 0$, which contradicts the second sentence of the proof. Thus, there exists a number s, $x < s < x+\delta$, such that $\boldsymbol{e}^+(x,s) \neq \mathbf{0}$.

Analogously, since $\det(\mathbf{u}^+(x,x+0) \pm \mathbf{u}^-(x,x+0)) \neq 0$ for each $x \in \mathbb{I}$, and $\mathbf{u}^+(x,x+0) \pm \mathbf{u}^-(x,x+0)$ are continuous extensions, there is a number $\delta_6 > 0$ such that

$$\det \begin{pmatrix} u_{11}^+(x,x+s_1) \pm u_{11}^-(x,x+s_1) & u_{12}^+(x,x+s_2) \pm u_{12}^-(x,x+s_2) \\ u_{21}^+(x,x+s_3) \pm u_{21}^-(x,x+s_3) & u_{22}^+(x,x+s_4) \pm u_{22}^-(x,x+s_4) \end{pmatrix} \neq 0$$

for all s_1, s_2, s_3, s_4 such that $0 < s_1, s_2, s_3, s_4 < \delta_6$. A investigation similar to the one above shows that for each $x \in (0,1)$, there exists a number s, $x < s < x+\delta$, such that $\boldsymbol{e}^+(x,s) \pm \boldsymbol{e}^-(x,s) \neq \mathbf{0}$. The proof is finished.

4 The Green functions

As mentioned in Section 1, the results of the theory of this paper have already been used by scientific community in a number of papers on direct and inverse scattering in complex, dispersive media, e.g., [7, 8, 15–17]. In these articles, the Green functions equations have been employed, rather than the similar and more natural canonical functions equations (2.2). For completeness, these equations are now presented and proved uniquely solvable in the general mismatch case.

There is a slight variation in the definition of the Green functions, $\mathbf{g}^\pm$, between different authors. In this paper, the following definition, closely related to the original one by Krueger-Ochs [18], is employed: $\mathbf{g}^\pm(x,s) := \partial_s \mathbf{u}^\pm(x,s)$, for all $(x,s) \in Q_{2n} \setminus L$, where $\mathbf{u}^\pm$ are the canonical functions in Theorem 2.2.

The relation between the Green functions and the split vector fields $\boldsymbol{e}^\pm$ is obtained by performing the differentiation in Eq. (3.1). The result is

$$\boldsymbol{e}^\pm(x,s) = \int_{-\infty}^{s-x} \mathbf{g}^\pm(x,s-s')\boldsymbol{e}^i(s')\,ds' + \sum_{k=k^\pm}^{\infty} [\mathbf{u}^\pm(x,\pm x+2k)]\boldsymbol{e}^i(s \mp x - 2k),$$

where $k^+ = 0$, $k^- = 1$. The properties of the Green functions are given by

Theorem 4.1. *Let* $\mathbf{a}$, $\mathbf{b}$, r_1, r_0, t_0, $L^\pm$, L, Q_{2n}, T_k, *and* $\mathbf{u}^\pm$ *be as in Theorem 2.2. Then, for each integer* $n \geq 0$, *the integro-differential equation defined in* $\mathbb{I} \times \mathbb{R}$ *by*

$$\begin{cases} \begin{pmatrix} (\partial_x + \partial_s)\mathbf{g}^+(x,s) \\ (\partial_x - \partial_s)\mathbf{g}^-(x,s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \mathbf{g}^+(x,s) \\ \mathbf{g}^-(x,s) \end{pmatrix} + \int_{-\infty}^{s} \mathbf{a}(x,s-s') \begin{pmatrix} \mathbf{g}^+(x,s') \\ \mathbf{g}^-(x,s') \end{pmatrix} ds' + \\ \qquad + \sum_{m=1}^{\infty} \begin{pmatrix} \mathbf{a}_{11}(x,s-x-2m+2) & \mathbf{a}_{12}(x,s+x-2m) \\ \mathbf{a}_{21}(x,s-x-2m+2) & \mathbf{a}_{22}(x,s+x-2m) \end{pmatrix} \begin{pmatrix} [\mathbf{u}^+(x,x+2m-2)] \\ [\mathbf{u}^-(x,-x+2m)] \end{pmatrix}, \\ \mathbf{g}^\pm(x,s) = \mathbf{0}, \quad s \leq 0, \\ t_0\mathbf{g}^+(+0,s) + r_0\mathbf{g}^-(+0,s) = \mathbf{0}, \quad \mathbf{g}^-(1-0,s) = r_1\mathbf{g}^+(1-0,s), \\ [\mathbf{g}^+(x,s)] = \mathbf{b}_{12}(x)[\mathbf{u}^-(x,s)]/2, \quad (x,s) \in L^-, \\ [\mathbf{g}^-(x,s)] = -\mathbf{b}_{21}(x)[\mathbf{u}^+(x,s)]/2, \quad (x,s) \in L^+, \end{cases} \tag{4.1}$$

where $\mathbf{a}(x,s) := \mathbf{0}$ *for* $s < 0$, *has a unique solution,* $\mathbf{g}^\pm \in \mathcal{C}_{2\times 2}(Q_{2n} \setminus L) = \mathcal{C}_{2\times 2}(\cup_{k=1}^{2n} T_k \cup Q_0)$, *in the weak sense of line integration along the characteristics within each triangle* T_k, $k \leq 2n$. *Thus, the derivatives are interpreted as derivatives* *with respect to the vectors* $(1, \pm 1)$*, respectively. The solution is given by the Green functions defined above. The finite jumps in* $\mathbf{g}^\pm = \partial_s \mathbf{u}^\pm$ *across* $L^\pm$*, respectively, are given by Eq. (2.7), and the jump conditions on the boundary by Eq. (2.9).*

As an immediate consequence of Eqs. (2.8) and (2.9), the 'initial values' of the Green functions are obtained:

$$\begin{cases} \mathbf{g}^+(x, x+0) = \mathbf{Q}_0^+(x) \left(\dfrac{r_0}{2t_0} \mathbf{b}_{21}(+0) + \displaystyle\int_0^x \mathbf{Q}_0^+(x')^{-1} \boldsymbol{\alpha}^+(x') \mathbf{Q}_0^+(x')\, dx' \right), \\ \mathbf{g}^-(x, x+0) = -\mathbf{b}_{21}(x) \mathbf{Q}_0^+(x)/2, \end{cases}$$

where $\mathbf{Q}_0^+(x) = \mathbf{u}^+(x, x+0)$ for $x \in \mathbb{I}$. Note that knowledge of the existence of a unique weak solution to the Green functions equations in the sense of the theorem above is sufficient for the numerical purposes in, e.g., [7, 8, 15, 16], since integration along the characteristics within each triangle T_k is the first step in the employed discretization method.

Proof of Theorem 4.1. By line integration of the canonical equations (2.2) along the characteristics within each triangle T_k, $k \leq 2n$, followed by differentiation with respect to s, the weak formulation of Eq. (4.1) is obtained, and the existence part follows from Theorem 2.2. The jump conditions are direct consequences of Eqs. (2.5) and (2.6). Suppose there is another weak solution to the problem (4.1) in the above sense. The difference between these solutions then satisfies Eq. (2.2), with input $\mathbf{0}$ instead of $\mathbf{I}$. By uniqueness in Theorem 2.1, this difference is zero, so there is a unique weak solution to Eq. (4.1) within each triangle T_k. The proof is finished.

Finally, the regularity of the Green functions on the curve $L^\pm$ in the mismatch cases, (1): $r_1 = 0$, and (2): $r_1 \neq 0$ and $r_0 = 0$, is commented upon. It is easy to obtain the explicit expressions for the jumps in $\mathbf{g}^\pm$ across L by combining various formulas in this paper; therefore, a quantitative discussion is sufficient.

(1): $\mathbf{g}^-$ is discontinuous across the line $s = 2 - x$, while $\mathbf{g}^+$ is continuous. Across the line $s = 2 + x$, $\mathbf{g}^+$ is discontinuous, and $\mathbf{g}^-$ is continuous. If also $r_0 = 0$, $\mathbf{g}^\pm$ are both continuous on this line. On the rest of L, $\mathbf{g}^\pm$ are both continuous.

(2): $\mathbf{g}^\pm$ and are both discontinuous across the line $s = 2 - x$. Across the line $s = 2 + x$, $\mathbf{g}^+$ is discontinuous, but $\mathbf{g}^-$ is continuous. Across the line $s = 4 - x$, $\mathbf{g}^-$ is discontinuous, and $\mathbf{g}^+$ is continuous. On the rest of L, $\mathbf{g}^\pm$ are both continuous.

5 A bi-isotropic example

In this section, a wave propagation problem for a dispersive, stratified, bi-isotropic (chiral) slab is formulated and analyzed. The analysis motivates the study of Eq. (1.1) in the previous sections.

The bi-isotropic slab is located between the planes $x_3 = 0$ and $x_3 = d$. The media outside the slab are homogeneous, isotropic, and without dispersion, and the permeability and permittivity of the medium to the right of the slab might differ from the corresponding properties of the medium to the left. The slab is excited by a transient transverse plane wave, incident from the left, and the incident electric field at the front wall at the time t, $\boldsymbol{E}^i(t)$, is presumed to be quiescent before a finite time T_1, i.e., $\boldsymbol{E}^i(t) = \mathbf{0}$ for all times $t < T_1$. Moreover, $\boldsymbol{E}^i$ is assumed to be continuously differentiable with bounded derivative, except for at most a finite number of points, $t_1 < \ldots < t_p$, where it is undefined. Note that the set of all incident electric fields with the above properties forms a linear space over the real numbers.

The constitutive relations for the bi-isotropic medium at the time t and at the point $\boldsymbol{r} \equiv (x_1, x_2, x_3) \equiv x_1\widehat{\mathbf{x}_1} + x_2\widehat{\mathbf{x}_2} + x_3\widehat{\mathbf{x}_3}$ are defined by the following relation between the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ on one hand, and the electric and magnetic flux densities, $\boldsymbol{D}$ and $\boldsymbol{B}$, respectively, on the other:

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon(x_3)\left(\boldsymbol{E}(\boldsymbol{r},t) + (\chi_{ee} * \boldsymbol{E})(\boldsymbol{r},t)\right) + c(x_3)^{-1}(\chi_{em} * \boldsymbol{H})(\boldsymbol{r},t), \\ \boldsymbol{B}(\boldsymbol{r},t) = c(x_3)^{-1}(\chi_{me} * \boldsymbol{E})(\boldsymbol{r},t) + \mu(x_3)\left(\boldsymbol{H}(\boldsymbol{r},t) + (\chi_{mm} * \boldsymbol{H})(\boldsymbol{r},t)\right), \end{cases} \tag{5.1}$$

where, e.g., $(\chi_{ee} * \boldsymbol{E})(\boldsymbol{r},t) = \int_{-\infty}^{t} \chi_{ee}(x_3, t-t')\boldsymbol{E}(\boldsymbol{r},t')\,dt'$. It is understood that the slab is initially quiescent, i.e., there is a time T, such that $\boldsymbol{E}(\boldsymbol{r},t) = \mathbf{0}$ for all $t \leq T$, and similarly for the magnetic field $\boldsymbol{H}(\boldsymbol{r},\cdot)$. Therefore, $\int_{-\infty}^{t}$ can be substituted for $\int_{T}^{t}$ in the convolutions above. The positive functions ϵ and μ are the non-dispersive parts of the permittivity and permeability, respectively, and $c := (\mu\epsilon)^{-1/2}$. All the functions χ_{ee}, χ_{em}, χ_{me}, and χ_{mm} have the same unit, s^{-1}, and are referred to as the susceptibility (integral) kernels. Clearly, the integral kernels χ_{ee} and χ_{mm} model the ordinary dispersive effects, while the chirality, $(\chi_{em} - \chi_{me})/2$, and the non-reciprocity, $(\chi_{em} + \chi_{me})/2$, are the characteristic properties of the bi-isotropic medium. The medium is reciprocal if $\chi_{em} + \chi_{me} = 0$, see Ref. [11].

The medium is assumed to be stratified with respect to depth, i.e., ϵ and μ depend on the spatial variable x_3, and the susceptibility kernels depend on x_3 and the time t. The functions ϵ and μ are continuously differentiable with bounded derivatives in the interval $(0,d)$, and the susceptibility kernels and their first and second time derivatives are assumed to be bounded and continuous functions in $(x_3,t) \in (0,d) \times (0,\infty)$. Due to causality in Eq. (5.1), the susceptibility kernels are equal to zero when $t < 0$, see Ref. [11]. These conditions guarantee similar regularity for the induced electro-magnetic fields (throughout space and time) as for the incident field, described in the second paragraph of this section. More precisely, the jump-discontinuities in the incident field will propagate along the characteristic curves. However, the number of discontinuity curves may be infinite, since wave impedance mismatch is allowed at the edges of the slab, i.e., there may be jump discontinuities in the permittivity and the permeability at the front and/or at the back wall.

The electro-magnetic field satisfies the source-free Maxwell equations:

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}, \qquad \nabla \cdot \boldsymbol{D} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{B} = \mathbf{0}. \tag{5.2}$$

Transverse solutions, independent of the transverse coordinates (x_1, x_2), are sought, i.e., $\boldsymbol{E}(\boldsymbol{r},t) = \widehat{\mathbf{x_1}}E_1(x_3,t) + \widehat{\mathbf{x_2}}E_2(x_3,t)$, and similarly for all the other electro-magnetic fields. Note that it is not necessary to assume that the 3-components of the vector fields vanish inside the bi-isotropic medium; the independence of the spatial variables (x_1, x_2) and the Maxwell equations (5.2) imply that D_3 and B_3 are both constant, and by the continuity at the walls, they are both equal to zero throughout space. The constitutive relations and the associative law for causal convolutions then imply that both $E_3(x_3,\cdot)$ and $H_3(x_3,\cdot)$ satisfy the equation $f +$

$(\chi_{ee}+\chi_{mm}+\chi_{ee}*\chi_{mm}-\chi_{em}*\chi_{me})*f=0$, which is a linear Volterra integral equation of the second kind, and therefore has the unique continuous solution $f=0$, see Ref. [13]. One arrives to the same conclusion if f has the regularity described in the third paragraph of this section.

With the transverse Ansatz above, the Maxwell equations (5.2) can be written

$$\partial_3 \boldsymbol{E}=\partial_t(\mathbf{J}\boldsymbol{B}), \qquad \partial_3(\mathbf{J}\boldsymbol{H})=\partial_t\boldsymbol{D}, \qquad \mathbf{J}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{5.3}$$

where a compact matrix notation, pertinent to the analysis of the propagation of electro-magnetic waves in the bi-isotropic slab, has been introduced. Put, $\boldsymbol{\chi}_{ee}:=\chi_{ee}\mathbf{I}$, $\boldsymbol{\chi}_{me}:=\chi_{me}\mathbf{J}$, $\boldsymbol{\chi}_{mm}:=\chi_{mm}\mathbf{I}$, $\boldsymbol{\chi}_{em}:=\chi_{em}\mathbf{J}$. By the constitutive relations (5.1), the flux densities $\boldsymbol{B}$ and $\boldsymbol{D}$ in Eq. (5.3) are eliminated, and a partial integro-differential equation in the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ is obtained:

$$\begin{aligned}\partial_3\begin{pmatrix}\boldsymbol{E}\\ \eta\mathbf{J}\boldsymbol{H}\end{pmatrix}&=\frac{\eta'}{\eta}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ \eta\mathbf{J}\boldsymbol{H}\end{pmatrix}+\\ &\quad+c^{-1}\partial_t\left(\begin{pmatrix}\boldsymbol{\chi}_{me}* & \mathbf{I}+\boldsymbol{\chi}_{mm}*\\ \mathbf{I}+\boldsymbol{\chi}_{ee}* & -\boldsymbol{\chi}_{em}*\end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ \eta\mathbf{J}\boldsymbol{H}\end{pmatrix}\right),\end{aligned} \tag{5.4}$$

where $\eta:=\sqrt{\mu/\epsilon}$ is the wave impedance. Next, the wave splitting,

$$\begin{pmatrix}\boldsymbol{E}^+\\ \boldsymbol{E}^-\end{pmatrix}=\mathbf{P}\begin{pmatrix}\boldsymbol{E}\\ \eta\mathbf{J}\boldsymbol{H}\end{pmatrix}, \qquad \mathbf{P}=\frac{1}{2}\begin{pmatrix}\mathbf{I} & -\mathbf{I}\\ \mathbf{I} & \mathbf{I}\end{pmatrix}, \qquad \mathbf{P}^{-1}=\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}, \tag{5.5}$$

is adopted. The wave splitting technique is now a well established method to solve direct and inverse scattering problems. For a recent survey of the technique, the reader is referred to Ref. [5]. Recent contributions to the solution of direct and inverse scattering problems in complex media can be found in Refs. [7, 8, 16, 17].

The form of the matrix $\mathbf{P}^{-1}$ shows that the electric field is the sum of the split vector fields, $\boldsymbol{E}^{\pm}$, and that the magnetic field is proportional to the difference (with a matrix as proportionality constant). Outside the slab, $\boldsymbol{E}^{\pm}$ represent the general right- and left going waves. More precisely, $\boldsymbol{E}^{\pm}(x_3,\cdot)$ are the incident and reflected electric fields at position x_3, respectively, if $x_3<0$. Analogously, $\boldsymbol{E}^-(x_3,\cdot)=\mathbf{0}$, and $\boldsymbol{E}^+(x_3,\cdot)$ is the transmitted electric field at position x_3, if $x_3>d$. In particular, it follows that the direct scattering problem is solved if the functions $\boldsymbol{E}^-(+0,\cdot)$ and $\boldsymbol{E}^+(d-0,\cdot)$ are known, since the continuity of (the tangential components of) the magnetic and electric fields $\boldsymbol{E}$ and $\boldsymbol{H}$ at the boundary implies that

$$\begin{aligned}\boldsymbol{E}^r(t)&=\frac{2\eta(-0)}{\eta(+0)+\eta(-0)}\boldsymbol{E}^-(+0,t)+\frac{\eta(+0)-\eta(-0)}{\eta(+0)+\eta(-0)}\boldsymbol{E}^i(t),\\ \boldsymbol{E}^t(t)&=\frac{2\eta(d+0)}{\eta(d+0)+\eta(d-0)}\boldsymbol{E}^+(d-0,t),\end{aligned}$$

at each time t. Here, $\boldsymbol{E}^r(t)$ is the electric field of the reflected transverse plane wave at the front wall, and $\boldsymbol{E}^t(t)$ is the electric field of the transmitted transverse plane wave at the back wall, both evaluated at time t. In the second formula, the fact that there is no incident field from the right has been used. In addition, Eq. (5.5) and the continuity of the magnetic and electric fields at the boundary yield

$$\begin{cases} \boldsymbol{E}^-(d-0,t) = r_1 \boldsymbol{E}^+(d-0,t), \\ 2\boldsymbol{E}^i(t) = t_0 \boldsymbol{E}^+(+0,t) + r_0 \boldsymbol{E}^-(+0,t), \end{cases} \tag{5.6}$$

where

$$r_1 = \frac{\eta(d+0)-\eta(d-0)}{\eta(d+0)+\eta(d-0)}, \quad t_0 = 1 + \frac{\eta(-0)}{\eta(+0)}, \quad r_0 = 1 - \frac{\eta(-0)}{\eta(+0)}.$$

The partial integro-differential equation for the split vector fields $\boldsymbol{E}^\pm$ is easily obtained from the wave equation (5.4) and the wave splitting (5.5). The result,

$$\begin{aligned} &\begin{pmatrix} (\partial_3 + c^{-1}\partial_t)\boldsymbol{E}^+ \\ (\partial_3 - c^{-1}\partial_t)\boldsymbol{E}^- \end{pmatrix} = \frac{\eta'}{2\eta} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} + \frac{1}{2c}\partial_t \left(\boldsymbol{\chi} * \begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} \right), \\ &\boldsymbol{\chi} := \begin{pmatrix} -\boldsymbol{\chi}_{ee} - \boldsymbol{\chi}_{mm} - \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} & -\boldsymbol{\chi}_{ee} + \boldsymbol{\chi}_{mm} + \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} \\ \boldsymbol{\chi}_{ee} - \boldsymbol{\chi}_{mm} + \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} & \boldsymbol{\chi}_{ee} + \boldsymbol{\chi}_{mm} - \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} \end{pmatrix}, \end{aligned} \tag{5.7}$$

is clearly equivalent to the Maxwell equations for the bi-isotropic medium. Moreover, since the slab is initially unexcited by the second and third paragraphs of this section, there is a time $T_0 := \min(T_1, T)$ such that

$$\begin{cases} \boldsymbol{E}^i(t) = \mathbf{0}, \quad t < T_0, \\ \boldsymbol{E}^\pm(x_3,t) = \mathbf{0}, \quad (x_3,t) \in (0,d) \times (-\infty, T_0]. \end{cases} \tag{5.8}$$

Introduce travel-time coordinates, (x,s), by

$$s(t) = \frac{t - T_0}{t_{slab}}, \qquad x(x_3) = \frac{1}{t_{slab}} \int_0^{x_3} \frac{dx_3'}{c(x_3')}, \qquad t_{slab} = \int_0^d \frac{dx_3'}{c(x_3')},$$

and put $\boldsymbol{e}^\pm(x,s) := \boldsymbol{E}^\pm(x_3(x), t(s))$ and $\boldsymbol{e}^i(s) := \boldsymbol{E}^i(t(s))$. By these substitutions of variables, Eqs. (5.7), (5.6), and (5.8) are transformed into the non-local hyperbolic initial-boundary value problem (1.1), where the functions $\mathbf{a}$ and $\mathbf{b}$ are defined by

$$\begin{cases} \mathbf{a}(x,s) = \dfrac{t_{slab}}{2} \partial_s \boldsymbol{\chi}\left(x_3(x), t_{slab} s\right), \qquad (x,s) \in \mathbb{I} \times \mathbb{R}_+ \equiv (0,1) \times (0,\infty), \\ \mathbf{b}(x) = \dfrac{t_{slab}}{2} \boldsymbol{\chi}(x_3(x), 0) + \dfrac{d}{dx} \ln \sqrt{\dfrac{\eta(x_3(x))}{\eta_0}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}, \qquad x \in \mathbb{I}, \end{cases}$$

and η_0 is the wave impedance in vacuum. From the third paragraph of this section, it is clear, that $\mathbf{a} \in \mathcal{C}_{4\times4}(\mathbb{I} \times \mathbb{R}_+)$ is differentiable with respect to time s, $\partial_s \mathbf{a} \in \mathcal{C}_{4\times4}(\mathbb{I} \times \mathbb{R}_+)$, and $\mathbf{b} \in \mathcal{C}_{4\times4}(\mathbb{I})$, so that all theorems in the previous sections are applicable to this wave propagation problem.

As a final remark, note that the solution to Eq. (1.1) in this bi-isotropic case is axially symmetric, i.e., if $\boldsymbol{e}^\pm$ is the solution corresponding to the input $\boldsymbol{e}^i$, and $\mathbf{R}$ is an arbitrary rotation matrix in the x_1-x_2-plane, then $\mathbf{R}\boldsymbol{e}^\pm$ is the solution corresponding to the input $\mathbf{R}\boldsymbol{e}^i$. This is not surprising since the constitutive relations for the bi-isotropic medium are isotropic. More generally, this happens for media such that $\mathbf{R}\mathbf{a}_{ij}\mathbf{R}^{-1} = \mathbf{a}_{ij}$ and $\mathbf{R}\mathbf{b}_{ij}\mathbf{R}^{-1} = \mathbf{b}_{ij}$, $1 \le i,j \le 2$, i.e., all the submatrices $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$ of $\mathbf{a}$ and $\mathbf{b}$, respectively, defined by the decompositions in Theorem 2.1, commute with every rotation matrix $\mathbf{R}$.

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