Nonlinear waveguides

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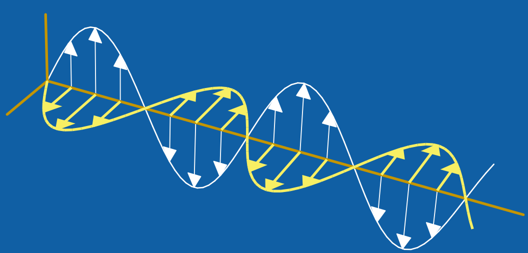

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Abstract

We investigate the propagation of electromagnetic waves in a waveguide filled with a nonlinear material. The electromagnetic field is expanded in the usual eigenmodes of the waveguide, and the coupling between the modes is quantified. We derive the wave equations governing each mode, with special emphasis on the situation with a dominant TE-mode. The result is a strictly hyperbolic system of nonlinear partial differential equations for the dominating mode, whereas the minor modes satisfy hyperbolic systems of linear, nonstationary, partial differential equations. A growth estimate is given for the minor modes.

1 Introduction

Electromagnetic waves can be guided in space by a hollow waveguide, where the walls are perfect electric conductors. This can be used to guide waves traveling from one point to another, and provide a controlled environment in which measurements can be made. In order to be able to interpret these measurements, we need to investigate what influence the waveguide structure and material or filling have on the wave propagation.

The waves can be decomposed in modes, which can be defined as the eigenfunctions of a transversal differential operator. These modes are orthogonal, and the wave equation for each mode decouples completely from the other modes for a linear, homogeneous filling. When the waveguide is filled with a nonlinear material, the equations no longer decouple, but it is still motivated to use the standard waveguide modes due to the possibility of using the resulting equations with a mode-matching algorithm in direct and inverse scattering problems [3, 16, 18].

In this paper we study the propagation of transient waves in nonlinear waveguides, *i.e.*, waves generated by an arbitrary signal. The theory of linear waveguides is well established since the major efforts during the second world war. The analysis is often made in the frequency domain, but since the nonlinear filling not only couples the modes but also induces a coupling between the different frequency components, we choose to treat the problem entirely in the time domain. The propagation of transient waves has been treated for linear materials in *e.g.*, [2, 6, 12], and some general references are [5, 15] and [10, Ch. 8].

There has been a number of papers on nonlinear waveguides. Some recent contributions consider the problem of self focusing, where the field energy inside the waveguide moves closer to the center as the wave propagates. A few early studies are found in [4, 11] and a more recent is [17]. The paper [19] discusses a problem similar to ours, where a modal expansion of the fields is attempted in a dielectric slab waveguide for a fixed frequency. The resulting equations are mainly used to determine where the energy will be localized.

This paper is organized as follows. In Section 2 we introduce the Maxwell equations and the instantaneous constitutive relations. The waveguide geometry is presented in Section 3, and the relevant expansion functions are derived. This is mostly established theory, but presented in a slightly different manner. Specifically, the starting point is not the Helmholtz' equation, but a system of vector-valued equations. The eigenvalue zero now plays a different rôle. We use the expansion functions to obtain wave equations for each mode in Section 4, and the explicit results for a parallel plate waveguide are calculated in Section 5. Since even this simple example proves very challenging, we make the reasonable assumption that almost all the energy is contained in one mode in Section 6, which enables us to derive a system of quasilinear, homogeneous, hyperbolic differential equations for the dominant mode, and a system of inhomogeneous hyperbolic equations with source terms for the minor modes. Some energy relations are derived for both the dominant and the minor modes, allowing an estimate of the growth of the minor modes. The final conclusions and discussions are given in Section 7.

2 Preliminaries

In this paper we use a slight modification of the Heaviside-Lorentz units [10, p. 781], where the electromagnetic fields are scaled so that they all have the physical dimension $\sqrt{\text{energy/volume}}$,

$$\begin{cases} \boldsymbol{E} = \sqrt{\epsilon_0}\,\boldsymbol{E}_{\text{SI}} \\ \boldsymbol{H} = \sqrt{\mu_0}\,\boldsymbol{H}_{\text{SI}}, \end{cases} \qquad \begin{cases} \boldsymbol{D} = 1/\sqrt{\epsilon_0}\,\boldsymbol{D}_{\text{SI}} \\ \boldsymbol{B} = 1/\sqrt{\mu_0}\,\boldsymbol{B}_{\text{SI}}, \end{cases} \tag{2.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field strength, respectively, and $\boldsymbol{D}$ and $\boldsymbol{B}$ are the electric and magnetic flux density, respectively. ϵ_0 is the permittivity of vacuum, and μ_0 is the permeability of vacuum. The speed of light in vacuum is $1/\sqrt{\epsilon_0\mu_0} = c_0$. We use instantaneous, isotropic constitutive relations, see [8, p. 231] and [20],

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = F_{\text{e}}(E(\boldsymbol{r},t)^2)\boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) = F_{\text{m}}(H(\boldsymbol{r},t)^2)\boldsymbol{H}(\boldsymbol{r},t), \end{cases} \tag{2.2}$$

where F_{e} and F_{m} are dimensionless functions of $E(\boldsymbol{r},t)^2 = |\boldsymbol{E}(\boldsymbol{r},t)|^2$ and $H(\boldsymbol{r},t)^2 = |\boldsymbol{H}(\boldsymbol{r},t)|^2$, respectively. We use the squared absolute values as arguments instead of the absolute values themselves, since this is beneficial in the final equations. The above constitutive relations imply that the time derivative of the fluxes can be written

$$\begin{cases} \partial_t\boldsymbol{D} = [F_{\text{e}}(E^2)\mathbf{I} + 2F_{\text{e}}'(E^2)\boldsymbol{E}\boldsymbol{E}]\cdot\partial_t\boldsymbol{E} = \boldsymbol{\epsilon}(\boldsymbol{E})\cdot\partial_t\boldsymbol{E} \\ \partial_t\boldsymbol{B} = [F_{\text{m}}(H^2)\mathbf{I} + 2F_{\text{m}}'(H^2)\boldsymbol{H}\boldsymbol{H}]\cdot\partial_t\boldsymbol{H} = \boldsymbol{\mu}(\boldsymbol{H})\cdot\partial_t\boldsymbol{H}. \end{cases} \tag{2.3}$$

We see that nonlinear, isotropic materials are described by non-diagonal dyadics $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$. However, we refrain from this formulation of the time derivative in this paper. We now write the source free Maxwell equations as

$$\begin{cases} -\nabla\times\boldsymbol{H} + \dfrac{1}{c_0}\partial_t\boldsymbol{D} = \mathbf{0} \\ \nabla\times\boldsymbol{E} + \dfrac{1}{c_0}\partial_t\boldsymbol{B} = \mathbf{0} \end{cases} \implies \begin{cases} -\nabla\times\boldsymbol{H} + \dfrac{1}{c_0}\partial_t\left(F_{\text{e}}(E^2)\boldsymbol{E}\right) = \mathbf{0} \\ \nabla\times\boldsymbol{E} + \dfrac{1}{c_0}\partial_t\left(F_{\text{m}}(H^2)\boldsymbol{H}\right) = \mathbf{0}. \end{cases} \tag{2.4}$$

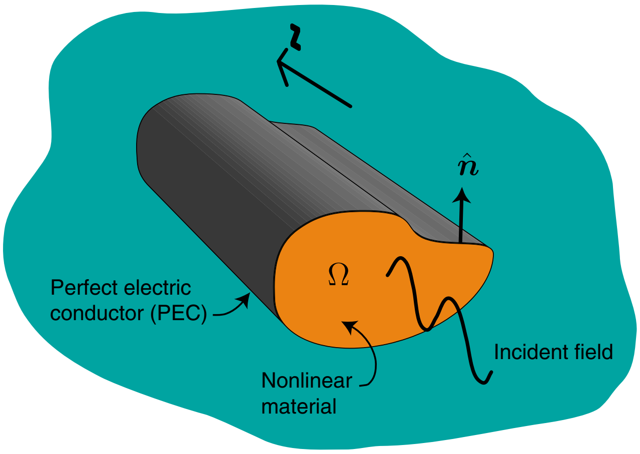

Figure 1: The geometry of a waveguide. The cross section of the cylinder is denoted by Ω, and the outward unit normal at the surface is denoted by $\hat{\boldsymbol{n}}$.

The purpose of the following section is to find suitable expansion functions for the fields $\boldsymbol{E}$ and $\boldsymbol{H}$, which simplifies the analysis of these equations.

3 Derivation of the expansion functions

The geometry of the waveguide is depicted in Figure 1. We wish to use the Maxwell equations to study propagation along the waveguide, *i.e.*, in the z-direction. To this end, we decompose the spatial differential operators and write the Maxwell equations as

$$\begin{cases} -\hat{\boldsymbol{z}} \times \partial_z \boldsymbol{H} + \dfrac{1}{c_0}\partial_t \left(F_{\mathrm{e}}(E^2)\boldsymbol{E}\right) = \nabla_{\mathrm{T}} \times \boldsymbol{H} \\ \hat{\boldsymbol{z}} \times \partial_z \boldsymbol{E} + \dfrac{1}{c_0}\partial_t \left(F_{\mathrm{m}}(H^2)\boldsymbol{H}\right) = -\nabla_{\mathrm{T}} \times \boldsymbol{E} \end{cases} \tag{3.1}$$

where $\nabla_{\mathrm{T}} = \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y$ and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ denote the unit vector in the x, y, and z direction, respectively. The goal is now to get rid of the transversal dependence of the fields, and we start by simplifying the right hand side of this system of equations. We do this by searching for eigenfunctions of the transverse curl operators, which is a diagonalization of the transverse differential operator.

3.1 Eigenfunctions of the transverse curl operators

We see that the right hand side of (3.1) can be formulated as a differential operator applied to the pair of fields $\boldsymbol{E}$ and $\boldsymbol{H}$. It is natural to look for eigenfunctions to this operator, and we formulate the eigenproblem

$$\begin{pmatrix} \mathbf{0} & -\mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} \\ \mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} = \lambda_n \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix}, \tag{3.2}$$

where we have included the imaginary unit i so that the operator is self-adjoint in the scalar product

$$\left(\begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}, \begin{pmatrix}\boldsymbol{E}_n\\ \boldsymbol{H}_n\end{pmatrix}\right) = \iint_\Omega (\boldsymbol{E}_m \cdot \boldsymbol{E}_n^* + \boldsymbol{H}_m \cdot \boldsymbol{H}_n^*)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.3}$$

The self-adjointness can be shown by straight-forward calculations using integration by parts and the boundary condition $\hat{\boldsymbol{n}} \times \boldsymbol{E} = \boldsymbol{0}$, which is the usual boundary condition for a perfect electric conductor. Once we have established that the transverse curl operator in (3.2) is self-adjoint, we also know that all eigenvalues λ_n are real.

3.1.1 Boundary conditions

The boundary condition $\hat{\boldsymbol{n}} \times \boldsymbol{E} = \boldsymbol{0}$ is the only one needed for the analysis of the three-dimensional Maxwell equations, but we must check that our eigenproblem (3.2) does not imply inconsistent boundary conditions for the magnetic field strength. In this section we investigate which boundary conditions are imposed by the Maxwell equations, and compare them with those imposed by the eigenproblem.

We first note that since we have $\nabla \cdot \boldsymbol{B} = 0$, we also have $\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0$ at the boundary. With $\boldsymbol{B} = F_\mathrm{m}(H^2)\boldsymbol{H}$, this implies $\hat{\boldsymbol{n}} \cdot \boldsymbol{H} = 0$ at the boundary. Ampère's law $\nabla \times \boldsymbol{H} - \partial_t \boldsymbol{D} = \boldsymbol{0}$ must be satisfied on the boundary of the waveguide, which implies

$$\hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{H}) - \hat{\boldsymbol{n}} \times \partial_t \boldsymbol{D} = \hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{H}) = \boldsymbol{0} \tag{3.4}$$

on the boundary, since $\hat{\boldsymbol{n}} \times \boldsymbol{D} = \hat{\boldsymbol{n}} \times F_\mathrm{e}(E^2)\boldsymbol{E} = \boldsymbol{0}$. A closer look at this equation reveals that the z component of $\hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{H}) = \boldsymbol{0}$ is

$$\hat{\boldsymbol{n}} \cdot \partial_z \boldsymbol{H} - (\hat{\boldsymbol{n}} \cdot \nabla_\mathrm{T}) H_z = 0 \quad \Rightarrow \quad (\hat{\boldsymbol{n}} \cdot \nabla_\mathrm{T}) H_z = 0, \tag{3.5}$$

where the implication follows from the fact that $\hat{\boldsymbol{n}}$ is independent of z and therefore $\hat{\boldsymbol{n}} \cdot \partial_z \boldsymbol{H} = 0$. The two conditions $\hat{\boldsymbol{n}} \cdot \boldsymbol{H} = 0$ and $(\hat{\boldsymbol{n}} \cdot \nabla_\mathrm{T}) H_z = 0$ are precisely the boundary conditions implied by our eigenvalue problem.

Note carefully that we still have no conditions on $\hat{\boldsymbol{n}} \cdot \boldsymbol{E}$ and $(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}) \cdot \boldsymbol{H}$. These components relate to the charge and current density on the walls of the waveguide, and can be used to study waveguides with a finite conductivity as in many textbooks, *e.g.*, [10, p. 366], [14, p. 317] and [5, p. 340]. This is not a problem we deal with in this paper.

3.1.2 Canonical problems

We show that the eigenproblem (3.2) implies the two-dimensional Helmholtz equation when $\lambda_n \neq 0$ and all necessary derivatives exist. By applying the operator three times, we have

$$
\begin{aligned}
\lambda_n^3 \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & -\mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} \\ \mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} & \mathbf{0} \end{pmatrix}^3 \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & -\mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} \\ \mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \nabla_{\mathrm{T}} \times \nabla_{\mathrm{T}} \times \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathrm{T}} \times \nabla_{\mathrm{T}} \times \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & -\mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} \\ \mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \nabla_{\mathrm{T}}\nabla_{\mathrm{T}} - \nabla_{\mathrm{T}}^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathrm{T}}\nabla_{\mathrm{T}} - \nabla_{\mathrm{T}}^2\mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} \\
&= -\nabla_{\mathrm{T}}^2 \begin{pmatrix} \mathbf{0} & -\mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} \\ \mathrm{i}\nabla_{\mathrm{T}} \times \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} \\
&= -\nabla_{\mathrm{T}}^2 \lambda_n \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix},
\end{aligned}
\tag{3.6}
$$

where we have used that the curl of any gradient is zero and that the Laplace operator ∇_{T}^2 commutes with the curl operator. Thus, when $\lambda_n \neq 0$ the eigenproblem (3.2) implies the usual Helmholtz equation for the cross section,

$$
\begin{cases}
\nabla_{\mathrm{T}}^2 \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} + \lambda_n^2 \begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} = 0 & \text{in } \Omega \\
\hat{\boldsymbol{n}} \times \boldsymbol{E} = \mathbf{0} & \text{on } \partial\Omega \\
\hat{\boldsymbol{n}} \cdot \boldsymbol{H} = 0 & \text{on } \partial\Omega \\
(\hat{\boldsymbol{n}} \cdot \nabla_{\mathrm{T}}) H_z = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.7}
$$

Since all components can be found from the z-components using the original eigenproblem (3.2), we now formulate the scalar canonical problems

$$
\begin{aligned}
&\text{TE} \quad \begin{cases} \nabla_{\mathrm{T}}^2 \phi_n^{\mathrm{TE}} + (\lambda_n^{\mathrm{TE}})^2 \phi_n^{\mathrm{TE}} = 0 & \text{in } \Omega \\ \dfrac{\partial \phi_n^{\mathrm{TE}}}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \\
&\text{TM} \quad \begin{cases} \nabla_{\mathrm{T}}^2 \phi_n^{\mathrm{TM}} + (\lambda_n^{\mathrm{TM}})^2 \phi_n^{\mathrm{TM}} = 0 & \text{in } \Omega \\ \phi_n^{\mathrm{TM}} = 0 & \text{on } \partial\Omega, \end{cases}
\end{aligned}
\tag{3.8}
$$

where the acronyms TE and TM stand for a solution with Transverse Electric field or Transverse Magnetic field, respectively. That is, ϕ_n^{TE} is associated with the z-component of the magnetic field strength $\boldsymbol{H}_n$, and ϕ_n^{TM} is associated with the z-component of the electric field strength $\boldsymbol{E}_n$.

What about the eigenfunctions when $\lambda_n = 0$? Since they satisfy $\nabla_{\mathrm{T}} \times \boldsymbol{E}_n = \nabla_{\mathrm{T}} \times \boldsymbol{H}_n = \mathbf{0}$, they can be written as gradients of a scalar function [1, p. 66]. But the canonical problems above supply us with a complete set of scalar functions on Ω, and after considering the appropriate boundary conditions we deduce $\boldsymbol{E}_n = \nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}}$ and $\boldsymbol{H}_n = \nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}}$ for these eigenfunctions.[1]

[1] The case $\lambda = 0$ is also associated with the TEM modes, but for a simply connected geometry as the hollow waveguide, these do not appear. The TEM modes are also gradients of a scalar function ϕ, but for these modes this function satisfies $\nabla_{\mathrm{T}}^2\phi = 0$ in Ω, which only has constant solutions in a simply connected geometry with homogeneous boundary conditions.

We normalize the canonical solutions ϕ_n^{TE} and ϕ_n^{TM} by requiring

$$\begin{cases} \iint_\Omega (\lambda_n^{\mathrm{TE}}\phi_n^{\mathrm{TE}})^2 \,\mathrm{d}x\,\mathrm{d}y = 1 \\ \iint_\Omega (\lambda_n^{\mathrm{TM}}\phi_n^{\mathrm{TM}})^2 \,\mathrm{d}x\,\mathrm{d}y = 1 \end{cases} \Leftrightarrow \begin{cases} \iint_\Omega |\nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y = 1 \\ \iint_\Omega |\nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}}|^2 \,\mathrm{d}x\,\mathrm{d}y = 1. \end{cases} \tag{3.9}$$

This implies that the scalar eigenfunctions are dimensionless. We summarize our results by expressing the vector eigenfunctions of the original transverse curl operator in the canonical, scalar functions ϕ_n^{TE} and ϕ_n^{TM},

$$\begin{pmatrix} \boldsymbol{E}_n \\ \boldsymbol{H}_n \end{pmatrix} = \begin{cases} \begin{pmatrix} -\hat{\boldsymbol{z}} \times \nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}} \\ \mathrm{i}\lambda_n^{\mathrm{TE}}\phi_n^{\mathrm{TE}}\hat{\boldsymbol{z}} \end{pmatrix}, & \begin{pmatrix} \boldsymbol{0} \\ \nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}} \end{pmatrix}, \\ \begin{pmatrix} \mathrm{i}\lambda_n^{\mathrm{TM}}\phi_n^{\mathrm{TM}}\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{z}} \times \nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}} \end{pmatrix}, & \begin{pmatrix} \nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}} \\ \boldsymbol{0} \end{pmatrix}, \end{cases} \tag{3.10}$$

where the rightmost eigenfunctions correspond to the solutions with $\lambda = 0$. We see that the complex conjugate of the above eigenfunctions are also eigenfunctions, corresponding to the change $\lambda \to -\lambda$. When we use the letter λ in the remainder of this paper, we mean $\lambda > 0$.

3.2 Real-valued expansion functions

The vector eigenfunctions in (3.10) constitute a complete system for expansion of electromagnetic fields in a waveguide [5, p. 329]. Though, since they are complex vectors, we need to use complex expansion coefficients in order to get real-valued fields. By explicitly writing out the real and imaginary values of the scalar expansion coefficients, we have the expansion

$$\begin{aligned} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_n & \frac{1}{2}(u_n^{\mathrm{TE}} - \mathrm{i}w_n^{\mathrm{TE}}) \begin{pmatrix} -\hat{\boldsymbol{z}} \times \nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}} \\ \mathrm{i}\lambda_n^{\mathrm{TE}}\phi_n^{\mathrm{TE}}\hat{\boldsymbol{z}} \end{pmatrix} + \frac{1}{2}v_n^{\mathrm{TE}} \begin{pmatrix} \boldsymbol{0} \\ \nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}} \end{pmatrix} \\ & + \frac{1}{2}(u_n^{\mathrm{TM}} - \mathrm{i}w_n^{\mathrm{TM}}) \begin{pmatrix} \mathrm{i}\lambda_n^{\mathrm{TM}}\phi_n^{\mathrm{TM}}\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{z}} \times \nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}} \end{pmatrix} + \frac{1}{2}v_n^{\mathrm{TM}} \begin{pmatrix} \nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}} \\ \boldsymbol{0} \end{pmatrix} \\ & + \text{complex conjugate terms}, \end{aligned} \tag{3.11}$$

where the expansion coefficients u_n^{TE}, w_n^{TE}, v_n^{TE}, u_n^{TM}, w_n^{TM} and v_n^{TM} are real-valued scalar functions of z and t. Note carefully that the functions $\nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}}$ and $\nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}}$ are real-valued, and are therefore multiplied with real-valued expansion coefficients w_n^{TE} and w_n^{TM}. The summation is taken over $n = 1, 2, \ldots$, with the eigenvalues arranged in ascending order, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. After adding the complex conjugate terms this is

$$
\begin{aligned}
\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} &= \sum_n u_n^{\mathrm{TE}}\begin{pmatrix}-\hat{\boldsymbol{z}}\times\nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}}\\ \mathbf{0}\end{pmatrix} + v_n^{\mathrm{TE}}\begin{pmatrix}\mathbf{0}\\ \nabla_{\mathrm{T}}\phi_n^{\mathrm{TE}}\end{pmatrix} + w_n^{\mathrm{TE}}\begin{pmatrix}\mathbf{0}\\ \lambda_n^{\mathrm{TE}}\phi_n^{\mathrm{TE}}\hat{\boldsymbol{z}}\end{pmatrix}\\
&\qquad + u_n^{\mathrm{TM}}\begin{pmatrix}\mathbf{0}\\ \hat{\boldsymbol{z}}\times\nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}}\end{pmatrix} + v_n^{\mathrm{TM}}\begin{pmatrix}\nabla_{\mathrm{T}}\phi_n^{\mathrm{TM}}\\ \mathbf{0}\end{pmatrix} + w_n^{\mathrm{TM}}\begin{pmatrix}\lambda_n^{\mathrm{TM}}\phi_n^{\mathrm{TM}}\hat{\boldsymbol{z}}\\ \mathbf{0}\end{pmatrix}\\
&= \sum_n u_n^{\mathrm{TE}}\boldsymbol{U}_n^{\mathrm{TE}} + v_n^{\mathrm{TE}}\boldsymbol{V}_n^{\mathrm{TE}} + w_n^{\mathrm{TE}}\boldsymbol{W}_n^{\mathrm{TE}}\\
&\qquad + u_n^{\mathrm{TM}}\boldsymbol{U}_n^{\mathrm{TM}} + v_n^{\mathrm{TM}}\boldsymbol{V}_n^{\mathrm{TM}} + w_n^{\mathrm{TM}}\boldsymbol{W}_n^{\mathrm{TM}},
\end{aligned}
\tag{3.12}
$$

where we have introduced the real-valued six-vector expansion functions $\boldsymbol{U}_n^{\mathrm{TE}}(x,y)$, $\boldsymbol{V}_n^{\mathrm{TE}}(x,y)$ etc. The fact that these expansion functions are derived from an eigenvalue problem, gives us strong orthogonality results, *i.e.*,

$$
\begin{cases}
(\boldsymbol{U}_m^{\mathrm{TE}},\boldsymbol{U}_n^{\mathrm{TE}}) = (\boldsymbol{V}_m^{\mathrm{TE}},\boldsymbol{V}_n^{\mathrm{TE}}) = (\boldsymbol{W}_m^{\mathrm{TE}},\boldsymbol{W}_n^{\mathrm{TE}}) = \delta_{m,n}\\
(\boldsymbol{U}_m^{\mathrm{TM}},\boldsymbol{U}_n^{\mathrm{TM}}) = (\boldsymbol{V}_m^{\mathrm{TM}},\boldsymbol{V}_n^{\mathrm{TM}}) = (\boldsymbol{W}_m^{\mathrm{TM}},\boldsymbol{W}_n^{\mathrm{TM}}) = \delta_{m,n}\\
\text{all other combinations} = 0,
\end{cases}
\tag{3.13}
$$

where $\delta_{m,n}$ denotes the Kronecker delta, $\delta_{m,m} = 1$, $\delta_{m,n} = 0$ for $m \neq n$. We also see that our expansion functions $\boldsymbol{U}_n^{\mathrm{TE}}(x,y)$ etc have the physical dimension $(\text{length})^{-1}$, and thus the expansion coefficients $u_n^{\mathrm{TE}}(z,t)$ etc have the physical dimension $\sqrt{\text{energy/length}}$.

4 Decomposition in modes

We continue the analysis by taking the scalar product of the expansion functions with the Maxwell equations (3.1), in order to remove the transverse dependence. If we denote an arbitrary expansion function by $\boldsymbol{\Psi}_m$, this means we wish to study the equation

$$
\begin{aligned}
\iint_\Omega \boldsymbol{\Psi}_m\cdot\Bigg[&\begin{pmatrix}\mathbf{0} & -\hat{\boldsymbol{z}}\times\mathbf{I}\\ \hat{\boldsymbol{z}}\times\mathbf{I} & \mathbf{0}\end{pmatrix}\partial_z\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} + \frac{1}{c_0}\partial_t\begin{pmatrix}F_{\mathrm{e}}(E^2)\boldsymbol{E}\\ F_{\mathrm{m}}(H^2)\boldsymbol{H}\end{pmatrix}\\
&+ \begin{pmatrix}\mathbf{0} & -\nabla_{\mathrm{T}}\times\mathbf{I}\\ \nabla_{\mathrm{T}}\times\mathbf{I} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}\Bigg]\,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned}
\tag{4.1}
$$

for each possible $\boldsymbol{\Psi}_m$.

4.1 Linear terms

We examine the expansion of the various terms in the Maxwell equations, starting with the z part of the curl operator,

$$
\begin{aligned}
\begin{pmatrix}\mathbf{0} & -\hat{\boldsymbol{z}}\times\mathbf{I}\\ \hat{\boldsymbol{z}}\times\mathbf{I} & \mathbf{0}\end{pmatrix}\partial_z\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} &= \sum_n \partial_z u_n^{\mathrm{TE}}\boldsymbol{V}_n^{\mathrm{TE}} + \partial_z v_n^{\mathrm{TE}}\boldsymbol{U}_n^{\mathrm{TE}}\\
&\qquad + \partial_z u_n^{\mathrm{TM}}\boldsymbol{V}_n^{\mathrm{TM}} + \partial_z v_n^{\mathrm{TM}}\boldsymbol{U}_n^{\mathrm{TM}},
\end{aligned}
\tag{4.2}
$$

which is proved by straight-forward calculations from the expansion (3.12). The last term is the transverse part of the curl operator, which is

$$\begin{pmatrix} \mathbf{0} & -\nabla_\mathrm{T}\times\mathbf{I} \\ \nabla_\mathrm{T}\times\mathbf{I} & \mathbf{0} \end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \sum_n \lambda_n^\mathrm{TE} u_n^\mathrm{TE}\boldsymbol{W}_n^\mathrm{TE} - \lambda_n^\mathrm{TE} w_n^\mathrm{TE}\boldsymbol{U}_n^\mathrm{TE} \\ + \lambda_n^\mathrm{TM} u_n^\mathrm{TM}\boldsymbol{W}_n^\mathrm{TM} - \lambda_n^\mathrm{TM} w_n^\mathrm{TM}\boldsymbol{U}_n^\mathrm{TM}. \tag{4.3}$$

So far, all our work is well known from the corresponding linear analysis, and the orthogonality relations (3.13) makes it easy to evaluate the linear terms in integral (4.1). We now turn our attention to the nonlinear term.

4.2 Nonlinear term

The only term left is the middle term in (4.1), which contains the nonlinear contribution. We remind that the integral to be evaluated is

$$\partial_t \iint_\Omega \boldsymbol{\Psi}_m \cdot \begin{pmatrix} F_\mathrm{e}(E^2)\boldsymbol{E} \\ F_\mathrm{m}(H^2)\boldsymbol{H}\end{pmatrix} \mathrm{d}x\,\mathrm{d}y. \tag{4.4}$$

For the time being, we ignore the time derivative and consider the exact form of this integral for each possible expansion function $\boldsymbol{\Psi}_m$, since many cross terms drop out immediately. For the TE modes this means

$$\begin{cases} \boldsymbol{U}_m^\mathrm{TE}: & \sum_n u_n^\mathrm{TE}(\boldsymbol{U}_m^\mathrm{TE}, F_\mathrm{e}(E^2)\boldsymbol{U}_n^\mathrm{TE}) + v_n^\mathrm{TM}(\boldsymbol{U}_m^\mathrm{TE}, F_\mathrm{e}(E^2)\boldsymbol{V}_n^\mathrm{TM}) \\ \boldsymbol{V}_m^\mathrm{TE}: & \sum_n u_n^\mathrm{TM}(\boldsymbol{V}_m^\mathrm{TE}, F_\mathrm{m}(H^2)\boldsymbol{U}_n^\mathrm{TM}) + v_n^\mathrm{TE}(\boldsymbol{V}_m^\mathrm{TE}, F_\mathrm{m}(H^2)\boldsymbol{V}_n^\mathrm{TE}) \\ \boldsymbol{W}_m^\mathrm{TE}: & \sum_n w_n^\mathrm{TE}(\boldsymbol{W}_m^\mathrm{TE}, F_\mathrm{m}(H^2)\boldsymbol{W}_n^\mathrm{TE}), \end{cases} \tag{4.5}$$

and for the TM modes

$$\begin{cases} \boldsymbol{U}_m^\mathrm{TM}: & \sum_n u_n^\mathrm{TM}(\boldsymbol{U}_m^\mathrm{TM}, F_\mathrm{m}(H^2)\boldsymbol{U}_n^\mathrm{TM}) + v_n^\mathrm{TE}(\boldsymbol{U}_m^\mathrm{TM}, F_\mathrm{m}(H^2)\boldsymbol{V}_n^\mathrm{TE}) \\ \boldsymbol{V}_m^\mathrm{TM}: & \sum_n u_n^\mathrm{TE}(\boldsymbol{V}_m^\mathrm{TM}, F_\mathrm{e}(E^2)\boldsymbol{U}_n^\mathrm{TE}) + v_n^\mathrm{TM}(\boldsymbol{V}_m^\mathrm{TM}, F_\mathrm{e}(E^2)\boldsymbol{V}_n^\mathrm{TM}) \\ \boldsymbol{W}_m^\mathrm{TM}: & \sum_n w_n^\mathrm{TM}(\boldsymbol{W}_m^\mathrm{TM}, F_\mathrm{e}(E^2)\boldsymbol{W}_n^\mathrm{TM}). \end{cases} \tag{4.6}$$

Since the scalar products contain the functions $F_\mathrm{e}(E^2)$ and $F_\mathrm{m}(H^2)$, the remaining terms do not simplify, and the different modes couple to each other. It seems as if our modal analysis breaks down, and of course it does if we want an exact result. Though, we argue that the nonlinearity has its strongest effects in the wave propagation, *i.e.*, it might be permissible to ignore the nonlinear effect over the cross section to some extent. We are able to do this in a manner that preserves some of the coupling between the modes.

An obvious approach is to expand $F_\mathrm{e}(E^2)$ and $F_\mathrm{m}(H^2)$ in a Taylor series and explicitly calculate the corresponding integrals. Since the expressions for $E^2(x,y,z,t)$ and $H^2(x,y,z,t)$ are rather complex, we wish to delay this approach for a while. For a hint of the expressions involved we refer to Appendix A. Instead we suggest to substitute $E^2(x,y,z,t)$ and $H^2(x,y,z,t)$ with some suitable functions independent of the transverse variables x and y, *i.e.*, $\tilde{E}^2(z,t)$ and $\tilde{H}^2(z,t)$,

$$\begin{cases} F_\mathrm{e}(E^2) = F_\mathrm{e}(\tilde{E}^2) + [F_\mathrm{e}(E^2) - F_\mathrm{e}(\tilde{E}^2)] \\ F_\mathrm{m}(H^2) = F_\mathrm{m}(\tilde{H}^2) + [F_\mathrm{m}(H^2) - F_\mathrm{m}(\tilde{H}^2)], \end{cases} \tag{4.7}$$

and treat the terms in square brackets as perturbations which are ignored. Since the factors $F_\mathrm{e}(\tilde{E}^2)$ and $F_\mathrm{m}(\tilde{H}^2)$ are independent of x and y, they can be pulled out of the scalar products in (4.5) and (4.6), and we can then use orthogonality. Though, as we show in Section 4.3, we must generally choose a different $\tilde{E}^2$ and $\tilde{H}^2$ for each expansion function $\boldsymbol{\Psi}_m$, which we denote by an index m, $\tilde{E}^2_m$ and $\tilde{H}^2_m$. The resulting equations are deduced from (4.2), (4.3), (4.5) and (4.6) as

$$\begin{cases} \boldsymbol{U}_m^\mathrm{TE}: & \partial_z v_m^\mathrm{TE} + \dfrac{1}{c_0}\partial_t\left(F_\mathrm{e}(\tilde{E}^2_m)u_m^\mathrm{TE}\right) - \lambda_m^\mathrm{TE} w_m^\mathrm{TE} = 0 \\ \boldsymbol{V}_m^\mathrm{TE}: & \partial_z u_m^\mathrm{TE} + \dfrac{1}{c_0}\partial_t\left(F_\mathrm{m}(\tilde{H}^2_m)v_m^\mathrm{TE}\right) = 0 \\ \boldsymbol{W}_m^\mathrm{TE}: & \dfrac{1}{c_0}\partial_t\left(F_\mathrm{m}(\tilde{H}^2_m)w_m^\mathrm{TE}\right) + \lambda_m^\mathrm{TE} u_m^\mathrm{TE} = 0 \end{cases} \tag{4.8}$$

for the TE-modes, and

$$\begin{cases} \boldsymbol{U}_m^\mathrm{TM}: & \partial_z v_m^\mathrm{TM} + \dfrac{1}{c_0}\partial_t\left(F_\mathrm{m}(\tilde{H}^2_m)u_m^\mathrm{TM}\right) - \lambda_m^\mathrm{TM} w_m^\mathrm{TM} = 0 \\ \boldsymbol{V}_m^\mathrm{TM}: & \partial_z u_m^\mathrm{TM} + \dfrac{1}{c_0}\partial_t\left(F_\mathrm{e}(\tilde{E}^2_m)v_m^\mathrm{TM}\right) = 0 \\ \boldsymbol{W}_m^\mathrm{TM}: & \dfrac{1}{c_0}\partial_t\left(F_\mathrm{e}(\tilde{E}^2_m)w_m^\mathrm{TM}\right) + \lambda_m^\mathrm{TM} u_m^\mathrm{TM} = 0 \end{cases} \tag{4.9}$$

for the TM-modes. We see that the equations are strictly hyperbolic[2], and the equation structure is exactly the same for both TE- and TM-modes, only the functions F_e and F_m must be interchanged.

In the next section we discuss the approximation leading to this result, but we must first consider an important detail. Each of the systems (4.8) and (4.9) must be supplemented by three initial conditions (three dependent variables, three equations, three conditions), but only two initial conditions can be chosen independent of each other. Since we know that $\nabla\cdot\boldsymbol{B} = \nabla\cdot\boldsymbol{D} = 0$ inside the waveguide, we must introduce the additional constraints

$$-\lambda_m^\mathrm{TE} F_\mathrm{m}(\tilde{H}^2_m)v_m^\mathrm{TE}(z,0) + \partial_z\left(F_\mathrm{m}(\tilde{H}^2_m)w_m^\mathrm{TE}(z,0)\right) = 0 \tag{4.10}$$

[2]A system of partial differential equations $\partial_t\boldsymbol{u} + \mathbf{A}(\boldsymbol{u})\partial_z\boldsymbol{u} = \mathbf{0}$ is strictly hyperbolic if all eigenvalues of the matrix $\mathbf{A}(\boldsymbol{u})$ are real and distinct [9, p. 573]. In our case, it can be shown that there is one positive eigenvalue, one negative eigenvalue and one zero eigenvalue.

and

$$-\lambda_m^{\mathrm{TM}} F_{\mathrm{e}}(\tilde{E}_m^2) v_m^{\mathrm{TM}}(z,0) + \partial_z \left(F_{\mathrm{e}}(\tilde{E}_m^2) w_m^{\mathrm{TM}}(z,0) \right) = 0 \tag{4.11}$$

to maintain compatibility with the original equations. Note that λ_m^{TE} and λ_m^{TM} correspond to the x and y derivatives. The above constraints can be reformulated as

$$\lambda_m^{\mathrm{TE}} v_m^{\mathrm{TE}}(z,0) = \partial_z w_m^{\mathrm{TE}}(z,0) + \partial_z \ln F_{\mathrm{m}}(\tilde{H}_m^2(z,0)) \tag{4.12}$$

and

$$\lambda_m^{\mathrm{TM}} v_m^{\mathrm{TM}}(z,0) = \partial_z w_m^{\mathrm{TM}}(z,0) + \partial_z \ln F_{\mathrm{e}}(\tilde{E}_m^2(z,0)), \tag{4.13}$$

which may be nontrivial to satisfy. These constraints can also be derived from the last two lines in (4.8) and (4.9), respectively.

4.3 Estimate of the approximation

To estimate the approximation we made in the previous section, we look at the electric field only. We use a Taylor expansion of the function $F_{\mathrm{e}}(\tilde{E}_m^2)$ in the vicinity of the (so far) unknown argument $\tilde{E}_m^2$. Since the explicit representation of $E^2 = \boldsymbol{E}\cdot\boldsymbol{E}$ is rather complicated if we use the expansion functions $\boldsymbol{U}_n^{\mathrm{TE}}$, $\boldsymbol{V}_n^{\mathrm{TM}}$ and $\boldsymbol{W}_n^{\mathrm{TM}}$ (see Appendix A), we formulate the expansion of the electric field in a somewhat more abstract, but compact, manner,

$$\boldsymbol{E} = \sum_n f_n(z,t)\boldsymbol{E}_n(x,y) \quad \Rightarrow \quad E^2 = \boldsymbol{E}\cdot\boldsymbol{E} = \sum_{kl} f_k f_l \boldsymbol{E}_k \cdot \boldsymbol{E}_l, \tag{4.14}$$

where $(\boldsymbol{E}_m, \boldsymbol{E}_n) = \delta_{mn}$, *i.e.*, we drop the distinction between functions having or not having a z-component. This means the index n also includes variations of TE- and TM-modes etc. The error we wish to estimate is formulated as the scalar product

$$(\boldsymbol{E}_m, [F_{\mathrm{e}}(E^2) - F_{\mathrm{e}}(\tilde{E}_m^2)]\boldsymbol{E}), \tag{4.15}$$

and upon expanding $F_{\mathrm{e}}(E^2)$ in a Taylor series we see that this term is at most $\mathcal{O}(F_{\mathrm{e}}''(\tilde{E}^2)\iint_\Omega (E^2 - \tilde{E}^2)^2 E \,\mathrm{d}x\,\mathrm{d}y)$ if

$$F_{\mathrm{e}}'(\tilde{E}^2)(\boldsymbol{E}_m, [E^2 - \tilde{E}_m^2]\boldsymbol{E}) = 0. \tag{4.16}$$

This can be accomplished by choosing

$$\tilde{E}_m^2 = \frac{1}{f_m}(\boldsymbol{E}_m, E^2\boldsymbol{E}) = \sum_{nkl} \frac{f_k f_l f_n}{f_m}(\boldsymbol{E}_m, \boldsymbol{E}_k\cdot\boldsymbol{E}_l\boldsymbol{E}_n), \tag{4.17}$$

which of course is the *exact* result for a Kerr material, *i.e.*, $F_{\mathrm{e}}(E^2) = 1 + E^2$. For other materials this represents an approximation to the first order in a Taylor series expansion of the constitutive function $F_{\mathrm{e}}(E^2)$, and we see that this choice of $\tilde{E}_m^2$ is

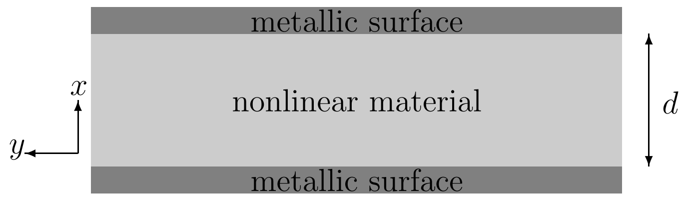

Figure 2: The parallel plate waveguide.

equivalent to the Taylor expansion suggested on page 9. However, by delaying the introduction of the Taylor expansion until this stage, we have gained an opportunity to make other choices of $\tilde{E}_m^2$, depending on the situation at hand.

It is clear that we in general must choose a different $\tilde{E}_m^2$ for different $\boldsymbol{E}_m$. We can calculate the scalar products $(\boldsymbol{E}_m, \boldsymbol{E}_k \cdot \boldsymbol{E}_l \boldsymbol{E}_n)$ analytically for a few geometries, especially the rectangular and the parallel plate waveguide. For general geometries, we must resort to numerical calculations of the scalar products. In the following section, we show that even when we can calculate everything analytically, the problem is still quite a challenge.

5 Parallel plate waveguide with non-magnetic material

We analyze one of the simplest possible waveguides, *i.e.*, the parallel plate waveguide filled with a non-magnetic material. The geometry is depicted in Figure 2, and the expansion functions are

$$\begin{cases} \boldsymbol{U}_n^{\mathrm{TE}} = \begin{pmatrix} -\sqrt{\frac{2}{d}}\sin(\frac{n\pi x}{d})\hat{\boldsymbol{y}} \\ \boldsymbol{0} \end{pmatrix}, & \boldsymbol{U}_n^{\mathrm{TM}} = \begin{pmatrix} \boldsymbol{0} \\ \sqrt{\frac{2}{d}}\cos(\frac{n\pi x}{d})\hat{\boldsymbol{y}} \end{pmatrix}, \\ \boldsymbol{V}_n^{\mathrm{TE}} = \begin{pmatrix} \boldsymbol{0} \\ \sqrt{\frac{2}{d}}\sin(\frac{n\pi x}{d})\hat{\boldsymbol{x}} \end{pmatrix}, & \boldsymbol{V}_n^{\mathrm{TM}} = \begin{pmatrix} \sqrt{\frac{2}{d}}\cos(\frac{n\pi x}{d})\hat{\boldsymbol{x}} \\ \boldsymbol{0} \end{pmatrix}, \\ \boldsymbol{W}_n^{\mathrm{TE}} = \begin{pmatrix} \boldsymbol{0} \\ \sqrt{\frac{2}{d}}\cos(\frac{n\pi x}{d})\hat{\boldsymbol{z}} \end{pmatrix}, & \boldsymbol{W}_n^{\mathrm{TM}} = \begin{pmatrix} \sqrt{\frac{2}{d}}\sin(\frac{n\pi x}{d})\hat{\boldsymbol{z}} \\ \boldsymbol{0} \end{pmatrix}. \end{cases} \tag{5.1}$$

If the two plates are held at different potentials, we also have the TEM-mode,

$$\boldsymbol{U}^{\mathrm{TEM}} = \begin{pmatrix} \sqrt{\frac{1}{d}}\hat{\boldsymbol{x}} \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{V}^{\mathrm{TEM}} = \begin{pmatrix} \boldsymbol{0} \\ \sqrt{\frac{1}{d}}\hat{\boldsymbol{y}} \end{pmatrix}, \quad \boldsymbol{W}^{\mathrm{TEM}} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \tag{5.2}$$

but we assume the TEM-mode to be absent in this section. The full expansion of the scalar products $(\boldsymbol{E}_m, \boldsymbol{E}_k \cdot \boldsymbol{E}_l \boldsymbol{E}_n)$ with arbitrary expansion functions is given in appendix A. The only non-zero scalar products we need to compute are

$$
\begin{aligned}
(\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}}) &= (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&= (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{W}_n^{\mathrm{TM}}) = (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{W}_n^{\mathrm{TM}}) \\
&= \frac{4}{d^2} \int_{x=0}^{d} \sin\frac{m\pi x}{d} \sin\frac{k\pi x}{d} \sin\frac{l\pi x}{d} \sin\frac{n\pi x}{d} \,\mathrm{d}x \\
&= \frac{1}{2d}[-\delta_{m,k+l+n} + \delta_{m,-k+l+n} + \delta_{m,k-l+n} - \delta_{m,-k-l+n} \\
&\qquad + \delta_{m,k+l-n} - \delta_{m,-k+l-n} - \delta_{m,k-l-n} + \delta_{m,-k-l-n}],
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
(\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) &= (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{W}_n^{\mathrm{TM}}) \\
&= \frac{4}{d^2} \int_{x=0}^{d} \sin\frac{m\pi x}{d} \cos\frac{k\pi x}{d} \cos\frac{l\pi x}{d} \sin\frac{n\pi x}{d} \,\mathrm{d}x \\
&= \frac{1}{2d}[\delta_{m,k+l+n} + \delta_{m,-k+l+n} + \delta_{m,k-l+n} + \delta_{m,-k-l+n} \\
&\qquad - \delta_{m,k+l-n} - \delta_{m,-k+l-n} - \delta_{m,k-l-n} - \delta_{m,-k-l-n}],
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
(\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TM}}) &= (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&= \frac{4}{d^2} \int_{x=0}^{d} \cos\frac{m\pi x}{d} \sin\frac{k\pi x}{d} \sin\frac{l\pi x}{d} \cos\frac{n\pi x}{d} \,\mathrm{d}x \\
&= \frac{1}{2d}[-\delta_{m,k+l+n} + \delta_{m,-k+l+n} + \delta_{m,k-l+n} - \delta_{m,-k-l+n} \\
&\qquad - \delta_{m,k+l-n} + \delta_{m,-k+l-n} + \delta_{m,k-l-n} - \delta_{m,-k-l-n}].
\end{aligned}
\tag{5.5}
$$

It is now clear that, even for the simplest cases, we have a formidable problem. For instance, we see that each mode generally couple to infinitely many others, sometimes with a nondecreasing coupling factor. This means that in order to proceed, we should impose some more restrictions on the problem.

6 Dominant TE mode in a non-magnetic material

In the previous section it is shown that (4.17) is quite complicated to handle explicitly. In this section, we look at a simplified case for a nonmagnetic material ($F_{\mathrm{m}}(H^2) = 1$), where almost all the energy is contained in the first TE-mode, which we denote TE_1. This mode is chosen since it generally corresponds to the lowest eigenvalue λ_1 [7, p. 410], and should be the easiest to generate. The assumption of a dominant TE_1 mode implies

$$
|u_1^{\mathrm{TE}}|, |v_1^{\mathrm{TE}}|, |w_1^{\mathrm{TE}}| \gg \text{all other expansion coefficients.} \tag{6.1}
$$

The expansion coefficients v_1^{TE} and w_1^{TE} are associated with the magnetic field, and do not enter our calculations below. Therefore we study only the case $|u_1^{\mathrm{TE}}| \to \infty$.

6.1 Leading terms for $\boldsymbol{U}_m^{\mathrm{TE}}$

With the above assumption on the relative sizes of the expansion coefficients, we deduce in Appendix A.3 that the leading term for the dominant mode $\boldsymbol{U}_1^{\mathrm{TE}}$ is

$$\tilde{E}_{\mathrm{u1}}^2 = (u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u1}}, \tag{6.2}$$

where the index "u1" is used to indicate the relation to the expansion function $\boldsymbol{U}_1^{\mathrm{TE}}$. For the other TE-modes, we have

$$\tilde{E}_{\mathrm{u}m}^2 = (u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u}m} + \frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}} \beta_{\mathrm{u}m} + \mathcal{O}(u_1^{\mathrm{TE}} u_m^{\mathrm{TE}}), \tag{6.3}$$

and the constants α_{u1}, $\alpha_{\mathrm{u}m}$ and $\beta_{\mathrm{u}m}$ are defined in Appendix A.3 as

$$\begin{cases} \alpha_{\mathrm{u1}} = \iint_\Omega |\nabla_{\mathrm{T}} \phi_1^{\mathrm{TE}}|^4 \,\mathrm{d}x\,\mathrm{d}y \\ \alpha_{\mathrm{u}m} = \iint_\Omega |\nabla_{\mathrm{T}} \phi_m^{\mathrm{TE}}|^2 |\nabla_{\mathrm{T}} \phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y + 2 \iint_\Omega |\nabla_{\mathrm{T}} \phi_m^{\mathrm{TE}} \cdot \nabla_{\mathrm{T}} \phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ \beta_{\mathrm{u}m} = \iint_\Omega \nabla_{\mathrm{T}} \phi_m^{\mathrm{TE}} \cdot \nabla_{\mathrm{T}} \phi_1^{\mathrm{TE}} |\nabla_{\mathrm{T}} \phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y \end{cases} \tag{6.4}$$

Notice that the first term in $\tilde{E}_{\mathrm{u}m}^2$ is one order less in u_1^{TE} than the second. This term is needed since the second will cancel in the wavespeed factor. As soon as $\beta_{\mathrm{u}m} \neq 0$, the second term can be very large for infinitesimal u_m^{TE}. This is the origin of the mode coupling, and causes the excitation of new modes which were not present in the beginning. This is made more clear in the sections to follow. Figure 3 shows the distribution of this factor for the rectangular and circular waveguides.

In Section 4.2 we ignored the time derivative in order to calculate a scalar product. It is now time to bring that operator back to life, and calculate the time derivative $\partial_t \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2) u_m^{\mathrm{TE}} \right)$. For the dominant mode this is

$$\begin{aligned} \partial_t \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u1}}^2) u_1^{\mathrm{TE}} \right) &= \partial_t \left(F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u1}}) u_1^{\mathrm{TE}} \right) \\ &= [F_{\mathrm{e}}(\tilde{E}_{\mathrm{u1}}^2) + 2F_{\mathrm{e}}'(\tilde{E}_{\mathrm{u1}}^2)(u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u1}}] \partial_t u_1^{\mathrm{TE}} \end{aligned} \tag{6.5}$$

and for the other TE-modes it is

$$\begin{aligned} \partial_t \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2) u_m^{\mathrm{TE}} \right) &= \partial_t \left(F_{\mathrm{e}} \left(\frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}} \beta_{\mathrm{u}m} + (u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u}m} \right) u_m^{\mathrm{TE}} \right) \\ &= [F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2) - F_{\mathrm{e}}'(\tilde{E}_{\mathrm{u}m}^2) \frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}} \beta_{\mathrm{u}m}] \partial_t u_m^{\mathrm{TE}} \\ &\quad + F_{\mathrm{e}}'(\tilde{E}_{\mathrm{u}m}^2)[3(u_1^{\mathrm{TE}})^2 \beta_{\mathrm{u}m} + 2u_1^{\mathrm{TE}} u_m^{\mathrm{TE}} \alpha_{\mathrm{u}m}] \partial_t u_1^{\mathrm{TE}}. \end{aligned} \tag{6.6}$$

Expanding $F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)$ in a Taylor series suggests that $F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2) - F_{\mathrm{e}}'(\tilde{E}_{\mathrm{u}m}^2) \frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}} \beta_{\mathrm{u}m} \approx F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u}m})$, *i.e.*, the factor multiplying $\partial_t u_m^{\mathrm{TE}}$ should be approximated without

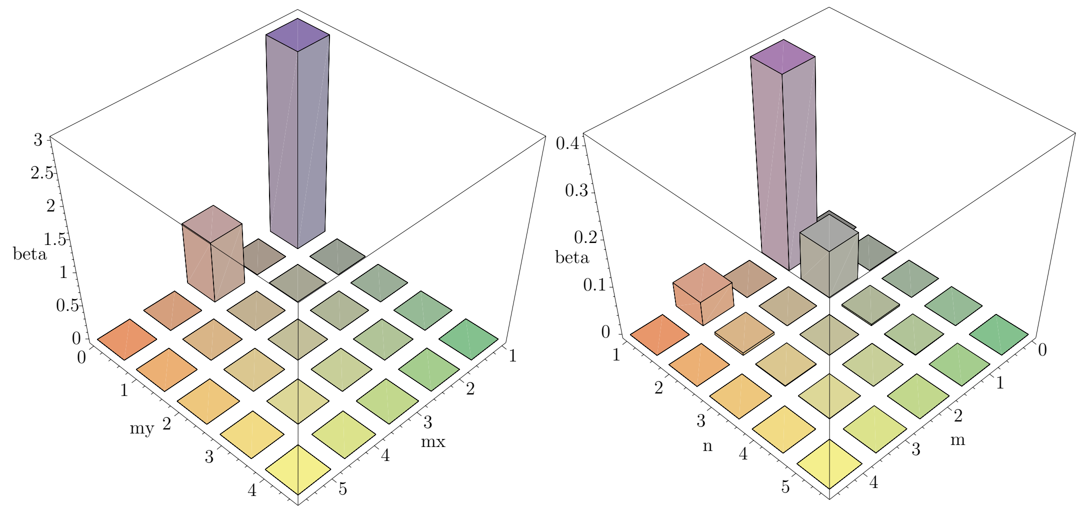

Figure 3: The distribution of the amplitude of the coupling factor $|\beta_{\mathrm{u}m}|$ for a waveguide with rectangular (left) and circular (right) cross section. The index m is in both cases composed of two indices (m_x and m_y for the rectangular waveguide, and m and n for the circular). The tall bars correspond to $\alpha_{\mathrm{u}1}$.

any dependence on u_m^{TE}. This implies

$$\partial_t \left(F_{\mathrm{e}}(\tilde{E}^2_{\mathrm{u}m}) u_m^{\mathrm{TE}} \right) = F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u}m}) \partial_t u_m^{\mathrm{TE}} \\ + F'_{\mathrm{e}}(\tilde{E}^2_{\mathrm{u}m})[3(u_1^{\mathrm{TE}})^2 \beta_{\mathrm{u}m} + 2 u_1^{\mathrm{TE}} u_m^{\mathrm{TE}} \alpha_{\mathrm{u}m}] \partial_t u_1^{\mathrm{TE}}, \quad (6.7)$$

i.e., the wave speed for TE-mode m depends only on the dominant mode. As is shown in the next section, this is valid for the TM-modes as well.

6.2 Leading terms for $\boldsymbol{V}_m^{\mathrm{TM}}$ and $\boldsymbol{W}_m^{\mathrm{TM}}$

In Appendix A.3 it is shown that the leading term for $\boldsymbol{V}_m^{\mathrm{TM}}$ has the exact same structure as for $\tilde{E}^2_{\mathrm{v}m}$,

$$\tilde{E}^2_{\mathrm{v}m} = (u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{v}m} + \frac{(u_1^{\mathrm{TE}})^3}{v_m^{\mathrm{TM}}} \beta_{\mathrm{v}m}, \quad (6.8)$$

where the constants $\alpha_{\mathrm{v}m}$ and $\beta_{\mathrm{v}m}$ are integrals similar to $\alpha_{\mathrm{u}m}$ and $\beta_{\mathrm{u}m}$. Calculations analogous to the previous subsection give

$$\partial_t \left(F_{\mathrm{e}}(\tilde{E}^2_{\mathrm{v}m}) v_m^{\mathrm{TM}} \right) = F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{v}m}) \partial_t v_m^{\mathrm{TM}} \\ + F'_{\mathrm{e}}(\tilde{E}^2_{\mathrm{v}m})[3(u_1^{\mathrm{TE}})^2 \beta_{\mathrm{v}m} + 2 u_1^{\mathrm{TE}} v_m^{\mathrm{TM}} \alpha_{\mathrm{v}m}] \partial_t u_1^{\mathrm{TE}}. \quad (6.9)$$

Appendix A.3 also shows that the leading term for $\boldsymbol{W}_m^{\mathrm{TM}}$ is

$$\tilde{E}^2_{\mathrm{w}m} = (u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{w}m}, \quad (6.10)$$

where the constants $\alpha_{\mathrm{w}m}$ are integrals similar to $\alpha_{\mathrm{u}m}$. Finally, we have the time derivative

$$\partial_t \left(F_{\mathrm{e}}(\tilde{E}^2_{\mathrm{wm}}) w_m^{\mathrm{TM}} \right) = F_{\mathrm{e}}(\tilde{E}^2_{\mathrm{wm}}) \partial_t v_m^{\mathrm{TM}} + 2F'_{\mathrm{e}}(\tilde{E}^2_{\mathrm{wm}}) u_1^{\mathrm{TE}} w_m^{\mathrm{TM}} \alpha_{\mathrm{wm}} \partial_t u_1^{\mathrm{TE}}. \tag{6.11}$$

Here, the factor multiplying $\partial_t w_m^{\mathrm{TM}}$ is explicitly independent of w_m^{TM}.

6.3 Resulting equations for a dominant TE mode

The equations for the dominant mode is

$$\begin{cases} \partial_z v_1^{\mathrm{TE}} + \dfrac{1}{c_0} \partial_t \left(F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u1}}) u_1^{\mathrm{TE}} \right) - \lambda_1^{\mathrm{TE}} w_1^{\mathrm{TE}} = 0 \\ \partial_z u_1^{\mathrm{TE}} + \dfrac{1}{c_0} \partial_t v_1^{\mathrm{TE}} = 0 \\ \dfrac{1}{c_0} \partial_t w_1^{\mathrm{TE}} + \lambda_1^{\mathrm{TE}} u_1^{\mathrm{TE}} = 0 \end{cases} \tag{6.12}$$

and

$$\begin{cases} \partial_z v_m^{\mathrm{TE}} + F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{um}}) \dfrac{1}{c_0} \partial_t u_m^{\mathrm{TE}} - \lambda_m^{\mathrm{TE}} w_m^{\mathrm{TE}} \\ \qquad = -F'_{\mathrm{e}}(\tilde{E}^2_{\mathrm{um}}) [3(u_1^{\mathrm{TE}})^2 \beta_{\mathrm{um}} + 2u_1^{\mathrm{TE}} u_m^{\mathrm{TE}} \alpha_{\mathrm{um}}] \dfrac{1}{c_0} \partial_t u_1^{\mathrm{TE}} \\ \partial_z u_m^{\mathrm{TE}} + \dfrac{1}{c_0} \partial_t v_m^{\mathrm{TE}} = 0 \\ \dfrac{1}{c_0} \partial_t w_m^{\mathrm{TE}} + \lambda_m^{\mathrm{TE}} u_m^{\mathrm{TE}} = 0 \end{cases} \tag{6.13}$$

for the rest of the TE-modes. For the TM-modes, we have

$$\begin{cases} \partial_z v_m^{\mathrm{TM}} + \dfrac{1}{c_0} \partial_t u_m^{\mathrm{TM}} - \lambda_m^{\mathrm{TM}} w_m^{\mathrm{TM}} = 0 \\ \partial_z u_m^{\mathrm{TM}} + F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{vm}}) \dfrac{1}{c_0} \partial_t v_m^{\mathrm{TM}} \\ \qquad = -F'_{\mathrm{e}}(\tilde{E}^2_{\mathrm{vm}}) [3(u_1^{\mathrm{TE}})^2 \beta_{\mathrm{vm}} + 2u_1^{\mathrm{TE}} v_m^{\mathrm{TM}} \alpha_{\mathrm{vm}}] \dfrac{1}{c_0} \partial_t u_1^{\mathrm{TE}} \\ F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{wm}}) \dfrac{1}{c_0} \partial_t w_m^{\mathrm{TM}} + \lambda_m^{\mathrm{TM}} u_m^{\mathrm{TM}} = -2F'_{\mathrm{e}}(\tilde{E}^2_{\mathrm{wm}}) u_1^{\mathrm{TE}} w_m^{\mathrm{TM}} \alpha_{\mathrm{wm}} \dfrac{1}{c_0} \partial_t u_1^{\mathrm{TE}} \end{cases} \tag{6.14}$$

Some conclusions from these equations are

- The model is suitable when the dominant mode is not affected by the minor modes. It is modeled by a system of quasi-linear partial differential equations without source terms, which can be reduced to the scalar problem $-\partial_z^2 u + \partial_t^2 \left(F(u^2)u\right) + \lambda^2 u = 0$.
- The minor modes travel through an inhomogeneous medium with source terms. Both the inhomogeneity and the source terms are induced by the dominant mode.

6.4 Energy for the dominant mode

By multiplying the first, second and third equation in (6.12) by u_1^{TE}, v_1^{TE} and w_1^{TE}, respectively, and adding the equations, we obtain

$$\partial_z(u_1^{\mathrm{TE}}v_1^{\mathrm{TE}}) + u_1^{\mathrm{TE}}\frac{1}{c_0}\partial_t\left(F_{\mathrm{e}}((u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u1}})u_1^{\mathrm{TE}}\right) + \frac{1}{c_0}\partial_t\left(\frac{(v_1^{\mathrm{TE}})^2}{2} + \frac{(w_1^{\mathrm{TE}})^2}{2}\right) = 0. \quad (6.15)$$

After some algebra, we find that this can be written as

$$\partial_z(u_1^{\mathrm{TE}}v_1^{\mathrm{TE}}) + \frac{1}{c_0}\partial_t\left(\frac{G((u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u1}})}{\alpha_{\mathrm{u1}}} + \frac{(v_1^{\mathrm{TE}})^2}{2} + \frac{(w_1^{\mathrm{TE}})^2}{2}\right) = 0, \quad (6.16)$$

if we introduce the energy density function

$$\begin{aligned} G((u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u1}}) &= \alpha_{\mathrm{u1}}\int_0^{u_1^{\mathrm{TE}}}[F_{\mathrm{e}}(u^2\alpha_{\mathrm{u1}}) + 2F_{\mathrm{e}}'(u^2\alpha_{\mathrm{u1}})u^2\alpha_{\mathrm{u1}}]u\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^{(u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u1}}}[F_{\mathrm{e}}(x) + 2F_{\mathrm{e}}'(x)x]\,\mathrm{d}x. \end{aligned} \quad (6.17)$$

This is exactly the energy density for propagation in an unbounded medium; the influence of the waveguide is reduced to a scaling constant α_{u1}. The energy density function can be calculated explicitly for some common cases. For the Kerr medium, where $F_{\mathrm{e}}(E^2) = 1 + E^2$, we have

$$G(u^2) = \frac{1}{2}u^2 + \frac{3}{4}u^4. \quad (6.18)$$

It is easy to show that for a medium where F_{e} can be expanded in a polynomial series, the energy density can be expressed in a related series. Another medium which has been used is the saturated Kerr medium [13, 21, 22], where $F_{\mathrm{e}}(E^2) = 1+E^2/(1+E^2)$. The energy density for this medium is

$$G(u^2) = \frac{1}{2}\left(\frac{2u^4}{1+u^2} + \ln(1+u^2)\right). \quad (6.19)$$

Integrating (6.16) over z and $0 \le t \le T$, we have

$$\begin{aligned} &\int\left(\frac{G((u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u1}})}{\alpha_{\mathrm{u1}}} + \frac{(v_1^{\mathrm{TE}})^2}{2} + \frac{(w_1^{\mathrm{TE}})^2}{2}\right)\mathrm{d}z\bigg|_{t=T} \\ &= \int\left(\frac{G((u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u1}})}{\alpha_{\mathrm{u1}}} + \frac{(v_1^{\mathrm{TE}})^2}{2} + \frac{(w_1^{\mathrm{TE}})^2}{2}\right)\mathrm{d}z\bigg|_{t=0} \end{aligned} \quad (6.20)$$

since the parts associated with the z derivative correspond to the field values at infinity, and can be assumed to disappear for finite times T. This means that the energy of the dominant mode is conserved in this approximation.

6.5 Estimate of the mode-spreading

We have shown that one of the most distinct features of nonlinearity in propagation of guided waves is that the modes are no longer independent, but rather couple in an intricate manner. In the case of a dominant mode, this coupling appears as creating inhomogeneities and source terms for the minor modes. In this section we estimate how fast these minor modes grow when the dominant mode is known.

We refrain from using the explicit representation of the source terms in equations (6.12), (6.13) and (6.14), and return to the generic case

$$\begin{cases} \partial_z v_m^{\mathrm{TE}} + \dfrac{1}{c_0}\partial_t\left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)u_m^{\mathrm{TE}}\right) - \lambda_m^{\mathrm{TE}} w_m^{\mathrm{TE}} = 0 \\ \partial_z u_m^{\mathrm{TE}} + \dfrac{1}{c_0}\partial_t v_m^{\mathrm{TE}} = 0 \\ \dfrac{1}{c_0}\partial_t w_m^{\mathrm{TE}} + \lambda_m^{\mathrm{TE}} u_m^{\mathrm{TE}} = 0 \end{cases} \tag{6.21}$$

where $\tilde{E}_{\mathrm{u}m}^2 = \alpha_{\mathrm{u}m}(u_1^{\mathrm{TE}})^2 + \beta_{\mathrm{u}m}\frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}}$. The differential energy equation is now

$$\partial_z(u_m^{\mathrm{TE}} v_m^{\mathrm{TE}}) + u_m^{\mathrm{TE}}\frac{1}{c_0}\partial_t\left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)u_m^{\mathrm{TE}}\right) + \frac{1}{c_0}\partial_t\left(\frac{(v_m^{\mathrm{TE}})^2}{2} + \frac{(w_m^{\mathrm{TE}})^2}{2}\right) = 0. \tag{6.22}$$

For the dominant mode, we found a total energy, *i.e.*, a total time derivative. This is no longer possible, but a step in the desired direction is

$$\begin{aligned} \partial_z\left(u_m^{\mathrm{TE}} v_m^{\mathrm{TE}}\right) + \frac{1}{c_0}\partial_t\left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\frac{(u_m^{\mathrm{TE}})^2}{2} + \frac{(v_m^{\mathrm{TE}})^2}{2} + \frac{(w_m^{\mathrm{TE}})^2}{2}\right) \\ = \frac{(u_m^{\mathrm{TE}})^2}{2}\frac{1}{c_0}\partial_t F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2). \end{aligned} \tag{6.23}$$

Integrating this equation over z, implies

$$\begin{aligned} &\partial_t \int \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\frac{(u_m^{\mathrm{TE}})^2}{2} + \frac{(v_m^{\mathrm{TE}})^2}{2} + \frac{(w_m^{\mathrm{TE}})^2}{2}\right)\mathrm{d}z \\ &\quad = \int \frac{(u_m^{\mathrm{TE}})^2}{2}\partial_t F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\,\mathrm{d}z \\ &\quad = \int F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\frac{(u_m^{\mathrm{TE}})^2}{2}\partial_t \ln F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\,\mathrm{d}z \\ &\quad \leq \sup_z |\partial_t \ln F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)| \int \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\frac{(u_m^{\mathrm{TE}})^2}{2} + \frac{(v_m^{\mathrm{TE}})^2}{2} + \frac{(w_m^{\mathrm{TE}})^2}{2}\right)\mathrm{d}z, \end{aligned} \tag{6.24}$$

where the last line follows only if we do not have any shocks, *i.e.*, the time derivative is bounded. If we assume the supremum is given by the dominating mode only, we can apply Grönwall's inequality (see *e.g.*, [9, p. 624]), to find

$$
\begin{aligned}
&\int \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2) \frac{(u_m^{\mathrm{TE}})^2}{2} + \frac{(v_m^{\mathrm{TE}})^2}{2} + \frac{(w_m^{\mathrm{TE}})^2}{2} \right) \mathrm{d}z \bigg|_{t=T} \\
&\quad \leq \exp\left(\int_0^T \sup_z |\partial_t \ln F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)| \,\mathrm{d}t \right) \\
&\quad \times \int \left(F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2) \frac{(u_m^{\mathrm{TE}})^2}{2} + \frac{(v_m^{\mathrm{TE}})^2}{2} + \frac{(w_m^{\mathrm{TE}})^2}{2} \right) \mathrm{d}z \bigg|_{t=0}.
\end{aligned}
\tag{6.25}
$$

This seems to imply that if we put no energy in this mode from the start, it will stay silent! Though, as we clearly see in the first line of (6.13), there is a source term which depends solely on u_1^{TE} and initiates the minor modes. The estimate (6.25) is simply not valid in the limit $u_m^{\mathrm{TE}} \to 0$, since

$$
\begin{aligned}
\left|\partial_t \ln F_{\mathrm{e}}(\tilde{E}_{\mathrm{u}m}^2)\right| &= \left| \frac{F_{\mathrm{e}}'(\alpha_{\mathrm{u}m}(u_1^{\mathrm{TE}})^2)}{F_{\mathrm{e}}(\alpha_{\mathrm{u}m}(u_1^{\mathrm{TE}})^2)} \right| \\
&\cdot \left| 2\alpha_{\mathrm{u}m} u_1^{\mathrm{TE}} \partial_t u_1^{\mathrm{TE}} + 3\beta_{\mathrm{u}m} \frac{(u_1^{\mathrm{TE}})^2}{u_m^{\mathrm{TE}}} \partial_t u_1^{\mathrm{TE}} - \beta_{\mathrm{u}m} \frac{(u_1^{\mathrm{TE}})^3}{(u_m^{\mathrm{TE}})^2} \partial_t u_m^{\mathrm{TE}} \right| \\
&\to \infty \quad \text{as} \quad u_m^{\mathrm{TE}} \to 0.
\end{aligned}
\tag{6.26}
$$

Thus, we can expect a relatively rapid growth when u_m^{TE} is very small and $\beta_{\mathrm{u}m} \neq 0$. In the case of the parallel plate waveguide, it is easily seen that the coupling factor is $\beta_{\mathrm{u}m} = C\delta_{m,3}$.

7 Conclusions and discussion

The coupling between the modes produces equations for a general mode analysis which are hard to analyze. By adding the assumption of a dominant mode, we obtain a tractable problem. The dominant mode is described by a nonlinear system of homogeneous partial differential equations, and the minor modes are described by a linear system of inhomogeneous partial differential equations. The equations describing the propagation of the dominant mode are inert with respect to the minor modes, and should be object for further studies.

There is *always* a mode spreading present. The mechanisms behind this must be examined further. Some open questions are: 1) is there an equilibrium in the mode distribution, 2) when is the mode spreading strong enough to influence the dominant mode, and 3) how fast do the minor modes grow? It should be stressed that the relative ease of implementing a finite difference algorithm for the three-dimensional analysis of a rectangular waveguide makes the numerical study of the "true" mode spreading possible.

Since we have used the common waveguide modes for the analysis, the equations derived in this paper may be of interest in a mode matching algorithm, especially for the inverse scattering problem. A remaining problem is the propagation through the boundary between a nonlinear material and, *e.g.*, vacuum.

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Appendix A Explicit representations of $\tilde{E}_m^2$

In Section 4.3 we derived the expression

$$\tilde{E}_m^2 = \sum_{kln} \frac{f_k f_l f_n}{f_m} (\boldsymbol{E}_m, \boldsymbol{E}_k \cdot \boldsymbol{E}_l \boldsymbol{E}_n) \tag{A.1}$$

if the electric field is expanded in orthonormal basis functions $\boldsymbol{E}_n$. Though, in order to derive the propagation equations for the modes, we use the expansion

$$\begin{aligned} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_n & u_n^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}} + v_n^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TE}} + w_n^{\mathrm{TE}} \boldsymbol{W}_n^{\mathrm{TE}} \\ & + u_n^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TM}} + v_n^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}} + w_n^{\mathrm{TM}} \boldsymbol{W}_n^{\mathrm{TM}}, \end{aligned} \tag{A.2}$$

where $\boldsymbol{U}_n^{\mathrm{TE}}$, $\boldsymbol{V}_n^{\mathrm{TM}}$ and $\boldsymbol{W}_n^{\mathrm{TM}}$ are associated with the electric field and $\boldsymbol{U}_n^{\mathrm{TM}}$, $\boldsymbol{V}_n^{\mathrm{TE}}$ and $\boldsymbol{W}_n^{\mathrm{TE}}$ are associated with the magnetic field. For the sake of completeness in our presentation, we give the formulas for $\tilde{E}_m^2$ related to each of these expansion functions in this appendix. We also replace the index m with the indices um, vm and wm, depending on if the basis function $\boldsymbol{E}_m$ is $\boldsymbol{U}_m^{\mathrm{TE}}$, $\boldsymbol{V}_m^{\mathrm{TE}}$ or $\boldsymbol{W}_m^{\mathrm{TE}}$.

A.1 Electric field

Expansion function $\boldsymbol{U}_m^{\mathrm{TE}}$:

$$
\begin{aligned}
\tilde{E}_{\mathrm{um}}^2 = &\sum_{kln} \frac{u_k^{\mathrm{TE}} u_l^{\mathrm{TE}} u_n^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} u_l^{\mathrm{TE}} u_n^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} v_l^{\mathrm{TM}} u_n^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} v_l^{\mathrm{TM}} u_n^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{w_k^{\mathrm{TM}} w_l^{\mathrm{TM}} u_n^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} u_l^{\mathrm{TE}} v_n^{\mathrm{TM}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} u_l^{\mathrm{TE}} v_n^{\mathrm{TM}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} v_l^{\mathrm{TM}} v_n^{\mathrm{TM}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} v_l^{\mathrm{TM}} v_n^{\mathrm{TM}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{w_k^{\mathrm{TM}} w_l^{\mathrm{TM}} v_n^{\mathrm{TM}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}).
\end{aligned}
\tag{A.3}
$$

Expansion function $\boldsymbol{V}_m^{\mathrm{TM}}$:

$$
\begin{aligned}
\tilde{E}^2_{\mathrm{vm}} = &\sum_{kln} \frac{u_k^{\mathrm{TE}} u_l^{\mathrm{TE}} u_n^{\mathrm{TE}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} u_l^{\mathrm{TE}} u_n^{\mathrm{TE}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} v_l^{\mathrm{TM}} u_n^{\mathrm{TE}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} v_l^{\mathrm{TM}} u_n^{\mathrm{TE}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{w_k^{\mathrm{TM}} w_l^{\mathrm{TM}} u_n^{\mathrm{TE}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} u_l^{\mathrm{TE}} v_n^{\mathrm{TM}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} u_l^{\mathrm{TE}} v_n^{\mathrm{TM}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} v_l^{\mathrm{TM}} v_n^{\mathrm{TM}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} v_l^{\mathrm{TM}} v_n^{\mathrm{TM}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{w_k^{\mathrm{TM}} w_l^{\mathrm{TM}} v_n^{\mathrm{TM}}}{v_m^{\mathrm{TM}}} (\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{V}_n^{\mathrm{TM}}).
\end{aligned}
\tag{A.4}
$$

Expansion function $\boldsymbol{W}_m^{\mathrm{TM}}$:

$$
\begin{aligned}
\tilde{E}^2_{\mathrm{wm}} = &\sum_{kln} \frac{u_k^{\mathrm{TE}} u_l^{\mathrm{TE}} w_n^{\mathrm{TM}}}{w_m^{\mathrm{TM}}} (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{W}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} u_l^{\mathrm{TE}} w_n^{\mathrm{TM}}}{w_m^{\mathrm{TM}}} (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{W}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{u_k^{\mathrm{TE}} v_l^{\mathrm{TM}} w_n^{\mathrm{TM}}}{w_m^{\mathrm{TM}}} (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{W}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{v_k^{\mathrm{TM}} v_l^{\mathrm{TM}} w_n^{\mathrm{TM}}}{w_m^{\mathrm{TM}}} (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{W}_n^{\mathrm{TM}}) \\
&+ \sum_{kln} \frac{w_k^{\mathrm{TM}} w_l^{\mathrm{TM}} w_n^{\mathrm{TM}}}{w_m^{\mathrm{TM}}} (\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{W}_k^{\mathrm{TM}} \cdot \boldsymbol{W}_l^{\mathrm{TM}} \boldsymbol{W}_n^{\mathrm{TM}}).
\end{aligned}
\tag{A.5}
$$

A.2 Magnetic field

Simply exchange TE for TM and vice versa in the formulas in the previous section to obtain $\tilde{H}^2$.

A.3 Derivation of leading terms for a dominant TE-mode

In Section 6 it was required to deduce the leading terms in the previous expressions when u_1^{TE} is much greater than the other expansion coefficients.

A.3.1 Expansion function $\boldsymbol{U}_m^{\mathrm{TE}}$

The terms in (A.3) proportional to at least $(u_1^{\mathrm{TE}})^2$ are

$$\begin{aligned}
\tilde{E}^2_{\mathrm{u}m} =& \frac{u_1^{\mathrm{TE}} u_1^{\mathrm{TE}} u_1^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}} \boldsymbol{U}_1^{\mathrm{TE}}) \\
&+ \sum_{k\neq 1} \frac{u_k^{\mathrm{TE}} u_1^{\mathrm{TE}} u_1^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}} \boldsymbol{U}_1^{\mathrm{TE}}) \\
&+ \sum_{l\neq 1} \frac{u_1^{\mathrm{TE}} u_l^{\mathrm{TE}} u_1^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_1^{\mathrm{TE}}) \\
&+ \sum_{n\neq 1} \frac{u_1^{\mathrm{TE}} u_1^{\mathrm{TE}} u_n^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}}) \\
&+ \sum_{k\neq 1} \frac{v_k^{\mathrm{TM}} u_1^{\mathrm{TE}} u_1^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{V}_k^{\mathrm{TM}} \cdot \boldsymbol{U}_1^{\mathrm{TE}} \boldsymbol{U}_1^{\mathrm{TE}}) \\
&+ \sum_{l\neq 1} \frac{u_1^{\mathrm{TE}} v_l^{\mathrm{TM}} u_1^{\mathrm{TE}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{V}_l^{\mathrm{TM}} \boldsymbol{U}_1^{\mathrm{TE}}) \\
&+ \sum_{n\neq 1} \frac{u_1^{\mathrm{TE}} u_1^{\mathrm{TE}} v_n^{\mathrm{TM}}}{u_m^{\mathrm{TE}}} (\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}} \boldsymbol{V}_n^{\mathrm{TM}}).
\end{aligned} \tag{A.6}$$

In Appendix B we show that in general these scalar products are small if the free index in the sums is much separated from m. Therefore, we expect the sums in the above expression for $\tilde{E}^2_m$ can be estimated by the terms given by $k = l = n = m$. For $m = 1$ this means

$$\begin{aligned}
\tilde{E}^2_{\mathrm{u}1} &= (u_1^{\mathrm{TE}})^2 (\boldsymbol{U}_1^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}} \boldsymbol{U}_1^{\mathrm{TE}}) \\
&= (u_1^{\mathrm{TE}})^2 \alpha_{\mathrm{u}1},
\end{aligned} \tag{A.7}$$

where

$$\alpha_{\mathrm{u}1} = \iint_\Omega |\nabla_{\mathrm{T}} \phi_1^{\mathrm{TE}}|^4 \,\mathrm{d}x\,\mathrm{d}y. \tag{A.8}$$

For $m \neq 1$ we have

$$\begin{aligned}\tilde{E}_{\mathrm{u}m}^2 =& \frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}}(\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}}\boldsymbol{U}_1^{\mathrm{TE}}) \\ &+ (u_1^{\mathrm{TE}})^2 \left[(\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}}\boldsymbol{U}_m^{\mathrm{TE}}) + 2(\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_m^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}}\boldsymbol{U}_1^{\mathrm{TE}})\right] \\ =& \frac{(u_1^{\mathrm{TE}})^3}{u_m^{\mathrm{TE}}}\beta_{\mathrm{u}m} + (u_1^{\mathrm{TE}})^2\alpha_{\mathrm{u}m},\end{aligned} \tag{A.9}$$

where

$$\alpha_{\mathrm{u}m} = \iint_\Omega |\nabla_{\mathrm{T}}\phi_m^{\mathrm{TE}}|^2|\nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y + 2\iint_\Omega |\nabla_{\mathrm{T}}\phi_m^{\mathrm{TE}} \cdot \nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y \tag{A.10}$$

and

$$\beta_{\mathrm{u}m} = \iint_\Omega \nabla_{\mathrm{T}}\phi_m^{\mathrm{TE}} \cdot \nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|\nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{A.11}$$

We see that $\alpha_{\mathrm{u}1}$ and $\alpha_{\mathrm{u}m}$ are positive, whereas $\beta_{\mathrm{u}m}$ is not that easily analyzed.

A.3.2 Expansion functions $\boldsymbol{V}_m^{\mathrm{TM}}$ and $\boldsymbol{W}_m^{\mathrm{TM}}$

A similar analysis as in the previous section implies that the leading terms for $\boldsymbol{V}_m^{\mathrm{TM}}$ are

$$\begin{aligned}\tilde{E}_{\mathrm{v}m}^2 =& \frac{(u_1^{\mathrm{TE}})^3}{v_m^{\mathrm{TM}}}(\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}}\boldsymbol{U}_1^{\mathrm{TE}}) \\ &+ (u_1^{\mathrm{TE}})^2 \left[(\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}}\boldsymbol{V}_m^{\mathrm{TM}}) + 2(\boldsymbol{V}_m^{\mathrm{TM}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{V}_m^{\mathrm{TM}}\boldsymbol{U}_1^{\mathrm{TE}}\right] \\ =& \frac{(u_1^{\mathrm{TE}})^3}{v_m^{\mathrm{TM}}}\beta_{\mathrm{v}m} + (u_1^{\mathrm{TE}})^2\alpha_{\mathrm{v}m}\end{aligned} \tag{A.12}$$

where

$$\alpha_{\mathrm{v}m} = \iint_\Omega |\nabla_{\mathrm{T}}\phi_m^{\mathrm{TM}}|^2|\nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y + 2\iint_\Omega |\nabla_{\mathrm{T}}\phi_m^{\mathrm{TM}} \times \nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y \tag{A.13}$$

and

$$\beta_{\mathrm{v}m} = \iint_\Omega \hat{\boldsymbol{z}} \cdot (\nabla_{\mathrm{T}}\phi_m^{\mathrm{TM}} \times \nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}})|\nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{A.14}$$

The leading term for $\boldsymbol{W}_m^{\mathrm{TM}}$ is

$$\begin{aligned}\tilde{E}_{\mathrm{w}m}^2 =& (u_1^{\mathrm{TE}})^2(\boldsymbol{W}_m^{\mathrm{TM}}, \boldsymbol{U}_1^{\mathrm{TE}} \cdot \boldsymbol{U}_1^{\mathrm{TE}}\boldsymbol{W}_m^{\mathrm{TM}}) \\ =& (u_1^{\mathrm{TE}})^2\alpha_{\mathrm{w}m},\end{aligned} \tag{A.15}$$

where

$$\alpha_{\mathrm{w}m} = \iint_\Omega |\lambda_m^{\mathrm{TM}}\phi_m^{\mathrm{TM}}|^2|\nabla_{\mathrm{T}}\phi_1^{\mathrm{TE}}|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{A.16}$$

Appendix B Decay of scalar product

In this paper, we have often come across scalar products of the form $(\boldsymbol{U}_m^{\mathrm{TE}}, \boldsymbol{U}_k^{\mathrm{TE}} \cdot \boldsymbol{U}_l^{\mathrm{TE}} \boldsymbol{U}_n^{\mathrm{TE}})$, where $\boldsymbol{U}_m^{\mathrm{TE}}$ is a real-valued expansion function satisfying Helmholtz' equation in the plane. To analyze the case with a dominant TE_1-mode, we must make some estimate of this scalar product when two of the indices k, l, and n are equal to one. The canonical problem for this consists in calculating the scalar products

$$\iint_\Omega \phi_1^2 \phi_m \phi_n \,\mathrm{d}x\,\mathrm{d}y \tag{B.1}$$

where the scalar functions ϕ_k satisfy the Helmholtz' equation

$$\nabla_{\mathrm{T}}^2 \phi_k + \lambda_k^2 \phi_k = 0, \tag{B.2}$$

with Dirichlet or Neumann boundary conditions. The λ_k:s are assumed to be non-degenerate, positive and arranged in ascending order, *i.e.*, $0 < \lambda_1 < \lambda_2 < \dots$ We choose the functions to be normalized as

$$\iint_\Omega |\phi_k|^2 \,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega \left|\frac{\nabla_{\mathrm{T}}\phi_k}{\lambda_k}\right|^2 \mathrm{d}x\,\mathrm{d}y = 1, \tag{B.3}$$

which is not the normalization used previously in the paper for the eigenfunctions ϕ_k^{TE}, but simplifies the notation in this appendix. If $m \neq n$ we use the Helmholtz' equation to conclude

$$\begin{aligned}
\left|\iint_\Omega \phi_1^2 \phi_m \phi_n \,\mathrm{d}x\,\mathrm{d}y\right| &= \frac{1}{|\lambda_m^2 - \lambda_n^2|}\left|\iint_\Omega \phi_1^2[-\phi_n \nabla_{\mathrm{T}}^2\phi_m + \phi_m \nabla_{\mathrm{T}}^2 \phi_n]\,\mathrm{d}x\,\mathrm{d}y\right| \\
&= \frac{1}{|\lambda_m^2 - \lambda_n^2|}\left|\iint_\Omega \nabla_{\mathrm{T}}\phi_1^2 \cdot [\phi_n \nabla_{\mathrm{T}}\phi_m - \phi_m \nabla_{\mathrm{T}}\phi_n]\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\leq \frac{\sup_\Omega |\nabla_{\mathrm{T}}\phi_1^2|}{|\lambda_m^2 - \lambda_n^2|}\iint_\Omega |\phi_n \nabla_{\mathrm{T}}\phi_m - \phi_m \nabla_{\mathrm{T}}\phi_n|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{B.4}$$

Using Cauchy's inequality the last line can be estimated by

$$\begin{aligned}
&\frac{\sup_\Omega |\nabla_{\mathrm{T}}\phi_1^2|}{|\lambda_m^2 - \lambda_n^2|}\left[\lambda_m \iint_\Omega \left|\phi_n \frac{\nabla_{\mathrm{T}}\phi_m}{\lambda_m}\right| \mathrm{d}x\,\mathrm{d}y + \lambda_n \iint_\Omega \left|\phi_m \frac{\nabla_{\mathrm{T}}\phi_n}{\lambda_n}\right| \mathrm{d}x\,\mathrm{d}y\right] \\
&\qquad \leq \frac{\sup_\Omega |\nabla_{\mathrm{T}}\phi_1^2|}{|\lambda_m^2 - \lambda_n^2|}\left[\frac{\lambda_m}{2}\iint_\Omega \left(|\phi_n|^2 + \left|\frac{\nabla_{\mathrm{T}}\phi_m}{\lambda_m}\right|^2\right)\mathrm{d}x\,\mathrm{d}y\right. \\
&\qquad\qquad \left. + \frac{\lambda_n}{2}\iint_\Omega \left(|\phi_m|^2 + \left|\frac{\nabla_{\mathrm{T}}\phi_n}{\lambda_n}\right|^2\right)\mathrm{d}x\,\mathrm{d}y\right] \\
&\qquad = \frac{\sup_\Omega |\nabla_{\mathrm{T}}\phi_1^2|}{|\lambda_m^2 - \lambda_n^2|}(\lambda_m + \lambda_n) \\
&\qquad = \frac{\sup_\Omega |\nabla_{\mathrm{T}}\phi_1^2|}{|\lambda_m - \lambda_n|}.
\end{aligned} \tag{B.5}$$

In two dimensions the eigenvalue λ_n grow approximately as the square root of the index n [7, p. 442], which implies the scalar product decays approximately as $1/|\sqrt{m} - \sqrt{n}|$ for large indices.

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