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Explicit Formulas for General Integrals of Motion for a Class of Mechanical Systems Subject to Virtual Constraints

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Abstract. The paper suggests an explicit form of a general integral of motion for some classes of dynamical systems including n-degree-of-freedom mechanical systems subject to $(n-1)$ virtual holonomic constraints. The computation of this integral opens several possibilities for generating and further exponential orbital stabilization of solutions in nonlinear feedback systems. An illustrative example is given.

I. INTRODUCTION AND MOTIVATING EXAMPLE

This paper considers a nonlinear dynamical system of the form

$$\alpha(q)\frac{d^2q}{dt^2} + \beta(q,\dot{q}^2) + \gamma(q) = 0, \qquad (1)$$

where $q \in R^1$; α, β, γ are C^1-scalar functions, with the objective to compute a general integral of motion of (1) for different cases of the functions α, β and γ.

Exploring the properties of system (1) is mainly motivated by the fact that any n-degree-of-freedom mechanical system with $(n-1)$ *virtual* holonomic constraints, that is, $(n-1)$ functions of generalized coordinate q made invariant along solutions by feedback control, can be rewritten in the form (1) with $\beta(q,\dot{q}^2) = \beta_1(q)\dot{q}^2$.

Knowledge of the general integral of (1) suggests a rigorous background for developing such control strategies. Indeed, an explicit form of the general integral of motion of (1), see Theorems 1–3, combined with a *generalized passivity relation*, Lemmas 2–3, enable us to introduce a linear control system. It is shown that the stabilization of the auxiliary system implies stabilization of the newly generated

motion for the original mechanical system. This way of motion generation and control design was elaborated in [2] for stabilization of periodic regimes in nonlinear systems. The current paper is a companion of [2] and focuses on revealing properties of (1).

As a motivating example, consider the ubiquitous cart-pendulum system

$$\begin{aligned}
(M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 &= f \\
ml\cos\theta\,\ddot{x} + ml^2\,\ddot{\theta} - mgl\sin\theta &= 0
\end{aligned} \qquad (2)$$

where $x \in R^1$ is the horizontal displacement of the cart, $\theta \in S^1$ is the angle between the pendulum rod and the vertical which is zero at the upright position; m, M are the masses of the pendulum and the cart respectively; l is the distance to the center of mass of the rod; f is the control variable (the force that could be applied to the cart).

Consider a problem of *a generation and further exponential orbital stabilization of periodic motion (approx. given frequency ω_c) of the cart-pendulum system (2) around*
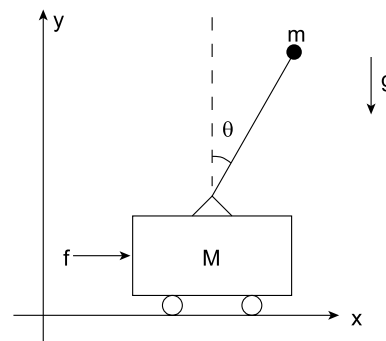

Fig. 1.   The cart pendulum system

the upright equilibrium when the cart is moving forward (backward) with a prescribed average velocity ($= V_x$) over the period of the pendulum.

To simplify calculation we will assume that the mass of the cart, the mass of the pendulum and the distance from the suspension of the pendulum to its center of mass are all equal to 1. Then the equations (2) are

$$2\ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = f \qquad (3)$$

$$\cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0 \qquad (4)$$

Consider the equation (*virtual constraint*) that relates the position of the cart x and the angle of the pendulum θ

$$x = V_x \cdot (t - t_0) - \left[1 + \frac{g}{\omega_c^2}\right] \ln\left(\frac{1 + \sin\theta}{\cos\theta}\right) \qquad (5)$$

where ω_c is a desired frequency of the pendulum oscillations; V_x is a desired speed of the cart; t is a time; t_0 is an initial time moment.

Suppose that there exists a controller[1] that ensures that (5) holds along solutions of the closed loop system, then we can rewrite the equations (3)–(4) into the new form

$$\ddot{x} + \left[1 + \frac{g}{\omega_c^2}\right]\frac{1}{\cos\theta}\ddot{\theta} + \left[1 + \frac{g}{\omega_c^2}\right]\frac{\sin\theta}{\cos^2\theta}\dot{\theta}^2 = 0 \quad (6)$$

$$\cos\theta\,\ddot{x} + \ddot{\theta} - g\sin\theta = 0 \quad (7)$$

where (6) comes from double differentiation of (5). Excluding $\ddot{x}$ from (6)–(7), one gets the equation w.r.t. θ

$$\ddot{\theta} + \left[1 + \frac{\omega_c^2}{g}\right]\frac{\sin\theta}{\cos\theta}\dot{\theta}^2 + \omega_c^2\sin\theta = 0. \qquad (8)$$

System (8) has the same structure as the general equation (1) with the functions

$$\alpha(\theta) = 1, \quad \beta(\theta, \dot{\theta}^2) = \left[1 + \frac{\omega_c^2}{g}\right]\frac{\sin\theta}{\cos\theta}\dot{\theta}^2, \quad \gamma(\theta) = \omega_c^2\sin\theta$$

One can readily check that the system (8) is neutrally stable[2] around the upward equilibrium $\theta = \dot{\theta} = 0$. Therefore it has some neighborhood of the equilibrium $\theta = \dot{\theta} = 0$ filled by periodic solutions. In Section III it will be shown how to orbitally stabilize any of these periodic solution.

The main contribution of this paper is the derivation of the explicit form of a general (full) integral I of the system (1) for some particular classes of the functions α, β, and γ. As a minor contribution, new differential equalities for I - the so-called *generalized passivity relations* – are found.

II. MAIN RESULTS

Lemma 1: Along any solution $[q(t), \dot{q}(t)]$ of the system (1), if exists, the following identity holds:

$$\frac{d^2}{dt^2}q(t) = \frac{d}{dq}\left(\frac{1}{2}\dot{q}^2(t)\right) \qquad (9)$$

Proof is based on the line of equalities

$$\frac{d^2}{dt^2}q(t) = \frac{d}{dt}\left(\dot{q}(t)\right) = \frac{\partial}{\partial q}\left(\dot{q}(t)\right)\frac{d}{dt}\left(q(t)\right)$$

$$= \frac{\partial}{\partial q}\left(\dot{q}(t)\right)\dot{q}(t) = \frac{d}{dq}\left(\frac{1}{2}\dot{q}^2(t)\right)$$

where the equality of the first and last expressions gives (9). ∎

The equality (9) allows to make the main step: to rewrite the second order differential equation (1) with respect to time into the first order differential equation

$$\frac{1}{2}\alpha(q)\frac{d}{dq}Y(q) + \beta(q, Y(q)) + \gamma(q) = 0 \qquad (10)$$

with respect to the variable q, where the function $Y(q)$ is defined as follows

$$Y(q(t)) = \left[\frac{dq(t)}{dt}\right]^2. \qquad (11)$$

The next three Theorems show classes of dynamical systems parameterized by functions α, β γ, for which the first order system (10) is integrable, and therefore, the general integral for the original system (1) can be explicitly found.

Theorem 1: [Case $\beta(q, \dot{q}^2) = \beta_1(q)\dot{q}^2$] Given an initial conditions $[q_0, \dot{q}_0]$, if the solution

$$\left[q(t), \dot{q}(t)\right] = \left[q(t, q_0), \dot{q}(t, \dot{q}_0)\right]$$

of system

$$\alpha(q)\frac{d^2q}{dt^2} + \beta_1(q)\dot{q}^2 + \gamma(q) = 0, \qquad (12)$$

exists for these initial conditions, then the function

$$I_0\left(q, \dot{q}, q_0, \dot{q}_0\right) = \dot{q}^2 - \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\dot{q}_0^2 + \qquad (13)$$

$$+ \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\int_{q_0}^{q}\exp\left\{2\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds$$

preserves its value along this solution. The last property holds irrespective of the boundedness of $\left[q(t), \dot{q}(t)\right]$. ∎

Proof. Based on (11) one can write an equivalent system to (12)

$$\frac{d}{dq}Y + \frac{2\beta_1(q)}{\alpha(q)}Y + \frac{2\gamma(q)}{\alpha(q)} = 0 \qquad (14)$$

which is now a linear differential equation with respect to the function Y. Its general solution looks as follows

$$Y(q) = \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}Y(q_0) - \qquad (15)$$

$$- \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\int_{q_0}^{q}\exp\left\{2\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds$$

$$(16)$$

Introducing the function $I_0 = I_0(q, \dot{q}, q_0, \dot{q}_0)$ as

$$I_0 = Y(q) - \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}Y(q_0)+$$

$$+\exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\int_{q_0}^{q}\exp\left\{2\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds$$

results in $I_0 = 0$ along the solution $\left[q(t), \dot{q}(t)\right]$. ∎

Theorem 2: [**Case** $\gamma(q) = 0$ **and** $\beta(q, \dot{q}^2) = \beta_1(q)\dot{q}^2 + \beta_2(q)\dot{q}^{2k}$, $k = 2, 3, \dots$] Given an initial conditions $[q_0, \dot{q}_0]$, if the solution

$$\left[q(t), \dot{q}(t)\right] = \left[q(t, q_0), \dot{q}(t, \dot{q}_0)\right]$$

of system

$$\alpha(q)\frac{d^2q}{dt^2} + \beta_1(q)\dot{q}^2 + \beta_2(q)\dot{q}^{2k} = 0, \quad k = 2, 3, \dots \quad (17)$$

exists for these initial conditions, then the function $I_1 = I_1\left(q, \dot{q}, q_0, \dot{q}_0\right)$ defined as

$$I_1 = \left[\dot{q}^2\right]^{1-k} - \exp\left\{2(k-1)\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\left[\dot{q}_0^2\right]^{1-k} -$$

$$- \exp\left\{2(k-1)\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\} \times \qquad (18)$$

$$\times \int_{q_0}^{q}\exp\left\{2(1-k)\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2(k-1)\beta_2(s)}{\alpha(s)}ds$$

preserves its value along this solution. The last property holds irrespective of the boundedness of $\left[q(t), \dot{q}(t)\right]$. ∎

Proof. Using (11), an equivalent system to (17) is

$$\frac{d}{dq}Y + \frac{2\beta_1(q)}{\alpha(q)}Y + \frac{2\beta_2(q)}{\alpha(q)}Y^k = 0 \qquad (19)$$

which is now the Bernoulli equation. As well known, by introducing a new function

$$Z = Y^{1-k} \qquad (20)$$

one gets the differential equation

$$\begin{aligned}\frac{d}{dq}Z &= (1-k)Y^{-k}\left[-\frac{2\beta_1(q)}{\alpha(q)}Y - \frac{2\beta_2(q)}{\alpha(q)}Y^{-k}\right]\\ &= 2(k-1)\frac{\beta_1(q)}{\alpha(q)}Y^{1-k} + 2(k-1)\frac{\beta_2(q)}{\alpha(q)}\\ &= 2(k-1)\frac{\beta_1(q)}{\alpha(q)}Z + 2(k-1)\frac{\beta_2(q)}{\alpha(q)}\end{aligned} \qquad (21)$$

which is linear with respect to the function Z. The general solution of (21) looks as

$$Z(q) = \exp\left\{2(k-1)\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}Z(q_0)+ \qquad (22)$$

$$+ \exp\left\{2(k-1)\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\} \times$$

$$\times \int_{q_0}^{q}\exp\left\{2(1-k)\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2(k-1)\beta_2(s)}{\alpha(s)}ds$$

If one introduces the function I_1 as the difference between $Z(q)$ and the right-hand side of the formula (22), one gets the function I_1 identically equal to zero along the solution. Coming back from Z to Y and further to $\dot{q}$, one gets the final formula for I_1. ∎

Remark 1: As readily seen, Theorem 1 is a particular case of Theorem 2. Indeed, the integral I_1 defined in (18) becomes I_0, check (13), when $k = 0$ and $\beta_2(q) = \gamma(q)$. The case when $k = 1$ has not explicitly mentioned in Theorem 2. In fact it is also covered and corresponds to Theorem 1 with $\beta(q, \dot{q}^2) = [\beta_1(q) + \beta_2(q)]\dot{q}^2$ and $\gamma(q) = 0$. ∎

Lemma 2: The time derivative of the function $I_1(q, \dot{q}, x, y)$ given in (18), with x, y being constants, along a solution of the system

$$\alpha(q)\frac{d^2q}{dt^2} + \beta_1(q)\dot{q}^2 + \beta_2(q)\dot{q}^{2k} = u \qquad (23)$$

with $k = 0, 2, 3, \dots$ has the form

$$\frac{d}{dt}I_1 = \frac{2(1-k)\dot{q}}{\alpha(q)}\left[\dot{q}^{-2k}u - \beta_1(q)I_1\right] \qquad (24)$$

Proof is based on the direct calculations. Indeed, along any solution of (23) the time derivative of I_1 is

$$\frac{d}{dt}I_1 = \frac{\partial I_1}{\partial q}\dot{q} + \frac{\partial I_1}{\partial \dot{q}}\ddot{q}, \qquad (25)$$

while the partial derivatives of I_1 w.r.t. $\dot{q}$ and q are

$$\frac{\partial I_1}{\partial \dot{q}} = 2(1-k)\dot{q}^{1-2k}$$

$$\frac{\partial I_1}{\partial q} = \exp\left\{2(k-1)\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2(1-k)\beta_1(q)}{\alpha(q)} \times$$

$$\times \left(\left[\dot{q}_0^2\right]^{1-k} + \int_{q_0}^{q}\exp\left\{2(1-k)\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2(k-1)\beta_2(s)}{\alpha(s)}ds\right)$$

$$+ \frac{2(1-k)\beta_2(q)}{\alpha(q)}$$

$$= \frac{2(1-k)\beta_1(q)}{\alpha(q)}\left(\left[\dot{q}^2\right]^{1-k} - I_1\right) + \frac{2(1-k)\beta_2(q)}{\alpha(q)}$$

Therefore, the time derivative of I_1 is

$$\frac{d}{d}I_1 = \frac{\partial I_1}{\partial q}\dot{q} + \frac{\partial I_1}{\partial \dot{q}}\ddot{q}$$

$$= \frac{2(1-k)\beta_1(q)}{\alpha(q)}\left(\left[\dot{q}^2\right]^{1-k} - I_1\right)\dot{q} +$$

$$+ \frac{2(1-k)\beta_2(q)}{\alpha(q)}\dot{q} +$$

$$+ 2(1-k)\dot{q}^{1-2k}\left[\frac{u}{\alpha(q)} - \frac{\beta_1(q)}{\alpha(q)}\dot{q}^2 - \frac{\beta_2(q)}{\alpha(q)}\dot{q}^{2k}\right]$$

$$= \frac{2(1-k)\dot{q}}{\alpha(q)}\left[\dot{q}^{-2k}u - \beta_1(q)I_1\right]$$

This is the differential relation (24). ∎

Theorem 3: [Case $\gamma(q) = 0$ and $\beta(q,\dot{q}^2) = \frac{\beta_1(q)f(\dot{q}^2) + \beta_2(q)}{f'(\dot{q}^2)}$]. Given an initial conditions $[q_0, \dot{q}_0]$ and a smooth scalar function f, if the solution

$$\left[q(t), \dot{q}(t)\right] = \left[q(t,q_0), \dot{q}(t,\dot{q}_0)\right]$$

of system

$$\alpha(q)\frac{d^2q}{dt^2} + \frac{\beta_1(q)f(\dot{q}^2) + \beta_2(q)}{f'(\dot{q}^2)} = 0 \qquad (26)$$

exists for these initial conditions, then the function $I_2 = I_2\left(q, \dot{q}, q_0, \dot{q}_0\right)$ defined as

$$I_2 = f\left(\dot{q}^2\right) - \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}f\left(\dot{q}_0^2\right) + \qquad (27)$$

$$+ \exp\left\{-2\int_{q_0}^{q}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\int_{q_0}^{q}\exp\left\{2\int_{q_0}^{s}\frac{\beta_1(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\beta_2(s)}{\alpha(s)}ds$$

preserves its value along this solution. The last property holds irrespective of the boundedness of $\left[q(t), \dot{q}(t)\right]$. ∎

Proof. Based on (11) one can write an equivalent system to (26)

$$\frac{d}{dq}Y + \frac{2\beta_1(q)f(Y) + 2\beta_2(q)}{\alpha(q)f'(Y)} = 0 \qquad (28)$$

Introducing the new variable $Z = f(Y)$ one gets

$$\frac{d}{dq}Z = f'(Y)\frac{dY}{dq} = -2\frac{\beta_1(q)}{\alpha(q)}f(Y) - 2\frac{\beta_2(q)}{\alpha(q)}$$

$$= -2\frac{\beta_1(q)}{\alpha(q)}Z - 2\frac{\beta_2(q)}{\alpha(q)}$$

The last equation is linear with respect to Z, finding its general solution and proceeding as in the previous theorems, gives the final expression for the general integral (27). ∎

Remark 2: It can be checked that I_2 defined in (27) is I_0, see (13), with $f(y) = y$, $\beta_2(q) = \gamma(q)$. ∎

Lemma 3: The time derivative of the function $I_2(q,\dot{q},x,y)$ given in (18), with x, y being constants, along a solution of the system

$$\alpha(q)\frac{d^2q}{dt^2} + \frac{\beta_1(q)f(\dot{q}^2) + \beta_2(q)}{f'(\dot{q}^2)} = u \qquad (29)$$

has the form

$$\frac{d}{dt}I_2 = \frac{2\dot{q}}{\alpha(q)}\left[f'(\dot{q}^2)u - \beta_1(q)I_2\right] \qquad (30)$$

Proof is based on the direct calculations. Indeed, along any solution of (23) the time derivative of I is

$$\frac{d}{dt}I_2 = \frac{\partial I_2}{\partial q}\dot{q} + \frac{\partial I_2}{\partial \dot{q}}\ddot{q} \qquad (31)$$

while the partial derivatives of I_2 w.r.t. q and $\dot{q}$ are

$$\frac{\partial I_2}{\partial \dot{q}} = 2f'(\dot{q}^2)\dot{q}$$

$$\frac{\partial I_2}{\partial q} = \frac{2\beta_1(q)}{\alpha(q)}\left[f(\dot{q}^2) - I_2\right] + \frac{2\beta_2(q)}{\alpha(q)}$$

Therefore $\dot{I}_2$ along a solution of (29) looks as

$$\dot{I}_2 = \frac{\partial I_2}{\partial q}\dot{q} + \frac{\partial I_2}{\partial \dot{q}}\ddot{q}$$

$$= \left\{\frac{2\beta_1(q)}{\alpha(q)}\left[f(\dot{q}^2) - I_2\right] + \frac{2\beta_2(q)}{\alpha(q)}\right\}\dot{q} +$$

$$+ 2f'(\dot{q}^2)\dot{q}\left\{\frac{u}{\alpha(q)} - \frac{\beta_1(q)f(\dot{q}^2) + \beta_2(q)}{\alpha(q)f'(\dot{q}^2)}\right\}$$

$$= \frac{2\dot{q}}{\alpha(q)}\left[f'(\dot{q}^2)u - \beta_1(q)I_2\right]$$

which proves the lemma. ∎

III. MOTIVATING EXAMPLE (CONT'D)

Here we proceed with developing controller for generation and further exponential orbital stabilization of periodic motion (approx. given frequency ω_c) of the cart-pendulum system, see Figure 1, around the upright equilibrium when the cart is moving forward (backward) with a prescribed velocity equal to V_x.

We have found on p. 3 that if one chooses the *virtual time-varying holonomic constraint* as in (5) and if there is a controller that makes this relation (5) valid along solutions of the closed loop system, then the angle θ of the pendulum is one of solutions of the system (8).

Let us check that the upward equilibrium

$$\theta_0 = 0, \quad \dot{\theta}_0 = 0 \qquad (32)$$

is a focus of the system (8). The reader is asked to verify that the following function, according to Theorem 1,

$$U(\theta, \dot{\theta}) = \frac{\dot{\theta}^2}{(\cos\theta)^{\left[1 + \frac{\omega_c^2}{g}\right]}} + \frac{2\omega_c^2}{1 + 2\frac{\omega_c^2}{g}}\frac{1}{(\cos\theta)^{\left[1 + 2\frac{\omega_c^2}{g}\right]}} \qquad (33)$$

is the first integral of the system (8). To prove that the equilibrium (32) of the system (8) is a focus, we can follow the Lagrange–Dirichlet theorem, which suggests to calculate the Hessian of $U(\theta, \dot{\theta})$ at this equilibrium. The straightforward calculations show that the Hessian

$$\left[\begin{array}{cc} \frac{\partial^2 U}{\partial \theta^2} & \frac{\partial^2 U}{\partial \theta \partial \dot{\theta}} \\ \frac{\partial^2 U}{\partial \theta \partial \dot{\theta}} & \frac{\partial^2 U}{\partial \dot{\theta}^2} \end{array} \right] \Bigg|_{\theta=0, \dot{\theta}=0} = \left[\begin{array}{cc} 2\omega_c^2 & 0 \\ 0 & 2 \end{array} \right]$$

is positive definite. The phase portrait and the solutions in time are shown in Figs. 2-3, when $\omega_c = \frac{2\pi}{5}$ [rad/sec].

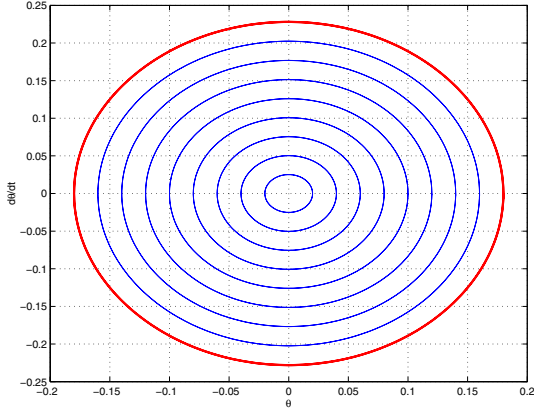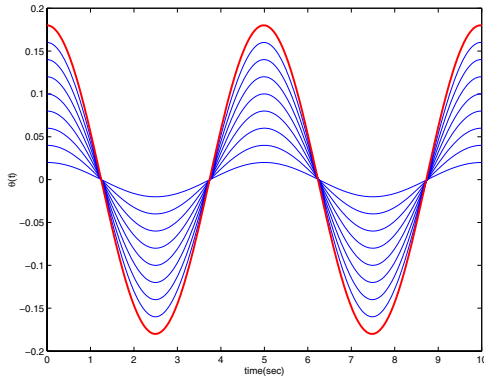Fig. 2. The phase portrait of (8) when $\omega_c = 2\pi/5$ [rad/sec].

Fig. 3. Solutions of (8) versus time when $\omega_c = 2\pi/5$ [rad/sec].

A. Partial Feedback Transformation

Let us introduce a variable y as follows

$$y = x + \left[1 + \frac{g}{\omega_c^2} \right] \ln\left(\frac{1 + \sin\theta}{\cos\theta} \right) - V_x \cdot (t - t_0) \quad (34)$$

Then the time derivative of y takes the form

$$\dot{y} = \dot{x} + \left[1 + \frac{g}{\omega_c^2} \right] \frac{1}{\cos\theta} \dot{\theta} - V_x \quad (35)$$

Taking a time derivative of $\dot{y}$ along a solution of (3)–(4), one gets

$$\ddot{y} = \ddot{x} + \left[1 + \frac{g}{\omega_c^2} \right] \frac{1}{\cos\theta} \ddot{\theta} + \left[1 + \frac{g}{\omega_c^2} \right] \frac{\sin\theta}{\cos^2\theta} \dot{\theta}^2 = v \quad (36)$$

where

$$v = \frac{1}{B(\theta)} \left[f - A(\theta, \dot{\theta}) \right] \quad (37)$$

$$A(\theta, \dot{\theta}) = \left[1 + \frac{g}{\omega_c^2} \right] \frac{\sin\theta}{\cos\theta} \dot{\theta}^2 + \quad (38)$$

$$+ \frac{\sin\theta}{1 + \sin^2\theta} \left(\dot{\theta}^2 - g\cos\theta + \left[1 + \frac{g}{\omega_c^2} \right] \frac{2g - \cos\theta\,\dot{\theta}^2}{\cos\theta} \right)$$

$$B(\theta) = -\frac{g}{\omega_c^2 (1 + \sin^2\theta)} \quad (39)$$

Furthermore, if one computes the expression for $\ddot{x}$ from the identity (36) and substitute it in Eq. (4), then it takes the form

$$-\frac{g}{\omega_c^2} \ddot{\theta} - \left[1 + \frac{g}{\omega_c^2} \right] \frac{\sin\theta}{\cos\theta} \dot{\theta}^2 - g\sin\theta = -\cos\theta \cdot v \quad (40)$$

The equations (36) and (40) represent the dynamics of the cart-pendulum system (3)–(4), when one introduces the variable y via the constraint (34) and makes the feedback transformation from the control variable f to a new one v.

B. Choice of Periodic Orbit

Our intention is to organize a periodic motion of the pendulum. Let us choose one $[\theta_*(t), \dot{\theta}_*(t)]$ of periodic solutions of the system (8), for example the solution shown in Fig. 3 in red, and consider the problem of its exponential orbital stabilization. It is seen from Fig. 3 that

$$\left[\theta_*(t), \dot{\theta}_*(t) \right] = \left[\theta_*(t + T), \dot{\theta}_*(t + T) \right], \quad \forall t \geq 0$$

with $T \approx 5$ sec.

C. Design of Stabilizing Controller

To design a stabilizing controller, let us, in addition to the cart-pendulum system (written in variables θ, y and v)

$$-\frac{g}{\omega_c^2} \ddot{\theta} - \left[1 + \frac{g}{\omega_c^2} \right] \frac{\sin\theta}{\cos\theta} \dot{\theta}^2 - g\sin\theta = -\cos\theta v \quad (41)$$

$$\ddot{y} = v \quad (42)$$

introduce its auxiliary representation

$$\dot{I}_0 = \frac{2\dot{\theta}}{g} \left\{ \omega_c^2 \cos\theta v - \left[1 + \frac{\omega_c^2}{g} \right] \frac{\sin\theta}{\cos\theta} I_0 \right\} \quad (43)$$

$$\ddot{y} = v \quad (44)$$

where I_0 is given in (13). The difference between (41)–(42) and (43)–(44) is in the first lines, where (43) represents the time derivative of the function I_0, introduced in (13) and evaluated along a solution of (41)[3].

So that the periodic solution to be stabilized is chosen, $[\theta_*(t), \dot{\theta}_*(t)]$, consider the system (43)–(44) when $\theta(t) = \theta_*(t)$, that is,

$$\dot{I} = \frac{2\dot{\theta}_*(t)}{g} \left\{ \omega_c^2 \cos\theta_*(t) v - \left[1 + \frac{\omega_c^2}{g} \right] \frac{\sin\theta_*(t)}{\cos\theta_*(t)} I \right\} \quad (45)$$

$$\ddot{y} = v \quad (46)$$

[3]Check (24) with $k = 0$.

The system (45)–(46) is a linear differential equation with periodic coefficients, and its state-space form is

$$\frac{d}{dt}\begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{2\dot{\theta}_*(t)}{g}\left[1+\frac{\omega_c^2}{g}\right]\frac{\sin\theta_*(t)}{\cos\theta_*(t)} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)}\begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix} +$$

$$+ \underbrace{\begin{bmatrix} \frac{2\dot{\theta}_*(t)\omega_c^2\cos\theta_*(t)}{g} \\ 0 \\ 1 \end{bmatrix}}_{B(t)} v \qquad (47)$$

The linear system (47) with periodic coefficients is completely controllable over the period $[0,5]$ sec. Then the LQR

$$v = -\Gamma^{-1}B(t)^T R(t)\begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix} \qquad (48)$$

with $R(t) = R(t+5) > 0$ being the stabilizing solution of associated Lur'e-Riccati equation

$$\dot{R}(t) + A(t)^T R(t) + R(t)A(t) + G = R(t)B(t)\Gamma^{-1}B(t)^T R(t)$$

and $G = G^T > 0$, $\Gamma > 0$, exponentially stabilizes the origin of the linear system (47), providing the minimum value for the performance index

$$\int_0^{+\infty}\left\{\begin{bmatrix} I(t) \\ y(t) \\ \dot{y}(t) \end{bmatrix}^T G \begin{bmatrix} I(t) \\ y(t) \\ \dot{y}(t) \end{bmatrix} + \Gamma v(t)^2\right\}dt$$

As proven in [2, Theorem 2] the following controller

$$v = -\Gamma^{-1}\left[\frac{2\dot{\theta}\omega_c^2\cos\theta}{g}, 0, 1\right]R(t)\begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix} \qquad (49)$$

makes the desired periodic motion of (41)–(42) exponentially orbitally stable.

IV. SIMULATION

In this section some simulation[4] results of the closed loop system [(3), (4), (34), (35), (37), (49)] are presented. The initial conditions were chosen as

$$\theta(0) = 0, \ \dot{\theta}(0) = 0, \ x(0) = 0, \ \dot{x}(0) = 0$$

While the desired frequency ω_c of oscillations for θ and the forward linear velocity V_x of the cart were chosen as

$$\omega_c = \frac{2\pi}{5} \text{ rad/s}, \quad V_x = 0.5[m/s]$$

Fig. 4 shows the phase portrait of θ and $d\theta/dt$ along the closed loop system solution. In turn, Fig. 5 shows the

[4]performed in Matlab/Simulink

position of the cart, that moves forward with the prescribed average speed $V_x = 0,5$ [m/s] over the period of oscillations The animations of this simulation could be found at [3].
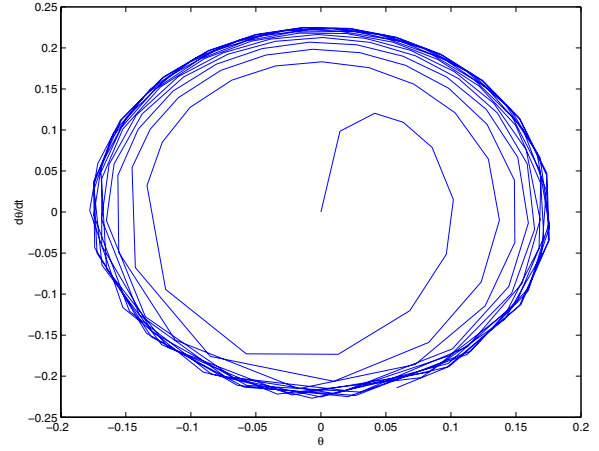

Fig. 4. The phase portrait of $[\theta, \dot{\theta}]$ along the closed loop system solution. The motion is converging to the orbit shown in red in Fig. 3
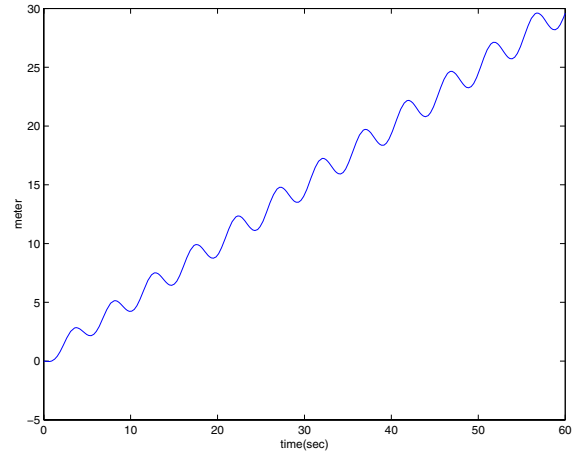

Fig. 5. The cart position x as a function of time. It travels forward with the prescribed average speed V_x over the period.

V. CONCLUSIONS

The paper is devoted to revealing explicit formulas of general integrals of motion for three classes of dynamical systems. One of consequences of these results is an ability of organizing interesting behavior in underactuated mechanical systems with further orbital stabilization of any of newly constructed motions.

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