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The Shannon Bit Error Limit for Linear Coded Modulation

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Abstract

We find the Shannon limit to bit error rate (BER) as a function of E_b/N_0 for linear coded modulations. Error performance of TCM and other schemes can be compared to this curve. As the bandwidth of the signals in Hz-s/data bit drops, capacity becomes sensitive to the stopband spectrum, and the limit moves strongly away from the BER of practical systems when the stopband rolloff is not severe.

1. Introduction

In this paper we find the Shannon limit to BER that corresponds to linear coded modulation over the AWGN channel. These modulations include those of the TCM type, and the limit is an interesting benchmark for TCM and other practical schemes. The starting point is the information carrying capacity of signals with bandwidth W positive Hz and power P watts over the AWGN channel. From Shannon theory this is $C_W = W\log_2[1 + P/WN_0]$ bits/s, where $N_0/2$ watts/Hz is the noise density. These signals have a "brickwall" power spectral density (PSD), which is either $P/2W$ or zero. Elementary calculus extends this result to signals with an arbitrary PSD $|H(f)|^2$; the outcome is

$$C_H = \int_0^\infty \log_2[1 + \frac{2|H(f)|^2}{N_0}]\,df \qquad \text{(bit/s)} \qquad (1)$$

in which P is now $\int |H(f)|^2 df$. (This classical result has appeared in e.g. Shamai *et al.*[2]).

Realizable lowpass signals are not brickwall, but instead have a passband inside a nominal bandwidth W and a stopband outside, where the PSD drops to zero. By dividing the coding rate in bits/s by W, we obtain a normalized bit density for the signaling in bits/Hz-s; the reciprocal of this is the bit-normalized spectrum of the signaling. The stopband will have a major effect on capacity and the Shannon BER limit in narrow bandwidth/high energy coding systems.

Next define a class of linear modulation codes, with signals of the form

$$s(t) = \sum_n a_n h(t - nT), \qquad (2)$$

where the pulses $h(t - nT)$, integer n, are not necessarily orthogonal and the a_n are information carrying values such as $\pm1, \pm3, \ldots$. If successive a_n are uncorrelated and zero-mean, the PSD of $s(t)$ has the same shape as the h power spectrum $|H(f)|^2$. Most practical schemes assume this condition and we will assume it here. It makes sense to compare these codes to the information carrying capacity C_H in (1).

In this paper we will also restrict the linear modulation class further to a generalized TCM class, which adds the condition that h is orthogonal. To avoid confusion, h will be denoted v in this case. In Shannon's 1949 formulation of this class, the a_n are without restriction; modern codes restrict them in various ways, the best known being Ungerboeck's idea of subsets. In ordinary pulse-amplitude modulation the a_n are directly the data values. In the case of orthogonal v, the nominal passband of v, hence s, is the band $[0, 1/2T]$ Hz, and the stopband is $[1/2T, (1+\alpha)/2T]$, for some $\alpha \geq 0$. Smaller α are more difficult to implement. Nyquist showed that a certain PSD symmetry condition implies orthogonality. The most common such pulse is the root raised-cosine (RC) pulse, with $\alpha = 0.1$–0.3. The narrowest such pulse $\mathrm{sinc}(t/T)$ has a brickwall stopband with $\alpha = 0$ and zero width.

For the orthogonal case, it can be shown that C_H in (1) always increases compared to the brickwall C_W when a Nyquist-orthogonal pulse $h = v$ with $\alpha > 0$ is substituted for $\mathrm{sinc}(t/T)$: The information carrying capacity of $\alpha > 0$ pulses is always *larger* than the brickwall spectrum case. For the same v shape, the capacity difference grows as the bit density of the signaling grows in bits/Hz-s. This can be shown by computing (1) with same-shape PSDs that have growing power P but narrowing width. For root RC pulses, the common value $\alpha = 0.3$ and density 10 bits/Hz-s ($\approx$ 25 kbit/s in a telephone modem), a given rate is carried with 4 dB less power than in the brickwall case.

The calculation may be repeated for general linear coded modulation with non-orthogonal h, with a similar outcome for h that are impulse responses of common filters (e.g., Butterworth) with a reasonable number of poles.

2. Computation of the Shannon Limit to BER

A limit of the Butman-McEliece type is the lowest BER of coding schemes with a given PSD, as a function of E_b/N_0. Consider a coded modulation system with PSD $|H(f)|^2$ that accepts R_{ber} data bits/s and decodes them at an error rate β. By the standard argument, rate-distortion theory tells us that rate R_{ber} can be compressed to C_H in the ratio

$$\frac{C_H}{R_{\text{ber}}} = 1 - h_B(\beta) \qquad (3)$$

with $h_B(\ )$ the binary entropy function. The coded modulation carries that information nearly perfectly over the channel at rate C_H in (1), where $R_{\text{ber}} \geq C_H$. Its transmission symbols need not be binary. A rate-distortion decoder then reproduces the R_{ber} bits/s data at BER β.

For a system working at a given R_{ber}, the Shannon limit can be displayed by scaling P/N_0 to different values, with P the power in the PSD, and computing C_H from (1); then E_b/N_0 is P/R_{ber} and β is found by inversing (3). The computation of C_H is the only deviation from the standard Butman-McEliece argument. The highest allowed P/N_0 is the one for which C_H in (1) is the given R_{ber}. To find this requires inversing (1).

Figure 1 shows the outcome for the generalized TCM case, when h in (2) is an orthogonal v. The pulse v is root RC with several α. There are two normalized signal bandwidths, 0.33 and 0.1 Hz-s/bit, on a scale where the Nyquist bandwidth $1/2T$ corresponds to 0.5. (The normalized bandwidth is $W/R_{\text{ber}} = 1/2TR_{\text{ber}}$. However, Fig. 1 does not in fact depend on the symbol time T and so it can be set to a convenient value such as 1/2; then R_{ber} is respectively 1/.33 and 10.)

The Shannon limits for brickwall signaling with $v(t) = \text{sinc}(t/T)$ at each bandwidth are the dashed lines. At 0.33 Hz-s/bit, actual 8- and 64-state TCM tests by Zhang, cited in [1], p. 168, are shown; these are for a rate 3 bit/2-dimensions scheme that does not use subsets. The 64-state scheme lies 4 dB from the brickwall limit at BER 10^{-5}, but 0.7 dB further from the limit based on $\alpha = .3$ pulses. At the narrower 0.1 Hz-s/bit bandwidth no BER tests are available to us. In any case, the $\alpha = 0.3$ limit is fully 4.2 dB inside the brickwall limit. This means that TCM schemes lying 3–5 dB above the brickwall limit lie 7–9 dB from the true Shannon limit for 0.3 pulses.

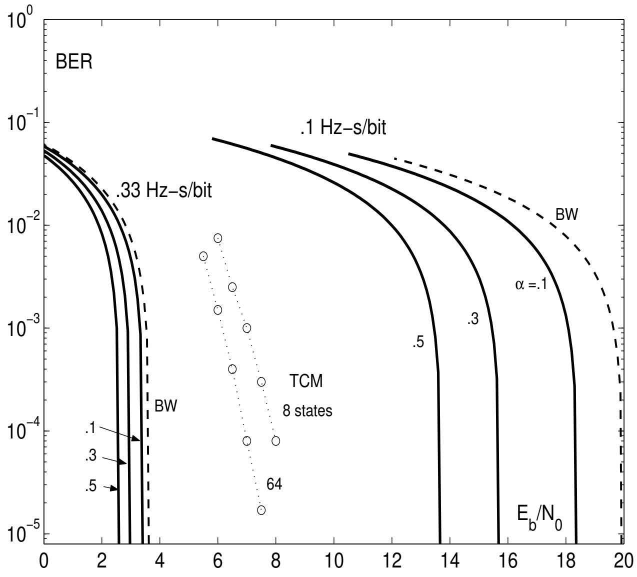

Figure 1: Shannon limit to BER versus data bit E_b/N_0 for signals based on root RC pulses with $\alpha = 0, .1, .3, .5$, with comparison to actual TCM tests. Normalized bandwidths 0.33 and 0.1 Hz-s/bit.

3. A Nonorthogonal Pulse Case

Similar plots can be made for the full linear coded modulation case where h is not orthogonal. At narrow normalized signal bandwidth (high bit density), the position of the Shannon BER limit continues to depend strongly on the severity of the bandlimitation and the stopband rate of decay.

Figure 2 (solid line) shows the calculation when the PSD has the shape of a Butterworth 8-pole filter and data is carried at the normalized density 9 databits/Hz-s (the passband width is taken as the cutoff frequency of the filter). This signal has strong intersymbol interference. Shown as circles are BER test results for M-algorithm detection of signals of form (2) when the alphabet size of $\{a_n\}$ is 4; this scheme retains a constant M signal paths while searching the tree of signal possibilities (for details see in ref. [4], this Proceedings). Even though the detector is near maximum likelihood and its BER is far better than that of simpler means of detecting the signal, the Shannon curve allows us to conclude that the outcome is 5 dB or more from Shannon performance. A concatenation method of forming signals with this PSD in fact performs several decibels closer to capacity (see [4]).

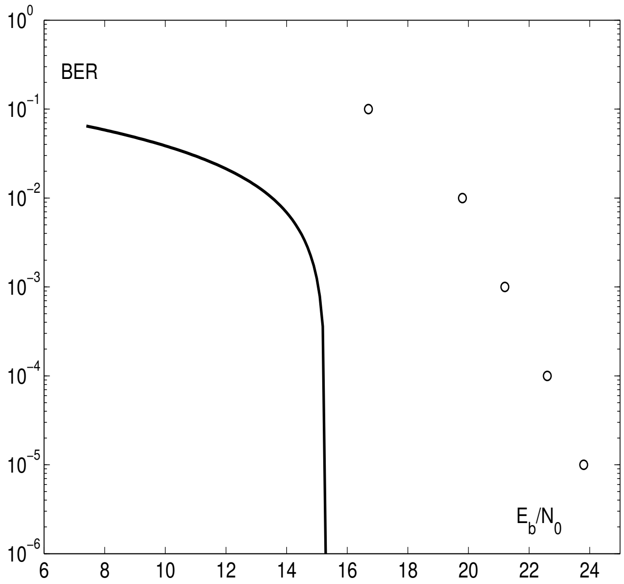

Figure 2: Shannon limit to BER versus data bit E_b/N_0 for $h(t)$ a 8-pole Butterworth impulse response. Comparison is made to actual M-algorithm decoding of 4-level signals of form (2). Normalized bandwidth 0.11 Hz-s/databit.

4. Conclusions and Some Implications

The strong effects shown here at narrow bandwidths are more than a theoretical oddity. In the realm of non-orthogonal h, we have found practical schemes that contradict a brickwall Shannon BER limit that is based on e.g. 3 dB or 99% power bandwidth; these schemes are based on concatenations of Butterworth filter responses with ordinary convolutional codes. As well, some of Said's optimal codes (ref. [3] or Table 6A.3 of [1]) violate a brickwall limit. At normalized bandwidths below 0.3 Hz-s/bit, we find that a capacity calculation based on a brickwall bandwidth is not a useful way to evaluate a practical system. Instead, we propose the Shannon BER alternative described here. We have found it informative and easy to use.

TCM-style codes with orthogonal v in (2), which have practical pulses and uncorrelated transmission symbols, apparently cannot achieve the capacity that applies to signals with their PSD. This weakness has been mentioned several times in the literature, but we will give an argument based on the foregoing root RC discussion: Practical schemes have a certain BER performance; this does not depend on α, only on the orthogonality of v; since α can be zero, these schemes must lie to the right of the *brickwall* Shannon limit to BER; but this is strictly to the right of the $\alpha > 0$ limit. Therefore TCM style coding with $\alpha > 0$ must be bounded away from the the Shannon limit for signals with its PSD.

It is apparently possible in principle to approach capacity by using a number of TCM style schemes in parallel. Each would have a narrow stopband and a nearly brickwall spectrum, and the composite spectrum would approach stepwise to the desired PSD. In this sense, linear transmission is possible near capacity for any PSD. But if a single signal is to be transmitted, some kind of concatenated code or nonlinear construction seems to be needed, as a number of researchers have observed.

References

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