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# Electromagnetic scattering by layered cylinders — low frequency reduction of RCS

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### Abstract

In this paper we present two different ways of reducing the radar cross section (RCS) of thin metallic wires at low frequencies. This reduction is achieved by adding a dielectric coating on the wire. The entire cylindrical structure is finally protected by a thin exterior non-magnetic dielectric layer (eagle skin). The contribution to the RCS has either an electric or a magnetic origin. To reduce the RCS it is possible to choose material parameters so that both the electric and the magnetic contributions vanish. This leads, however, to non-physical material parameter values. The underlying principle in the second method is to choose material parameters in the dielectric coating such that the electric and magnetic contributions annihilate each other. Fortunately, this annihilation can be obtained with physically realistic material parameters. Explicit numerical examples illustrate both ideas.

## 1 Introduction

The main question in this paper is to explore means of reducing the radar cross section (RCS) of long metallic wires. Specifically, is it possible to add one or two layers of cladding in order to reduce the RCS? The methods presented here apply at low frequencies, or, more correctly, to wires with a diameter that is small compared to the wavelength. At higher frequencies, other methods have to be employed.

We focus on a geometry where the perfectly electric conducting (PEC) wire has two exterior dielectric layers (claddings), see Figure 1. The outer cladding is a protection layer, usually thin and typically non-magnetic,  $\mu = 1$ , and with permeability  $\epsilon \approx 3$ –4.

The literature on scattering by layered cylinders is extensive. Some of the major and early contributions with both analytical and experimental results include [2, 3, 4, 6, 7]. We have no ambition to list all the contributions in this paper, and the present reduction of RCS seems not to have been reported in the open literature before.

## 2 Two layered coated PEC cylinder

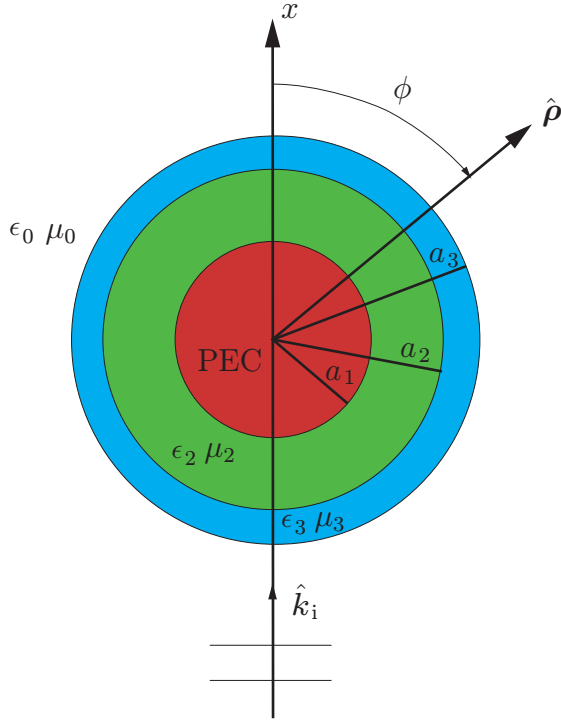
Let the radii of the individual layers of the infinitely long cylinder be  $a_1 < a_2 < a_3$ . Inside  $\rho < a_1$  is a perfectly conducting material, and in  $a_1 < \rho < a_2$  and  $a_2 < \rho < a_3$ , the material parameters (relative permittivity and permeability) are  $\epsilon_2, \mu_2$  and  $\epsilon_3, \mu_3$ , respectively. The region outside the cylinder  $\rho > a_3$  is vacuum, see Figure 1. We treat only the  $H$ -wave (TE).<sup>1</sup>

The appropriate expansion of the incident magnetic field, impinging along the  $x$  axis, is

$$H_z^i(\rho, \phi) = e^{ik_0x} = \sum_{m=-\infty}^{\infty} i^m J_m(k_0\rho) e^{im\phi} = \sum_{m=-\infty}^{\infty} a_m J_m(k_0\rho) e^{im\phi}$$

---

<sup>1</sup>A reduction of RCS with the other polarization (TM) seems not to be tractable with the method presented in this paper.



**Figure 1:** The geometry of the scattering problem.

where  $a_m = i^m$ , and  $\rho = \sqrt{x^2 + y^2}$  and  $\phi$  are the cylindrical polar coordinates. The wave number in vacuum is denoted  $k_0 = \omega/c_0$  (angular frequency  $\omega$  and the speed of light in vacuum  $c_0$ ), and we employ the time convention  $\exp\{-i\omega t\}$ .

The total magnetic fields in the relevant regions are

$$H_z(\rho, \phi) = \begin{cases} \sum_{m=-\infty}^{\infty} \alpha_m (J_m(k_2\rho) + t_m^{\text{PEC}} H_m^{(1)}(k_2\rho)) e^{im\phi}, & a_1 < \rho < a_2 \\ \sum_{m=-\infty}^{\infty} \beta_m (J_m(k_3\rho) + t_m H_m^{(1)}(k_3\rho)) e^{im\phi}, & a_2 < \rho < a_3 \\ \sum_{m=-\infty}^{\infty} a_m (J_m(k_0\rho) + T_m H_m^{(1)}(k_0\rho)) e^{im\phi}, & a_3 < \rho \end{cases} \quad (2.1)$$

where the wave number  $k_n = k_0 (\epsilon_n \mu_n)^{1/2}$ ,  $n = 2, 3$ , and, as above,  $a_m = i^m$ . The coefficients  $T_m$ , the transition matrix entries, are the sought quantities, and they determine the scattering properties of the wire. All scattering properties can be written in these coefficients, see *e.g.*, RCS in Section 2.2.

The boundary conditions on each layer, continuity of the tangential electric and magnetic fields, determine the unknown expansion coefficients in (2.1), *i.e.*, conti-

nuity of  $H_z$  and  $\epsilon^{-1} \frac{\partial}{\partial \rho} H_z$  on each interface. The boundary conditions imply

$$\begin{cases} J'_m(k_2 a_1) + t_m^{\text{PEC}} H_m^{(1)'}(k_2 a_1) = 0 \\ \alpha_m (J_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)}(k_2 a_2)) = \beta_m (J_m(k_3 a_2) + t_m H_m^{(1)}(k_3 a_2)) \\ \eta_2 \alpha_m (J'_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)'}(k_2 a_2)) = \eta_3 \beta_m (J'_m(k_3 a_2) + t_m H_m^{(1)'}(k_3 a_2)) \\ \beta_m (J_m(k_3 a_3) + t_m H_m^{(1)}(k_3 a_3)) = a_m (J_m(k_0 a_3) + T_m H_m^{(1)}(k_0 a_3)) \\ \eta_3 \beta_m (J'_m(k_3 a_3) + t_m H_m^{(1)'}(k_3 a_3)) = a_m (J'_m(k_0 a_3) + T_m H_m^{(1)'}(k_0 a_3)) \end{cases}$$

where the relative wave impedance  $\eta_n = (\mu_n/\epsilon_n)^{1/2}$ ,  $n = 2, 3$  has been used. The coefficients  $a_m$  are known and the transition matrix entries  $T_m$  are sought. Division eliminates the unknown coefficients. We get

$$\begin{cases} \frac{J_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)}(k_2 a_2)}{\eta_2 (J'_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)'}(k_2 a_2))} = \frac{J_m(k_3 a_2) + t_m H_m^{(1)}(k_3 a_2)}{\eta_3 (J'_m(k_3 a_2) + t_m H_m^{(1)'}(k_3 a_2))} \\ \frac{J_m(k_3 a_3) + t_m H_m^{(1)}(k_3 a_3)}{\eta_3 (J'_m(k_3 a_3) + t_m H_m^{(1)'}(k_3 a_3))} = \frac{J_m(k_0 a_3) + T_m H_m^{(1)}(k_0 a_3)}{(J'_m(k_0 a_3) + T_m H_m^{(1)'}(k_0 a_3))} \end{cases}$$

We write the solution as an iteration scheme as

$$\begin{cases} t_m^{\text{PEC}} = -\frac{J'_m(k_2 a_1)}{H_m^{(1)'}(k_2 a_1)} = -\frac{1}{2} - \frac{1}{2} \frac{H_m^{(2)'}(k_2 a_1)}{H_m^{(1)'}(k_2 a_1)} \\ t_m = -\frac{\eta_3 J'_m(k_3 a_2) - J_m(k_3 a_2) \eta_2 \frac{J'_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)'}(k_2 a_2)}{J_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)}(k_2 a_2)}}{\eta_3 H_m^{(1)'}(k_3 a_2) - H_m^{(1)}(k_3 a_2) \eta_2 \frac{J'_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)'}(k_2 a_2)}{J_m(k_2 a_2) + t_m^{\text{PEC}} H_m^{(1)}(k_2 a_2)}} \\ T_m = -\frac{J'_m(k_0 a_3) - J_m(k_0 a_3) \eta_3 \frac{J'_m(k_3 a_3) + t_m H_m^{(1)'}(k_3 a_3)}{J_m(k_3 a_3) + t_m H_m^{(1)}(k_3 a_3)}}{H_m^{(1)'}(k_0 a_3) - H_m^{(1)}(k_0 a_3) \eta_3 \frac{J'_m(k_3 a_3) + t_m H_m^{(1)'}(k_3 a_3)}{J_m(k_3 a_3) + t_m H_m^{(1)}(k_3 a_3)}} \end{cases} \quad (2.2)$$

More layers can easily be added following the same iteration scheme. In fact, for each new layer added, the new transition matrix entries are obtained as a Möbius transformation [5] on the transition matrix entries of the inner layers. We refrain from pursuing this analysis further in this paper.

For a given geometry and material parameters, equation (2.2) solves the scattering problem, and we now concentrate on a low frequency expansion of the relevant quantities, *i.e.*, the transition matrix entries  $T_m$ . We also observe that, due to the fact that  $J_{-m}(z) = (-1)^m J_m(z)$  and  $H_{-m}^{(1)}(z) = (-1)^m H_m^{(1)}(z)$  [1], the transition matrix entries satisfy  $T_{-m} = T_m$ , and only positive values of  $m$  need to be considered.

## 2.1 Low frequency expansion

For small arguments, the dominant contributions of the Bessel and Hankel functions are [1]

$$\begin{aligned} J_m(z) &= \frac{1}{m!} \left(\frac{z}{2}\right)^m (1 + O(z^2)) \\ J'_0(z) &= -J_1(z) = -\frac{z}{2} (1 + O(z^2)) \\ J'_m(z) &= \frac{1}{2(m-1)!} \left(\frac{z}{2}\right)^{m-1} (1 + O(z^2)), \quad m > 0 \end{aligned}$$

and

$$\begin{aligned} H_0^{(1)}(z) &= \frac{2i}{\pi} \left(\ln\left(\frac{z}{2}\right) + \gamma\right) (1 + O(z^2)) \\ H_m^{(1)}(z) &= -i \frac{(m-1)!}{\pi} \left(\frac{z}{2}\right)^{-m} (1 + O(z^2)), \quad m > 0 \\ H_0^{(1)'}(z) &= -H_1^{(1)}(z) = \frac{2i}{\pi z} (1 + O(z^2)) \\ H_m^{(1)'}(z) &= i \frac{m!}{2\pi} \left(\frac{z}{2}\right)^{-m-1} (1 + O(z^2)), \quad m > 0 \end{aligned}$$

The first order term in frequency,  $k_2 a_1$ , for the transition matrix entry  $t_0^{\text{PEC}}$  of the PEC surface then is

$$t_0^{\text{PEC}} = -\frac{J'_0(k_2 a_1)}{H_0^{(1)'}(k_2 a_1)} = -\frac{i\pi}{4} (k_2 a_1)^2 + O((k_2 a_1)^4)$$

and the expansion of the higher modes become

$$t_m^{\text{PEC}} = -\frac{J'_m(k_2 a_1)}{H_m^{(1)'}(k_2 a_1)} = \frac{i\pi}{2^{2m} m! (m-1)!} (k_2 a_1)^{2m} + O((k_2 a_1)^{2m+2}), \quad m > 0$$

Lowest order contribution to  $T_0$  then becomes after some algebra

$$T_0 = -\frac{i\pi}{4} \left( \frac{a_1^2}{a_3^2} \mu_2 + \frac{a_2^2}{a_3^2} (\mu_3 - \mu_2) + 1 - \mu_3 \right) (k_0 a_3)^2 + O((k_0 a_3)^4) \quad (2.3)$$

Similarly, the lowest order contribution to  $T_1$  is

$$T_1 = \frac{i\pi}{4} \frac{\frac{a_2^2}{a_3^2} \left( \frac{a_1^2}{a_3^2} - \frac{1-\epsilon_3}{1+\epsilon_3} \right) + \frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \left( \frac{a_1^2}{a_3^2} \frac{1-\epsilon_3}{1+\epsilon_3} - \frac{a_2^4}{a_3^4} \right)}{\frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \left( \frac{a_2^4}{a_3^4} \frac{1-\epsilon_3}{1+\epsilon_3} - \frac{a_1^2}{a_3^2} \right) + \frac{a_2^2}{a_3^2} \left( 1 - \frac{a_1^2}{a_3^2} \frac{1-\epsilon_3}{1+\epsilon_3} \right)} (k_0 a_3)^2 + O((k_0 a_3)^4) \quad (2.4)$$

Higher  $m$  values contribute with higher order terms, *e.g.*,

$$T_m = O((k_0 a_3)^{2m}), \quad m > 1$$

Notice that  $T_0$  only contains magnetic material parameters, and  $T_1$  only electric material parameters.

## 2.2 RCS

The scattering amplitude in 2D has the form

$$P(\phi) = \lim_{\rho \rightarrow \infty} 2\pi\rho |H_z^s(\rho, \phi)|^2 = \frac{4}{k} \left| \sum_{m=-\infty}^{\infty} e^{im\phi} T_m \right|^2$$

In particular, the back-scattering amplitude is

$$P_{\text{RCS}} = P(\pi) = \frac{4}{k} \left| \sum_{m=-\infty}^{\infty} (-1)^m T_m \right|^2 \quad (2.5)$$

As seen from the low frequency expansions of the transition matrix above, the general behavior at low frequencies is  $P_{\text{RCS}} = O((k_0 a_3)^4)$ .

These relations solve the scattering problem. We now focus on two ways to reduce the low frequency behavior of the RCS.

## 3 Reduction of RCS

We investigate whether it's possible to eliminate the lowest order contribution to the RCS, *i.e.*, an elimination of the  $(k_0 a_3)^4$  term. The origin of the  $(k_0 a_3)^4$  contribution is the two lowest  $m$  values,  $m = 0$  and  $m = \pm 1$ , each contributing to the  $(k_0 a_3)^2$  term.

We investigate two means of reduction. The first, in Section 3.1, we find the explicit values of the material parameters that kill the leading contributions from  $m = 0$  and  $m = \pm 1$ . In Section 3.2, we explore the possibility to let the  $m = 0$  and  $m = \pm 1$  terms annihilate each other.

### 3.1 Elimination of the low frequency contributions I

If we can find material parameters such that the lowest order terms in  $T_0$  and  $T_1$  vanish, the RCS would behave as  $P_{\text{RCS}} = O((k_0 a_1)^8) = O((k_0 a_3)^8)$  instead of the general behavior as  $P_{\text{RCS}} = O((k_0 a_1)^4) = O((k_0 a_3)^4)$ .

We start with the simplest  $m = 0$ , and we get to lowest order  $T_0 = 0$  provided, see (2.3)

$$\frac{a_1^2}{a_3^2} \mu_2 + \frac{a_2^2}{a_3^2} (\mu_3 - \mu_2) + 1 - \mu_3 = 0$$

For given geometry, the optimal permeability  $\mu_2$  of the inner cladding in terms of the permeability of the outer cladding has to be

$$\mu_{2\text{opt}} = \frac{a_2^2 \mu_3 + a_3^2 (1 - \mu_3)}{a_2^2 - a_1^2}$$

This permeability satisfies  $\mu_{2\text{opt}} > 1$ , provided  $0 < a_1 < a_2 < a_3$  and

$$\mu_3 < \frac{a_3^2 + a_1^2 - a_2^2}{a_3^2 - a_2^2} = 1 + \frac{a_1^2}{a_3^2 - a_2^2}$$



If  $\mu_3 = 1$  (outer cladding non-magnetic), we get

$$\mu_{2\text{opt}} = \frac{a_2^2}{a_2^2 - a_1^2} > 1$$

This leads to an interval of realistic values of the optimal permeability of the outer and inner cladding.

Proceeding with the next term,  $T_1$ , that also contributes to the lowest order contribution of the RCS, we get  $T_1 = 0$  provided, see (2.4)

$$\frac{a_2^2}{a_3^2} \left( \frac{a_1^2}{a_3^2} - \frac{1 - \epsilon_3}{1 + \epsilon_3} \right) + \frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \left( \frac{a_1^2}{a_3^2} \frac{1 - \epsilon_3}{1 + \epsilon_3} - \frac{a_2^4}{a_3^4} \right) = 0$$

For a given geometry, the optimal permittivity  $\epsilon_2$  of the inner cladding in terms of the permittivity of the outer cladding has to be

$$\epsilon_{2\text{opt}} = \epsilon_3 \frac{a_2^2 - a_1^2 a_2^2 (1 + \epsilon_3) + a_3^2 (1 - \epsilon_3)}{a_2^2 + a_1^2 a_2^2 (1 + \epsilon_3) - a_3^2 (1 - \epsilon_3)}$$

This permittivity satisfies  $\epsilon_{2\text{opt}} > 1$ , provided  $0 < a_1 < a_2 < a_3$  and

$$\frac{a_3^2 - a_2^2}{a_3^2 + a_2^2} < \epsilon_3 < \frac{-a_1^2 (a_3^2 + a_2^2) + \sqrt{a_2^8 + 4a_1^4 a_2^2 a_3^2 - 2a_2^6 a_3^2 + a_2^4 a_3^4}}{(a_2^2 - a_1^2)(a_3^2 - a_2^2)}$$

In Appendix A, we prove that, for  $0 < a_1 < a_2 < a_3$ , the left-hand side is greater than zero, and the right-hand side always is less than one, *i.e.*,

$$0 < \epsilon_3 < 1$$

Nevertheless, as seen in Figure 2, these fictitious material parameters give a significant reduction in the RCS.

The conclusion is that the elimination of the lowest order contribution in  $T_0$  and  $T_1$  leads to unrealistic material parameters. The reduction from  $P_{\text{RCS}} = O((k_0 a_1)^4)$  to  $P_{\text{RCS}} = O((k_0 a_1)^8)$  is therefore not possible to obtain in practice unless the use of special material. In the next subsection, we proceed with another approach.

### 3.2 Elimination of the low frequency contributions II

The transition matrix behaves at low frequencies as, see (2.3) and (2.4)

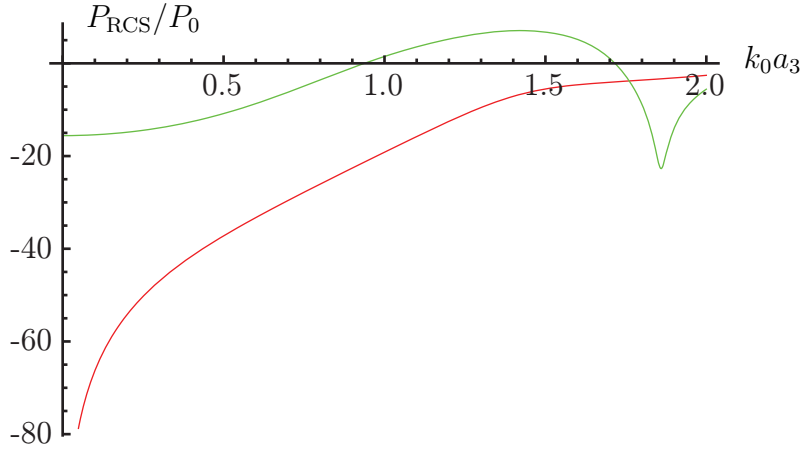
$$T_0 = t_0 (k_0 a_3)^2 + O((k_0 a_3)^4), \quad T_1 = t_1 (k_0 a_3)^2 + O((k_0 a_3)^4)$$

where

$$t_0 = -\frac{i\pi}{4} \left( \frac{a_2^2}{a_3^2} \mu_2 + \frac{a_2^2}{a_3^2} (\mu_3 - \mu_2) - \mu_3 + 1 \right)$$

and

$$t_1 = \frac{i\pi}{4} \frac{\frac{a_2^2}{a_3^2} \left( \frac{a_1^2}{a_3^2} - \frac{1 - \epsilon_3}{1 + \epsilon_3} \right) + \frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \left( \frac{a_1^2}{a_3^2} \frac{1 - \epsilon_3}{1 + \epsilon_3} - \frac{a_2^4}{a_3^4} \right)}{\frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \left( \frac{a_1^2}{a_3^2} \frac{1 - \epsilon_3}{1 + \epsilon_3} - \frac{a_2^4}{a_3^4} \right) + \frac{a_2^2}{a_3^2} \left( 1 - \frac{a_2^2}{a_3^2} \frac{1 - \epsilon_3}{1 + \epsilon_3} \right)}$$



**Figure 2:** The RCS,  $P_{\text{RCS}}$ , for a thin outer cladding,  $a_2/a_3 = 0.95$ , and a typical value of the permittivity and permeability of the outer cladding  $\epsilon_3 = 3$ ,  $\mu_3 = 1$ . The inner cladding has optimal permittivity and permeability,  $\epsilon_{2\text{opt}} = 0.37$ ,  $\mu_{2\text{opt}} = 1.66$ , (red curve), and perturbed values,  $\epsilon_2 = 1.37$ ,  $\mu_2 = 2.66$ , (green curve). The curves are in logarithmic scale, and scaled with the RCS,  $P_0$ , of a metallic wire of exterior radius  $a_3$ . The thickness of the claddings are  $a_1/a_3 = 0.6$ .

We see that at low frequencies, the sum in (2.5) behaves as

$$\sum_{m=-\infty}^{\infty} (-1)^m T_m = (t_0 - 2t_1)^2 (k_0 a_3)^2 + O((k_0 a_3)^4)$$

Provided the material parameters and the radii are chosen such that  $t_0 = 2t_1$ , the RCS in (2.5) reduces to  $\sigma_{\text{RCS}} \sim O((k_0 a_3)^8)$ .

The constraint,  $t_0 = 2t_1$ , leads to a rather complex relation between geometry and material parameters, and there are several ways to proceed depending on what is given — either the geometry is fixed, and we find the material parameters that satisfy this constraint, or the material parameters are fixed, and we find the geometry that satisfy the constraint. We adopt the former approach, where the geometry is given, in this paper.

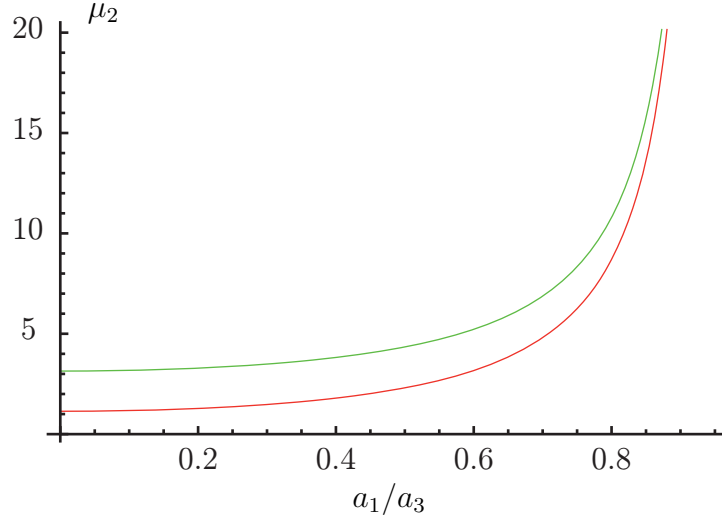
If the geometry is fixed, we proceed by looking for the optimal material parameters for given radii  $a_i$ ,  $i = 1, 2, 3$ . Moreover, we specify  $\epsilon_3$  and  $\mu_3$  and look for the optimal permittivity and permeability of the inner cladding,  $\epsilon_2$  and  $\mu_2$ .

This leads to

$$\epsilon_{2\text{opt}} = \frac{a_2^2 - a_1^2 N}{a_1^2 + a_2^2 D}$$

where

$$N = -\epsilon_3 \left( a_2^2 (a_1^2 (-1 + \epsilon_3) \mu_2 + a_3^2 (1 + \mu_2 + \epsilon_3 (3 + \mu_2 - 2\mu_3))) \right. \\ \left. - a_2^4 (-1 + \epsilon_3) (\mu_2 - \mu_3) + a_3^2 (-a_1^2 (1 + \epsilon_3) \mu_2 + a_3^2 (1 + \epsilon_3 (-3 + \mu_3) + \mu_3)) \right)$$



**Figure 3:** The highest (green curve) and lowest value (red curve) of  $\mu_2$  as a function of  $a_1/a_3$  in order to fulfill  $\epsilon_2 > 1$ . The quotient between the outer radii is  $a_2/a_3 = 0.95$ , and typical values of the permittivity and permeability of the outer cladding,  $\epsilon_3 = 3$  and  $\mu_3 = 1$ , are assumed.

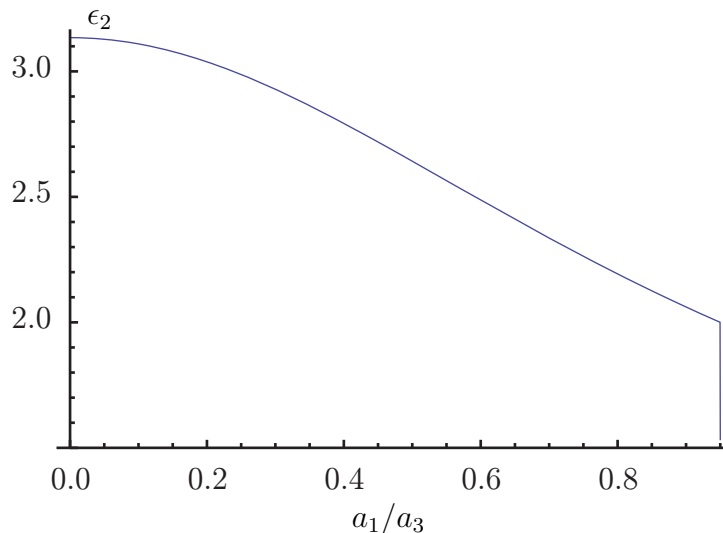
and

$$D = a_2^2(-a_1^2(-1 + \epsilon_3)\mu_2 + a_3^2(-1 + \epsilon_3(-3 + \mu_2) + \mu_2 - 2\mu_3)) \\ + a_2^4(-1 + \epsilon_3)(\mu_2 - \mu_3) + a_3^2(-a_1^2(1 + \epsilon_3)\mu_2 + a_3^2(1 + \epsilon_3(-3 + \mu_3) + \mu_3))$$

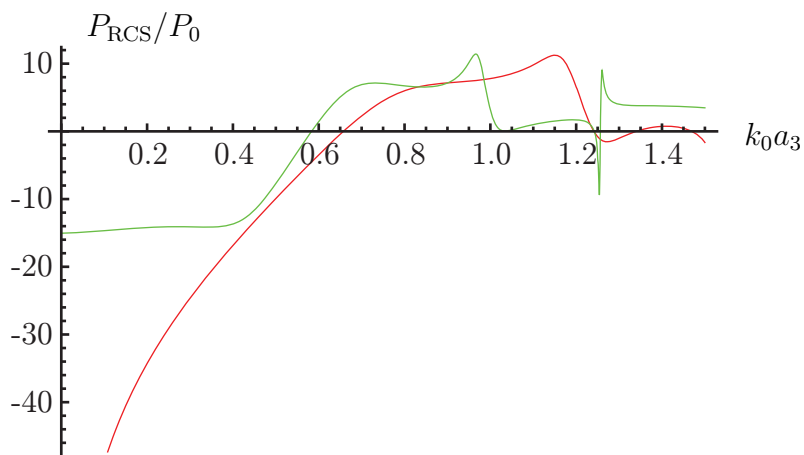
To proceed, we reduce the parameter space and let  $\epsilon_3 = 3$  and  $\mu_3 = 1$ . These values are realistic values of the outer cladding. The permittivity satisfies  $\epsilon_{2\text{opt}} > 1$ , provided  $0 < a_1 < a_2 < a_3$  and  $\mu_2$  lies between the red and green curves in Figure 3, which shows the permitted interval of the permeability  $\mu_2$  as a function of  $a_1/a_3$ . The value of  $\epsilon_2$  as a function of  $a_1/a_3$  for the mean value between the red and green curved in Figure 3 is depicted in Figure 4. We observe that both material parameters can be chosen physically realizable. For at typical value of  $a_1/a_3 = 0.6$ , we get the optimal permittivity of the inner cladding  $\epsilon_2 = 2.60$ , and the optimal permeability of the inner cladding  $\mu_2 = 4.24$ . This gives a significant reduction in the RCS as seen in Figure 5. Notice also that the effect is robust against changes in the material parameters of the inner cladding.

## 4 Conclusion and summary

In this paper we have analyzed two ways of reducing the radar cross section of thin perfectly conducting wire at low frequencies. The first method, eliminating each of the leading low-frequency contributions in the transition matrix entries leads to non-physical material parameters, and for this reason the method is not realistic for implementation. The other method — annihilating the combination of the leading low-frequency contributions — leads to realizable material parameters, and this



**Figure 4:** The value of  $\epsilon_2$  as a function of  $a_1/a_3$  for the mean value between the red and green curves in Figure 3. The quotient between the outer radii is  $a_2/a_3 = 0.95$ , and a typical value of the permittivity and permeability of the outer cladding  $\epsilon_3 = 3$ ,  $\mu_3 = 1$ .



**Figure 5:** The RCS,  $P_{\text{RCS}}$ , for a thin outer cladding,  $a_2/a_3 = 0.95$ , and a typical value of the permittivity and permeability of the outer cladding  $\epsilon_3 = 3$ ,  $\mu_3 = 1$ . The inner cladding has optimal permittivity and permeability,  $\epsilon_{2\text{opt}} = 2.60$ ,  $\mu_{2\text{opt}} = 4.24$ , (red curve), and perturbed values,  $\epsilon_2 = 3.60$ ,  $\mu_2 = 5.24$ , (green curve). The curves are in logarithmic scale, and scaled with the RCS,  $P_0$ , of a metallic wire of exterior radius  $a_3$ . The thickness of the claddings are  $a_1/a_3 = 0.6$ .

approach shows more potential. A preliminary investigation of how sensitive the reduction in RCS is, due to variation in the material parameters of the inner cladding, show that the method is fairly robust.

## Appendix A Inequalities

We let  $0 < a_1 < a_2 < a_3$ .

We first conclude that

$$0 < \frac{a_3^2 - a_2^2}{a_3^2 + a_2^2}$$

and then we prove that

$$\frac{-a_1^2(a_3^2 + a_2^2) + \sqrt{a_2^8 + 4a_1^4 a_2^2 a_3^2 - 2a_2^6 a_3^2 + a_2^4 a_3^4}}{(a_2^2 - a_1^2)(a_3^2 - a_2^2)} < 1$$

To see this, rewrite using

$$(a_2^8 + 4a_1^4 a_2^2 a_3^2 - 2a_2^6 a_3^2 + a_2^4 a_3^4) - (a_3^2 a_1^2 + a_2^2 a_1^2)^2 = (a_2^4 - a_1^4) (a_3^2 - a_2^2)^2 > 0$$

which gives

$$\begin{aligned} & \frac{-a_1^2(a_3^2 + a_2^2) + \sqrt{a_2^8 + 4a_1^4 a_2^2 a_3^2 - 2a_2^6 a_3^2 + a_2^4 a_3^4}}{(a_2^2 - a_1^2)(a_3^2 - a_2^2)} \\ &= -\frac{a_1^2}{a_2^2 - a_1^2} \frac{a_3^2 + a_2^2}{a_3^2 - a_2^2} + \sqrt{\frac{a_2^2 + a_1^2}{a_2^2 - a_1^2} + \frac{a_1^4}{(a_2^2 - a_1^2)^2} \frac{(a_3^2 + a_2^2)^2}{(a_3^2 - a_2^2)^2}} \end{aligned}$$

This motivates us to study the function

$$f(x, y) = -x(1 + 2y) + \sqrt{1 + 2x + x^2(1 + 2y)^2}, \quad x, y \geq 0$$

where  $x = a_1^2/(a_2^2 - a_1^2)$  and  $y = a_2^2/(a_3^2 - a_2^2)$ . At the origin  $f(0, 0) = 1$ , and along the boundaries in the first quadrant in the  $x$ - $y$  plane  $f(0, y) = 1$  and  $f(x, 0) = 1$ . The partial derivatives in the first quadrant are non-positive

$$\begin{aligned} f_x(x, y) &= -(1 + 2y) + \frac{1 + x(1 + 2y)^2}{\sqrt{1 + 2x + x^2(1 + 2y)^2}} \\ &= (1 + 2y) \left( \frac{1 + x(1 + 2y)^2}{\sqrt{(1 + 2y)^2 - 1 + (1 + x(1 + 2y)^2)^2}} - 1 \right) \\ &\leq (1 + 2y) \left( \frac{1 + x(1 + 2y)^2}{\sqrt{(1 + x(1 + 2y)^2)^2}} - 1 \right) = 0 \end{aligned}$$

and

$$f_y(x, y) = -2x + \frac{2x^2(1 + 2y)}{\sqrt{1 + 2x + x^2(1 + 2y)^2}} \leq -2x + \frac{2x^2(1 + 2y)}{\sqrt{x^2(1 + 2y)^2}} = 0$$

This implies that the function  $f(x, y)$  is non-increasing in the first quadrant, and

$$0 \leq f(x, y) \leq 1, \quad x, y \geq 0$$

## Appendix References

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