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Comparison and Simplification of Uncertain Models

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Abstract

Mathematical models describing the behavior of physical systems are important in many areas. The required accuracy of the model largely depends on the purpose for which the model is intended.

In control engineering there is usually a trade off between model simplicity and model accuracy. Simulation and controller design based on complex models may result in time consuming computations, numerical difficulties and a designed controller which usually is complex and therefore may be expensive and difficult to implement. Models with low accuracy, on the other hand, may give rise to incorrect simulations, as well as control loops with low performance or even instability.

For these reasons there is a need to use a hierarchy of models, each equipped with a quality measure. The most appropriate should then be used for each task. Computation of such quality measures is the objective of this thesis. We analyze the importance of different components in a model. We then simplify the description of the less important components to a suitable level of accuracy. The resulting model is obtained together with a quality measure. This approach is a generalization of the Balanced truncation method for linear time-invariant models.

The components in the model may contain dynamics, nonlinearities and uncertainty and are described using integral quadratic constraints. The information captured by such constraints allows us to calculate positive values to each of the components, such that the simplification error always is bounded by the sum of these values for the simplified components. The computations are based on convex optimization and the bound is defined in terms of the induced 2-norm.

Key words
Model simplification, Model comparison, Uncertain models, Nonlinear models, Integral quadratic constraints, Linear matrix inequalities, Error bounds

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1

Introduction

1.1 Background

Mathematical models describing the behavior of physical systems are important in many areas. The required accuracy of the model largely depend on the purpose for which the model is intended.

In control engineering there is usually a trade off between model simplicity and model accuracy. Simulation and controller design based on complex models may result in time consuming computations as well as numerical difficulties. The designed controller usually also becomes complex and may therefore be expensive and difficult to implement. Models with low accuracy, on the other hand, may give rise to incorrect simulations, as well as control loops with low performance or even instability.

For these reasons there is a need to use a hierarchy of models, each equipped with a quality measure. The most appropriate should then be used for each task. Computation of such quality measures is the objective of this thesis. We analyze the importance of different components in a model. We then simplify the description of the less important components to a suitable level of accuracy. The resulting model is obtained together with a quality measure.

The choice of modeling framework restricts the type of behaviors that can be described. Mathematical models describing only the linear part of a system may, in more difficult design problems, result

in control loops with low performance or even instability. This has lead to the development of the robust control framework, see *e.g.* Zhou *et al.* (1996). In this framework the model may include an uncertainty description. These are described using constraints, such as norm bounds. Recently a unified framework for describing uncertainty and also nonlinearities has been developed, using integral quadratic constraints, see Megretski and Rantzer (1995). This is the modeling framework chosen in this thesis.

1.2 Related work

Model simplification has been studied for many years and there is a large amount of work done in this area. The most common simplification method is linearization, where a linear, usually time-invariant, model is obtained by simplifying a non-linear model. The linear time-invariant model is relatively easy to analyze and use in controller design. However, it is still important to keep the model order low to avoid numerical difficulties in the controller design and to obtain fast simulations and fast control algorithms. Below we review one well-known reduction method for linear models and a generalization applicable to models with norm-bounded uncertainty. These results are closely related to the results in this thesis. We emphasize, however, that the reviewed results only are a very small part of all the work done in the area.

State-space models

A lot of work has been done, and is still being done on simplification of state-space models

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{1.1}$$

Two well-known reduction methods for stable models are balanced truncation, see Moore (1981), Glover (1984), Enns (1984), Pernebo and Silverman (1982), and singular perturbation approximation, see Fernando and Nicholson (1982b), Liu and Anderson (1989), Fernando and Nicholson (1982a).

The first step in these two reduction methods is to transform the given realization into a balanced realization, which is a realization where each of the states is equally controllable and observable in an energy perspective. The controllability Grammian $P > 0$ is the solution to the Lyapunov equality

$$AP + PA^T + BB^T = 0$$

and the observability Grammian $Q > 0$ the solution to

$$QA + A^TQ + C^TC = 0.$$

Solutions exist if the model is stable. It is then always possible to find a realization where the states are equally controllable and observable, $P = Q = \Sigma$. Such a realization satisfies

$$\begin{aligned} \bar{A}\Sigma + \Sigma\bar{A}^T + \bar{B}\bar{B}^T &= 0 \\ \Sigma\bar{A} + \bar{A}^T\Sigma + \bar{C}^T\bar{C} &= 0 \end{aligned}$$

for some diagonal matrix

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \geq 0.$$

The last step is to eliminate the chosen states. In balanced truncation, as the name indicates, we truncate the parts of A, B and C corresponding to the chosen states. If we assume that we choose to eliminate the last states and partition the balanced realization correspondingly as

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} & D, \end{aligned}$$

then the truncated model is given by

$$\begin{aligned} \hat{A} &= A_{11} & \hat{B} &= B_1 \\ \hat{C} &= C_1 & \hat{D} &= D. \end{aligned}$$

This method results in no error at very high frequencies. In singular perturbation approximation the reduced order model is instead given by

$$\begin{aligned} \hat{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} & \hat{B} &= B_1 - A_{12}A_{22}^{-1}B_2 \\ \hat{C} &= C_1 - C_2A_{22}^{-1}A_{21} & \hat{D} &= D - C_2A_{22}^{-1}B_2. \end{aligned}$$

This method has the advantage that there is no error ar zero frequency (stationarity).

The calculated matrix Σ provides an easy way to calculate an upper bound on the reduction error. If we let $G(s)$ be the transfer function for the original model and $\hat{G}(s)$ be the transfer function for the simplified model then

$$\|G(s) - \hat{G}(s)\|_\infty \leq \sum 2\sigma_k,$$

where the sum should be taken over the eliminated states. Small elements in Σ thus indicate states with little importance. The Σ-matrix is therefore also an important guide when we select which of the states to eliminate.

The above results have been presented for continuous time models but similar results also exist for discrete time models, see Hinrichsen and Pritchard (1990) in addition to the previous references. The model is

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}$$

and the Lyapunov equalities

$$\begin{aligned} APA^T + P + BB^T &= 0 \\ A^TQA + Q + C^TC &= 0. \end{aligned}$$

Finally, we remind the reader that there exist many other results for simplification of linear time-invariant models, than those presented above.

Closed loop models

Simplification of a model results in an approximation error. It is important to be able to measure if this error satisfies the specified accuracy. This accuracy is in controller design usually specified for the closed loop model. However, the closed-loop approximation error depends not only on the open-loop error but also on the controller to be designed. The relation between the open-loop error and the closed-loop error, for a given controller, is in addition non-linear. This means that a small open loop error at some frequencies may result in a large closed loop error while the same open loop error at another frequency results in a small closed loop error. We illustrate this with two examples taken from Skelton (1989). First we show that arbitrarily small modeling errors may result in arbitrarily bad closed-loop performance.

EXAMPLE 1.1—SENSITIVE MODELING PROBLEM
Consider the system

$$G(s) = \frac{1}{(s+1)(s\varepsilon+1)}$$

controlled using a P-controller. The closed loop system will then be given by

$$G_{cl}(s) = \frac{K/\varepsilon}{s^2 + (1+1/\varepsilon)s + (K+1)/\varepsilon} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2},$$

where ε is a small value. The parameter $\zeta \geq 0$ is the damping of the closed loop system. When ζ is close to zero then the system is oscillative and when $\zeta \geq 1$ then the system is well damped. We, thus, see that using a large value on K the closed loop system gets arbitrary oscillative.

Now assume that the design is based on the model

$$\hat{G}(s) = \frac{1}{s+1}$$

which has a small error at all frequencies,

$$\|G - \hat{G}\|_\infty < \varepsilon.$$

The closed loop model using a P-controller is

$$\hat{G}_{cl}(s) = \frac{K}{s + K + 1}.$$

The closed loop model will therefore be well damped for all values on K. □

We now show that large open-loop modeling errors do not necessarily lead to large closed-loop prediction errors.

EXAMPLE 1.2—ROBUST MODELING PROBLEM
Consider the system

$$G(s) = \frac{1}{s + 1}$$

controlled using a P-controller. The closed loop system will then be given by

$$G_{cl}(s) = \frac{K}{s + K + 1}.$$

Now assume that the design is based on the model

$$\hat{G}(s) = \frac{1}{s}$$

which has a infinite approximation error at low frequencies. The closed loop model using a P-controller is

$$G_{cl}(s) = \frac{K}{s + K}.$$

The closed loop model will therefore be close to the closed loop system when K is large. □

To avoid the above described type of problem, frequency weighting was introduced in Enns (1984). To apply this method one has to choose the weights depending on what frequency ranges that are important for the closed loop model.

It is clear from the discussion above that the required accuracy of the open-loop model is different at different frequencies. This may in fact also be the case for the closed-loop model. Frequency dependent error bounds are therefore important, and are the subject of study in Chapter 2. These frequency dependent error-bounds may also be important in other applications than controller design.

Models with perturbation

In modern robust analysis and control design it is common to describe a system using a feedback interconnection according to the relations

$$x = \Delta z$$

and

$$\begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix}, \tag{1.2}$$

as illustrated in Figure 1.1. Here Δ represents nonlinearities and model uncertainty, and M represent linear time-invariant dynamics. The per-

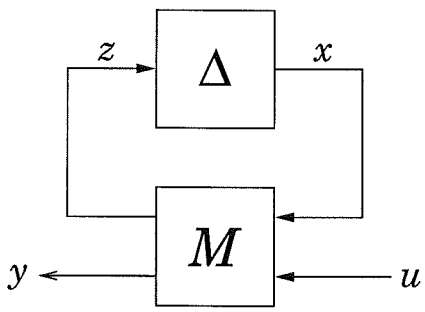

Figure 1.1 Feedback interconnection representing a model with nonlinearities and uncertainty.

turbation Δ is usually block diagonal,

$$\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_r).$$

and M is a transfer matrix partitioned consistently with the signals as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Simplification of the above described type of model has only been studied for a small class of Δ. For discrete time models where the uncertainty is norm bounded, linear and time-varying and the spatial structure is

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}), \quad \|\delta_k\| \leq 1, \quad k = 1, \ldots, r$$

and M is a frequency independent matrix, a truncation method similar to balanced truncation has been developed, see Wang *et al.* (1991), Beck *et al.* (1996), Beck (1996). The Lyapunov equalities are now replaced by Lyapunov inequalities

$$\begin{aligned} A\Sigma A^T + \Sigma + BB^T &< 0 \\ A^T\Sigma A + \Sigma + C^TC &< 0. \end{aligned}$$

Both the transformation into a new realization and the Σ-matrix are in this case non-unique. To obtain good error bounds we have to solve for a realization as well as a Σ so that Σ has small elements. This is not a convex optimization problem. A suboptimal algorithm has been proposed which minimizes the trace of $P > 0$ and $Q > 0$ for the following Lyapunov inequalities

$$\begin{aligned} APA^T + P + BB^T &< 0 \\ A^TQA + Q + C^TC &< 0, \end{aligned}$$

followed by a (unique) transformation. Each minimization is a convex problem involving linear matrix inequalities (LMIs). Efficient algorithms for solving such problems have been developed, see Nesterov and Nemirovski (1993), and reliable software packages exist, see for example Gahinet *et al.* (1995). Example 3.5 illustrates the described method.

Similar results where special consideration is taken if the uncertain part does not include dynamics have been presented in Helmersson (1995).

1.3 Contribution

This thesis provides error-bounds, that are numerically obtainable using convex optimization, for comparison and simplification of a large

class of nonlinear and uncertain models.

We will consider the feedback interconnection in Figure 1.1. The nonlinearities and uncertainties will be described in the IQC-framework by frequency dependent matrices $\Pi(i\omega)$. The Σ-matrix is obtained as a solution to the inequalities

$$\begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* \Sigma^2 \Pi(i\omega) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix} + C(i\omega)^* C(i\omega) < 0.$$

For linear time-invariant, possibly uncertain, models the error-bounds may be obtained separately for each frequency. Then Σ will be frequency dependent.

We propose a general class of reduction methods, which include truncation and singular perturbation approximation as special cases.

The thesis consist of two main chapters that may be read independently of each other.

Chapter 2 (Frequency dependent Error Bounds) considers uncertain linear time-invariant models, with the uncertainty described using quadratic constraints. Model truncation is first studied. Then we study model comparison and propose a class of reduction methods, where truncation is a special case. The error-bounds are obtained separately at each frequency. Chapter 2 is based on the following publications:

ANDERSSON, L. and A. RANTZER (1997a): "Frequency dependent error bounds for uncertain linear models." To appear in American Control Conference, Albuquerque, New Mexico.

ANDERSSON, L. and A. RANTZER (1997b): "Frequency dependent error bounds for uncertain linear models." Journal article under preparation.

Chapter 3 (Comparison and Simplification using IQCs) considers the same problem as the previous chapter; model comparison and simplification. However, the class of models is generalized to include both time-variability and nonlinearities. The nonlinearities and

model uncertainty are now described using integral quadratic constrains (IQCs), instead of quadratic constraints, and the error-bounds are no longer obtained at separate frequencies. Effort has been paid to give an extensive comparison with the results for models with norm-bounded uncertainty presented in Wang *et al.* (1991), which is a special case of the results in this chapter. Chapter 3 is based on the following publications:

RANTZER, A. (1995): "Error bounds for nonlinear model truncation." Presented at Bernoulli Workshop, Groningen.

ANDERSSON, L. and C. BECK (1996): "Model comparison and simplification." In *35th IEEE CDC Proceedings*, Kobe, Japan.

ANDERSSON, L., A. RANTZER, and C. BECK (1996): "Model comparison and simplification." Submitted to the Int. Journal on Robust and Nonlinear Control.

We conclude the thesis with some remarks in Chapter 4.

1.4 Alternative approach

An alternative approach to calculating upper bounds would be to calculate the error between the original model and all those models that we consider for simplification.

Each error can, at least approximately, be calculated using the H_∞-norm, μ-analysis or stability theory based on integral quadratic constraints.

One advantage with direct calculation of the error is that we find a better upper bound. A disadvantage is that the computation time depends exponentially on the number of blocks instead of polynomially.

EXAMPLE 1.3
Assume that we have a model with

$$\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3)$$

and that we would like to truncate some of these three blocks. We can then either calculate upper bounds using σ-values or the error

Δ_1	Δ_2	Δ_3	error	upper bound
			0	0
x			e_1	$2\sigma_1$
	x		e_2	$2\sigma_2$
x	x		e_{12}	$2(\sigma_1 + \sigma_2)$
		x	e_3	$2\sigma_3$
x		x	e_{13}	$2(\sigma_1 + \sigma_3)$
	x	x	e_{23}	$2(\sigma_2 + \sigma_3)$
x	x	x	e_{123}	$2(\sigma_1 + \sigma_2 + \sigma_3)$

Figure 1.2 The errors and upper bounds when we simplify different parts of Δ in Figure 1.1. The blocks in Δ that we simplify are marked with x. The number of σ-values are equal to the number of blocks while the number of errors grow exponentially.

directly for the different possible simplified models. Figure 1.2 shows that we have to calculate $2^3 = 8$ errors or 3 σ-values. Each of these computations are done in polynomial time *w.r.t.* the number of errors and σ-values respectively. □

2

Frequency dependent error bounds

Abstract

In this chapter, mainly consisting of the paper Andersson and Rantzer (1997b), we study frequency dependent error bounds for approximation and truncation of linear dynamic models with uncertainty. The uncertainty is described by quadratic constraints and the error bounds are calculated based on solutions to linear matrix inequalities.

2.1 Introduction

In modern robust control design it is common to model both the system dynamics and uncertainty. This often results in models that have high state order and complicated uncertainty descriptions. These models may be difficult to analyze and the subsequent controller design, based on these models, may be both difficult and time consuming. The resulting controller usually also become complex and may therefore be expensive and difficult to implement.

For these reasons there is a need to develop methods to analyze the importance of the uncertainty description as well as the states. In many situations, such as in controller design, the required accuracy of the model is different at different frequencies. It is therefore desirable to do the analysis frequency by frequency.

For linear time-invariant models without uncertainty there exist well-known order reduction methods and associated error bounds. Two such methods are balanced truncation, see Moore (1981), Glover (1984), Enns (1984) and singular perturbation approximation, see Fernando and Nicholson (1982b), Liu and Anderson (1989), Fernando and Nicholson (1982a). The balanced truncation method has been generalized to models with norm-bounded uncertainty, see Wang *et al.* (1991), Beck *et al.* (1996), Beck (1996).

In this thesis we generalize these results to include a more general class of uncertainty descriptions as well as nonlinearities. This chapter will focus on uncertain linear time-invariant models, for which frequency dependent error bounds can be found. Error bounds for models with time-variation and nonlinearities will be obtained in Chapter 3.

The chapter is organized as follows. We start, in Section 2.2, by describing the modeling framework and stating the problem. In Section 2.3 we describe model truncation and present the associated frequency dependent error bounds. An application example is then given in Section 2.4. Model comparison as well as model reduction using a general reduction method is described in Section 2.5. A numerical example is given in Section 2.6 and the application example is continued in Section 2.7. Numerical calculation of the error bounds are described in Section 2.8 and we conclude the paper in Section 2.9 with some remarks.

2.2 Preliminaries

In this section we describe the modeling framework and state the problem. Everything in this chapter is done in continuous time, even though similar results also hold in discrete time.

Notation

Let $\mathbf{R}$ denote the real numbers. Subscripts of a matrix denote submatrices and parentheses in the subscripts denote submatrices of submatrices, *e.g.*

$$A_{11} = \begin{bmatrix} A_{11(11)} & A_{11(12)} \\ A_{11(21)} & A_{11(22)} \end{bmatrix}.$$

The hermitian conjugate of a matrix A is denoted A^* and the symmetric transpose A^T. The space of square integrable functions of dimension p, is denoted $\mathbf{L}_2^p[0,\infty)$ and the space of functions that are square integrable over any finite time $\mathbf{L}_{2e}^p[0,\infty)$. The gain of a transfer matrix is defined by the maximum singular value as

$$\|G(i\omega)\| = \bar{\sigma}(G(i\omega)).$$

Note that the gain is frequency dependent.

Model description

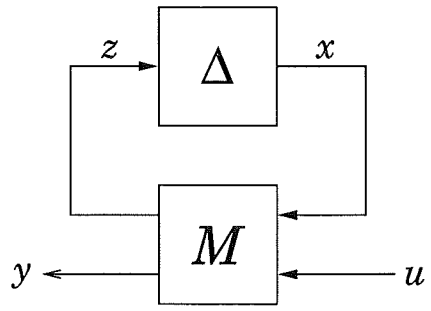

Figure 2.1 Feedback interconnection representing a model with uncertainty.

The modeling framework considered in this thesis, is commonly used in modern robustness analysis and control design, see *e.g.* the book by Zhou *et al.* (1996), and is defined by the interconnection of a pair (Δ, M) according to the relations

$$x = \Delta z$$

and

$$\begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix},$$

as illustrated in Figure 2.1.

We will usually assume that Δ and M are stable proper transfer matrices and that at least one of them is strictly proper. The transfer

matrix Δ is used to represent uncertain dynamics as well as known dynamics considered for simplification while M represent the remaining part of the model.

The transfer matrix Δ is also assumed to have a block diagonal structure $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r)$, where each of the blocks satisfy a constraint, *e.g.* a norm bound or some other quadratic constraint. These quadratic constraints can also be used to describe uncertain dynamics.

The signals in the interconnection are the input $u \in \mathbf{L}_{2e}^m[0,\infty)$, the output $y \in \mathbf{L}_{2e}^p[0,\infty)$ and the internal signals $z \in \mathbf{L}_{2e}^n[0,\infty)$ and $x \in \mathbf{L}_{2e}^n[0,\infty)$.

The transfer matrix M is partitioned consistently with the signal dimensions as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.1}$$

The input-output mapping of the interconnection in Figure 2.1 is then defined by the Redheffer star product

$$y = (\Delta \star M)u = (D + C\Delta(I - A\Delta)^{-1}B)u.$$

EXAMPLE 2.1—UNCERTAIN SPRING CONSTANT

A mass and spring system can be described using the equation

$$m\ddot{\xi} + d\dot{\xi} + k\xi = F$$

where ξ is the position, F an external force, m the mass, d the damping and k the spring constant. Assume that the spring constant is known with only 10% accuracy. This may be written as

$$k = k_0(1 + 0.1\delta)$$

where k_0 is the nominal spring constant and δ a real unknown constant satisfying $|\delta| \leq 1$. The equation then becomes

$$m\ddot{\xi} + d\dot{\xi} + k_0\xi = F - 0.1k_0\delta\xi$$

The described model can be written on standard form using

$$M(s) = \frac{1}{ms^2 + ds + k_0}\begin{bmatrix} -0.1k_0 & 1 \\ -0.1k_0 & 1 \end{bmatrix},$$

$$\Delta = \delta.$$

□

Problem formulation

The problem considered in this chapter is to analyze the importance of the different blocks in Δ for different frequencies. In particular, we would like to find upper bounds on the error,

$$e = y - \hat{y} = (\Delta \star M - \hat{\Delta} \star M)u,$$

or more precisely the norm

$$\|(\Delta \star M - \hat{\Delta} \star M)(i\omega)\|,$$

between two models. We will assume that the difference between the two models is that some of the blocks in the second model have been truncated, simplified or changed in some other way. These results can then be used for model comparison, model simplification and model reduction.

Quadratic constraints

We use quadratic constraints as a general framework to describe uncertain dynamics as well as known dynamics considered for simplification. This framework includes a number of well-known constraints such as passivity and norm bounds.

We say that the matrix Δ satisfies the quadratic constraint defined by the hermitian matrix Π if

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0. \tag{2.2}$$

The matrix Π will be called a multiplier. The following two properties are useful:

- Assume that Δ satisfies the quadratic constraints defined by the multipliers $\Pi_1, \ldots, \Pi_n$ then Δ also satisfies the quadratic constraint defined by

$$\Pi = \sum_{k=1}^{n} \alpha_k \Pi_k$$

for any $\alpha_k \geq 0$, $k = 1, \ldots, n$.

- Assume that Δ has a block diagonal structure,

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r),$$

and that Δ_k satisfies the quadratic constraints defined by Π_k, for $k = 1, \ldots, r$. Then Δ satisfies the quadratic constraint defined by

$$\Pi = \mathrm{daug}(\Pi_1, \ldots, \Pi_r),$$

where

$$\mathrm{daug}(\Pi_1, \ldots, \Pi_r) = \left[\begin{array}{ccc|ccc} \Pi_{1(11)} & & 0 & \Pi_{1(12)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(11)} & 0 & & \Pi_{r(12)} \\ \hline \Pi_{1(21)} & & 0 & \Pi_{1(22)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(21)} & 0 & & \Pi_{r(22)} \end{array}\right].$$

Below is a list of some multipliers.

EXAMPLE 2.2—MULTIPLIERS

- Let $\Delta(i\omega)$ be any unity norm bounded transfer matrix. Then $\Delta(i\omega)$ satisfies all quadratic constraints defined by

$$\begin{bmatrix} x(\omega)I & 0 \\ 0 & -x(\omega)I \end{bmatrix}$$

where $x(\omega) \geq 0$.

- Let $\Delta(i\omega)$ be any passive (positive real) transfer matrix. Then it satisfies all quadratic constraints defined by

$$\begin{bmatrix} 0 & x(\omega)I \\ x(\omega)I & 0 \end{bmatrix}$$

where $x(\omega) \geq 0$.

- Let $\Delta(i\omega) = \delta I$, where δ is a constant real scalar satisfying $|\delta| \leq 1$, then $\Delta(i\omega)$ satisfies all quadratic constrains defined by

$$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$$

  where $X(i\omega) = X(i\omega)^* \geq 0$ and $Y(i\omega) + Y(i\omega)^* = 0$.

- Let $\Delta(i\omega) = \delta I$, where δ is a constant real scalar satisfying $\delta \geq 0$, then $\Delta(i\omega)$ satisfies all quadratic constraints defined by

$$\begin{bmatrix} 0 & Y(i\omega) \\ Y(i\omega)^* & 0 \end{bmatrix}$$

  where $Y(i\omega) + Y(i\omega)^* \geq 0$.

- A given transfer matrix $\Delta(i\omega)$ satisfies all quadratic constraints defined by multipliers $\Pi(i\omega)$ satisfying

$$\begin{bmatrix} I \\ \Delta(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} I \\ \Delta(i\omega) \end{bmatrix} \geq 0.$$

□

The same Δ satisfies the quadratic constraints defined by many different multipliers Π as shown in Figure 2.2.

It will be shown later that the multipliers $\Pi(i\omega)$ play a fundamental role when we compute the error bounds between two models. The particular choice of multiplier will influence the error bounds. In order to obtain less conservative error bounds it is therefore important to choose the multiplier Π resulting in the lowest error bound. Restricting the set of multipliers beforehand may therefore result in more conservative error bounds than necessary.

EXAMPLE 2.3—QUADRATIC CONSTRAINTS FOR AN INTEGRATOR
Assume that $\Delta(s) = 1/s$. Then $\Delta(i\omega)$ satisfies the quadratic constraint defined by $\Pi(i\omega)$ if and only if $\Pi(i\omega)$ satisfies

$$\begin{bmatrix} 1 \\ 1/i\omega \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} 1 \\ 1/i\omega \end{bmatrix} \geq 0.$$

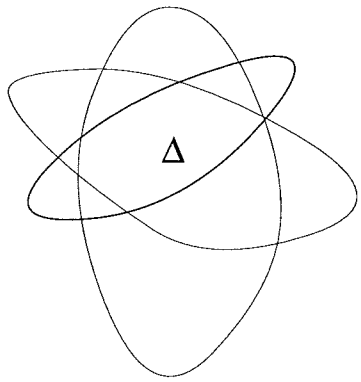

Figure 2.2 The same Δ satisfies the quadratic constraints defined by many different multipliers Π. The figure shows some sets corresponding to different multipliers.

A subset of these multipliers are multipliers of the form

$$\begin{bmatrix} 0 & x(\omega) \\ x(\omega) & 0 \end{bmatrix},$$

where $x(\omega) \geq 0$ is a real-valued function. Multipliers of this form define quadratic constraints that are satisfied for any passive transfer function. Thus, restricting the multiplier to have this form, may lead to conservative error bounds, since the error bounds will hold not only if Δ is an integrator but also if Δ is any passive transfer function. $\square$

2.3 Model truncation

In this section we present frequency dependent upper bounds on the error when parts of the model is truncated. This is a special case of the more general results presented Section 2.5.

Model truncation is illustrated in Figure 2.3. We assume, without loss of generality, that the model is partitioned in such a way that the lower blocks of Δ and the corresponding part of M should be truncated.

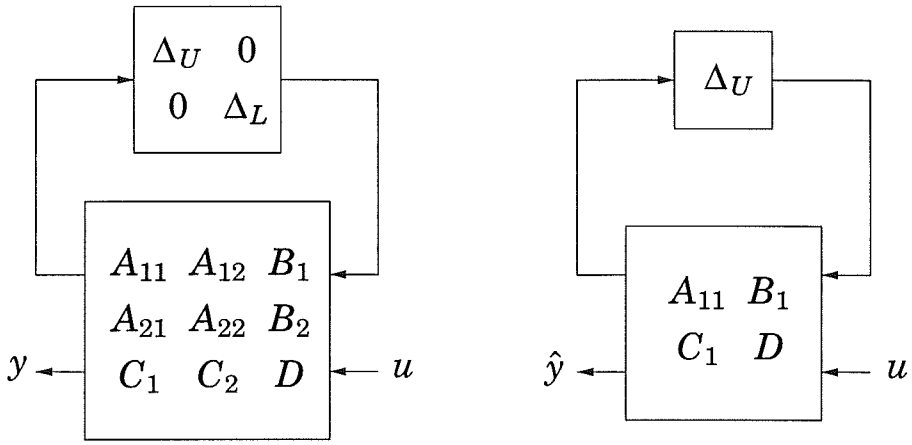

Figure 2.3 Model truncation.

We partition the model consistently as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \Delta_L \end{bmatrix},$$

where

$$\begin{aligned} \Delta_U &= \operatorname{diag}(\Delta_1, \ldots, \Delta_{\hat{r}}), \\ \Delta_L &= \operatorname{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r). \end{aligned}$$

The truncated model is then given by

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}, \qquad \hat{\Delta} = \Delta_U.$$

We assumed above that we had specified the parts to truncate beforehand. This is not always the case. Instead we can make this specification based on positive real-valued functions $\sigma_k(\omega)$ that we assign to each of the blocks in Δ, as illustrated in Figure 2.4. Each function $\sigma_k(\omega)$ gives a measure on the importance of the corresponding block Δ_k at each frequency. When the model is truncated then the error at each frequency, is bounded by two times the sum of the σ_k-functions corresponding to the truncated blocks. This is stated in the following theorem, which follows from the more general result presented later in Theorem 2.2.

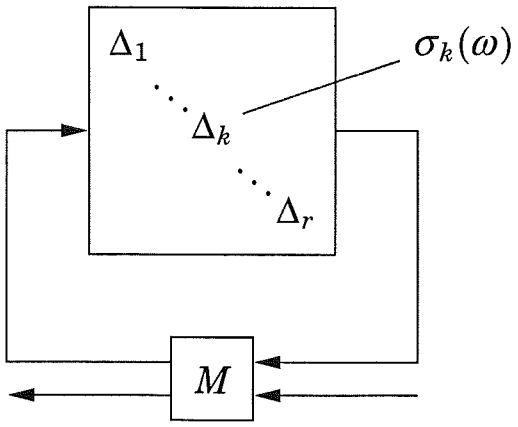

Figure 2.4 Each block Δ_k has an associated function $\sigma_k(\omega)$ that measure the importance of the block at each frequency.

THEOREM 2.1
Assume that Δ and M are stable proper transfer matrices and at least one of them is strictly proper. Let $\Pi_k(i\omega)$, for $k = 1, \ldots, r$, be bounded measurable functions taking hermitian values. Assume for all $\omega \in \mathbf{R}$ that

- $\tau\Delta_k(i\omega)$, for $k = 1, \ldots, r$, and all $\tau \in [0, 1]$ satisfies the quadratic constraint defined by $\Pi_k(i\omega)$
- $\Pi_{k(11)}(i\omega) \geq 0$, for $k = \hat{r} + 1, \ldots, r$

If there exist real valued functions $\sigma_1(\omega), \ldots, \sigma_r(\omega) > 0$ such that

$$\begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix}^* \operatorname{daug}(\Pi_1(i\omega), \ldots, \Pi_r(i\omega)) \begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* \operatorname{daug}(\sigma_1^2\Pi_1(i\omega), \ldots, \sigma_r^2\Pi_r(i\omega)) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix} + C(i\omega)^*C(i\omega) < 0,$$

for all $\omega \in [0, \infty]$, then both the original and truncated model are stable and

$$\|(\Delta \star M - \hat{\Delta} \star \hat{M})(i\omega)\| \leq 2\sigma_{\hat{r}+1}(\omega) + \cdots + 2\sigma_r(\omega).$$

□

Remark 1 How to compute σ_k-functions satisfying these inequalities is described in Section 2.8.

Remark 2 Note that the σ_k-functions not are unique.

Remark 3 We describe Δ using quadratic constraints. It is not necessary to know the specific transfer matrix Δ. This makes the result applicable to models where Δ is uncertain. One must then find a quadratic constraint that is satisfied for all Δ in the uncertainty set.

Remark 4 The assumptions on Δ and M to be stable and proper may be replaced with the assumption that both the original and truncated models are stable. Then it is also sufficient that $\Delta_k \in \mathrm{IQC}(\Pi_k)$ for $\tau = 1$ and not for all $\tau \in [0, 1]$. This version of the assumptions is useful when Δ is unstable, for example when Δ contains integrators.

We first apply the theorem to state space models without uncertainty.

EXAMPLE 2.4—TRUNCATION OF STATE SPACE MODELS
A state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

may be written on the standard form using

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and $\Delta = I_n/s$. The internal signal x in Figure 2.1 may in this case be interpreted as the state and $z = \dot{x}$ as the state derivatives. We let each integrator $1/s$ be regarded as a block of Δ. Each such block satisfies the quadratic constraint defined by any multiplier Π_k satisfying

$$\begin{bmatrix} 1 \\ 1/i\omega \end{bmatrix}^* \Pi_k(i\omega) \begin{bmatrix} 1 \\ 1/i\omega \end{bmatrix} \geq 0.$$

For application of Theorem 2.1, it must in addition hold that

$$\Pi_{k(11)}(i\omega) \geq 0.$$

We consider the special case where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$C = [1 \quad 1/2 \quad 1/3] \qquad D = 0,$$

with the corresponding transfer function

$$G(s) = \frac{3s^2 + 32s + 65}{(s+1)(s+5)(s+10)}.$$

Based on a numerical solution, see Section 2.8, we find the functions $\sigma_k(\omega)$ shown in Figure 2.5. If we decide to truncate the last state we obtain a state space model with

$$\hat{A} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\hat{C} = [1 \quad 1/2] \qquad \hat{D} = 0.$$

The error between these two models is at each frequency, ω, bounded by $2\sigma_3(\omega)$. □

The example shows two important features; we do not need a certain type of realization (such as a balanced realization) and the error bounds are given at each frequency. An other important point is that the results are applicable to models with uncertainty. An example illustrating this is given in the next section.

2.4 Application example; part I

In this section we consider simplification of models for the flexible servo in Figure 2.6. First we state a nominal model, then we extend this model to include either an uncertain spring constant or an uncertain moment of inertia. The importance of including an uncertainty description in these two cases is analyzed.

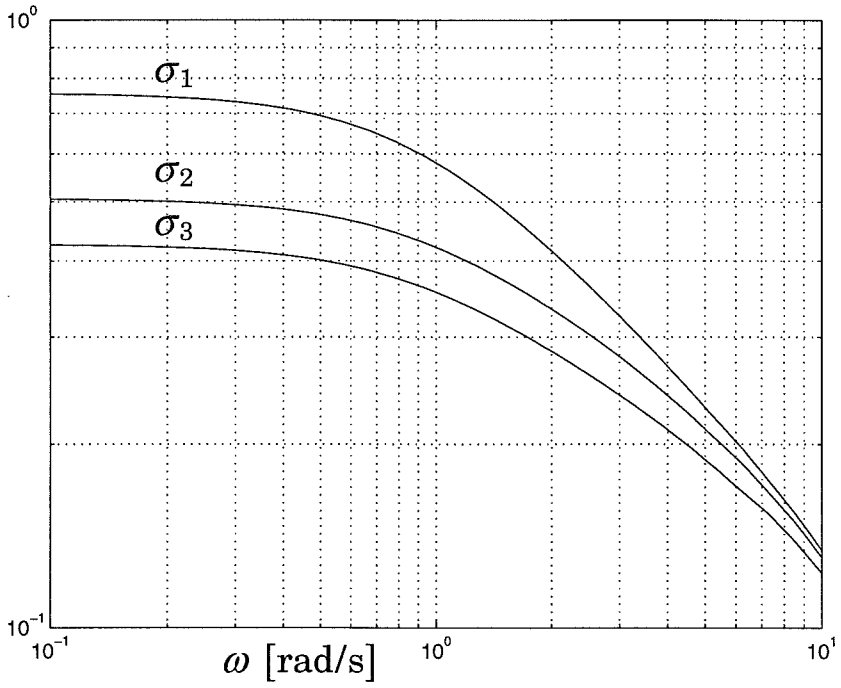

Figure 2.5 The functions $\sigma_k(\omega)$ measuring the importance of the different states (integrators) in the example. The error, at frequency ω, for truncating state k is bounded by $2\sigma_k(\omega)$.

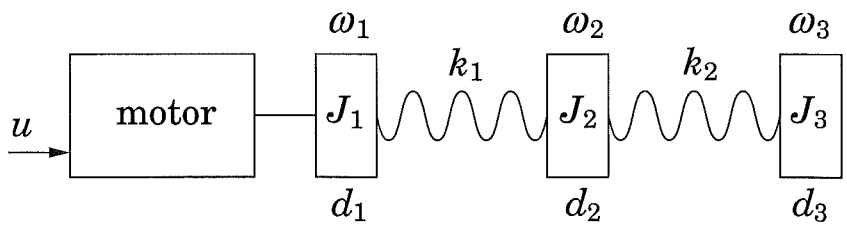

Figure 2.6 A flexible servo.

A simple model for the servo in Figure 2.6 is given by the following

equations

$$
\begin{aligned}
J_1\dot{\omega}_1 &= -k_1(\phi_1 - \phi_2) - d_1\omega_1 + k_u u \\
J_2\dot{\omega}_2 &= k_1(\phi_1 - \phi_2) - k_2(\phi_2 - \phi_3) - d_2\omega_2 \\
J_3\dot{\omega}_3 &= k_2(\phi_2 - \phi_3) - d_3\omega_3 \\
\dot{\phi}_1 &= \omega_1 \\
\dot{\phi}_2 &= \omega_2 \\
\dot{\phi}_3 &= \omega_3 \\
y &= k_\omega \omega_1
\end{aligned}
$$

where ω_k denotes angular velocity and ϕ_k the corresponding angle. We assume that the parameters are

$$
\begin{aligned}
J_1 &= 50 \cdot 10^{-6}\ \text{kgm}^2 \\
J_2 &= 20 \cdot 10^{-6}\ \text{kgm}^2 \\
J_3 &= 55 \cdot 10^{-6}\ \text{kgm}^2 \\
d_1 &= d_2 = d_3 = 30 \cdot 10^{-6}\ \text{Nm/rad/s} \\
k_u &= 25 \cdot 10^{-3}\ \text{Nm/V} \\
k_\omega &= 0.1\ \text{V/rad/s.}
\end{aligned}
$$

Introducing the state vector

$$
v^T = [\,\omega_1 \quad \omega_2 \quad \omega_3 \quad \phi_1 - \phi_2 \quad \phi_2 - \phi_3\,]
$$

results in the state space model,

$$
\begin{aligned}
\dot{v} &= \begin{bmatrix} -d_1/J_1 & 0 & 0 & -k_1/J_1 & 0 \\ 0 & -d_2/J_2 & 0 & k_1/J_2 & -k_2/J_2 \\ 0 & 0 & -d_3/J_3 & 0 & k_2/J_3 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} v + \begin{bmatrix} k_u/J_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\
y &= [\,k_\omega \quad 0 \quad 0 \quad 0 \quad 0\,]\,v.
\end{aligned}
$$

This model may be written in a more compact way as

$$
\begin{aligned}
\dot{v} &= \bar{A}v + \bar{B}u \\
y &= \bar{C}v.
\end{aligned}
$$

The Bode diagram for the model is given in Figure 2.7. We see that there are resonances at 9 rad/s and 22 rad/s and notches at 6 rad/s and 21 rad/s.

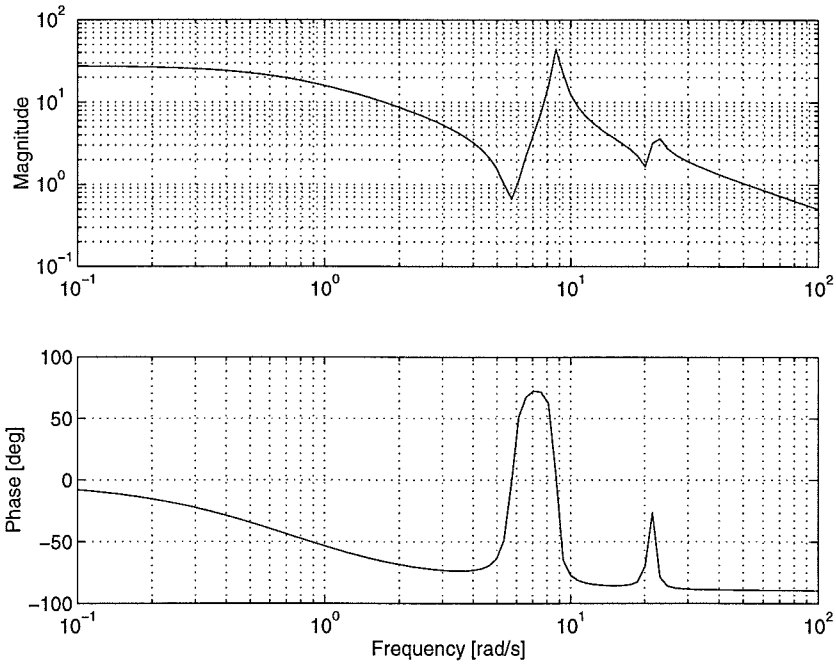

Figure 2.7 Bode diagram for the flexible servo. There are one resonance at 9 rad/s and one at 22 rad/s.

Uncertain spring constant

In this first part of the example we assume that the spring constant k_1 is known only with 10% accuracy, and that we are interested in analyzing in what frequency ranges the nominal model is a sufficient description of the uncertain model. We model the unknown spring constant $\bar{k}_1$ as

$$\bar{k}_1 = k_1(1 + 0.1\delta_k)$$

where $|\delta_k| \leq 1$ and k_1 is the nominal spring constant. Using this description we find that

$$\dot{v} = \bar{A}v + \bar{A}[0 \quad 0 \quad 0 \quad 1 \quad 0]^T \delta_k \cdot 0.1 \cdot [0 \quad 0 \quad 0 \quad 1 \quad 0]v + \bar{B}u$$

and get the model (Δ, M) where $\Delta = \delta_k$ and

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where

$$\begin{aligned}
A &= 0.1 \cdot [0 \quad 0 \quad 0 \quad 1 \quad 0](sI - \bar{A})^{-1}\bar{A}[0 \quad 0 \quad 0 \quad 1 \quad 0]^T \\
B &= 0.1 \cdot [0 \quad 0 \quad 0 \quad 1 \quad 0](sI - \bar{A})^{-1}\bar{B} \\
C &= \bar{C}(sI - \bar{A})^{-1}\bar{A}[0 \quad 0 \quad 0 \quad 1 \quad 0]^T \\
D &= \bar{C}(sI - \bar{A})^{-1}\bar{B}.
\end{aligned}$$

The uncertainty is in this model represented by a norm bounded real scalar δ_k. Such a scalar satisfies the quadratic constraint defined by

$$\Pi(i\omega) = \begin{bmatrix} x(\omega) & y(i\omega) \\ y(i\omega)^* & -x(\omega) \end{bmatrix}$$

where $x(\omega) \geq 0$ is real and $\operatorname{Re} y(i\omega) = 0$. Applying Theorem 2.1 using this multiplier and numerical optimization, see Section 2.8, we find the upper bound $2\sigma(\omega)$, shown in Figure 2.8 with a solid line. The true upper bound (the exact error can not be determined since the original model is uncertain), see Section 2.7, is shown with a dashed line. We see that the obtained upper bound is close to true upper bound. The relative error at different frequencies is shown in Figure 2.9. We see that the uncertainty description is important close to resonances and notches but is of very little importance at other frequencies.

2.5 Model comparison and simplification

In this section we describe model comparison and how it can be used for model simplification and model reduction.

Model comparison

Consider comparison of two models that are identical except for some of the blocks in Δ, see Figure 2.10. We assume, without loss of generality, that the upper blocks in Δ, denoted $\Delta_U = \operatorname{diag}(\Delta_1, \ldots, \Delta_{\hat{r}})$, are

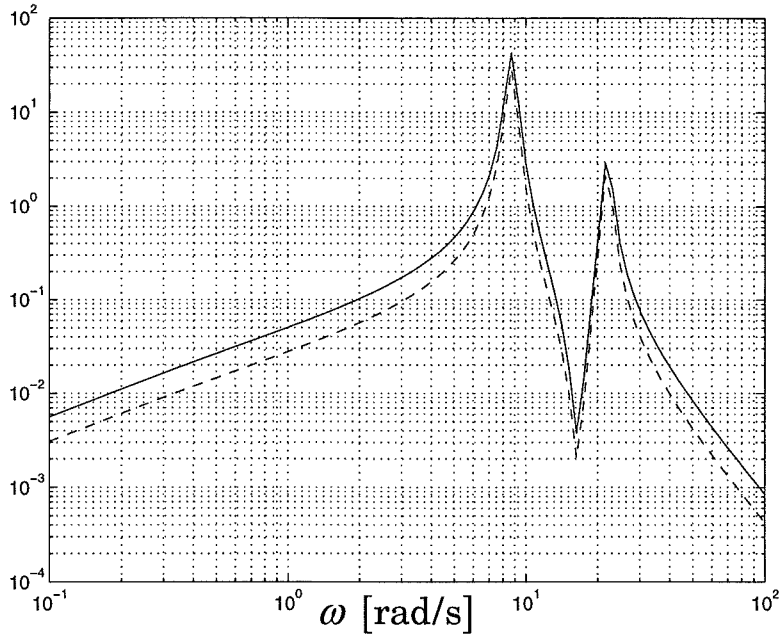

Figure 2.8 This figure shows upper bounds on the error when the uncertainty in the spring constant k_1 is neglected. The solid line shows the error bound obtained using our method while the dashed line shows the true upper bound.

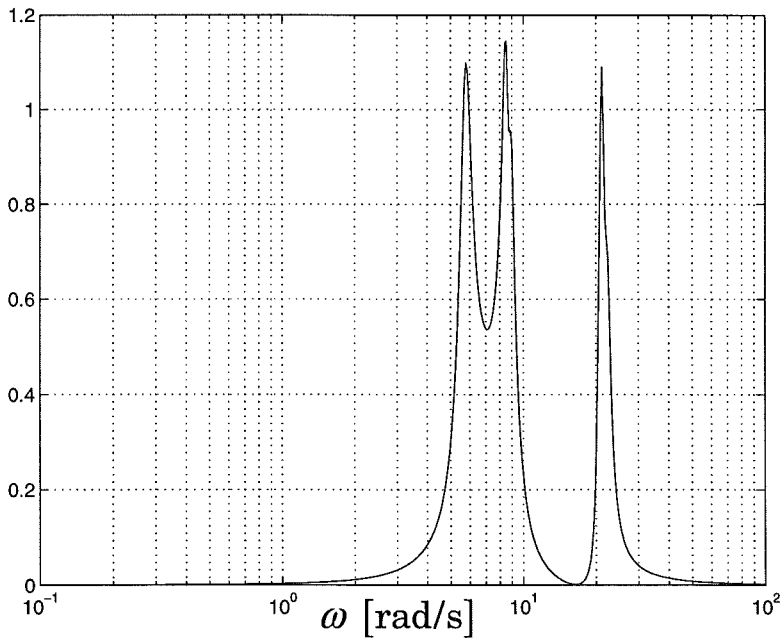

Figure 2.9 Upper bound on the relative error when the uncertainty in the spring constant k_1 is neglected. It is seen that the uncertainty description is important close to resonances and notches.

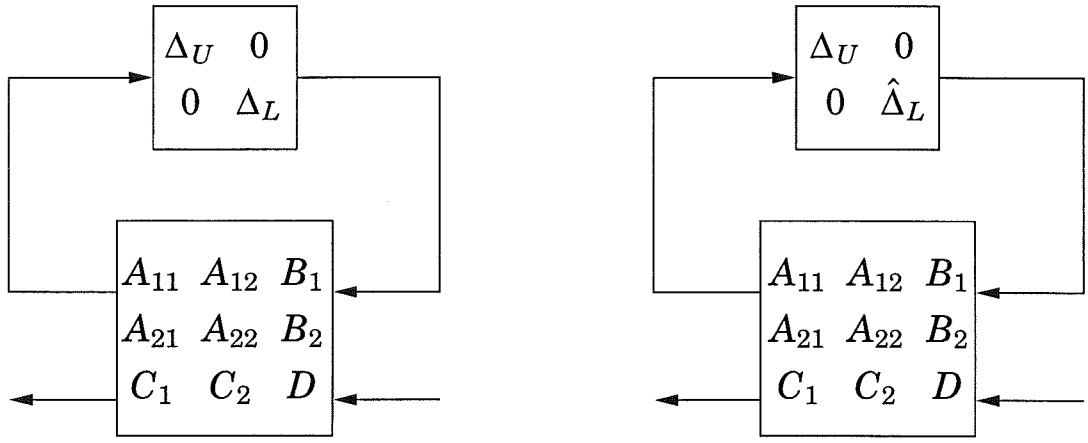

Figure 2.10 The two models that we compare.

identical while the lower blocks denoted $\Delta_L = \mathrm{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r)$ and $\hat{\Delta}_L = \mathrm{diag}(\hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r)$, respectively, are nonidentical. We partition the two models consistently as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \Delta_L \end{bmatrix}, \qquad \hat{\Delta} = \begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix}.$$

To compare the two models we assign positive real-valued functions $\sigma_k(\omega)$ to each of the blocks in Δ, as illustrated in Figure 2.11. Each function $\sigma_k(\omega)$ gives a measure on the importance of the corresponding block Δ_k. The error between the two models, is bounded by two times the sum of the σ_k-functions corresponding to the nonidentical blocks. Note, that the σ_k-functions will depend on the selection of a set, described using quadratic constraints, that both Δ_k and $\hat{\Delta}_k$ belong to. Figure 2.12 illustrates a case where Δ_k is an uncertain transfer matrix, with a value in the grey set, and $\hat{\Delta}_k$ is a known transfer matrix.

The results in this thesis do not take inter-dependence between the blocks in Δ into account. A priori applied coordinate transformations may therefore be valuable in order to obtain less conservative error bounds, but do at the same time change the structure of the model. This change of coordinates may be undesirable, for example if the elements in M have a physical interpretation.

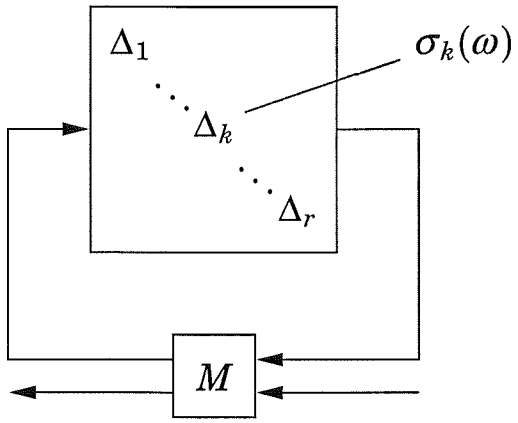

Figure 2.11 Each block Δ_k has an associated function $\sigma_k(\omega)$ that measure the importance of the block at each frequency.

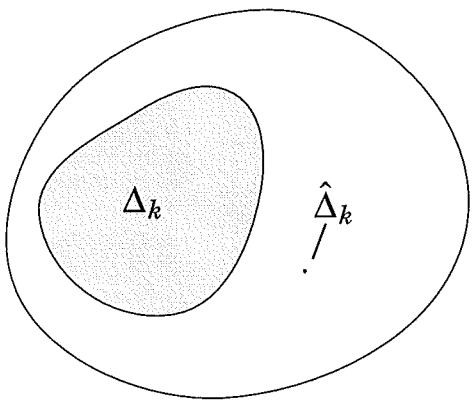

Figure 2.12 Both the block Δ_k and $\hat{\Delta}_k$ must satisfy the same quadratic constraint (which describes the large set). We illustrate the case where Δ_k is an uncertain transfer matrix with a value in the grey set and $\hat{\Delta}_k$ is a known transfer matrix.

Model simplification and reduction

The described comparison of models may be used for model simplification. The second model is in this case considered as a simplification of the first model. To choose which of the blocks in Δ to simplify, we may look at the σ_k-functions since they indicate for which of the blocks in Δ simplification is cheapest. The simplification is then done by replacing Δ_L with a fixed transfer matrix $\hat{\Delta}_L$, with the same block structure as Δ_L. We choose the replacing transfer matrix to be frequency inde-

pendent to avoid unnecessary dynamics, and choose it close to Δ_L to obtain a simplified model close to the original model; the last statement is motivated by the following result.

Assume that (Δ, M) is stable then for every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$\|\Delta \star M - \hat{\Delta} \star M\| < \varepsilon \qquad \text{when} \qquad \|\Delta - \hat{\Delta}\| < \delta.$$

This follows since

$$\Delta \star M - \hat{\Delta} \star M = C\Delta(I - A\Delta)^{-1}B - C\hat{\Delta}(I - A\hat{\Delta})^{-1}B$$

is a continuous function of the matrix elements, and no singularities occur due to the stability assumption.

After the described simplification, we find an upper bound on the error by summing the σ_k-functions corresponding to the replaced blocks.

The spatial dimension of the simplified model may be reduced. This follows by observing that

$$\begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix} \star M = \Delta_U \star \hat{M}, \tag{2.3}$$

where

$$\hat{M} = \begin{pmatrix} A_{11} & B_1 \\ C_1 & D \end{pmatrix} + \begin{pmatrix} A_{12} \\ C_2 \end{pmatrix} \hat{\Delta}_L (I - A_{22}\hat{\Delta}_L)^{-1} \begin{pmatrix} A_{21} & B_2 \end{pmatrix}.$$

This is illustrated in Figure 2.13. Note that it is crucial for this reduction that $\hat{\Delta}_L$ is a fixed transfer matrix, and not a set of transfer matrices, so that $\hat{M}$ is a fixed transfer matrix. In some cases it is also useful to have M frequency independent. To maintain this property after the reduction, we use a frequency independent $\hat{\Delta}_L$.

EXAMPLE 2.5—TRUNCATION
Choosing the fixed matrix $\hat{\Delta}_L = 0$ results in a reduced model with

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}.$$

This model is simply a truncation of the original model. This was the special case considered in Section 2.3. □

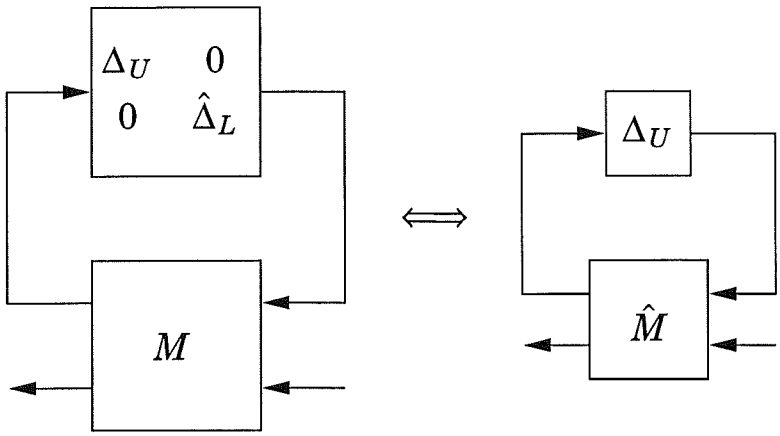

Figure 2.13 Model reduction. A fixed transfer matrix $\hat{\Delta}_L$ may be captured in M. The new model $(\Delta_U, \hat{M})$ will then have a lower, *i.e.* reduced, spatial dimension.

EXAMPLE 2.6—SINGULAR PERTURBATION APPROXIMATION
A state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

may be written on the standard form using

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and $\Delta = I_n/s$. Reduction of balanced state-space models using singular perturbation approximation, see *e.g.* Liu and Anderson (1989), is well-known. If we assume that (A, B, C, D) is a balanced realization, then the reduced order model obtained using singular perturbation approximation in continuous time is given by

$$\hat{M} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{bmatrix}.$$

This reduced order model is obtained by letting $\hat{\Delta}_L \to \infty \cdot I$ in the reduction algorithm above. □

Error bounds

We now present the frequency dependent error bounds associated with model comparison and simplification.

THEOREM 2.2
Assume that M is a stable proper (strictly proper) transfer matrix and that we have two stable strictly proper (proper) block diagonal transfer function matrices

$$\begin{aligned} \Delta &= \operatorname{diag}(\Delta_1, \ldots, \Delta_r), \\ \hat{\Delta} &= \operatorname{diag}(\Delta_1, \ldots, \Delta_{\hat{r}}, \hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r). \end{aligned}$$

Let $\Pi_k(i\omega)$, for $k = 1, \ldots, r$, be bounded measurable functions taking hermitian values. Assume for all $\omega \in \mathbf{R}$ and all $\tau \in [0, 1]$ that

- $\tau\Delta_k(i\omega)$, for $k = 1, \ldots, r$, satisfies the quadratic constraint defined by $\Pi_k(i\omega)$
- $\tau\hat{\Delta}_k(i\omega)$, for $k = \hat{r} + 1, \ldots, r$, satisfies the quadratic constraint defined by $\Pi_k(i\omega)$

If there exist real-valued functions $\sigma_1(\omega), \ldots, \sigma_r(\omega) > 0$ such that

$$\begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix}^* \operatorname{daug}(\Pi_1(i\omega), \ldots, \Pi_r(i\omega)) \begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tag{2.4}$$

$$\begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* \operatorname{daug}(\sigma_1^2\Pi_1(i\omega), \ldots, \sigma_r^2\Pi_r(i\omega)) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix} + C(i\omega)^* C(i\omega) < 0 \tag{2.5}$$

for all $\omega \in [0, \infty]$. Then both (Δ, M) and $(\hat{\Delta}, \hat{M})$ are stable and

$$\|(\Delta \star M - \hat{\Delta} \star M)(i\omega)\| \leq 2\sigma_{\hat{r}+1}(\omega) + \cdots + 2\sigma_r(\omega).$$

Let $(\Delta_U, \hat{M})$ be the reduced order model obtained using the reduction formula (2.3). Then

$$\|(\Delta \star M - \Delta_U \star \hat{M})(i\omega)\| \leq 2\sigma_{\hat{r}+1}(\omega) + \cdots + 2\sigma_r(\omega).$$

Remark 1 The interpretation of this theorem is that, if the transfer matrices $\Delta_k, k = \hat{r} + 1, \ldots, r$ are replaced by different transfer matrices, usually constant matrices, satisfying the same constraints, then the error is bounded by positive frequency dependent functions corresponding to the replaced transfer matrices. The reduction algorithm give an equivalent model with lower spatial dimension.

Remark 2 Model truncation corresponds to $\hat{\Delta}_k = 0$. The only constraint on the multiplier Π_k obtained from this replacing transfer matrix is $\Pi_{k(11)} \geq 0$. This condition is found in Theorem 2.1.

Remark 3 We describe Δ using quadratic constraints. It is therefore not necessary to know the specific transfer matrix Δ. This makes the result applicable to models where Δ is uncertain. One must then find a quadratic constraint that is satisfied for all Δ in the uncertainty set.

Remark 4 The assumptions on $\Delta, \hat{\Delta}$ and M to be stable and proper may be replaced with the assumption that both the original and truncated models are stable. Then it is also sufficient that $\Delta_k, \hat{\Delta}_k \in \mathrm{IQC}(\Pi_k)$ for $\tau = 1$ and not for all $\tau \in [0, 1]$. This version of the assumptions is useful when Δ is stable, for example when Δ contains integrators.

Proof We start by proving stability. This proof is only given for the model (Δ, M) since the proof for $(\hat{\Delta}, M)$ is identical.

We will consider the model where Δ is replaced with $\tau\Delta$ and show that this model is stable for $\tau = 1$. Stability means that all the poles are in the open left half plain. This holds for $\tau = 0$ since both Δ and M are stable. If no pole crosses the imaginary axis as we let τ increase from zero to one, then the model (Δ, M) will be stable. It therefore remains to check that there are no imaginary poles for any value on $\tau \in [0, 1]$, *i.e.*

$$\det(I - A(i\omega)\tau\Delta(i\omega)) \neq 0, \quad \omega \in [0, \infty], \quad \tau \in [0, 1].$$

This holds if

$$\begin{bmatrix} I & A(i\omega) \\ \tau\Delta(i\omega) & I \end{bmatrix}$$

is invertible, *i.e.* the columns are linearly independent. If we assume that they are linearly dependent then

$$\begin{bmatrix} A \\ I \end{bmatrix} w_1 = \begin{bmatrix} I \\ \tau\Delta \end{bmatrix} w_2, \qquad w_1 \neq 0, w_2 \neq 0.$$

This implies that

$$w_1^* \begin{bmatrix} A \\ I \end{bmatrix}^* \Pi \begin{bmatrix} A \\ I \end{bmatrix} w_1 = w_2^* \begin{bmatrix} I \\ \tau\Delta \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \tau\Delta \end{bmatrix} w_2$$

which is a contradiction since

$$\begin{bmatrix} A \\ I \end{bmatrix}^* \Pi \begin{bmatrix} A \\ I \end{bmatrix} < 0$$

and

$$\begin{bmatrix} I \\ \tau\Delta \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \tau\Delta \end{bmatrix} \geq 0.$$

This proves stability.

We now prove the error bounds for the case with $\hat{r} = 1$ and $r = 2$. The case with $\hat{r} = r = 1$ is almost identical, but less pedagogical. We partition the matrix M as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}.$$

Since we have two models (Δ, M) and $(\hat{\Delta}, M)$ we also have two feedback interconnections and also two sets of signals. We introduce the following notation for the laplace transformed signals:

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, \qquad \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{y} \end{bmatrix} = M \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ u \end{bmatrix}.$$

We assume that the initial conditions are zero. This is realistic since the two models are stable. Note that the input signal u is the same for both models.

Multiplying the inequality (2.4) with the vector $(x_1 + \hat{x}_1, x_2 + \hat{x}_2, 2u)$ from the right and its complex conjugate from the left, and adding this to the inequality obtained by multiplying the inequality (2.5) with the column vector $\sigma_2^{-1}(x_1 - \hat{x}_1, x_2 - \hat{x}_2)$ from the right and its complex conjugate from the left, we then get by noting that $z - \hat{z} = A(x - \hat{x})$, $y - \hat{y} = C(x - \hat{x})$ and using some matrix manipulations, the following inequality

$$\begin{aligned} 0 \quad > \quad & \begin{bmatrix} z_1 + \hat{z}_1 \\ x_1 + \hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1 + \hat{z}_1 \\ x_1 + \hat{x}_1 \end{bmatrix} \\ & + \sigma_1^2 \sigma_2^{-2} \begin{bmatrix} z_1 - \hat{z}_1 \\ x_1 - \hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1 - \hat{z}_1 \\ x_1 - \hat{x}_1 \end{bmatrix} \\ & + 2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix} \\ & + \sigma_2^{-2} |y - \hat{y}|^2 - 4|u|^2. \end{aligned}$$

The first and second term are nonnegative since $\Delta_1 = \hat{\Delta}_1$ satisfies the quadratic constraint defined by Π_1 and

$$x_1 + \hat{x}_1 = \Delta_1(z_1 + \hat{z}_1).$$

The third and fourth term are nonnegative since Δ_2 and $\hat{\Delta}_2$ both satisfy the quadratic constraint defined by Π_2. This completes the proof for the case $\hat{r} + 1 = r = 2$.

More generally one can generate a sequence of new models by replacing $\Delta_r, \ldots, \Delta_{\hat{r}+1}$ one at a time. At each step, the above argument can be used and the total error bound becomes

$$|(y - \hat{y})(i\omega)| \leq 2(\sigma_r(\omega) + \cdots + \sigma_{\hat{r}+1}(\omega))|u(i\omega)|.$$

□

Some optimality results

The error bounds obtained from Theorem 2.2 are in some cases optimal, *i.e.* equal to the true upper bound. The true upper bound is obtained by taking the worst possible error at each frequency. This means that there for each frequency exist values on $\Delta(i\omega)$ and $\hat{\Delta}(i\omega)$ such that the error is equal to the true upper bound.

The following theorems are given without proofs (standard but tedious calculations).

THEOREM 2.3
Let Δ and $\hat{\Delta}$ be non-negative constant real scalars, then they satisfy any quadratic constraint defined by

$$\Pi(i\omega) = \begin{bmatrix} 0 & y(i\omega) \\ y(i\omega)^* & 0 \end{bmatrix},$$

where $\operatorname{Re} y \geq 0$. The inequalities (2.4) and (2.5) become equivalent to

$$\sigma(\omega) > \begin{cases} \dfrac{|C(i\omega)| \cdot |B(i\omega)|}{2|A(i\omega)|}, & \operatorname{Re} A(i\omega) \leq 0 \\ \dfrac{|C(i\omega)| \cdot |B(i\omega)|}{2|\operatorname{Im} A(i\omega)|}, & \operatorname{Re} A(i\omega) > 0 \end{cases}$$

using the freedom in these multipliers.

Assume that the assumptions of Theorem 2.2 are satisfied and let $\sigma_{\text{opt}}(\omega) = \inf \sigma(\omega)$, then the true error bound is given by $2\sigma_{\text{opt}}(\omega)$. □

THEOREM 2.4
Let Δ and $\hat{\Delta}$ be constant real scalars with $|\Delta| \leq 1$ and $|\hat{\Delta}| \leq 1$, then they satisfy any quadratic constraint defined by

$$\Pi(i\omega) = \begin{bmatrix} x(\omega) & y(i\omega) \\ y(i\omega)^* & -x(\omega) \end{bmatrix},$$

where $x(\omega) \geq 0$ is real and $\operatorname{Re} y(i\omega) = 0$. The inequalities (2.4) and (2.5) become equivalent to

$$\sigma(\omega) > \begin{cases} \dfrac{|C(i\omega)| \cdot |B(i\omega)|}{\sqrt{(1 - |A(i\omega)|^2)^2 + 4(\operatorname{Im} A(i\omega))^2}}, & |A(i\omega)| \leq 1 \\ \dfrac{|C(i\omega)| \cdot |B(i\omega)|}{2|\operatorname{Im} A(i\omega)|}, & |A(i\omega)| > 1 \end{cases}$$

using the freedom in these multipliers.

Assume that the assumptions of Theorem 2.2 are satisfied and let $\sigma_{\text{opt}}(\omega) = \inf \sigma(\omega)$, then the true error bound is given by $2\sigma_{\text{opt}}(\omega)$. □

2.6 Numerical example

In this section we give an example where we illustrate both the use of other reduction methods than truncation and the importance of not restricting the set of multipliers more than necessary.

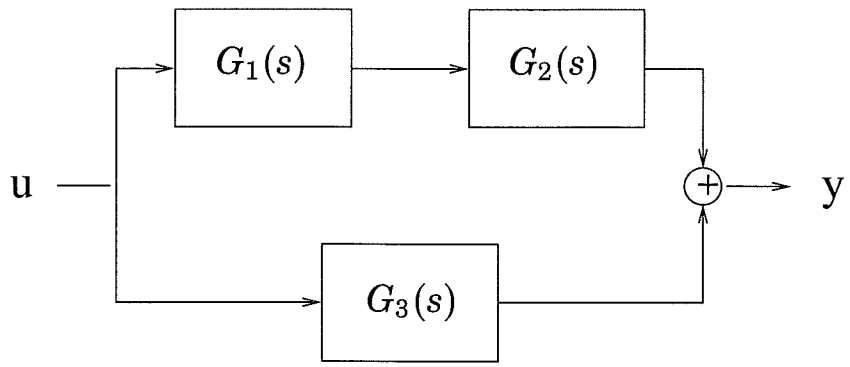

Figure 2.14 The model in the example.

Consider the model in Figure 2.14 where the transfer functions are

$$G_k(s) = \frac{1}{s\tau_k + 1}, \quad k = 1, 2, 3.$$

The time constants are $\tau_1 = 1$, $\tau_2 = 0.1$ and $\tau_3 = 0.01$. We assume that we are interested in knowing how large the error would be if we neglect the dynamics for some of the transfer functions, *i.e.* assumes that $G_k(i\omega) \approx 1$. This problem can not be studied using the results in Section 2.3 where only truncation corresponding to the approximation $G_k(i\omega) \approx 0$ is studied.

To be able to apply the results in this thesis we rewrite the model on the form (Δ, M). We let Δ contain the transfer functions considered for simplification and M the remaining part of the model. For example if we consider simplification of G_2 and G_3 then

$$\Delta = \text{diag}(\delta_2, \delta_3) = \text{diag}(G_2(s), G_3(s))$$

and

$$M(s) = \begin{bmatrix} 0 & 0 & G_1(s) \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

To simplify the dynamics corresponding to G_k we use $\hat{\delta}_k = \hat{G}_k = 1$ in the reduction formula (2.3).

We will now consider two cases. In the first case we describe the transfer functions as known transfer functions and in the second case we describe them as unity norm-bounded transfer functions.

In the first case, the multiplier Π_k describing the transfer function G_k, should satisfy both

$$\begin{bmatrix} 1 \\ G_k(i\omega) \end{bmatrix}^* \Pi_k(i\omega) \begin{bmatrix} 1 \\ G_k(i\omega) \end{bmatrix} \geq 0$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^* \Pi_k(i\omega) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq 0.$$

The first constraint corresponds to the original transfer function and the second to the simplified transfer function. These constraints on the multipliers are necessary and sufficient.

We now consider simplification of one of the transfer functions at a time. Using numerical calculations, see Section 2.8, we find the σ-functions in Figure 2.15. The corresponding error bounds $2\sigma_k$ are in these cases equal to the true error. The true error has in this simple case been calculated as the difference between the transfer functions corresponding to the original and simplified simplified models.

It is easy to understand why the error curves have the form shown in Figure 2.15. The approximation $G_3(s) \approx 1$ is accurate at low frequencies where $G_3(s) \approx 1$ but not at higher frequencies where $G_3(s) \approx 0$. The same idea can be used to understand why the curves for $\sigma_1(\omega)$ and $\sigma_2(\omega)$ are small at low frequencies. At high frequencies, however, the small values are explained by the low-pass character of the transfer functions in the series connection.

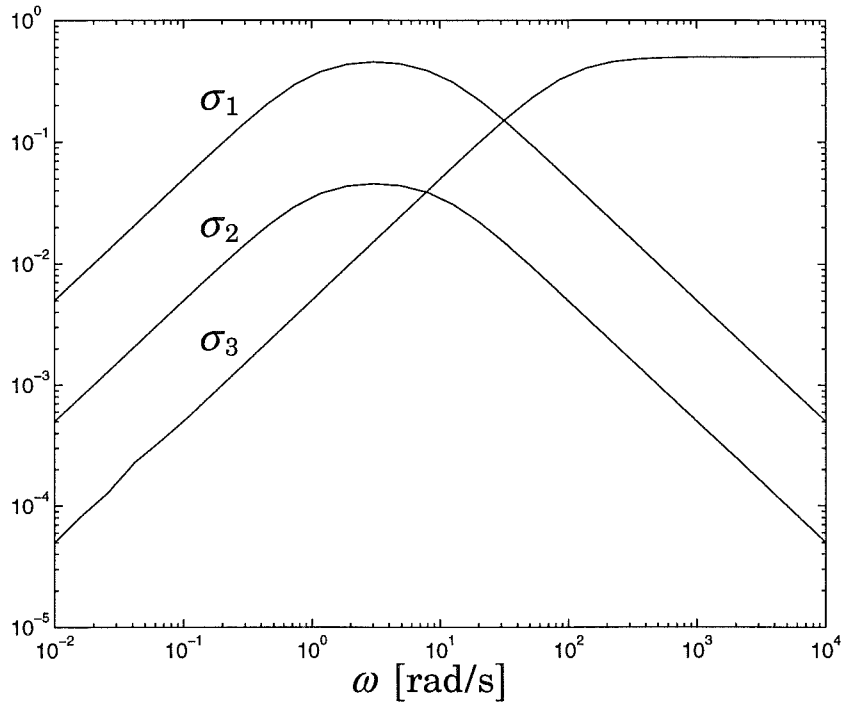

Figure 2.15 The σ_k-functions corresponding to the dynamics in $G_k(s)$. If the dynamics in $G_k(s)$ is neglected then the approximation error will be bounded by $2\sigma_k(\omega)$.

In the second case we describe the transfer function G_k using the multiplier

$$\Pi_k(i\omega) = \begin{bmatrix} x(\omega) & 0 \\ 0 & -x(\omega) \end{bmatrix}$$

where $x(\omega) \geq 0$ is a real valued function. This multiplier defines quadratic constraints satisfied by any unity norm-bounded transfer function and is, thus, a conservative description of our transfer function. Note that the simplified transfer function $\hat{\delta}_k = \hat{G}_k = 1$, also satisfies the quadratic constraint defined by this multiplier.

Using this multiplier when considering simplification of one block at the time we obtain the σ-functions in Figure 2.16. Comparing with Figure 2.15 we see that the σ-functions in Figure 2.16 give more conservative error bounds. This shows that it is important to allow as much freedom in the multiplier Π as possible, in order to obtain good error bounds.

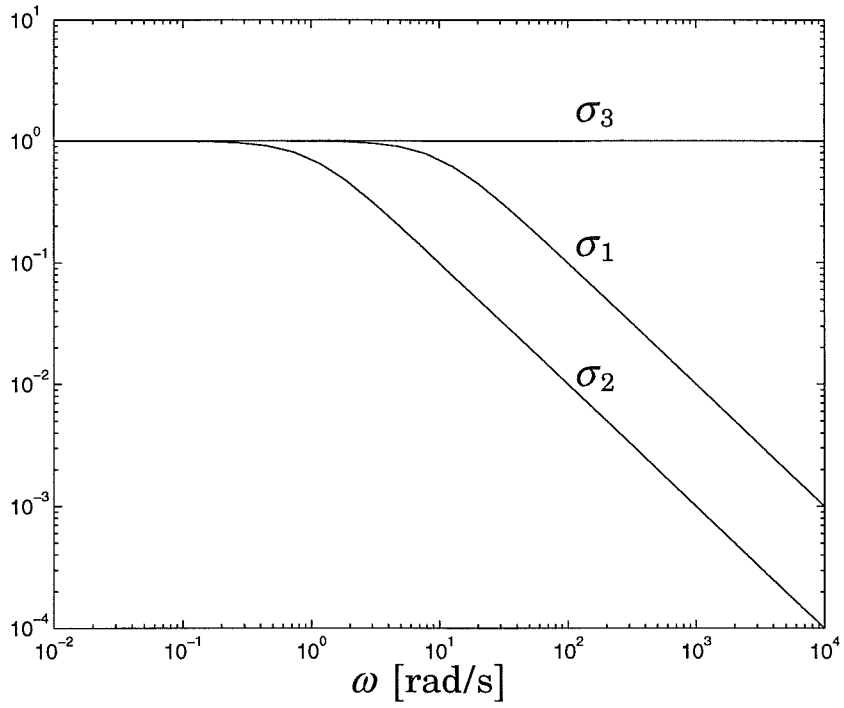

Figure 2.16 The σ_k-functions corresponding to the dynamics in $G_k(s)$, when a more conservative description is used.

We now return to the more accurate description and let Δ contain both the transfer functions G_2 and G_3 at the same time. Using the two step procedure, described in Section 2.8 we find the σ-functions in Figure 2.17. These σ-functions have higher values and thus result in more conservative error bounds when we consider simplification of one transfer function at the time, than the previously obtained. One possible reason for getting higher values is that the suboptimal two-step algorithm not necessarily give us the the best possible solution to the inequalities in Theorem 2.2, as it does in the one-block case. An other reason is that the σ_k values are expected to increase with the number of blocks in Δ when there is an interdependence between the blocks. The reason for this is as follows. If we simplify one block then the error will be bounded by $2\sigma_k$, this must hold independent of how many blocks there are in Δ. We therefore do not expect σ_k to decrease with the number of blocks in Δ. On the other hand if we increase the number of blocks in Δ then the function σ_k should be used to calculate more upper bounds. The σ_k-values may then have to be increased. The following example illustrates the idea.

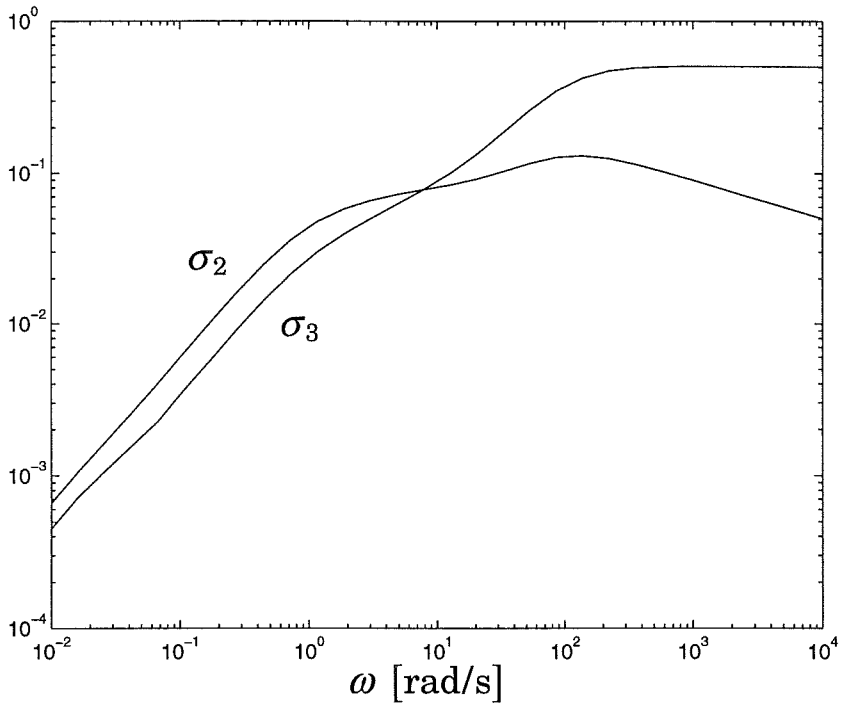

Figure 2.17 The σ-functions corresponding to the dynamics in $G_2(s)$ and $G_3(s)$, respectively. If the dynamics in both these transfer functions are neglected then the approximation error will be bounded by $2(\sigma_2(\omega) + \sigma_3(\omega))$.

EXAMPLE 2.7
Consider the function

$$f(\delta_1, \delta_2, \delta_3) = (\delta_1 - \delta_2)(\delta_1 - \delta_3),$$

defined for $|\delta_k| \leq 1$. Think of each δ_k as a transfer function and the function f as the input-output relation of a model. Assume that the nominal model is given by $G(\delta) = f(\delta, \delta, \delta) = 0$ and consider the following three simplified models,

$$\begin{aligned}
\hat{G}_2(\delta) &= f(\delta, \hat{\delta}_2, \delta) = 0, \\
\hat{G}_3(\delta) &= f(\delta, \delta, \hat{\delta}_3) = 0, \\
\hat{G}_{23}(\delta) &= f(\delta, \hat{\delta}_2, \hat{\delta}_3) = (\delta - \hat{\delta}_2)(\delta - \hat{\delta}_3).
\end{aligned}$$

We see that replacing only one of the transfer functions δ_2 and δ_3 does not result in any input-output error. If we on the other hand replace

both δ_2 and δ_3 then the error satisfies

$$|G - \hat{G}_{23}| = |\hat{G}_{23}| \leq 4,$$

where the bound is reached for $\hat{\delta}_2 = \hat{\delta}_3 = -\delta = \pm 1$.

This shows that the error when we replace more than one component of a model may be larger than the sum of the errors for replacing one component. This indicates that is may be necessary to increase the value on σ_k when the number of blocks considered for replacement is increased. □

Finally we let Δ contain all the transfer functions at the same time. The suboptimal σ-functions are in this case shown in Figure 2.18. We see that the σ-functions in this case have even higher values.

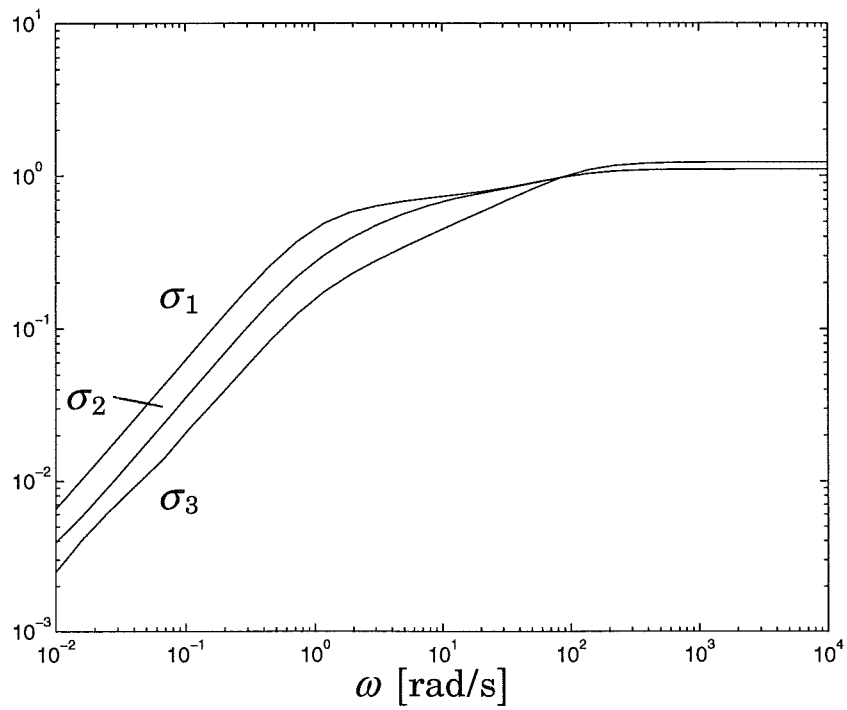

Figure 2.18 The σ-functions corresponding to the dynamics in $G_1(s)$, $G_2(s)$ and $G_3(s)$, respectively. If the dynamics in all these transfer functions are neglected then the approximation error will be bounded by $2(\sigma_1(\omega) + \sigma_2(\omega) + \sigma_3(\omega))$.

The conclusions from this section are that it is important to describe the transfer function Δ as accurately as possible, *i.e.* use as much freedom as possible when the multipliers Π_k are chosen. Also one should

only include in Δ the dynamics that are intended to be simplified. If these recommendations are violated the error bounds may be more conservative than necessary. On the other hand, if these recommendations are followed the method gives good error bounds for this example.

2.7 Application example; part II

In this section we continue with the flexible servo example introduced in Section 2.4. Neglecting the moment of inertia naturally leads to the use of an other reduction method than truncation.

Uncertain moment of inertia

In this second part of the example we consider the moment of inertia for the second mass as unknown, and denote it by $\bar{J}_2$. We assume that $\bar{J}_2 \leq J_2$ which we model as

$$\bar{J}_2 = \frac{J_2}{1 + \delta_J},$$

where $\delta_J \geq 0$. We are now interested in analyzing whether the moment of inertia, $\bar{J}_2$, may be neglected or not.

Including the uncertainty description in the model we find that the state equation becomes

$$\dot{v} = \bar{A}v + [0 \quad 1 \quad 0 \quad 0 \quad 0]^T \delta_J [0 \quad 1 \quad 0 \quad 0 \quad 0]\bar{A}v + \bar{B}u.$$

This model may be written on the form (Δ, M) used in this paper by letting $\Delta = \delta_J$ and

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$\begin{aligned}
A &= [0 \quad 1 \quad 0 \quad 0 \quad 0]\bar{A}(sI - \bar{A})^{-1}[0 \quad 1 \quad 0 \quad 0 \quad 0]^T \\
B &= [0 \quad 1 \quad 0 \quad 0 \quad 0]\bar{A}(sI - \bar{A})^{-1}\bar{B} \\
C &= \bar{C}(sI - \bar{A})^{-1}[0 \quad 1 \quad 0 \quad 0 \quad 0]^T \\
D &= \bar{C}(sI - \bar{A})^{-1}\bar{B}.
\end{aligned}$$

The uncertainty is in this model represented by the real uncertain scalar $\delta_J \geq 0$. Such a scalar satisfies the quadratic constraint defined by

$$\Pi(i\omega) = \begin{bmatrix} 0 & y(i\omega) \\ y(i\omega)^* & 0 \end{bmatrix}$$

where $\operatorname{Re} y \geq 0$. Using this multiplier in the inequalities (2.4) and (2.5) give after numerical optimization, see Section 2.8, the error bound $2\sigma(\omega)$, shown in Figure 2.19. It can be shown that this upper bound is the true upper bound, see the end of this section. The relative

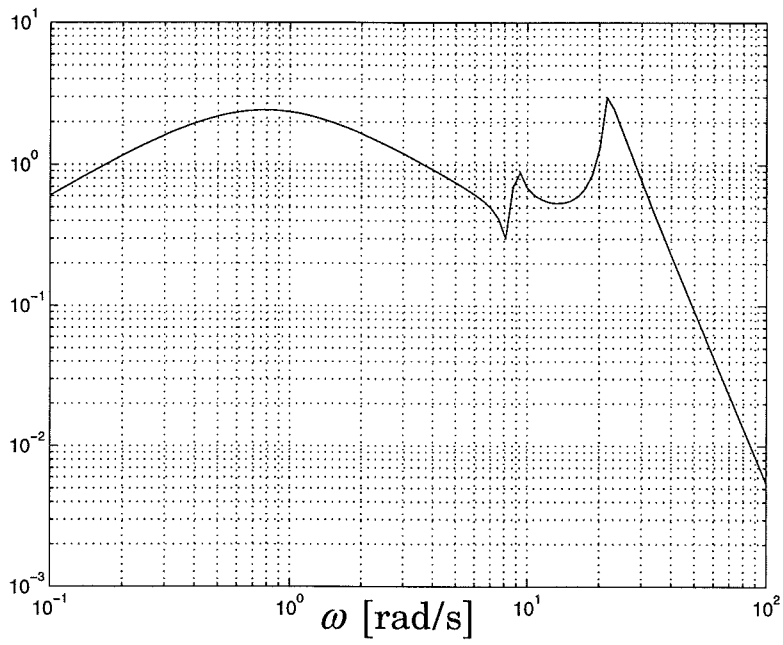

Figure 2.19 The solid line shows the upper bound on the error when the moment of inertia for the middle mass is neglected.

error at different frequencies is shown in Figure 2.20. We see that the uncertainty description is important close to resonances and notches.

To obtain the model where the moment of inertia for the middle mass is neglected we use $\hat{\delta}_J \to \infty$ in the reduction formula (2.3). This

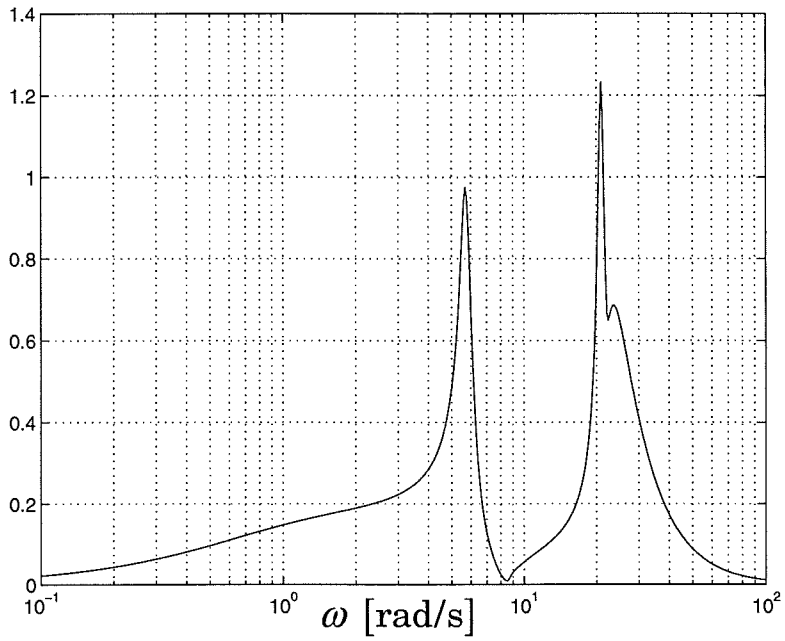

Figure 2.20 Upper bound on the relative error when the uncertainty in the moment of inertia J_2 is neglected. It is seen that the uncertainty description is important close to resonances and notches.

corresponds to $\bar{J}_2 = 0$. We then get the simplified model

$$\begin{aligned}
J_1\dot{\omega}_1 &= -k_1(\phi_1 - \phi_2) - d_1\omega_1 + k_u u \\
J_3\dot{\omega}_3 &= k_2(\phi_2 - \phi_3) - d_3\omega_2 \\
\dot{\phi}_1 &= \omega_1 \\
\dot{\phi}_2 &= \omega_2 = \tfrac{k_1}{d_2}(\phi_1 - \phi_2) - \tfrac{k_2}{d_2}(\phi_2 - \phi_3) \\
\dot{\phi}_3 &= \omega_3 \\
y &= k_\omega \omega_1.
\end{aligned}$$

This model can also be obtained in a more direct way. We then use $\bar{J}_2 = 0$ to eliminate the state equation for $\dot{\omega}_2$. This results in the following constraint

$$k_1(\phi_1 - \phi_2) - k_2(\phi_2 - \phi_3) - d_2\omega_2 = 0$$

which we can solve for ω_2 and then substitute into the remaining dynamic equations.

Note that the simplified model was obtained by letting $\bar{J}_2 = 0$ while d_2 remained unchanged. From a physical point of view this means that the inertia for the middle rotating mass has been neglected but not the damping.

Exact error bounds

The error bound obtained in the two cases studied for the flexible servo has been obtained numerically in the way described in Section 2.8. However, in these simple cases it is also possible to obtain analytical expressions for the best error bounds obtainable using Theorem 2.2.

The uncertainty in the spring constant δ_k is a constant real scalar satisfying $|\delta_k| \leq 1$. We then find an analytic expression for the error bound using Theorem 2.4. This upper bound is the same as the one found by numerical optimization and shown in Figure 2.8. This upper bound is optimal if we consider reduction using any $|\hat{\delta}_k| \leq 1$. In the studied example we only considered truncation ($\hat{\delta}_k = 0$, then the optimal error bound can be shown to be

$$\begin{cases} \dfrac{|C(i\omega)| \cdot |B(i\omega)|}{\sqrt{(1 - |\operatorname{Re} A(i\omega)|^2)^2 + (\operatorname{Im} A(i\omega))^2}}, & |\operatorname{Re} A(i\omega)| \leq 1 \\ \dfrac{|C(i\omega)| \cdot |B(i\omega)|}{|\operatorname{Im} A(i\omega)|}, & |\operatorname{Re} A(i\omega)| > 1. \end{cases}$$

which is the dashed line in Figure 2.8.

The uncertainty in the moment of inertia δ_J is a constant real scalar satisfying $\delta_J \geq 0$. We then find an analytic expression for the error bound using Theorem 2.3. This upper bound is the same as the one found by numerical optimization and shown in Figure 2.19. The theorem also state that this upper bound is optimal if we consider reduction using any $\hat{\delta}_J \geq 0$. In the considered example we only use $\hat{\delta}_J \to \infty$, but it can be shown that the worst error bound is found for $\hat{\delta}_J \to \infty$ (and $\hat{\delta}_J = 0$) so that the error bound is equal to the true error bound, in this case.

2.8 Numerical computation

In this section we describe how the error bounds, or more precisely the $\sigma_k(\omega)$-functions, can be obtained by numerical computation.

We simplify the notation by collecting the $\sigma_k(\omega)$-functions functions in the real matrix-valued function

$$\Sigma(\omega) = \operatorname{diag}(\sigma_1(\omega)I_{n_1}, \ldots, \sigma_r(\omega)I_{n_r}) > 0.$$

The error bound inequalities can then be written as

$$\begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix}^* \operatorname{daug}(\Pi_1(i\omega), \ldots, \Pi_r(i\omega)) \begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* \Sigma^2(\omega) \operatorname{daug}(\Pi_1(i\omega), \ldots, \Pi_r(i\omega)) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix} + C(i\omega)^* C(i\omega) < 0.$$

The problem is to find $\Sigma(\omega)$ that solves these inequalities, and where in addition $\Sigma(\omega)$ is as small as possible. We have to distinguish between the following two cases:

- Π given
- Π constrained by linear matrix inequalities

The error bound inequalities are in the first case linear matrix inequalities in $\Sigma^2(\omega)$. An optimal $\Sigma(\omega)$ can then be found numerically, for one frequency at the time, using for example the LMI control toolbox, Gahinet *et al.* (1995). In the second case, where we would like to optimize over both Π and Σ, the inequalities usually become nonconvex. The resulting optimization problem is in general difficult. We therefore propose the following suboptimal two step algorithm.

Step 1 Find $\Pi(i\omega)$ that minimizes $\gamma(\omega)$ under the constraints

$$\gamma^2(\omega) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* \Sigma_0^2 \Pi(i\omega) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* + C^*(i\omega) C(i\omega) < 0,$$

$$\begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix}^* \operatorname{daug}(\Pi_1(i\omega), \ldots, \Pi_r(i\omega)) \begin{bmatrix} A(i\omega) & B(i\omega) \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

where Π in addition must satisfy the multiplier constraints used to describe Δ. The matrix Σ_0 is a guess for the optimal value of Σ. If no information is available we choose $\Sigma_0 = I$.

Step 2 Find $\Sigma(\omega) = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ that minimizes $\operatorname{tr} W\Sigma^2$ under the constraint

$$\begin{bmatrix} A(i\omega) \\ I \end{bmatrix}^* \Sigma^2(\omega)\Pi(i\omega) \begin{bmatrix} A(i\omega) \\ I \end{bmatrix} + C^*(i\omega)C(i\omega) < 0,$$

where Π is the solution in Step 1. The matrix $W = \operatorname{diag}(\alpha_1, \ldots, \alpha_r)$ is a weight on the relative importance of different elements in Σ. If they are equally important we choose $W = I$.

Remark 1 The initial guess Σ_0 and the weighting matrix W may be frequency dependent.

Remark 2 When there is only one block in Δ then there is also only one σ value to be calculated. In this case this algorithm gives the best possible value on σ. This does, however, not necessarily mean that we find the optimal error bound.

Remark 3 The two-step algorithm may be used for iteration; then Σ obtained in Step 2 replaces Σ_0 in Step 1 when a new iteration begins.

2.9 Conclusions

In this chapter we have found upper bounds on the error between two uncertain linear time-invariant models. The results can be used to analyze the importance of different components of a model and for model reduction. A class of model reduction methods has been introduced.

The obtained error bounds are frequency dependent which is an important feature in many applications such as controller design where different frequencies have different importance.

3

Comparison and simplification using IQCs

Abstract

In this chapter, which mainly consist of the paper Andersson *et al.* (1996), we consider simplification of dynamical models, using a general reduction algorithm. The model may contain nonlinearities as well as uncertainty. The uncertainty and nonlinearities are described using integral quadratic constraints (IQCs). The proposed reduction algorithm includes truncation and singular perturbation approximation as special cases. The reduction error is defined in terms of the L_2-induced gain. It is shown that each component can be assigned a positive value, computable by convex optimization, such that the reduction error is always bounded by the sum of these values for the simplified components.

3.1 Introduction

In modern robust control design it is common to model both the system dynamics and uncertainty. This often results in models that have high state order and complicated uncertainty descriptions. These models may be difficult to analyze and the subsequent controller design, based on these models, may be both difficult and time consuming. The resulting controller usually also become complex and may therefore be expensive and difficult to implement. For these reasons there is a need

to develop methods to simplify the descriptions of the nonlinearity and uncertainty in the model as well as the state order.

For standard linear time-invariant models there exist well-known state-order reduction methods and associated error-bounds. Two of these reduction methods are balanced truncation, see Moore (1981), Glover (1984), Enns (1984), and singular perturbation approximation, see Fernando and Nicholson (1982b), Liu and Anderson (1989), Fernando and Nicholson (1982a). The balanced truncation method has been generalized to models with norm-bounded uncertainty, see Wang *et al.* (1991), Beck *et al.* (1996), Beck (1996).

In this chapter we show that similar error bounds can be obtained for a general class of dynamical models and a general reduction algorithm. The proposed reduction algorithm includes truncation and singular perturbation approximation as special cases. The models may contain both nonlinearities and uncertainty, and are described using integral quadratic constraints. This latter framework was outlined in Megretski (1993) and further developed in Rantzer and Megretski (1994), Megretski and Rantzer (1995), and leads to noticeably simpler proofs.

The chapter is organized as follows. We start, in Section 3.2, by describing the modeling framework and stating the comparison problem. The corresponding error bounds are given and proved in Section 3.3. In Section 3.4 we apply the method to models with norm-bounded uncertainty. Section 3.6 contains some concluding remarks.

3.2 Preliminaries

In this section we first describe the notation and the modeling framework. The modeling framework takes the uncertainty structure into account as proposed independently in Safonov (1982) and Doyle (1982). This is done using integral quadratic constraints as outlined in Megretski (1993) and further developed in Rantzer and Megretski (1994) and Megretski and Rantzer (1995). We conclude the section with a statement of the problem.

Notation

We use $\mathbf{R}$ to denote the real numbers. We use subscripts to denote submatrixes, and parentheses in the subscripts to denote submatrices of submatrices, *e.g.* $A_{1(12)}$. The complex conjugate of a matrix A is denoted by A^* and the transpose by A^T.

We let $\mathbf{L}_2^p[0,\infty)$ denote the space of square integrable vector signals of dimension p, with corresponding inner product defined as

$$\langle v, w\rangle = \int_0^\infty v(t)^T w(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty v(i\omega)^* w(i\omega)d\omega,$$

where the last equality is known as the Parseval formula. Note that the same notation is used for time and frequency signals. The signal norm is defined as $||v|| = \sqrt{\langle v, v\rangle}$.

An operator on $\mathbf{L}_2^n[0,\infty)$ is a map $\Delta : \mathbf{L}_2^n[0,\infty) \to \mathbf{L}_2^n[0,\infty)$; the operator gain is given by the induced norm

$$||\Delta|| = \sup_{\substack{z\in\mathbf{L}_2^n[0,\infty)\\ z\neq 0}} \frac{||\Delta z||}{||z||}.$$

When Δ is linear, we use Δ^* to denote the adjoint operator of Δ defined by

$$\langle \Delta z_1, z_2\rangle = \langle z_1, \Delta^* z_2\rangle.$$

We let $\mathbf{RH}_\infty^{p\times m}$ denote the set of proper rational matrix functions with real coefficients and without poles in the closed right half plane.

Although most results are given in continuous time they also hold for discrete time using the signal space $l_2^p[0,\infty)$ with inner product

$$\langle v, w\rangle = \sum_{k=0}^\infty v_k^T w_k.$$

Model description

The modeling framework considered in this chapter is defined by the interconnection of a pair (Δ, M) according to the relations

$$x = \Delta z$$

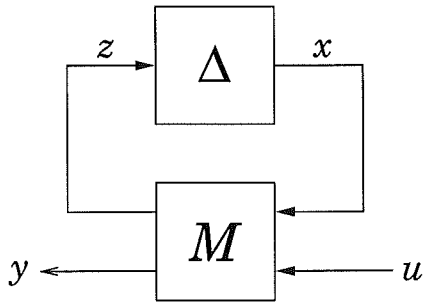

Figure 3.1 Feedback interconnection representing a model with nonlinearities and uncertainty.

and

$$\begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix},$$

as illustrated in Figure 3.1.

We let Δ represents nonlinearities, uncertainty, and dynamic elements, and M represents linear time-invariant dynamics.

Typically, M is the nominal system model, which is often assumed to consist of a transfer matrix including model weighting functions. However, as will be seen in Section 3.4, it is often advantageous to consider M constant, *i.e.*, $M \in \mathbf{R}^{(n+p)\times(n+m)}$. The delay or integral operator is, in this case, included in Δ and M will no longer be a nominal model. Throughout the paper, we will use M to denote both constant matrices and transfer function matrices, where the interpretation should be clear from the context.

The operator Δ is assumed to have a block diagonal structure

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r),$$

where each of the blocks satisfies a constraint, *e.g.* a norm bound or more generally an integral quadratic constraint. The specific operator Δ need not be known, thus Δ is particularly useful for describing uncertainty.

The signals in the interconnection are the input $u \in \mathbf{L}_2^m[0,\infty)$, the output $y \in \mathbf{L}_2^p[0,\infty)$ and the internal signals $z \in \mathbf{L}_2^n[0,\infty)$ and $x \in \mathbf{L}_2^n[0,\infty)$.

The transfer matrix $M \in \mathbf{RH}_\infty^{(n+p)\times(n+m)}$ is partitioned consistently with the signal dimensions as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{3.1}$$

The input-output mapping of the interconnection in Figure 3.1 is then defined by the Redheffer star product

$$y = (\Delta \star M)u = [D + C\Delta(I - A\Delta)^{-1}B]u.$$

For simplicity we assume that the model is causal, well-defined and stable.

DEFINITION 3.1
A model (Δ, M) is *well-posed* if the operator $I - A\Delta$ is causally invertible on $\mathbf{L}_2^n[0,\infty)$. If in addition $(I - A\Delta)^{-1}$ is bounded then the model is *stable*. □

Integral quadratic constraints

As a general framework to describe the nonlinearities and uncertainty of the model, we will use integral quadratic constraints (IQCs), see Megretski and Rantzer (1995). This framework includes a number of well-known constraints such as passivity and norm bounds.

DEFINITION 3.2
Let $\Pi : i\mathbf{R} \to \mathbf{C}^{2n\times 2n}$ be a bounded measurable function, taking Hermitian values. The operator Δ is said to *satisfy the integral quadratic constraint (IQC) defined by the multiplier* Π, written $\Delta \in \mathrm{IQC}(\Pi)$ if

$$\int_{-\infty}^{\infty} \begin{bmatrix} z(i\omega) \\ x(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} z(i\omega) \\ x(i\omega) \end{bmatrix} d\omega \quad \geq \quad 0, \tag{3.2}$$

for all vectors $z, x = \Delta z \in \mathbf{L}_2^n[0,\infty)$. □

The next two properties are used in the sequel.

Property 1 Assume that Δ satisfies the IQCs defined by $\Pi_1, \ldots, \Pi_k$ then Δ also satisfies the IQC defined by $\sum_{k=1}^n \alpha_k \Pi_k$ for any $\alpha_k \geq 0$, $k = 1, \ldots, n$.

Property 2 Assume that Δ has a block diagonal structure,

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r)$$

and that $\Delta_k \in \mathrm{IQC}(\Pi_k), k = 1, \ldots, r$. Then Δ satisfies the IQC defined by $\Pi = \mathrm{daug}(\Pi_1, \ldots, \Pi_r)$, where the operator daug is defined as

$$\left[\begin{array}{ccc|ccc} \Pi_{1(11)} & & 0 & \Pi_{1(12)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(11)} & 0 & & \Pi_{r(12)} \\ \hline \Pi_{1(21)} & & 0 & \Pi_{1(22)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(21)} & 0 & & \Pi_{r(22)} \end{array}\right].$$

EXAMPLE 3.1—MULTIPLIERS
This example illustrates that the more we restrict the operator Δ the more freedom we get in the the choice of multiplier Π.

- Let Δ be any operator with gain (induced two-norm) less than one. Then Δ satisfies all IQCs defined by

$$\begin{bmatrix} xI & 0 \\ 0 & -xI \end{bmatrix},$$

  where $x \geq 0$ is a constant number.

- Let Δ be any linear time-invariant operator with gain (H_∞ norm) less than one. Then Δ satisfies all IQCs defined by

$$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix},$$

  where $x(i\omega) \geq 0$ is a bounded measurable function.

- Let Δ be defined as multiplication with a time-varying real scalar with absolute value less than one. Then Δ satisfies all IQCs defined by

$$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$$

, where $X = X^T \geq 0$ and $Y = -Y^T$ are real matrices.

- Let Δ be defined as multiplication with a constant real scalar with absolute value less than one. Then Δ satisfies all IQCs defined by

$$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$$

where $X(i\omega) = X(i\omega)^* \geq 0$ and $Y(i\omega) = -Y(i\omega)^*$ are bounded measurable matrix functions.

□

Problem formulation

In this thesis we study model comparison and simplification. Consider comparison of two models that are identical except for some of the blocks in Δ, see Figure 3.2. We assume, without loss of general-

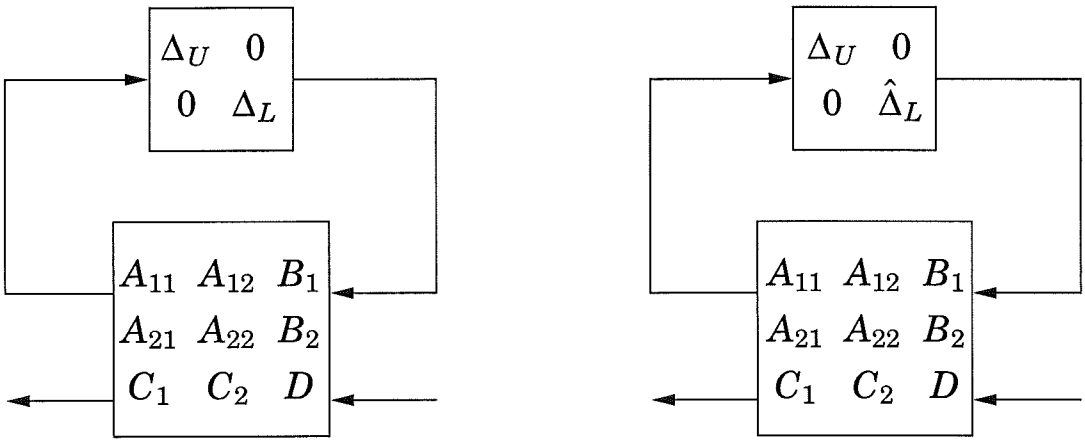

Figure 3.2 The two models that we compare.

ity, that the upper blocks in Δ, denoted $\Delta_U = \mathrm{diag}(\Delta_1, \ldots, \Delta_{\hat{r}})$, are identical while the lower blocks denoted $\Delta_L = \mathrm{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r)$ and $\hat{\Delta}_L = \mathrm{diag}(\hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r)$, respectively, are nonidentical. We partition the two models consistently as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \Delta_L \end{bmatrix}, \qquad \hat{\Delta} = \begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix}.$$

To compare the two models denoted (Δ, M) and $(\hat{\Delta}, M)$, respectively, we consider the output difference

$$e = y - \hat{y} = (\Delta \star M)u - (\hat{\Delta} \star M)u,$$

or more precisely the norm

$$||\Delta \star M - \hat{\Delta} \star M||.$$

Note that we do not consider allowable model transformations, such as similarity transformations; that is, we maintain the given model structure. This may be important if the matrix elements have a physical interpretation, but may also result in conservative bounds.

The above described comparison of models may be used for model simplification. The second model is in this case considered as a simplification of the first model and obtained by replacing Δ_L with a fixed transfer matrix $\hat{\Delta}_L = \mathrm{diag}(\hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r)$. If this fixed matrix is chosen within a given set then we can compute an upper bound on the approximation error using the results in the next section. Note, however, that we will not give any guidance among the possible selections for fixed $\hat{\Delta}_L$.

The spatial dimension of the simplified model may be reduced. This follows by observing that

$$\begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix} \star M = \Delta_U \star \hat{M}, \tag{3.3}$$

where

$$\hat{M} = \begin{pmatrix} A_{11} & B_1 \\ C_1 & D \end{pmatrix} + \begin{pmatrix} A_{12} \\ C_2 \end{pmatrix} \hat{\Delta}_L (I - A_{22}\hat{\Delta}_L)^{-1} \begin{pmatrix} A_{21} & B_2 \end{pmatrix}.$$

This thus gives us $(\Delta_U, \hat{M})$ as the reduced dimension model. Note that it is crucial for this reduction that $\hat{\Delta}_L$ is a fixed operator, and not a set of operators, so that $\hat{M}$ is a fixed operator. In some cases it is also useful to have M frequency independent. To maintain this property after the reduction, we use a constant matrix $\hat{\Delta}_L$.

EXAMPLE 3.2—TRUNCATION
Choosing for the fixed matrix $\hat{\Delta}_L = 0$ results in a reduced model with

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}.$$

This model is simply a truncation of the original model. □

EXAMPLE 3.3—SINGULAR PERTURBATION APPROXIMATION
If standard state-space models are written in the (Δ, M) form used in this thesis, see Example 3.4, then

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

consists of the realization matrices, and Δ the dynamics. Reduction of balanced state-space models using singular perturbation approximation, see *e.g.* Liu and Anderson (1989), is well-known. If we assume that (A, B, C, D) is a balanced realization, then the reduced model obtained using singular perturbation approximation in discrete time is

$$\hat{M} = \begin{bmatrix} A_{11} + A_{12}(I - A_{22})^{-1}A_{21} & B_1 + A_{12}(I - A_{22})^{-1}B_2 \\ C_1 + C_2(I - A_{22})^{-1}A_{21} & D + C_2(I - A_{22})^{-1}B_2 \end{bmatrix}$$

and in continuous time

$$\hat{M} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{bmatrix}.$$

These reduced models are obtained using $\hat{\Delta}_L = I$ and by letting

$$\hat{\Delta}_L \to \infty \cdot I,$$

respectively. □

3.3 Error bounds

In this section we present the error bounds associated with the comparison presented in the previous section. To obtain these error bounds we use integral quadratic constraints to describe Δ.

THEOREM 3.1
Assume that both (Δ, M) and $(\hat{\Delta}, M)$ are stable and that the operators

$$\begin{aligned} \Delta &= \operatorname{diag}(\Delta_1, \ldots, \Delta_r), \\ \hat{\Delta} &= \operatorname{diag}(\Delta_1, \ldots, \Delta_{\hat{r}}, \hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r). \end{aligned}$$

are linear. Also assume that $\Delta_k \in \mathrm{IQC}(\Pi_k)$ for $k = 1, \ldots, r$, and that $\hat{\Delta}_k \in \mathrm{IQC}(\Pi_k)$ for $k = \hat{r} + 1, \ldots, r$.

If there exist real numbers $\sigma_1, \ldots, \sigma_r > 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \operatorname{daug}(\Pi_1, \ldots, \Pi_r) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tag{3.4}$$

$$\begin{bmatrix} A \\ I \end{bmatrix}^* \operatorname{daug}(\sigma_1^2 \Pi_1, \ldots, \sigma_r^2 \Pi_r) \begin{bmatrix} A \\ I \end{bmatrix} + C^* C < 0 \tag{3.5}$$

for all $\omega \in [0, \infty]$ then

$$\|\Delta \star M - \hat{\Delta} \star M\| \leq 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r.$$

Let $(\Delta_U, \hat{M})$ be the reduced order model obtained using the reduction algorithm in the previous section then

$$\|\Delta \star M - \Delta_U \star \hat{M}\| \leq 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r.$$

Remark 1 Note that the σ_k-values not are unique.

Remark 2 Stability for the original and simplified model essentially follows from inequality (3.4) or inequality (3.5). We refer to Jönsson (1996) for details on proving stability for models in our framework.

Proof We start by proving the case $\hat{r} + 1 = r = 2$. The case with $\hat{r} = r = 1$ is almost identical, but less pedagogical. We partition the matrix M as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}.$$

Since we have two models (Δ, M) and $(\hat{\Delta}, M)$ we also have two feedback interconnections and also two sets of signals. We introduce the following notation for the (frequency) signals.

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, \qquad \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{y} \end{bmatrix} = M \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ u \end{bmatrix}.$$

We assume that the initial conditions are zero. This is realistic since the two models are stable. Note that the input signal u is the same for both models.

Multiplying the inequality (3.4) with the vector $(x_1 + \hat{x}_1, x_2 + \hat{x}_2, 2u)$ from the right and its complex conjugate from the left, and adding this to the inequality obtained by multiplying the inequality (3.5) with the column vector $\sigma_2^{-1}(x_1 - \hat{x}_1, x_2 - \hat{x}_2)$ from the right and its complex conjugate from the left, we then get the following inequality by noting that $z - \hat{z} = A(x - \hat{x})$, $y - \hat{y} = C(x - \hat{x})$ and using some matrix manipulations:

$$\begin{aligned} 0 \quad > \quad & \begin{bmatrix} z_1 + \hat{z}_1 \\ x_1 + \hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1 + \hat{z}_1 \\ x_1 + \hat{x}_1 \end{bmatrix} \\ & + \sigma_1^2 \sigma_2^{-2} \begin{bmatrix} z_1 - \hat{z}_1 \\ x_1 - \hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1 - \hat{z}_1 \\ x_1 - \hat{x}_1 \end{bmatrix} \\ & + 2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix} \\ & + \sigma_2^{-2} |y - \hat{y}|^2 - 4|u|^2. \end{aligned} \tag{3.6}$$

Integrating the expression on the right hand side of the inequality along the frequency axis causes the first four terms to become nonnegative. In the linear case, the first and second term become nonnegative

since $\Delta_1 = \hat{\Delta}_1 \in \mathrm{IQC}(\Pi_1)$ and are linear. (In the nonlinear case, discussed later, this follows from the additional assumptions that the IQC is satisfied incrementally and that $\Delta_1, \hat{\Delta}_1$ are odd.) The third and fourth term become nonnegative since $\Delta_2, \hat{\Delta}_2 \in \mathrm{IQC}(\Pi_2)$. This completes the proof for the case $\hat{r} + 1 = r = 2$.

More generally one can generate a sequence of new models by replacing $\Delta_r, \ldots, \Delta_{\hat{r}+1}$ one at a time. At each step, the above argument can be used and the total error bound becomes

$$\|y - \hat{y}\| \leq 2(\sigma_r + \cdots + \sigma_{\hat{r}+1})\|u\|.$$

□

The interpretation of this theorem is that if the operators Δ_k are replaced by different operators $\hat{\Delta}_k$ satisfying the same constraints, then the error is bounded by the sum of positive values corresponding to the operators which were replaced. This gives a solution to the comparison problem. If we then reduce the model to an equivalent model with lower spatial dimension as in 3.3, then the same error bound holds. This gives a solution to the model dimension reduction problem.

The error bounds may be improved if some of the σ_k-values are equal.

THEOREM 3.2
Equal σ_k values in the error bound need only be added once.

Proof If there are equal σ_k-values then the corresponding blocks can be collected in one block and the corresponding Π_k matrices collected in one larger Π_k matrix. The σ_k-value corresponding to this larger Δ_k is the same as for the smaller ones. □

The problem of finding upper bounds on the error has been rewritten in the theorem as a problem of finding values σ_k that satisfy the inequalities (3.4) and (3.5). If the multiplier Π is given then these inequalities become linear matrix inequalities in Σ. Efficient algorithms for solving such problems using interior point methods have recently been developed, see Nesterov and Nemirovski (1993), Vandenberghe and Boyd (1996) for details and further references; a reliable software package is *e.g.* the LMI control toolbox, Gahinet *et al.* (1995). If more freedom in the choice of Π exists then the error bounds may be

improved and the resulting optimization problem is, in general, non-convex. A suboptimal solution may then be obtained using the following two step algorithm, where Π_Δ is used to denote a convex set of multipliers Π that define integral quadratic constraints that are satisfied by Δ.

Step 1 Find $\Pi \in \Pi_\Delta$ that satisfies inequality (3.4) and minimizes γ in

$$\gamma^2 \begin{bmatrix} \Sigma_0 A \\ \Sigma_0 \end{bmatrix}^* \Pi \begin{bmatrix} \Sigma_0 A \\ \Sigma_0 \end{bmatrix}^* + C^* C < 0.$$

The matrix Σ_0 is a guess for the optimal value of Σ. If no information is available we choose $\Sigma_0 = I$.

Step 2 Find $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r})$ that minimizes $\mathrm{tr}\, W\Sigma^2$ for

$$\begin{bmatrix} A \\ I \end{bmatrix}^* \Sigma^2 \Pi \begin{bmatrix} A \\ I \end{bmatrix} + C^* C < 0$$

where Π is the solution in Step 1. The matrix $W = \mathrm{diag}(\alpha_1, \ldots, \alpha_r)$ serves to weight the relative importance of different elements in Σ. If they are of equal significance we choose $W = I$.

Remark 1 The two-step algorithm may also be used for iteration; then Σ obtained in Step 2 replaces Σ_0 in Step 1 when a new iteration begins.

Remark 2 When $\gamma\Sigma_0 = \Sigma$ then the inequality in Step 2 is equivalent to the inequality in Step 1 and inequality (3.5). This follows from the equalities $\Sigma^2\Pi = \Sigma\Pi\Sigma = \mathrm{daug}(\sigma_1^2\Pi_1, \ldots, \sigma_r^2\Pi_r)$.

3.4 Special cases

In this section we consider the case where Δ represents norm-bounded uncertainty, and has a repeated scalar structure. Error bounds for truncation of such uncertainty descriptions have been studied during recent years, see *e.g.* Wang *et al.* (1991), Beck (1996). We will show that the same error bounds follow from Theorem 3.1, but then also provide additional reduction methods. We will also consider the closely related case where the scalars are passive operators. Finally, we try to give a deeper understanding of Theorem 3.1 using these cases.

Uncertain norm bounded components

We start by introducing the same model structure as used in Wang *et al.* (1991), Beck (1996). The matrix M is constant and Δ is linear with a repeated scalar structure

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_r I_{n_r}),$$

where the scalar operators are norm bounded, that is,

$$\|\delta_k\| \leq 1, \; k = 1, \ldots, r.$$

This type of model is typically used to describe discrete time systems with norm-bounded uncertainty. The known linear time-invariant dynamics are modeled by letting one of the δ_k-s be the delay operator. The following example illustrates this idea.

EXAMPLE 3.4—STATE-SPACE MODELS
Discrete time state-space models

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}$$

with signals in l_2 and no uncertainty are formed by letting $\Delta = z^{-1}I$, where z^{-1} is the transfer function for the delay operator. □

The idea of capturing the linear time-invariant dynamics of the model in Δ is used to rewrite a model with a frequency-dependent M as a model with a constant M.

Applying Theorem 3.1 gives the following result.

COROLLARY 3.3
Let

$$\begin{aligned} \Delta &= \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}), \\ \hat{\Delta} &= \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_{\hat{r} I_{n_{\hat{r}}}}, \hat{\delta}_{\hat{r}+1} I_{n_{\hat{r}+1}}, \ldots, \hat{\delta}_r I_{n_r}). \end{aligned}$$

Assume that $\|\delta_k\| \leq 1$ for $k = 1, \ldots, r$ and $\|\hat{\delta}_k\| \leq 1$ for $k = \hat{r} + 1, \ldots, r$ and that there exists a real matrix $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0$, that

solves the Lyapunov inequalities

$$\begin{aligned} A\Sigma A^* - \Sigma + BB^* &< 0 \\ A^*\Sigma A - \Sigma + C^*C &< 0 \end{aligned} \tag{3.7}$$

then

$$||\Delta \star M - \hat{\Delta} \star M|| \leq 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r.$$

Furthermore, if $(\Delta_U, \hat{M})$ is the reduced dimension model obtained using the proposed reduction algorithm, then

$$||\Delta \star M - \Delta_U \star \hat{M}|| \leq 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r.$$

Remark 1 Note that the error bound given in Wang *et al.* (1991) is obtained by assuming that the repeated scalars are different scalars, *i.e.* $n_1 = \ldots, n_r = 1$.

Remark 2 The model (Δ, M) is stable if the inequalities in (3.7) are satisfied. This has been proven in *e.g.* Beck *et al.* (1996), but a more elegant proof is obtained using the stability results presented in Megretski and Rantzer (1995). These results are based on integral quadratic constraints.

Proof The multiplier

$$\Pi = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -\Sigma^{-1} \end{bmatrix}$$

defines an integral quadratic constraint that is satisfied for any Δ with the above structure. Using this multiplier we find that inequality (3.5) simplifies to $A^T\Sigma A - \Sigma + C^TC < 0$ and inequality (3.4) can be simplified

in the following way

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -\Sigma^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} \Sigma^{-1} [\, A \quad B \,] < \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} \Sigma^{1/2} A^T \Sigma^{-1/2} \\ B^T \Sigma^{-1/2} \end{bmatrix} [\, \Sigma^{-1/2} A \Sigma^{1/2} \quad \Sigma^{-1/2} B \,] < \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$[\, \Sigma^{-1/2} A \Sigma^{1/2} \quad \Sigma^{-1/2} B \,] \begin{bmatrix} \Sigma^{1/2} A^T \Sigma^{-1/2} \\ B^T \Sigma^{-1/2} \end{bmatrix} < I$$

$$A \Sigma A^T - \Sigma + B B^T < 0.$$

This completes the proof. □

The positive number σ_k gives a measure on the importance of the repeated scalar block $\delta_k I_{n_k}$. The difference between the original model and the model where this block is replaced by $\hat{\delta}_k I_{n_k}$ is bounded by $2\sigma_k$.

The error bounds hold for a number of different reduced dimension models. These models are obtained using different fixed operators, satisfying $\|\hat{\delta}_k\| \leq 1$. To obtain reduced dimension models where $\hat{M}$ is a constant matrix, we restrict the fixed operators $\hat{\delta}_{\hat{r}+1}, \ldots, \hat{\delta}_r$ to be constant scalars in the interval $[-1, 1]$. Note that the choice $\hat{\delta}_k = 0$ corresponds to truncation *i.e.* $\hat{x}_k = 0$ and the choice $\hat{\delta}_k = 1$ corresponds to singular perturbation approximation *i.e.* $\hat{x}_{k+1} = \hat{x}_k$.

The following simple example will illustrate how the result have been used in Wang *et al.* (1991). Note that we in addition know that the error bounds hold for a larger class of reduction methods than truncation.

EXAMPLE 3.5—MINIMAL REALIZATIONS

Consider the model in Figure 3.3 with norm bounded uncertainty $\|\delta\| \leq 1$. This model is written on the form (Δ, M) using

$$M = \begin{bmatrix} 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

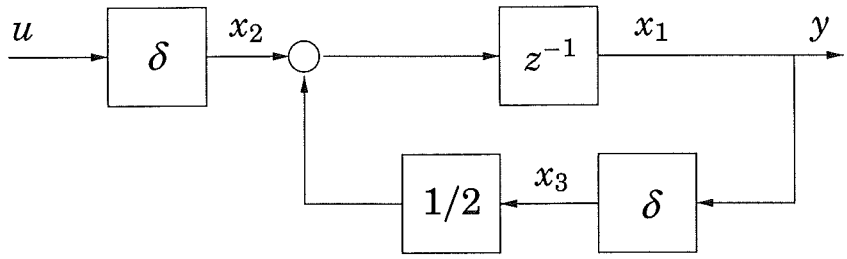

Figure 3.3 The model in Example 3.5.

and

$$\Delta = \mathrm{diag}(z^{-1}, \delta, \delta).$$

Since there is an interdependence between the elements in Δ it is possible to use a coordinate transformation,

$$\hat{x} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix},$$

without changing the input-output properties of the model. This is used to obtain improved error bounds. We then first solve the inequalities

$$\begin{aligned} APA^* - P + BB^* &< 0, \\ A^*QA - Q + C^*C &< 0, \end{aligned}$$

for block structured matrices $P > 0$ and $Q > 0$ and then based on these find a coordinate transformation, which always exist, such that the new realization satisfies inequalities (3.7). In order to obtain a small Σ-matrix it is shown in Beck (1996) that it is reasonable to minimize the trace of P and Q. Doing so we obtain

$$M = \begin{bmatrix} 0 & 1.00 & 0.0022 & 0 \\ 0.500 & 0 & 0 & 1.00 \\ 0.134 & 0 & 0 & 0.090 \\ 1.00 & 0 & 0 & 0 \end{bmatrix},$$

and Δ as before. This new model satisfies inequalities (3.7) for

$$\Sigma = \mathrm{diag}(1.33, 1.33, 0.03).$$

This shows that reduction of the last entry in δ will result in an error less than $2\sigma_3 = 0.06$, which is neglectable. The reduced dimension model is visualized in Figure 3.4. This model has the same input-output properties as the original model in Figure 3.3. The original model may therefore be called non-minimal. Minimality for models with norm-bounded uncertainty is studied in Beck and Doyle (1996), Beck (1996). □

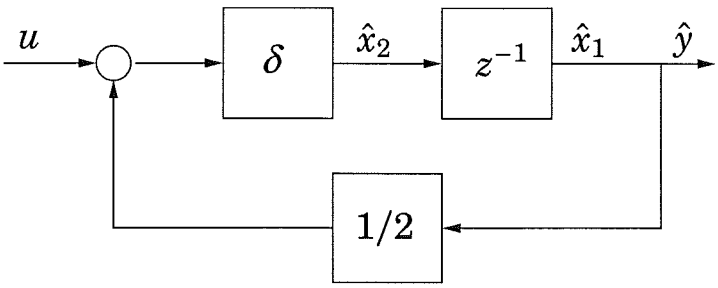

Figure 3.4 A minimal description of the model in Example 3.5.

The next example will illustrate that the actual error usually depends on the fixed operator used in the reduction algorithm.

EXAMPLE 3.6—REDUCTION METHODS
Consider a discrete time state-space model with

$$A = \begin{bmatrix} 0.9638 & 0.0784 & 0.0082 \\ -0.0784 & 0.5261 & -0.1824 \\ 0.0082 & 0.1824 & 0.1602 \end{bmatrix},$$
$$B = \begin{bmatrix} 1.3639 \\ 1.3762 \\ -0.1835 \end{bmatrix}, \; C = \begin{bmatrix} 1.3639 & -1.3762 & -0.1835 \end{bmatrix}.$$

This model has a zero in $z = 0$ and poles in $z = 0.95, z = 0.4$ and $z = 0.3$ and can be written in standard form using $\Delta = z^{-1}I_3$ as described in Example 3.4.

Assume that we would like to find a first order approximation of this model together with error bounds. We then choose $\hat{\Delta} = \mathrm{diag}(q^{-1}, \hat{\delta}_2, \hat{\delta}_3)$ with $\hat{\delta}_2, \hat{\delta}_3 \in [-1, 1]$, and note that the inequalities (3.7) are satisfied for

$$\Sigma = \mathrm{diag}(26.4, 2.85, 0.134),$$

so that the upper-bound is given by

$$||\Delta \star M - \hat{\Delta}_U \star \hat{M}|| \leq 2(\sigma_2 + \sigma_3) = 5.97.$$

Figure 3.5 shows the actual error in the case where $\hat{\delta}_2 = \hat{\delta}_3 = \delta$. It is seen that we obtain a smaller error by chosing $\delta = 0.5$ than we do with truncation, $\delta = 0$, and singular perturbation approximation $\delta = 1$. This shows that the proposed reduction algorithm may result in a smaller error than that for either truncation or singular perturbation approximation. The choice for δ_k resulting in the smallest error is, however, not directly given by the reduction algorithm.

To see why the error has it's smallest value for $\delta = 0.5$ we consider the Nyquistplots in Figure 3.6. We see that Singular perturbation approximation $\delta = 1$ has no error for low frequences (to the right) but has a large error as $\omega \to \infty$. We also see that $\delta = 0.5$ has a medium error for most frequencies, and thus not a large error at any frequency. □

Uncertain passive components

We now consider the closely related case where the scalars are passive operators, and show that error bounds, not previously published, easily are found using Theorem 3.1. We also give a further indication of the importance of the unified reduction algorithm.

We still assume that Δ is linear with a repeated scalar structure

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_r I_{n_r}),$$

but now assume that the scalar operators are passive, that is,

$$\delta_k + \delta_k^* \geq 0, \; k = 1, \ldots, r.$$

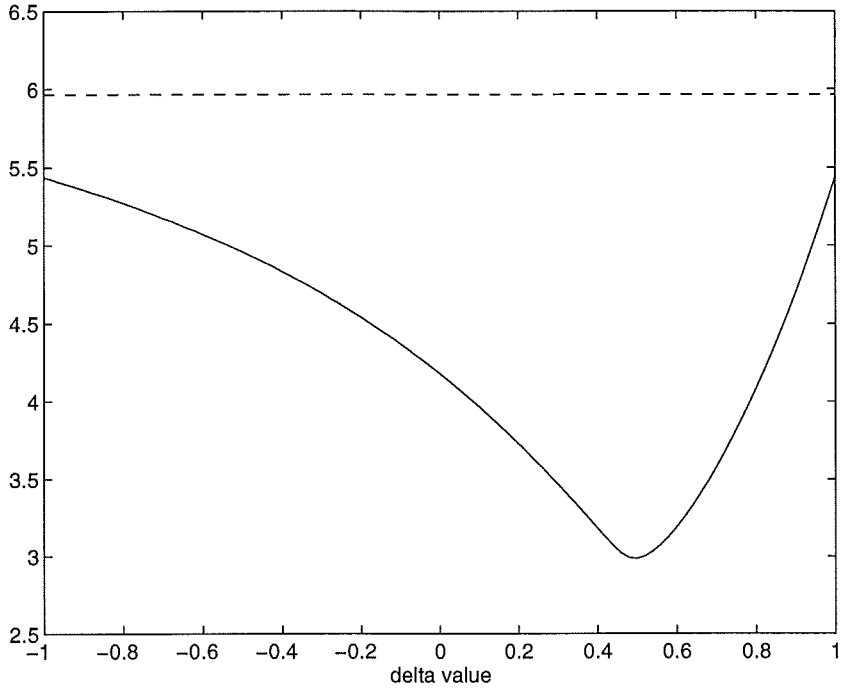

Figure 3.5 Actual error as a function of the replacing scalar δ. The upper bound is shown with a dashed line.

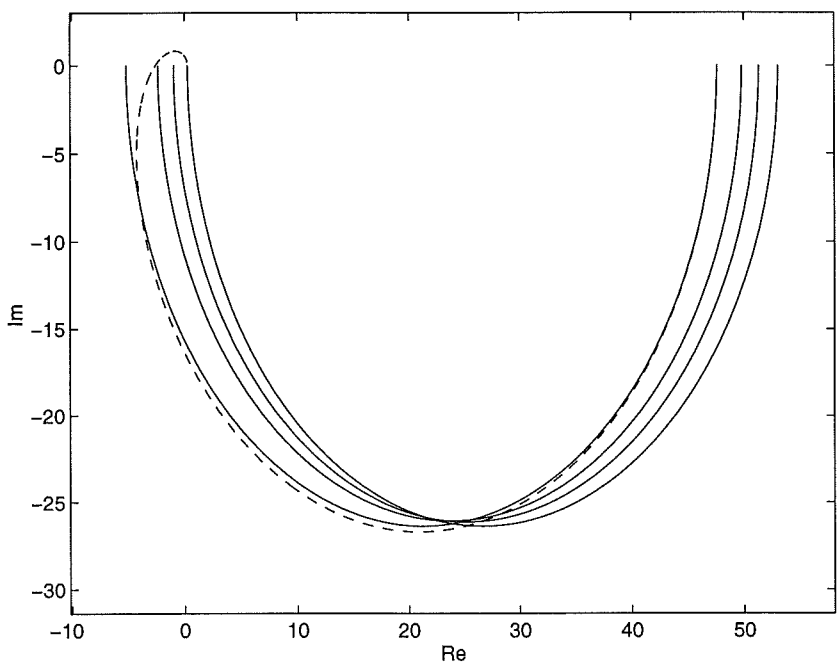

Figure 3.6 Nyquist plots for the simplified models obtained using $\delta = 1, 0.5, 0, -1$ are shown in that order with solid lines and the original model with a dashed line.

This type of model may be used to describe continuous-time systems with passive uncertainty. The known linear time-invariant dynamics are modeled by letting one of the δ_k be the integral operator.

Applying Theorem 3.1 gives the following result.

COROLLARY 3.4
Let

$$\begin{aligned} \Delta &= \operatorname{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}) \\ \hat{\Delta} &= \operatorname{diag}(\delta_1 I_{n_1}, \ldots, \delta_{\hat{r} I_{n_{\hat{r}}}}, \hat{\delta}_{\hat{r}+1} I_{n_{\hat{r}+1}}, \ldots, \hat{\delta}_r I_{n_r}). \end{aligned}$$

Assume $\delta_k + \delta_k^* \geq 0$ for $k = 1, \ldots, r$ and $\hat{\delta}_k + \hat{\delta}_k^* \geq 0$ for $k = \hat{r}+1, \ldots, r$ and that there exists a real matrix $\Sigma = \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0$, that solves the Lyapunov inequalities

$$\begin{aligned} A\Sigma + \Sigma A^* + BB^* &< 0, \\ A^*\Sigma + \Sigma A + C^*C &< 0, \end{aligned} \tag{3.8}$$

then

$$\|\Delta \star M - \hat{\Delta} \star M\| \leq 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r.$$

In particular, if $(\Delta_U, \hat{M})$ is the reduced dimension model obtained using the proposed reduction algorithm, then

$$\|\Delta \star M - \Delta_U \star \hat{M}\| \leq 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r.$$

Remark In this case we choose $\hat{\delta}_k \in [0, \infty)$ in order that $\hat{M}$ be a constant matrix.

Proof The multiplier

$$\Pi = \begin{bmatrix} 0 & \Sigma^{-1} \\ \Sigma^{-1} & 0 \end{bmatrix}$$

defines an integral quadratic constraint that is satisfied for every Δ with the above structure. Using this multiplier we find that inequality (3.4) becomes

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & \Sigma^{-1} \\ \Sigma^{-1} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} < 0,$$

which is equivalent to

$$\Sigma A^T + A\Sigma + BB^T \quad < \quad 0.$$

Furthermore, (3.5) becomes

$$A^T\Sigma + \Sigma A + C^T C \quad < \quad 0.$$

Applying Theorem 3.1 completes the proof. □

These error bounds have not been published before; the error bounds for truncation of passive operators can not be proven by applying the usual bilinear transformation to the results published in Wang *et al.* (1991). This is a consequence of the fact that the bilinear transformation takes a norm-bounded operator Δ_n to a passive operator $\Delta_p = (I + \Delta_n)^{-1}(I - \Delta_n)$. In particular $\hat{\Delta} = 0$ is in the interior of the set of unit norm bounded operators, but on the boundary of the set of passive operators. Thus truncation, corresponding to $\hat{\Delta} = 0$, for norm-bounded operators does not correspond to truncation for passive operators. Note, however, that the results in this paper hold not only for truncation but for an entire set of reduction methods. Since here $\Delta_p + \Delta_p^* \geq 0 \Leftrightarrow \|\Delta_n\| \leq 1$ we have been able to prove the passive case results using the more general norm-bounded case results presented in this paper.

Further generalization

We now try to give a deeper understanding of Theorem 3.1. This is done by generalizing the results for norm-bounded uncertainty. Similar results also hold when the uncertainty is passive.

We consider a linear Δ where

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_r I_{n_r}),$$

and the scalars are norm bounded

$$\|\delta_k\| \leq 1, \; k = 1, \ldots, r.$$

We will apply Theorem 3.1 to find σ-values that we collect in a matrix as

$$\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0.$$

We keep in mind that the positive number σ_k gives a measure on the importance of the repeated scalar block $\Delta_k = \delta_k I_{n_k}$ and that the corresponding error bound $2\sigma_k$ holds if we replace the block by $\hat{\Delta}_k = \hat{\delta}_k I_{n_k}$.

In the proof of Corollary 3.3 we used the assumption that Δ satisfies any integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -\Sigma^{-1} \end{bmatrix}, \qquad \Sigma = \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0, \tag{3.9}$$

where Σ is a real matrix. In fact Δ satisfies any integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}, \qquad X = X^T = \operatorname{diag}(X_1, \ldots, X_r) > 0, \tag{3.10}$$

where X is a real matrix. Note that $X = \Sigma^{-1}$ give us the previous multiplier (3.9). One expects that the extra freedom can be exploited to improve the error bounds.

The additional freedom in the multiplier (3.10) compared to the multiplier (3.9) has an interpretation. Consider the following Lemma.

LEMMA 3.5
Let Π have the form (3.10). The inequalities in Theorem 3.1 are then equivalent to

$$\begin{aligned} \hat{A}\Sigma\hat{A}^T - \Sigma + \hat{B}\hat{B}^T &< 0, \\ \hat{A}^T\Sigma\hat{A} - \Sigma + \hat{C}^T\hat{C} &< 0, \end{aligned}$$

where

$$\hat{A} = TAT^{-1}, \qquad \hat{B} = TB, \qquad \hat{C} = CT^{-1},$$

and T is any invertible matrix of the form

$$T = \operatorname{diag}(T_1, \ldots, T_r).$$

Proof Factor $X = T^T\Sigma^{-1}T$ and use the steps in the proof of Corollary 3.3. □

This shows that the additional freedom consists of similarity transformations

$$x = T\hat{x}, \qquad z = T\hat{z}$$

from the old internal signals x and z to the new signals $\hat{x}$ and $\hat{z}$. This type of transformations do not change the input-output relation of the model since $T\Delta T^{-1} = \Delta$.

Note that the given realization has no importance since similarity transformations are captured by the inequalities. This statement is, however, only true if we have a method that finds an optimal solution to the inequalities. If we only can find suboptimal solutions, then these solutions may depend on the given realization.

EXAMPLE 3.7
Consider a discrete time state space model with

$$A = \begin{bmatrix} 0.3 & 0 & 0 \\ 1 & 0.4 & 0 \\ 0 & 1 & 0.95 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

This model has a zero at $z = 0$ and poles at $z = 0.95, z = 0.4$ and $z = 0.3$ and can be written in standard form using $\Delta = z^{-1}I_3$ as described in Example 3.4.

If we consider simplification of the whole block, then no dynamics will remain after the simplification. Applying Theorem 3.1 using the multiplier (3.10) we find $\Sigma = 41.9 \cdot I_3$ as the optimal value. The corresponding error-bound is given by $2\sigma = 83.8$. If we on the other hand try to use the multiplier (3.9) then no error bound will be found. The reason is that there does not exist a σ that solves the inequalities in Theorem 3.1 or equivalently the inequalities (3.7), in this case. □

Now consider simplification of individual entries in Δ and not blocks. We then have to consider each element in $\delta_k I_{n_k}$ as different scalar

operators $\delta_{k(1)}, \ldots, \delta_{k(n_k)}$. In this case each entry of Σ is a measure of the importance of the corresponding entry in Δ. This means that we throw away information about the repeated scalar structure in Δ, in order to find error bounds for the separate entries.

EXAMPLE 3.8
We return to Example 3.7. If we are interested in reduction of only some of the states then we must consider the delay operators in Δ as independent operators. Applying Theorem 3.1 using the multiplier (3.10) we find

$$\Sigma = \operatorname{diag}(22.9, 45.7, 34.0).$$

The error bound for reduction of all the states in this case will be $2(\sigma_1 + \sigma_2 + \sigma_3) = 205$ which is much larger than when we considered the whole block at the same time. Once again we fail to find upper bounds if we try to use the multiplier (3.9). □

To obtain error bounds for the separate entries we threw away information about the repeated scalar structure in Δ. This information still exist and can be used a priori *e.g.* by similarity transformations. Note, however, that the model structure will change, which not always is acceptable.

EXAMPLE 3.9
Once again return to Example 3.7 and Example 3.8. If the particular realization is not important then we can improve the results in Example 3.8 by a priori applied similarity transformations. Consider the following realization, which is the realization used in Example 3.6,

$$A = \begin{bmatrix} 0.9638 & 0.0784 & 0.0082 \\ -0.0784 & 0.5261 & -0.1824 \\ 0.0082 & 0.1824 & 0.1602 \end{bmatrix},$$
$$B = \begin{bmatrix} 1.3639 \\ 1.3762 \\ -0.1835 \end{bmatrix}, \quad C = \begin{bmatrix} 1.3639 & -1.3762 & -0.1835 \end{bmatrix}.$$

This realization was found using the ideas in Example 3.5. Solving

the inequalities in Theorem 3.1 for Σ using either of the two multipliers (3.9) and (3.10) gives us the solution

$$\Sigma = \mathrm{diag}(26.4, 2.85, 0.134).$$

The error when all the states are reduced will be bounded by

$$2(\sigma_1 + \sigma_2 + \sigma_3) = 58.7$$

which is lower than in both the previous cases. □

The examples show that the result in Corollary 3.3 is relatively useless without a priori applied coordinate transformations. Either there does not exist σ-values satisfying the inequalities or they may be very large. The results in Wang *et al.* (1991) therefore depend critically on the use of a coordinate transformation. The results in this thesis, however, may be used without coordinate transformations. One should then keep in mind that we analyze the importance of states in a given realization, where all states may be important. This should be compared to a Balanced realization where some of the states usually are less important and others are very important.

3.5 Nonlinear models

The reduction results presented in this thesis, so far, assume that the operator Δ is linear. In this section we show how these results may me generalized to include nonlinearities.

To extend the results to the nonlinear case we require the following definitions. An operator Δ is *odd* if

$$\Delta(-z) = -\Delta z.$$

An integral quadratic constraint is said to be satisfied *incrementally* if

$$\int_{-\infty}^{\infty} \begin{bmatrix} z_2 - z_1 \\ x_2 - x_1 \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} z_2 - z_1 \\ x_2 - x_1 \end{bmatrix} d\omega \geq 0,$$

holds $\forall z_1, z_2, x_1 = \Delta z_1, x_2 = \Delta z_2 \in \mathbf{L}_2^n[0, \infty)$. These two properties are sufficient for the results to hold also in the nonlinear case.

THEOREM 3.6
Assume that all the assumptions in Theorem 3.1, except the linearity assumption, hold. Also assume that Δ_k is odd and satisfies the integral quadratic constraint incrementally for $k < r$. Then

$$||\Delta \star M - \hat{\Delta} \star M|| \leq 2(\sigma_{\hat{r}+1} + \cdots + \sigma_r).$$

Let $(\Delta_U, \hat{M})$ be the reduced dimension model obtained using the proposed reduction algorithm. Then

$$||\Delta \star M - \Delta_U \star \hat{M}|| \leq 2(\sigma_{\hat{r}+1} + \cdots + \sigma_r).$$

Proof This case is discussed in the proof of Theorem 3.1. □

Remark Similar results hold also if Δ is non-causal.

The following examples will illustrate what the incrementality-condition means for specific static nonlinearities.

EXAMPLE 3.10

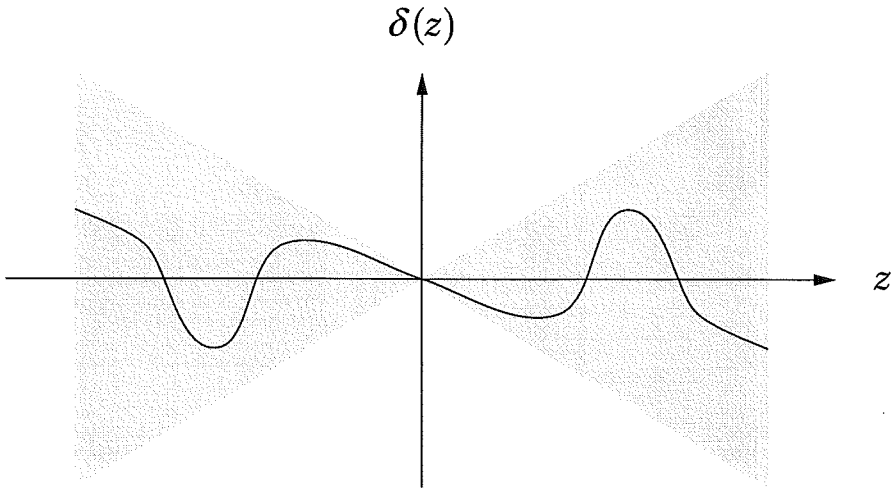

Figure 3.7 Static non-linearity in a sector.

Consider a static non-linearity in the sector $||\delta|| \leq 1$, see Figure 3.7. This operator satisfies the integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The incrementality-condition is in this case satisfied if

$$|\delta(z_2) - \delta(z_1)| \leq |z_2 - z_1|,$$

which is a *Lipschitz condition* satisfied if *e.g.*

$$\left|\frac{d\delta(z)}{dz}\right| \leq 1.$$

□

EXAMPLE 3.11

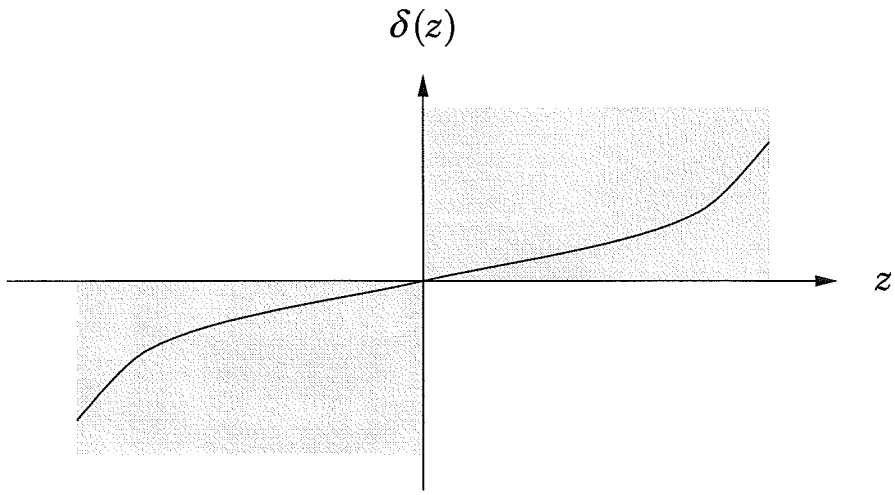

Figure 3.8 Static non-linearity in the first and third quadrant.

Consider a static non-linearity in the first and third quadrant

$$z \cdot \delta(z) \geq 0,$$

see Figure 3.8. This operator satisfies the integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The incrementality-condition is satisfied if

$$[\delta(z_2) - \delta(z_1)] \cdot [z_2 - z_1] \geq 0,$$

which is a *Monotonicity condition* satisfied if *e.g.*

$$\frac{d\delta(z)}{dz} \geq 0.$$

□

3.6 Conclusions

In this chapter we have shown that error bounds can be found for a general class of dynamical models and a general reduction algorithm, using convex optimization. The proposed reduction algorithm includes truncation and singular perturbation approximation. The models may contain both nonlinearities and uncertainty, and are described using integral quadratic constraints. Special attention has been paid to compare the results with those in Wang *et al.* (1991) which are a special case of the results in this thesis.

4

Concluding remarks

In this thesis we have shown that error bounds can be found for a general class of dynamical models and a general reduction algorithm, using convex optimization. The proposed reduction algorithm includes truncation and singular perturbation approximation as special cases. The models may contain both nonlinearities and uncertainty, and are described using integral quadratic constraints. This framework leads to quite simple proofs.

The proposed reduction algorithm allows for a choice of the replacing operators. One such choice corresponds to truncation and another to singular perturbation approximation. The optimal choice, however, is not indicated by the method. The type and properties of the reduction methods corresponding to different replacing operators need to be further investigated.

An important aspect in model reduction is frequency dependent error bounds. This is important in many applications, such as controller design where different frequencies have different importance. Frequency dependent error bounds were obtained in Chapter 2.

A feature of the method proposed is that error bounds are found for the given model structure without any need for model transformations, such as similarity transformations. This may be important when the model parameters have a physical interpretation; when model transformations are acceptable then the error bounds may be improved. Optimal transformations for different types of models are, however, difficult to find and remains as an area of further study.

In fact, it still remains a lot of interesting work to be done before

the results in this thesis can be used efficiently on a wide range of problems. Larger application examples have to be studied to further investigate the strength and weakness of the method. Comparisons with direct calculation of the errors as discussed in Section 1.4 has to be done. Theoretical results for special areas of application such as control oriented simplification must be obtained. Finally, user friendly software with efficient computations is needed.

5

References

ANDERSSON, L. and C. BECK (1996): "Model comparison and simplification." In *35th IEEE CDC Proceedings*, Kobe, Japan.

ANDERSSON, L. and A. RANTZER (1997a): "Frequency dependent error bounds for uncertain linear models." To appear in American Control Conference, Albuquerque, New Mexico.

ANDERSSON, L. and A. RANTZER (1997b): "Frequency dependent error bounds for uncertain linear models." Journal article under preparation.

ANDERSSON, L., A. RANTZER, and C. BECK (1996): "Model comparison and simplification." Submitted to the Int. Journal on Robust and Nonlinear Control.

BECK, C. (1996): *Model Reduction and Minimality for Uncertain Systems.* PhD thesis, California Institute of Technology.

BECK, C. and J. DOYLE (1996): "A necessary and sufficient minimality condition for uncertain systems." Submitted to IEEE Trans. on Automatic Control.

BECK, C., J. DOYLE, and K. GLOVER (1996): "Model reduction of multdimensional and uncertain systems." *IEEE Transaction on Automatic Control*, **41:10**, pp. 1466–1477.

DOYLE, J. (1982): "Analysis of feedback systems with structured uncertainties." In *IEE Proceedings*, volume D-129, pp. 242–251.

ENNS, D. (1984): *Model Reduction for Control System Design.* PhD thesis, Stanford University.

FERNANDO, K. and H. NICHOLSON (1982a): "Singular perturbational model reduction in the frequency domain." *IEEE Transactions on Automatic Control,* **27**, pp. 969–970.

FERNANDO, K. and H. NICHOLSON (1982b): "Singular perturbational model reduction of balanced systems." *IEEE Transactions on Automatic Control,* **27**, pp. 466–468.

GAHINET, G., A. NEMIROVSKI, A. LAUB, and M. CHILALI (1995): *LMI Control Toolbox.* The Math Works Inc.

GLOVER, K. (1984): "All optimal hankel-norm approximations of linear multivariable systems and their L^∞-error bounds,." *Int. J. Control.,* **39**, pp. 1115–1193.

HELMERSSON, A. (1995): *Methods for Robust Gain Scheduling.* PhD thesis, Linköping University.

HINRICHSEN, D. and A. PRITCHARD (1990): "An improved error estimate for reduced order models of discrete-time systems." *IEEE trans. on Automatic Control,* **35**, pp. 317–320.

JÖNSSON, U. (1996): *Robustness Analysis of Uncertain and Nonlinear Systems.* PhD thesis ISRN LUTFD2/TFRT--1047--SE, Department of Automatic Control, Lund Inst. of Technology.

LIU, Y. and B. D. O. ANDERSON (1989): "Singular perturbation approximation of balanced systems." *Int. Journal of Control,* **50**, pp. 1379–1405.

MEGRETSKI, A. (1993): "Power distribution approach in robust cotrol." In *Proceedings of the IFAC Congress,* pp. 399–402.

MEGRETSKI, A. and A. RANTZER (1995): "System analysis via integral quadratic constraints, part I." Technical Report, Department of Automatic Control, Lund Institute of Technology.

MOORE, B. (1981): "Principal component analysis in linear systems: Controllability, observability and model reduction." *IEEE Trans. on Automatic Control,* **26**, pp. 17–32.

NESTEROV, Y. and A. NEMIROVSKI (1993): "Interior point polynomial methods in convex programming." *Studies in Applied Mathematics, SIAM*, **13**.

PERNEBO, L. and L. SILVERMAN (1982): "Model reduction via balanced state space representations." *IEEE Trans. on Automatic Control*, **27:2**, pp. 382–387.

RANTZER, A. (1995): "Error bounds for nonlinear model truncation." Presented at Bernoulli Workshop, Groningen.

RANTZER, A. and A. MEGRETSKI (1994): "System analysis via integral quadratic constraints." In *Proceedings of the IEEE Conference on Decision and Control*, volume 3, pp. 3062–3067, Lake Buena Vista, Florida.

SAFONOV, M. (1982): "Stability margins of diagonally perturbed multivariable feedback systems." In *IEE Proceedings*, volume 129 of *6*, pp. 251–256.

SKELTON, R. (1989): "Model error concepts in control design." *Int. J. Control*, **49:5**, pp. 1725–1753.

VANDENBERGHE, L. and S. BOYD (1996): "Semidefinite programming." *SIAM Review*, **38**, pp. 49–95.

WANG, W., J. DOYLE, C. BECK, and K. GLOVER (1991): "Model reduction of LFT systems." In *30th CDC Proceedings*.

ZHOU, K., J. DOYLE, and K. GLOVER (1996): *Robust and Optimal Control*. Prentice-Hall.