Linear Control and Estimation Using Operator Factorization

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ABSTRACT

The filtering, prediction and smoothing problems as well as the linear quadratic control problems can very generally be formulated as operator equations using basic linear algebra.

The equations are of Fredholm type II and difficult to solve directly.

It is shown how the operator can be factorized into two Volterra operators using a matrix Riccati equation. Recursive solution of these triangular operator equations is then obtained by two initialvalue differential equations.

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1. INTRODUCTION.

Linear estimation and linear quadratic control problems formulated in linear function spaces require the solution of integral equation of a certain structure

$$(I+HH^{*})x = y \tag{1.1}$$

where H is a Volterra operator and H^{*} its adjoint.

The solutions to the estimation and control problems are known to be simple and possible to obtain recursively. Thus the integral equation (1.1) must have a simple solution. It will here be shown how factorization of the operator using a matrix Riccati equation can be used to prove the well-known results.

A Fredholm equation of the second kind

$$(I+A)x = y$$

where A is an integral operator is sometimes solved using successive approximations giving a resolvent kernel or by discretizing and using matrix inversion.

In (1.1) H is a Volterra operator, i.e. its kernel $h(t,s)$ is zero for $s > t$, and furthermore the kernel is degenerated

$$h(t,s) = \begin{cases} C(t)\phi(t,s)B(s) & t \geq s \\ 0 & t < s \end{cases}$$

where $\phi(t,s)$ is the fundamental matrix of an ordinary differential equation.

Then the symmetric solution $R(t)$ of a Riccati equation can decompose $I+HH^{*}$

$$I + HH^{*} = (I+HR)(I+RH^{*}) \tag{1.2}$$

R is here regarded as an operator.

The solution of (1.1) is now obtained by recursive solution of the Volterra equations

$$(I+RH^{*})z = y$$

and

$$(I+HR)x = z$$

The idea of decomposition, which very much simplifies calculations, comes from the theory of systems of linear equations, and is here applied to the theory of linear dynamic systems.

The essence of the decomposition will be given in the next chapter, where also the main results are presented. Chapter 3 contains a rigorous treatment of the operator definitions and their major characteristics. In the following three chapters the fundamentals of linear dynamic systems: the estimation, the optimization, and the separation theorem are deduced by means of the operator decomposition technique.

2. BASIC IDEA.

In this chapter I want to demonstrate the main facts that are exploited throughout the thesis.

Consider the functional equation

$$(I + LL^*)x = y \tag{2.1}$$

where x and y are continuous vector functions on (t_0,t_1), and the operator L is defined by

$$z = Lu, \; z(t) = \int_{t_0}^{t} \phi(t,s)u(s)ds, \; t \in [t_0,t_1] \tag{2.2}$$

with the kernel $\phi(t,s)$ satisfying the differential equation

$$\begin{cases} \frac{d}{dt}\phi(t,s) = A\phi(t,s) \\ \phi(s,s) = I \end{cases}$$

L^* means the adjoint operator of L:

$$z = L^*u, \; z(t) = \int_{t}^{t_1} \phi^T(s,t)u(s)ds, \; t \in [t_0,t_1]$$

Since the upper limit of the integral in (2.2) is t, L is a Volterra operator, a triangular operator. Its kernel vanishes for s > t. LL^* can also be considered as an integral operator, but it is no longer triangular. The integral operator LL^* now corresponds to a two point boundary value problem instead of an initial value problem and thus (2.1) illustrates a quite complicated relationship between the functions x and y. My objective is now to offer a procedure to solve (2.1) for x without trying to invert $(I + LL^*)$.

This can easily be accomplished if it is possible to write (2.1) as

$$(I + LR)(I + LR)^* x = y \qquad (2.3)$$

with R simple enough.

In order to demonstrate that R can be chosen merely as a multiplication with a time varying matrix P, consider the matrix Riccati equation

$$\begin{cases} \frac{d}{dt} P(t) = AP + PA^T + I - P^2 \\ P(t_0) = 0 \end{cases}$$

which has a unique, symmetric, nonnegative definite solution for all $t \in [t_0, t_1]$.

Regard this as an operator equality:

$$\begin{cases} \frac{d}{dt} P - P \frac{d}{dt} = AP + PA^T + I - P^2 \\ P(t_0) = 0 \end{cases}$$

where the operator P is defined by

$$y = Px, \; y(t) = P(t)x(t) \qquad P(t) \text{ symmetric}$$

Now let L operate from the left and L^* from the right:

$$L\left(\frac{d}{dt} - A\right)PL^* + LP\left(-\frac{d}{dt} - A^T\right)L^* + LP^2L^* = LL^* \qquad (2.4)$$

Look at

$$z = L\left(\frac{d}{dt} - A\right)Px$$

and

$$y = \left(-\frac{d}{dt} - A^T\right)L^* x$$

Then

$$z(t) = \int_{t_0}^{t} \phi(t,s)\left(\frac{d}{ds} - A\right)P(s)x(s)ds =$$

$$= \phi(t,t)P(t)x(t) - \phi(t,t_0)P(t_0)x(t_0) =$$

$$= P(t)x(t)$$

by partial integration noticing that

$$\frac{d}{ds}\phi(t,s) = -\phi(t,s)A$$

and that

$$P(t_0) = 0$$

Likewise

$$\frac{d}{dt}\phi^T(q,t) = -A^T\phi^T(q,t)$$

implies that

$$y(t) = \left(-\frac{d}{dt} - A^T\right) \int_t^{t_1} \phi^T(q,t)x(q)dq =$$

$$= \phi^T(t,t)x(t) = x(t)$$

Thus the equation (2.4) reduces to

$$PL^* + LP + LP^2L^* = LL^*$$

and summarized:

Theorem 2.1: The operator

$$I + LL^*$$

operating in the space of continuous functions on (t_0,t_1) with L defined by (2.2) can be decomposed into

$$I + LL^* = (I + LP)(I + LP)^*$$

where the operator P is a multiplication with the symmetric solution of the matrix Riccati equation

$$\begin{cases} \dot{P} = AP + PA^T + I - P^2 \\ P(t_0) = 0 \end{cases} \qquad \square$$

Now that the decomposition is done equation (2.1) is easier to solve. Introducing z by

$$(I + LP)z = y \tag{2.5}$$

then x is the solution of

$$(I + LP)^{*} x = z \tag{2.6}$$

Since L is triangular, I + LP is also triangular because of the very simple structure, and since L corresponds to an ordinary differential equation, y is the solution of a linear dynamic system with boundary value at the initial time. z is the solution to the adjoint system with its boundary value at the final time.

Notice that instead of somehow trying to solve the resolvent kernel of (2.1) and then computing the function x by an integral operation on y, the problem is decomposed by solving the Riccati equation and then compute z from a dynamic system. The function x is then obtained from the stored z by the adjoint system. Both storage and computer time is reduced enormously.

Notice also the resemblance to the Gauss decomposition of linear equations (Forsythe-Moler [3]). The major work, the decomposition, the solution of the Riccati equation, can be done once for all, and the minor work, the recursive solution of the two triangular systems of equations, is done for each right hand side y. The words "wherever an inverse appears, something must be wrong" is even more true in the continuous case. Notice that the special structure of (2.1) very much simplifies the decomposition.

Theorem 2.1 or modification of it will in the sequel be used to solve the estimation and optimization problems as well as the separation theorem for linear dynamic systems, starting with very basic linear algebra lemmas. See for instance Brockett [1] , Luenberger [4].

The basis and the results are now formulated as lemmas and theorems (all well-known). The new proofs appear in the later sections.

Lemma 2.1 (projection theorem):

Let x and y be stochastic variables with zero mean, and let x have the components x_i and y the components y_j. Then the minimum variance, linear estimate $\hat{x}$ of x given y is the orthogonal projection of x on the linear subspace, spanned by the y_j:s. By minimum variance of a vector is meant minimum variance of each component. The orthogonality is defined by the scalar product

$$(x_i, x_k) = Ex_i x_k$$

If the y_j:s are linear independent, $\hat{x}$ is given by

$$\hat{x} = R_{xy} R_y^{-1} y \qquad \square$$

This now gives the following three estimation theorems for the dynamic system

$$dx = Axdt + de$$

$x(t_0)$ having zero mean and covariance R_0, with the observation process

$$dy = Cxdt + dv$$

e and v independent Wiener processes with incremental covariance $R_1 dt$ and $R_2 dt$, $R_2 > 0$.

Theorem 2.2 (= Theorem 4.5):
The best filtering estimate $\hat{x}_f$ is given by

$$\begin{cases} d\hat{x}_f = A\hat{x}_f dt + PC^T R_2^{-1}(dy - C\hat{x}_f dt) \\ \hat{x}_f(t_0) = 0 \end{cases}$$

with P from

$$\begin{cases} \dot{P} = AP + PA^T + R_1 - PC^T R_2^{-1} CP \\ P(t_0) = R_0 \end{cases} \qquad \square$$

<u>Corollary</u>: The best predictor $\hat{x}(t+T|t)$ of $x(t+T)$ by use of the outputs up to time t is given by

$$\hat{x}(t+T|t) = \phi(t+T,t)\hat{x}_f(t)$$

where $\phi(t,s)$ is the fundamental matrix for the system. □

<u>Theorem 2.3</u> (= Theorem 4.6):

The best smoothing estimate $\hat{x}(t|t_1)$ of $x(t)$ with information up to t_1, $t \leq t_1$, is given by

$$\hat{x}(t|t_1) = \hat{x}_f(t) + P(t)\lambda(t)$$

with $\hat{x}_f$ and P from Theorem 2.2 and λ defined by the adjoint equation

$$\begin{cases} -d\lambda = (A^T - C^T R_2^{-1} CP)\lambda dt + C^T R_2^{-1}(dy - C\hat{x}_f dt) \\ \lambda(t_1) = 0 \end{cases} \qquad \square$$

Theorem 2.4 (= Theorem 4.7):

The smoothing estimation error, $\tilde{x}$, has the covariance function

$$r_{\tilde{x}}(t,s) = \begin{cases} \Psi(t,s)P(s) - P(t)\Lambda(t)\Psi(t,s)P(s) & t \geq s \\ P(t)\Psi^T(s,t) - P(t)\Psi^T(s,t)\Lambda(s)P(s) & t \leq s \end{cases}$$

where the fundamental matrix Ψ is defined by

$$\begin{cases} \frac{d}{dt}\Psi(t,s) = (A - PC^T R_2^{-1} C)\Psi(t,s) \\ \Psi(s,s) = I \end{cases}$$

P defined in Theorem 2.2 and Λ by

$$\begin{cases} -\frac{d}{dt}\Lambda = (A - PC^T R_2^{-1} C)^T \Lambda + \Lambda(A - PC^T R_2^{-1} C) + C^T R_2^{-1} C \\ \Lambda(t_1) = 0 \end{cases}$$ □

Lemma 2.2:

Minimize the quadratic functional

$$V = x \cdot Px + 2r \cdot x + b$$

where P is a self adjoint, positive definite linear operator. Then the minimizing x is a solution to the linear equation

$$Px = -r$$

The solution x is unique. □

<u>Lemma 2.3:</u>

Minimization of

$$J = x \cdot Px$$

under the linear constraint

$$Ux = y$$

where U is not invertible, has, if the compound operator $UP^{-1}U^{*}$ is invertible, the solution

$$x = P^{-1}U^{*}(UP^{-1}U^{*})^{-1}y \qquad \square$$

These two lemmas give the following two theorems on minimizing

$$V = \int_{t_0}^{t_1} x^TQ_1xdt + x^T(t_1)Q_0x(t_1) + \int_{t_0}^{t_1} u^TQ_2udt, \; Q_2 > 0$$

for the system

$$\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = a \end{cases}$$

Theorem 2.5 (= Theorem 5.2)

Let S be the solution of

$$\begin{cases} -\dfrac{dS}{dt} = A^T S + SA + Q_1 - SBQ_2^{-1}B^T S \\ S(t_1) = Q_0 \end{cases}$$

Then V is minimized for the strategy

$$u = - Q_2^{-1}B^T Sx$$

giving the closed loop system

$$\begin{cases} \dfrac{d}{dt} x = (A - BQ_2^{-1}B^T S)x \\ x(t_0) = a \end{cases}$$

□

Theorem 2.6 (= Theorem 5.5)

Under the additional constraint

$$[I \quad 0]x(t_1) = c$$

V is minimized for the strategy

$$u = - Q_2^{-1}B^T Sx + Bv$$

where

$$v = - Q_2^{-1}B^T \xi, \qquad y = c - [I \quad 0]\Psi(t_1,t_0)a$$

$$\begin{cases} -\dot{\xi} = (A - BQ_2^{-1}B^TS)^T\xi \\ \xi(t_1) = \begin{bmatrix} I \\ 0 \end{bmatrix} W^{-1}(t_0)y \end{cases}$$

provided that

$$W(t_0) = [I \quad 0] \int_{t_0}^{t_1} \Psi(t_1,s)BQ_2^{-1}B^T\Psi^T(t_1,s)ds \begin{bmatrix} I \\ 0 \end{bmatrix}$$

is invertible.

$\Psi(t,s)$ is the fundamental matrix for the closed loop system in Theorem 2.5, and I is the (n-q) dimensional identity matrix. □

Further use of the operator technique gives the separation theorem for minimizing EV when disturbances are accounted for in the system

$$dx = Axdt + Budt + dv$$

and the observation process is given:

$$dy = Cxdt + de$$

The available information when choosing u(t) is the observations up to time t.

Theorem 2.7 (= Theorem 6.2):

The loss function has its minimum

$$EV = m^T S(t_0)m + \text{tr } S(t_0)R_0 + \int_{t_0}^{t_1} \text{tr } S(t)R_1(t)dt +$$

$$+ \int_{t_0}^{t_1} \text{tr } PSBQ_2^{-1}B^T Sdt$$

(with the information above available) for the strategy

$$u = - Q_2^{-1}B^T S\hat{x}_f$$

where $\hat{x}_f$ is the filter estimate of Theorem 2, and where P and S are defined by the dual Riccati equations in Theorems 2.2 and 2.5 respectively. □

Exactly the same program can be performed for the discrete time systems starting with the same three lemmas. The concepts are only much easier since the spaces involved are finite dimensional. The differential equations are replaced by difference equations and the trouble with the Wiener process calculus is avoided, since it is possible to use discrete white noise.

The operators R_y and $(Q_2 + L^* Q_1 L)$ (see Chs. 4-5) are decomposed by the two quadratic difference equations

$$\begin{cases} P(t+1) = \Phi P(t)\Phi^T + R_1 - \Phi P(t)\Theta^T[\Theta P(t)\Theta^T + R_2]^{-1} \cdot \\ \qquad \cdot \Theta P(t)\Phi^T \\ P(t_0) = R_0 \end{cases}$$

and

$$\begin{cases} S(t-1) = \Phi^T S(t)\Phi + Q_1 - \Phi^T S(t)\Gamma[Q_2 + \Gamma^T S(t)\Gamma]^{-1} \cdot \\ \qquad \cdot \Gamma^T S(t)\Phi \\ S(t_1) = Q_0 \end{cases}$$

giving the results in the form of discrete dynamic systems.

If the operator

$I + A$

cannot be written as

$I + LL^*$

but only as

$I + GH^*$

then a corresponding decomposition into

$(I + GR)(I + HR)^*$

might be possible, but further conditions must be imposed in order to guarantee a global solution to the corresponding Riccati equation.

3. DEFINITIONS, NOTATIONS AND OPERATOR CHARACTERISTICS.

3.1. Introduction.

In this chapter I present the notation I am using in the analysis. I also introduce the mathematical framework that I have found suitable for this purpose.

3.2. Notations and Definitions, Scalar Products and Adjoints.

The elements I am interested to describe are functions of time on a finite interval $[t_0,t_1]$. Sometimes I will let the limits vary.

In the filtering problem the basic elements are actually Wiener processes and some concepts have to be generalized to handle these, but as I restrict myself by assuming normality or by additional structuring of the allowed filters to linear, most characteristics and objectives can be described by the functions mean value and covariance.

The state space approach is utilized, i.e.

$$\begin{cases} \dot{x} = Ax + Bu & x(t_0) = a \\ y = Cx + Du & t \in [t_0,t_1] \end{cases} \qquad (3.1)$$

where x(t), u(t) and y(t) are elements of the usual euclidian vector spaces R^n, R^p and R^m respectively.

It is now convenient to consider x as a continuous vector function of time, i.e. an element of a space $L_2[t_0,t_1]$ with the root of the square integral as norm. Bu is also an element of the same space.

No new notation is introduced to cover the space of Wiener processes in the case of a noisy system.

The description (3.1) of a dynamic system is in some respects inconvenient! Many theoretical deductions need closed expressions for the state variables.

If the initial value is zero, the solution x of

$$\dot{x} = Ax + z \tag{3.2}$$

can be viewed as the result of a linear operation in the space $L_2[t_0,t_1]$

$$x = Lz; \quad x(t) = \int_{t_0}^{t} \phi(t,s)z(s)ds \tag{3.3}$$

where $\phi(t,s)$ is the fundamental matrix of (3.2).

L is thus a Volterra operator with its kernel being $\phi(t,s)$ when $t \geq s$ and 0 when $t < s$.

The adjoint operator.

Introduce in $L_2[t_0,t_1]$ a scalar product by

$$x \cdot y = \int_{t_0}^{t_1} x^T(t)y(t)dt$$

Then it is possible to define the adjoint of L by

$$y \cdot Lz = L^* y \cdot z$$

with

$$x = L^{*}y \qquad x(t) = \int_{t}^{t_1} \phi^T(s,t)y(s)ds \tag{3.4}$$

The corresponding adjoint differential equation is

$$-\dot{x} = A^T x + y \tag{3.5}$$

with

$$x(t_1) = 0$$

<u>Initial values</u>.

In order to introduce initial values it is convenient to define operators g and h from R^n to $\mathbb{L}_2[t_0,t_1]$:

$$v = ga: \quad v(t) = \phi(t,t_0)a \tag{3.6}$$

and

$$w = hb: \quad w(t) = \phi^T(t_1,t)b \tag{3.7}$$

The solution of (3.2) with $x(t_0) = a$ will be

$$x = \mathbb{L}z + ga$$

and of (3.5) with $x(t_1) = b$

$$x = L^{*}y + hb$$

Operator $\bar{L}$ and its adjoint.

An ordinary differential equation like (3.2) can be viewed as an operation on an initial value in R^n and on a forcing function in $L_2[t_0,t_1]$, i.e. on $L_2[t_0,t_1]xR^n$, giving the solution in $L_2[t_0,t_1]$.

It is favourable, however, for the sake of symmetry to give special emphasis to the value of the solution at the end point t_1. Thus the differential equation (3.2) implies an operation, say $\bar{L}$, in the space $L_2[t_0,t_1]xR^n$. Let (z,a) and (x,b) be elements in $L_2[t_0,t_1]xR^n$, then $(x,b) = \bar{L}(z,a)$ means

$$\begin{cases} x = Lz + ga \\ b = x(t_1) \end{cases} \tag{3.8}$$

With the scalar product in $L_2[t_0,t_1]xR^n$ defined as the sum of the scalar products in $L_2[t_0,t_1]$ and R^n

$$(z,a) \cdot (x,b) = \int_{t_0}^{t_1} z^T(t)x(t)dt + a^Tb = z \cdot x + a \cdot b$$

it is possible to express an adjoint to $\bar{L}$ by

$$(y,c) = \bar{L}^*(x,b)$$

$$\begin{cases} y = L^*x + hb \\ c = y(t_0) \end{cases} \tag{3.9}$$

Operators T_0 and T_1.

Because of the special emphasis on the end points of the time interval, I have found it suitable with auxiliary operators from $L_2[t_0,t_1]$ to R^n:

$$a = T_0x; \quad a = x(t_0)$$

$$b = T_1x; \quad b = x(t_1)$$

Example

T_0 and T_1 can for instance be used for relating the adjoints of g and h to L^* and L. With the usual scalar product in R^n

$$g^*z = \int_{t_0}^{t_1} \phi^T(s,t_0)z(s)ds = T_0L^*z$$

and

$$h^*z = \int_{t_0}^{t_1} \phi(t_1,s)z(s)ds = T_1Lz$$

Thus it is possible to avoid the operators g and h,

$$(T_0L^*)^* = g, \quad (T_1L)^* = h$$

but I have decided to use them for the sake of simplicity in notation.

3.3. Inversion

Next problem to focus is that of inversion and solution of linear equations.

In the finite dimensional case, when an operator corresponds to a matrix, an equation is especially easy to solve if the matrix is triangular. The solution can be obtained by recursion.

The operator L is triangular, and the recursion when solving (3.3) for z corresponds to an operation containing differentiation.

(3.2) gives

$$z(t) = \left(\frac{d}{dt} - A\right)x(t)$$

Obviously

$$\left(\frac{d}{dt} - A\right)L = I$$

that is d/dt - A is a left hand inverse of L.

Inverse of L.

The range of L is not the whole space $L_2[t_0,t_1]$. It is restricted by the zero initial condition and by the fact that x from x = Lz is differentiable. This defines a linear manifold in $L_2[t_0,t_1]$. When restricted to this manifold

$$\frac{d}{dt} - A$$

is also a right hand inverse of L, i.e.

$$x(t) = \int_{t_0}^{t} \phi(t,s)\left(\frac{d}{ds} - A\right)x(s)ds$$

Therefore introduce the notation

$$L^{-1} = \left(\frac{d}{dt} - A\right) \tag{3.10}$$

although it is dangerous and must be handled with greatest care, since

$$LL^{-1} = I - gT_0 \tag{3.11}$$

if the initial value is different from zero.

Ambiguity of (3.2).

If the initial value condition is not imposed then the solution of (3.2) is not unique, which can be seen from the fact that the homogeneous equation has nontrivial solutions

$$\left(\frac{d}{dt} - A\right)x = 0$$

$$\Rightarrow x(t) = \phi(t,t_0)x(t_0)$$

Thus, with no boundary restrictions, the solution x of

$$\left(\frac{d}{dt} - A\right)x(t) = z(t)$$

can be written like

$$x(t) = \int_{t_0}^{t} \phi(t,s)z(s)ds + \phi(t,t_0)a$$

or

$$x = Lz + ga$$

where a is an arbitrary element of R^n.

This means that $z = L^{-1}x \Leftrightarrow x = Lz + ga$ if there are no restrictions on x!

<u>Inversion of $\bar{L}$.</u>

The operator $\bar{L}$ made it possible to handle boundary conditions to (3.2) and offered a logical way of introducing adjoint boundary conditions. There exists an inverse to $\bar{L}$ too in the same sense as to L.

$$(x,b) = \bar{L}(z,a) \tag{3.8}$$

$$\begin{cases} x = Lz + ga \\ b = T_1 x \end{cases}$$

gives

$$\begin{cases} z(t) = \left(\dfrac{d}{dt} - A\right)x(t) \\ a = x(t_0) = T_0 x \end{cases}$$

defining a left hand inverse to $\bar{L}$.

On the linear manifold in $L_2[t_0,t_1]xR^n$ restricted by x differentiable and $b = T_1 x$ this also defines a right hand inverse.

$$\bar{L}\,\bar{L}^{-1}(x,b) = (x,b), \qquad \bar{L}^{-1}(x,b) = (z,a)$$

$$\begin{cases} z = L^{-1}x \\ a = T_0 x \end{cases}$$

$$\begin{cases} x(t) = \int_{t_0}^{t} \phi(t,s)\left(\frac{d}{ds} - A\right)x(s)ds + \phi(t,t_0)T_0 x \\ b = T_1 x \end{cases}$$

Also the notation $\bar{L}^{-1}$ is somewhat dangerous. If no restriction $b = T_1 x$:

$$(z,a) = \bar{L}^{-1}(x,b) \iff (x,b) = \bar{L}(z,a) + (0,c)$$

with c arbitrary in R^n!

The adjoint operators L^* and $\bar{L}^*$ have inverses under adjoint boundary conditions!

$$L^{*-1} = \left(-\frac{d}{dt} - A^T\right)$$

and

$$(\bar{L}^*)^{-1}(y,c) = (x,b)$$

means

$$\begin{cases} x = L^{*-1}y \\ b = T_1 y \end{cases}$$

Inversion of a dynamic system.

Next problem to be considered is that of inverting a dynamical system.

$$\begin{cases} \dot{x} = Ax + Bu & x(t_0) = a \\ y = Cx + Du & t \in (t_0, t_1) \end{cases} \qquad (3.1)$$

In the introduced notation this can be written

$$y = CLBu + Du + Cga \qquad (3.12)$$

Now specialize to a = 0 giving

$$y = (CLB + D)u \qquad (3.13)$$

Since y(t) and u(t) have different dimensions it is generally not meaningful to speak about the inverse of (CLB + D). A left hand inverse might, however, exist as an operator producing u from y (and a = 0). This is the question of invertability described for instance by Silverman [7] and Sain & Massey [6]. Note that the operator is not necessarily unique. If, however, y(t) and u(t) belong to the same spaces then the existence of a left hand inverse and a right hand inverse are dual problems and the same criteria are applicable [6].

Consider the special case when D is a regular matrix. Then the inversion is always possible and quite simple:

$$\begin{cases} \dot{x} = (A - BD^{-1}C)x + BD^{-1}y & x(t_0) = 0 \\ u = D^{-1}y - D^{-1}Cx \end{cases}$$

which written in operator notion provides the inverse of (3.13)

$$u = (CLB + D)^{-1}y = \\ = \left(D^{-1} - D^{-1}C(L^{-1} + BD^{-1}C)^{-1}BD^{-1}\right)y \qquad (3.14)$$

where the notation

$$(L^{-1} + BD^{-1}C)$$

must be regarded as a differential operator with initial value zero, so that its inverse is uniquely defined. Notice that (3.14) is exactly an operator equivalence to the well-known matrix lemma.

If D is singular or even zero, the equation (3.13) might still be unique. It will, however, contain derivatives of some components of y. For instance in the special case B = C = I, D = 0

$$y = Lu \quad \Rightarrow \quad u = \left(\frac{d}{dt} - A\right)y$$

Another special case is the single input, single output case where CLB is invertible if its kernel, $C\phi(t,s)B$, is not identically zero.

4. FILTERING.

4.1. Introduction, Problem Formulation.

First application of the "triangularization lemma" is made in the field of linear filtering.

Best estimate of the state vector, as a function of time, is found for the linear stochastic system

$$\begin{cases} dx = Axdt + de \\ dy = Cxdt + dv \end{cases} \tag{4.1}$$

where x_0 has mean value m and covariance R_0, and where e and v are independent, zero mean Wiener processes with incremental covariance R_1dt and R_2dt respectively. When otherwise not explicitely pointed out it is assumed that R_2 is nonsingular.

Since the system (4.1) is linear and the deterministic effect of the initial mean value m can be added by superposition, it is no restriction to specialize to

$$m = 0$$

Essential is, however, the initial arbitrariness described by R_0.

My object is to find the best estimate of the function x based on measurements of the function y in the time interval $[t_0,t_1]$. If possible the solution should be made using a minimum of storage. By best estimate is understood any linear estimate minimizing the variance of a linear combination of the estimation error.

This problem can be recognized as the dual of an optimization problem, which opens one way for solution [8]. I will here present another solution where x and y are regarded to be stochastic variables over a function space, and the best estimate is found directly by a generalization of the projection theorem.

4.2. The Projection Theorem.

I will first formulate the projection theorem for stochastic variables over real numbers, and then I will generalize [4 Chapter 4].

Let x and y be two stochastic variables with zero mean. By introducing the scalar product

$$\langle x,y\rangle = \mathrm{cov}(x,y)$$

they can be regarded as elements of a vector space. Since $||x|| = \langle x,x\rangle = 0$ requires $x = 0$, equality is defined in the mean square sense.

Theorem 4.1:

The minimum variance estimate $\hat{x}$ of x with $\hat{x} = \alpha y$ is the orthogonal projection of x on the line spanned by y. Thus

$$\langle x - \hat{x},y\rangle = 0$$

and if $\langle y,y\rangle \neq 0$

$$\hat{x} = \{\langle x,y\rangle/\langle y,y\rangle\}y \tag{4.2}$$

$$||x - \hat{x}||^2 = ||x||^2 - \langle x,y\rangle^2/||y||^2 \qquad \square \qquad (4.3)$$

This can be extended to give an estimate of x as a linear combination of m stochastic variables y_j. $\hat{x}$ is then the orthogonal projection of x on the linear subspace spanned by the y_j:s, and the orthogonality condition says

$$\langle x - \hat{x}, y_j\rangle = 0 \qquad j = 1, \ldots, m$$

Now define

$$y = (y_1, \ldots, y_m)^T$$

and let x consist of n components

$$x = (x_1, \ldots, x_n)^T$$

then by repeating this for each component of x.

<u>Corollary 4.1</u>:

The estimate $\hat{x}$ of x, with $\hat{x} = \alpha y$, minimizing the estimation variance in each component is the orthogonal projection of x (or to be correct of its components) on the subspace spanned by the y_j:s. Thus

$$\langle x_i - \hat{x}_i, y_j\rangle = 0 \qquad i = 1, \ldots, n;\ j = 1, \ldots, m$$

and if the y_j:s are linear independent

$$\hat{x} = R_{xy}R_y^{-1}y \qquad \square$$

Remark 1:

It is obvious that $a^T\hat{x}$ is the minimum variance estimate of a^Tx for all a.

Remark 2:

If the y_j:s were linearly dependent some of them could be eliminated and the dimension of the problem reduced. Computationally this can be done using a pseudoinverse instead.

Even further generalization is necessary. Let y and x be stochastic vector functions on the intervals

$$I_1 = [t_0, t_1] \quad \text{and} \quad I_2 = [t_0, t_2]$$

respectively, i.e. they are indexed both by some natural numbers and by an interval of real numbers.

Corollary 4.2:

The estimate $\hat{x}$ of x with

$$\hat{x}(t) = \int_{t_0}^{t_1} \alpha(t,s)dy(s) \qquad t \in I_2$$

minimizing the estimation variance in each component, i, and at each time, t, is the orthogonal projection of x (or to be correct $x_i(t)$) on the closed linear subspace spanned by $dy_j(s)$, $s \in I_1$, $j = 1, \ldots, m$.

Thus

$$\langle x_i(t) - \hat{x}_i(t), dy_j(s)\rangle = 0 \qquad \begin{array}{l} i = 1, \ldots, n; \\ j = 1, \ldots, m \\ t \in I_2;\ s \in I_1 \end{array}$$

which gives for $dy_j(s)$ linear independent

$$\hat{x} = R_{xdy}R_{dy}^{-1}dy \qquad \square \qquad (4.4)$$

<u>Remark 3</u>:

Also equation (4.3) of Theorem 4.1 can be extended:

$$E(x-\hat{x})(x-\hat{x})^T = R_x - R_{xdy}R_{dy}^{-1}R_{dyx} \qquad (4.5)$$

The meaning of R_{xdy}, R_{dy} and R_{dyx} of (4.4) and (4.5) must be defined properly.

Let

$$r_{xdy}(t,s) = Ex(t)dy^T(s) \qquad t \in I_2,\ s \in I_1$$

$$r_{dy}(t,s) = Edy(t)dy^T(s) \qquad t,s \in I_1$$

and

$$r_{dyx}(t,s) = Edy(t)x^T(s) \qquad t \in I_1,\ s \in I_2$$

Then

$$z = R_{xdy}u$$

means

$$z(t) = \int_{t_0}^{t_1} r_{xdy}(t,s)u(s) \qquad t \in I_2$$

$$dz = R_{dy}u$$

means

$$dz(t) = \int_{t_0}^{t_1} r_{dy}(t,s)u(s) \qquad t \in I_1$$

and

$$z = R_{dyx}u$$

means

$$z(t) = \int_{t_0}^{t_2} r_{dyx}(t,s)u(s) \qquad t \in I_1$$

where the integrals exist since the covariance functions contain differentials. The covariance matrices of for instance Corollary 4.1 correspond to linear operators in finite dimensional spaces. The inverse in (4.4) and (4.5) must be interpreted with care. (4.4) means for instance that there exists a kernel h(t,s) such that

$$r_{xdy}(t,s) = \int_{t_0}^{t_1} h(t,q)r_{dy}(q,s) \qquad t \in I_2,\ s \in I_1$$

giving

$$\hat{x}(t) = \int_{t_0}^{t_1} h(t,s)dy(s) \qquad t \in I_2$$

4.3. Derivation of the Covariance Operators.

In order to apply the results of 4.2, when x and y are stochastic processes - i.e. stochastic variables with their values in a function space - connected by the stochastic system equations (4.1), it is necessary to develop the covariances. As mentioned above the correspondence to a covariance matrix as a linear transformation in the finite dimensional space is the covariance integral operator with the covariance function as its kernel.

Therefore state the following well-known theorem.

Theorem 4.2:

Let the stochastic process x be defined by a differential equation driven by a Wiener process v.

$$dx(t) = Ax(t)dt + dv, \qquad Ex(t_0) = m, Var[x(t_0)] = R_0$$

where v has the incremental covariance R_1dt.

Then the covariance function is

$$r_x(t,s) = \begin{cases} \phi(t,s)r_x(s,s) & t \geq s \\ r_x(t,t)\phi^T(s,t) & t \leq s \end{cases}$$

with

$$r_x(t,t) = \int_{t_0}^{t} \phi(t,q)R_1\phi^T(t,q)dq + \phi(t,t_0)R_0\phi^T(t,t_0)$$

and $\phi(t;s)$ being the fundamental matrix. □

An immediate consequence of

$$dy = Cxdt + de$$

where e is a zero mean Wiener process with incremental covariance R_2dt and independent of v and x_0, is now:

Corollary 4.3:

The covariance functions for the system (4.1) are

$$r_{xdy}(t,s) = r_x(t,s)C^T(s)ds$$

$$r_{dy}(t,s) = \begin{cases} C(t)r_x(t,s)C^T(s)dtds & \text{if } t \neq s \\ C(t)r_x(t,t)C^T(t)dtdt + R_2dt & \text{if } t = s \end{cases}$$ □

It is now possible to express the operators defined in 4.2 in the formalism of Chapter 3. Notice that when operating on Wiener processes it is sometimes necessary to interpret the operators with stochastic integrals.

Theorem 4.3:

Let $I_2 \leq I_1$, that is $t_2 \leq t_1$, then the operators R_{xdy} and R_{dy} can be written as

$$R_{xdy} = (LR_1 + gT_0R_0)L^*C^T \tag{4.6}$$

$$R_{dy} = dt\left\{C(LR_1 + gT_0R_0)L^*C^T + R_2\right\} \tag{4.7}$$

For $t_2 > t_1$

$$z = R_{xdy}u$$

defines $z(t)$ by (4.6) for $t \in (t_0,t_1)$, and by

$$z(t) = \phi(t,t_1)T_1(LR_1 + gT_0R_0)L^*C^Tu$$

for $t \in (t_1,t_2)$.

Proof:

Assume first $t_2 \leq t_1$, and $t \in I_2 = (t_0,t_2)$.

$$z = R_{xdy}u$$

means

$$z(t) = \int_{t_0}^{t_1} r_{xdy}(t,s)u(s) =$$

$$= \int_{t_0}^{t} \left\{ \int_{t_0}^{s} \phi(t,q)R_1\phi^T(s,q)dq + \right.$$

$$+ \phi(t,t_0)R_0\phi^T(s,t_0)\Big\}C^T u(s)ds +$$

$$+ \int_t^{t_1} \Big\{\int_{t_0}^{t} \phi(t,q)R_1\phi^T(s,q)dq +$$

$$+ \phi(t,t_0)R_0\phi^T(s,t_0)\Big\}C^T u(s)ds =$$

= [by changing the integration order] =

$$= \int_{t_0}^{t} \phi(t,q)R_1\Big\{\int_q^t \phi^T(s,q)C^T u(s)ds\Big\}dq +$$

$$+ \int_{t_0}^{t} \phi(t,q)R_1\Big\{\int_t^{t_1} \phi^T(s,q)C^T u(s)ds\Big\}dq +$$

$$+ \phi(t,t_0)R_0\Big\{\int_{t_0}^{t} \phi^T(s,t_0)C^T u(s)ds +$$

$$+ \int_t^{t_1} \phi^T(s,t_0)C^T u(s)ds\Big\} =$$

$$= \int_{t_0}^{t} \phi(t,q)R_1\Big\{\int_q^{t_1} \phi^T(s,q)C^T u(s)ds\Big\}dq +$$

$$+ \phi(t,t_0)R_0 \int_{t_0}^{t_1} \phi^T(s,t_0)C^T u(s)ds$$

Thus

$$z = R_{xdy}u = LR_1L^*C^Tu + gR_0T_0L^*C^Tu =$$

$$= (LR_1 + gT_0R_0)L^*C^Tu$$

Similarly

$$dz(t) = \int_{t_0}^{t_1} r_{dy}(t,s)u(s) =$$

$$= dtC(t)\int_{t_0}^{t_1} r_x(t,s)C^T(s)u(s)ds + dtR_2u(t) =$$

$$= dtR_2u(t) + dtC\int_{t_0}^{t} \phi(t,q)R_1\left\{\int_{q}^{t_1} \phi^T(s,q)C^Tu(s)ds\right\}dq +$$

$$+ dtC\phi(t,t_0)R_0\int_{t_0}^{t_1} \phi^T(s,t_0)C^Tu(s)ds$$

and

$$dz = R_{dy}u = dt\left\{C(LR_1 + gT_0R_0)L^*C^T + R_2\right\}u$$

Now assume $t_2 > t_1$.

Note that R_{dy} is not affected.

$$z = R_{xdy}u$$

means

$$z(t) = \int_{t_0}^{t_1} r_{xdy}(t,s)u(s) \qquad t \in I_2 = (t_0,t_2)$$

which for $t \in I_1 = (t_0,t_1)$ is evaluated above and for $t \in (t_1,t_2)$ can be rewritten like

$$z(t) = \Phi(t,t_1) \int_{t_0}^{t_1} r_{xdy}(t_1,s)u(s) \qquad \square$$

4.4. Triangularization of the Two Point Boundary Value Problem.

I will now further calculate the estimate of the state of the stochastic system (4.1). The operators from 4.3

$$R_{xdy} = (LR_1 + gT_0R_0)L^*C^T \tag{4.6}$$

$$R_{dy} = dt\left\{C(LR_1 + gT_0R_0)L^*C^T + R_2\right\} \tag{4.7}$$

inserted in the expression for the estimate

$$\hat{x} = R_{xdy}R_{dy}^{-1}dy \tag{4.4}$$

yield

$$\hat{x} = (LR_1 + gT_0R_0)L^*C^T\left\{dt\left(C(LR_1 + gT_0R_0)L^*C^T + R_2\right)\right\}^{-1}dy \tag{4.8}$$

This expression is unusable as it stands! It contains the inversion of a complicated Fredholm integral operator.

However, there is great resemblance to the operators in Chapter 2, which opens up possibilities for a solution via the Riccati equation.

Theorem 4.4:

Let $P(t)$, $t \in I = (t_0, t_1)$, be the unique solution of the matrix Riccati equation

$$\begin{cases} \dot{P} = AP + PA^T + R_1 - PC^T R_2^{-1} CP \\ P(t_0) = R_0 \end{cases} \tag{4.9}$$

If P is interpreted as an operator in $C[t_0, t_1]$, $y = Px$, meaning $y(t) = P(t)x(t)$, $t \in [t_0, t_1]$, then the following triangularizations are possible:

$$C(LR_1 + gT_0R_0)L^*C^T + R_2 = (R_2 + CLPC^T)R_2^{-1}(R_2 + CPL^*C^T) \tag{4.10}$$

and

$$(LR_1 + gT_0R_0)L^*C^T = PL^*C^T + LPC^TR_2^{-1}(R_2 + CPL^*C^T) \tag{4.11}$$

Proof:

Regarding the solution P of (4.9) as an operator P, the commutator between P and d/dt can be written:

$$\frac{d}{dt}P - P\frac{d}{dt} = AP + PA^T + R_1 - PC^TR_2^{-1}CP$$

Now let L operate from the left and L^* from the right.

$$L\left(\frac{d}{dt} - A\right)PL^* + LP\left(-\frac{d}{dt} - A^T\right)L^* + LPC^TR_2^{-1}CPL^* = LR_1L^*$$

In Chapter 3 I showed that

$$L\left(\frac{d}{dt} - A\right) = I - gT$$

and

$$\left(-\frac{d}{dt} - A^T\right)L^* = I$$

$P(t_0) = R_0$ implies $T_0P = T_0R_0$, thus

$$PL^* - gT_0R_0L^* + LP + LPC^TR_2^{-1}CPL^* = LR_1L^*$$

or

$$LR_1L^* + gT_0R_0L^* = PL^* + LP + LPC^TR_2^{-1}CPL^*$$

Consequently

$$(LR_1L^* + gT_0R_0L^*)C^T = PL^*C^T + LPC^TR_2^{-1}(R_2 + CPL^*C^T)$$

and

$$C(LR_1L^* + gT_0R_0L^*)C^T + R_2 = (R_2 + CLPC^T)R_2^{-1}(R_2 + CPL^*C^T)$$

which completes the proof. □

This theorem gives the nucleus of my argumentation. The complicated operator R_{dy} is decomposed into two "triangular" operators and the inversion of R_{dy} will be simplified.

Rewrite (4.8) by means of (4.10) and (4.11).

$$\hat{x} = \left\{PL^{*}C^{T} + LPC^{T}R_2^{-1}(R_2 + CPL^{*}C^{T})\right\} \cdot$$

$$\cdot \left\{(R_2 + CPL^{*}C^{T})^{-1}R_2(R_2dt + dtCLPC^{T})^{-1}\right\}dy$$

Consequently

$$\hat{x} = \left[LPC^{T} + PL^{*}C^{T}(R_2 + CPL^{*}C^{T})^{-1}R_2\right] \cdot$$

$$\cdot (R_2dt + dtCLPC^{T})^{-1}dy \qquad (4.12)$$

Notice that the analysis of Chapter 3 concerning dynamic systems implies that $(R_2 + CLPC^{T})$ and $(R_2 + CPL^{*}C^{T})$ are invertible at least if R_2 is a nonsingular matrix.

Notice also that some of the operators must be interpreted as stochastic integrals!

A sufficient condition for R_{dy} to be invertible is now obvious. When R_2 is nonsingular then, according to Chapter 3, both the factors of the right hand side of (4.10) are invertible and consequently also the left hand side and R_{dy}. Eq. (4.12) is a main result. In the following it will be simplified in different special cases.

4.5. Specializing to $t = t_1$. Filtering.

The result of 4.4, eq. (4.12), is very general. It tells how to obtain the best estimate of the function x in the whole interval $I_2 = (t_0,t_2)$ on basis of measurements during the interval $I_1 = (t_0,t_1)$ (if $t_2 \leq t_1$). This is the general smoothing problem.

What is more interesting is that the upper limits of the intervals I_2 and I_1 have no special significance! Therefore it is possible to let them vary and study the effects on the estimation function.

In this section I will specialize to $t = t_1$, that is I am only interested in $\hat{x}(t_1)$. This is the filtering problem. I want to estimate the current state relying on the measurements up to now.

The equation (4.12) is rewritten

$$\hat{x} = LPC^T(R_2dt + dtCLPC^T)^{-1}dy +$$

$$+ PL^*C^T(R_2 + CPL^*C^T)^{-1}R_2(R_2dt + dtCLPC^T)^{-1}dy$$

Notice that the last term will vanish.
Call it PL^*Xdy
or

$$P(t)\int_t^{t_1} \phi^T(s,t)\ X\ ds$$

which is obviously zero for $t=t_1$.

Denote

$$z = (R_2dt + dtCLPC^T)^{-1}dy \tag{4.13}$$

meaning

$$(R_2 dt + dtCLPC^T)z = dy$$

and

$$zdt = R_2^{-1}(dy - CdtLPC^Tz) \tag{4.14}$$

Introduce $\hat{x}_f = LPC^T(R_2dt + dtCLPC^T)^{-1}dy$

and $\hat{x}(t|t) = \hat{x}_f(t)$

Thus

$$\hat{x}_f = LPC^Tz$$

with z from the inverse dynamic system (4.13).
Now the term LPC^Tz is recognized in the expression (4.14) for z and

$$zdt = R_2^{-1}(dy - Cdt\hat{x}_f)$$

Thus instead of solving the two dynamic systems giving first z then $\hat{x}_f$ it is possible to solve $\hat{x}_f$ directly

$$\hat{x}_f = LPC^TR_2^{-1}\left(\frac{dy}{dt} - C\hat{x}_f\right)$$ understood as a stochastic integral

which is equivalent to the differential

$$\begin{cases} d\hat{x}(t|t) = A\hat{x}(t|t)dt + PC^TR_2^{-1}(dy - C\hat{x}(t|t)dt) = \\ \qquad = (A - PC^TR_2^{-1}C)\hat{x}(t|t)dt + PC^TR_2^{-1}dy \\ \hat{x}(t_0|t_0) = 0 \end{cases} \tag{4.15}$$

Theorem 4.5:

The best linear filter estimate for (4.1) is given by the differential (4.15).

Remark 1: In this deduction there are three crucial points.

1. The original Fredholm equation (4.8) is separated into two Volterra equations (4.12) by the Riccati equations.
2. Half of the problem, the one with boundary conditions at final time vanishes for $t = t_1$.
3. The two systems

$$\hat{x}_f = LPC^T z$$

and

$$(R_2 dt + dtCLPC^T)z = dy$$

can be united into one dynamic system (4.15).

Using the matrix analogy this means that instead of first solving z with (3.14), the identity

$$LPC^T(R_2 + CLPC^T)^{-1} = (L^{-1} + PC^T R_2{}^{-1}C)^{-1} PC^T R_2{}^{-1}$$

could be used to obtain $\hat{x}_f$ directly. The inverse on the right hand side must be understood with zero initial condition.
Notice also that the inverse system giving z obtained by (3.14) has the same dynamics as (4.15). In fact $\hat{x}_f$ would be equal to the state variables of the z-system.

Remark 2:

Note that with the definition

$$d\tilde{y}(t) = dy(t) - C\hat{x}(t|t)dt$$

it can be proven that $d\tilde{y}(t)$ and $d\tilde{y}(s)$ are uncorrelated (mutually orthogonal and linear independent), if

$t \neq s$, and since $\hat{x}$ is a linear function of dy, $d\tilde{y}$ is an orthogonal base of the subspace spanned by dy. For this reason $d\tilde{y}$ are often called the innovations.

The equation (4.15) can be regarded as another system representation for the output $\{y(t)\}$ of (4.1), the innovations representation

$$\begin{cases} d\hat{x}(t|t) = A\hat{x}(t|t)dt + PC^T R_2^{-1} d\tilde{y}, \quad x(t_0|t_0) = 0 \\ dy = C\hat{x}(t|t)dt + d\tilde{y} \end{cases}$$

The incremental covariance of the Wiener process $\tilde{y}$ can be proven to be $R_2 dt$.

4.6. Prediction $t > t_1$.

When having solved the filtering problem, it is easy to solve the prediction problem, that is to estimate x(t) on basis of measurement up to t_1 for $t > t_1$. The notation $\hat{x}(t|t_1)$ is suitable.

This case was not covered in 4.4, but the expression for R_{xdy}, given in 4.3, can easily be used, giving

$$\hat{x}(t|t_1) = \phi(t,t_1)T_1(LR_1 + gT_0R_0)L^* C^T R_{dy}^{-1} dy$$

and

$$\hat{x}(t|t_1) = \phi(t,t_1)\hat{x}(t_1|t_1), \quad t \geq t_1 \qquad (4.16)$$

The interpretation of this is natural. Since no extra information is available compared with the filtering case, nothing better can be done than to assume that $\hat{x}(t_1|t_1)$ is correct and than to let it follow the given state equation with zero noise.

It is also here possible to let t_1 increase, and for instance predict the state h time units ahead of this

$$\hat{x}(t_1 + h|t_1) = \phi(t_1 + h, t_1)\hat{x}(t_1|t_1), \qquad h \geq 0$$

where $\hat{x}(t_1|t_1)$ is obtained recursively according to 4.5.

4.7. Smoothing.

Next application of equation (4.12) is smoothing. x(t) is estimated on basis of measurements up to t_1 with t less than t_1. The symbol $\hat{x}(t|t_1)$ will be used.

The simplification from filtering and prediction, that the term containing backward integration is zero, is no longer valid. The full effect of the triangularization will be seen.

Still referring to (4.12)

$$\hat{x} = LPC^T(R_2dt + dtCLPC^T)^{-1}dy + PL^*C^T(R_2 + CPL^*C^T)^{-1}R_2 \cdot$$

$$\cdot (R_2dt + dtCLPC^T)^{-1}dy$$

Introduce the function λ by

$$\lambda = L^*C^T(R_2 + CPL^*C^T)^{-1}R_2(R_2dt + dtCLPC^T)^{-1}dy$$

and

$$\hat{x} = \hat{x}_f + P\lambda$$

In the same way as in 4.5 it is possible to modify the inverses:

$$L^{*}C^{T}(R_2 + CPL^{*}C^{T})^{-1}R_2 = (L^{*-1} + C^{T}R_2^{-1}CP)^{-1}C^{T}$$

and

$$C^{T}(R_2 dt + dtCLPC^{T})^{-1}dy = C^{T}R_2^{-1}\left(\frac{dy}{dt} - C\hat{x}_f\right)$$

Thus

$$\begin{cases} -\,d\lambda(t) = (A^{T} - C^{T}R_2^{-1}CP)\lambda(t)dt + C^{T}R_2^{-1}(dy - C\hat{x}_f dt) \\ \lambda(t_1) = 0 \end{cases}$$

and the following theorem can be formulated.

Theorem 4.6:

The smoothing estimate $\hat{x}(t|t_1)$ for the system (4.1) is obtained by first calculating the filtering estimate $\hat{x}(t|t)$ from the information available at time t:

$$\begin{cases} d\hat{x}(t|t) = (A - PC^{T}R_2^{-1}C)\hat{x}(t|t)dt + PC^{T}R_2^{-1}dy(t) \\ \hat{x}(t_0|t_0) = 0 \end{cases} \qquad (4.15)$$

The innovations $d\tilde{y} = dy - C\hat{x}_f dt$ during (t,t_1) are then calculated by further integration of (4.15) up to t_1.

The adjoint equation

$$\begin{cases} - d\lambda(t) = (A^T - C^T R_2^{-1} CP)\lambda(t)dt + C^T R_2^{-1} d\tilde{y}(t) \\ \lambda(t_1) = 0 \end{cases} \quad (4.17)$$

is then solved from t_1 and backwards.

Finally

$$\hat{x}(t|t_1) = \hat{x}(t|t) + P(t)\lambda(t) \quad \square \quad (4.18)$$

Remark:

This is one of many formulations, and it has the great advantage that both the integrations are stable under mild observability conditions. It is also reasonable to start to find a rough estimate $\hat{x}(t|t)$, which is then improved by adding the presumably small correction term $P(t)\lambda(t)$.

The limit t_1 can also here be varied and formulas giving best estimate of $x(t)$, t fixed or $x(t_1 - T)$ for example can be deduced from (4.18).

4.8. The Covariance Function of the Estimation Error.

The projection theorem in 4.2 also gives an expression for the covariance operator for the estimation error $\tilde{x}$.

$$R_{\tilde{x}} = R_x - R_{xdy} R_{dy}^{-1} R_{dyx}$$

Eliminate the prediction case by assuming $I_1 = I_2$, $t_1 = t_2$. In 4.3 was shown

$$R_{xdy} = (LR_1 + gT_0R_0)L^{*}C^T$$

$$R_{dy} = dt\left[C(LR_1 + gT_0R_0)L^{*}C^T + R_2\right]$$

and similarly

$$R_x = (LR_1 + gT_0R_0)L^{*}\frac{1}{ds}$$

$$R_{dyx} = dtC^T(LR_1 + gT_0R_0)L^{*}\frac{1}{ds}$$

$$=> R_{\tilde{x}}ds = (LR_1 + gT_0R_0)L^{*} - (LR_1 + gT_0R_0)L^{*}C^T \cdot$$

$$\cdot \left[C(LR_1 + gT_0R_0)L^{*}C^T + R_2\right]^{-1} C^T(LR_1 + gT_0R_0)L^{*}$$

Using the decomposition of 4.4:

$$LR_1L^{*} + gT_0R_0L^{*} = PL^{*} + LP + LPC^TR_2^{-1}CPL^{*}$$

the following expression is obtained:

$$R_{\tilde{x}}ds = LP + PL^{*} + LPC^TR_2^{-1}CPL^{*} -$$

$$- \left[LPC^T + PL^{*}C^T(I + R_2^{-1}CPL^{*}C^T)^{-1}\right] \cdot$$

$$\cdot \left[(R_2 + CLPC^T)^{-1}CLP + R_2^{-1}CPL^{*}\right] =$$

$$= PL^{*}\{I - C^T(I + R_2^{-1}CPL^{*}C^T)^{-1} \cdot$$

$$\cdot\ [(R_2 + CLPC^T)^{-1}CLP + R_2^{-1}CPL^*]\Big\} +$$

$$+ LP - LPC^T(R_2 + CLPC^T)^{-1}CLP =$$

$$= LP - LPC^T(R_2 + CLPC^T)^{-1}CLP +$$

$$+ PL^* - PL^*C^T(R_2 + CPL^*C^T)^{-1}CPL^* -$$

$$- PL^*C^T(R_2 + CPL^*C^T)^{-1}R_2(R_2 + CLPC^T)^{-1}CLP =$$

$$= (L^{-1} + PC^TR_2^{-1}C)^{-1}P + P(L^{*-1} + C^TR_2^{-1}CP)^{-1} -$$

$$- P(L^{*-1} + C^TR_2^{-1}CP)^{-1}C^TR_2^{-1}C(L^{-1} + PC^TR_2^{-1}C)^{-1}P \tag{4.19}$$

where the last equality is obtained by the matrix lemma. The inverses must be properly understood as integral operators having the fundamental matrix

$$\begin{cases} \frac{d}{dt}\Psi(t,s) = (A - PC^TR_2^{-1}C)\Psi(t,s) \\ \Psi(s,s) = I \end{cases} \tag{4.20}$$

and its transpose as kernels (cf. Chapter 3).

Now interpret $R_{\tilde{x}}$ as an integral operator with the covariance function of $\tilde{x}$ as the kernel:

$$z = R_{\tilde{x}}du, \quad z(t) = \int_{t_0}^{t_1} r_{\tilde{x}}(t,s)du(s)$$

giving

Theorem 4.7:

The smoothing estimation error for the system (4.1) has the covariance function

$$r_{\tilde{x}}(t,s) = \begin{cases} \Psi(t,s)P(s) - P(t)\Lambda(t)\Psi(t,s)P(s) & t > s, \leq t_1 \\ P(t)\Psi^T(s,t) - P(t)\Psi^T(s,t)\Lambda(s)P(s) & t < s, \leq t_1 \end{cases} \quad (4.21)$$

where the fundamental matrix Ψ is defined by (4.20) and where $\Lambda(t)$ satisfies the differential equation

$$\begin{cases} -\frac{d}{dt}\Lambda(t) = (A - PC^TR_2^{-1}C)^T\Lambda(t) + \\ \qquad + \Lambda(t)(A - PC^TR_2^{-1}C) + C^TR_2^{-1}C \\ \Lambda(t_1) = 0 \end{cases} \quad (4.22)$$

□

Remark 1:

An interesting special case is $t = t_1$, the filtering case. Then $\Lambda = 0$ and the covariance function is the well-known $\Psi(t,s)P(s)$, $t \geq s$. The second term of (4.21) describes how the extra information in the smoothing case, obtained from (4.17), decreases the estimation variance.

Remark 2:

The form of the covariance function (4.21) is possible to obtain directly from the structure of the estimate, (4.18), and the estimation error

$\tilde{x} = \tilde{x}_f - P\lambda$

Since λ is a linear function of $d\tilde{y}(s)$, $s > t$, and since the projection theorem construction says that $\tilde{x}$ and $d\tilde{y}$ are uncorrelated (orthogonal) for all times, it follows that λ and $\tilde{x}$ are also uncorrelated and

$$r_{\tilde{x}}(t,s) + r_{\lambda}(t,s) = r_{\tilde{x}_f}(t,s)$$

with r_λ being the covariance function of the stochastic process defined by (4.17). Noting that $\tilde{y}$ has the incremental covariance R_2dt the theorem follows immediately.

4.9. Correlated e and v.

One major restriction on the original problem formulation is in the model assumption

$$\begin{cases} dx = Axdt + de \\ dy = Cxdt + dv \end{cases} \qquad (4.1)$$

$$x(t_0) = x_0$$

where x_0 has mean value $m = 0$ and covariance R_0 and where e and v are independent.

This will now be generalized so that correlation between e and v is allowed

$$E\, dedv^T = R_{12}dt$$

Just as in 4.3 R_{xdy} and R_{dy} can be derived and expressed in the operator notation of Chapter 3.

$$R_{xdy} = gT_0R_0L^*C^T + LR_1L^*C^T + LR_{12}$$

$$R_{dy} = dt(R_2 + CgT_0R_0L^*C^T + CLR_1L^*C^T + CLR_{12} + R_{12}^TL^*C^T)$$

When calculating

$$\hat{x} = R_{xdy}R_{dy}^{-1}dy$$

a Riccati equation should be determined in order to triangularize the problem.

Theorem 4.8:

Let $P(t)$, $t \in (t_0,t_1)$, be the solution of the matrix Riccati equation

$$\begin{cases} \dot{P} = AP + PA^T + R_1 - [R_{12} + PC^T]R_2^{-1}[R_{12} + PC^T]^T \\ P(t_0) = R_0 \end{cases} \qquad (4.23)$$

and interpret P as an operator in $L_2[t_0,t_1]$, then the following decompositions are valid:

$$C(LR_1 + gT_0R_0)L^*C^T + CLR_{12} + R_{12}^TL^*C^T + R_2 =$$

$$= \left(R_2 + CL(PC^T + R_{12})\right)R_2^{-1}\left(R_2 + (R_{12} + PC^T)^TL^*C^T\right)$$

and

$$(LR_1 + gT_0R_0)L^{*}C^T + LR_{12} =$$

$$= PL^{*}C^T + L(PC^T + R_{12})R_2^{-1}\left[R_2 + (R_{12} + PC^T)^T L^{*} C^T\right]$$

The proof is equivalent to that of Section 4.3.

The estimate is now

$$\hat{x} = L(PC^T + R_{12})\left\{R_2 dt + dtCL(PC^T + R_{12})\right\}^{-1} dy +$$

$$+ PL^{*}C^T\left[R_2 + (R_{12} + PC^T)^T L^{*} C^T\right]^{-1} R_2\left[R_2 dt +\right.$$

$$\left. + dtCL(PC^T + R_{12})\right]^{-1} dy \qquad (4.24)$$

and like in 4.5

$$\hat{x}_f = L(PC^T + R_{12})\left\{R_2 dt + dtCL(PC^T + R_{12})\right\}^{-1} dy$$

Thus

$$\hat{x}_f = (L^{-1} + PC^TR_2^{-1} + R_{12}R_2^{-1}C)^{-1}(PC^TR_2^{-1} + R_{12}R_2^{-1})\,\frac{dy}{dt}$$

or

$$\begin{cases} d\hat{x}(t|t) = (A - PC^TR_2^{-1}C - R_{12}R_2^{-1}C)\hat{x}(t|t)dt + \\ \qquad + (PC^TR_2^{-1} + R_{12}R_2^{-1})dy \\ \qquad = A\hat{x}(t|t)dt + (PC^TR_2^{-1} + R_{12}R_2^{-1})\left[dy - C\hat{x}(t|t)dt\right] \\ \hat{x}(t_0|t_0) = 0 \end{cases} \qquad (4.25)$$

I want to emphasize that P is now defined by (4.23). The prediction and smoothing could be obtained just as for the special case with independent e and v.

The above calculation only showed the possibility to generalize the decomposition with an extended Riccati equation.

There is, however, an interesting special case, that gives further insight in the nature of the Riccati equation decomposition, and that has consequences for the system description.

It is the case when de = Kdv (in mean square sense!).

$$\begin{cases} dx = Axdt + Kdv \\ dy = Cxdt + dv \end{cases} \qquad (4.26)$$

$$x(t_0) = x_0, \quad Ex_0 = 0, \quad Ex_0x_0^T = R_0$$

This implies

$$R_1 = KR_2K^T \quad \text{and} \quad R_{12} = KR_2$$

giving

$$\begin{aligned} R_{xdy} &= LKR_2(I + K^TL^{*}C^T) + gT_0R_0L^{*}C^T \\ R_{dy} &= dt\left\{(I + CLK)R_2(I + K^TL^{*}C^T) + CgT_0R_0L^{*}C^T\right\} \end{aligned} \qquad (4.27)$$

Note the special structure if $R_0 = 0$.

The decomposition of the estimation expression now requires the Riccati equation

$$\begin{cases} \dot{P} = AP + PA^T + KR_2K^T - (KR_2 + PC^T)R_2^{-1}(KR_2 + PC^T)^T \\ P(t_0) = R_0 \end{cases}$$

or rewritten

$$\begin{cases} \dot{P} = (A - KC)P + P(A - KC)^T - PC^TR_2^{-1}CP \\ P(t_0) = R_0 \end{cases} \tag{4.28}$$

Note the solution $P(t) \equiv 0$ for $R_0 = 0$.

The correspondence of (4.24) is

$$\hat{x} = L(PC^T + KR_2)\left\{R_2dt + dtCL(PC^T + KR_2)\right\}^{-1}dy +$$
$$+ PL^{*}C^T\left[R_2 + (KR_2 + PC^T)^TL^{*}C^T\right]^{-1}R_2 \cdot$$
$$\cdot \left\{R_2dt + dtCL(PC^T + KR_2)\right\}^{-1}dy \tag{4.29}$$

and

$$\hat{x}_f = L(K + PC^TR_2^{-1})\left[Idt + dtCL(K + PC^TR_2^{-1})\right]^{-1}dy$$

Thus

$$\begin{cases} d\hat{x}(t|t) = (A - KC - PC^TR_2^{-1}C)\hat{x}(t|t)dt + (K + PC^TR_2^{-1})dy = \\ \qquad = A\hat{x}(t|t)dt + (K + PC^TR_2^{-1})\left(dy - C\hat{x}(t|t)\right) \\ \hat{x}(t_0|t_0) = 0 \end{cases} \tag{4.30}$$

(4.29) also implies

$$\hat{x} = \hat{x}_f + P\lambda$$

with

$$\lambda = L^{*}C^{T}\left(R_2 + KR_2 + PC^{T})^{T}L^{*}C^{T}\right)^{-1}R_2 \cdot$$

$$\cdot \left(R_2 dt + dtCL(PC^{T} + KR_2)\right)^{-1} dy$$

or

$$\lambda = (L^{*-1} + C^{T}K^{T} + C^{T}R_2^{-1}CP)^{-1}C^{T}R_2^{-1}\left(\frac{dy}{dt} - C\hat{x}_f\right)$$

Now state:

<u>Theorem 4.9</u>:

The best linear estimate of the state of (4.26) is given by

$$\hat{x}(t|t_1) = \hat{x}(t|t) + P(t)\lambda(t)$$

with P obtained from (4.28), $\hat{x}(t|t)$ from (4.30) and $\lambda(t)$ given by the adjoint equation

$$\begin{cases} -d\lambda(t) = (A - KC - PC^{T}R_2^{-1}C)^{T}\lambda(t)dt + C^{T}R_2^{-1}\left(dy - C\hat{x}(t|t)\right) \\ \lambda(t_1) = 0 \end{cases}$$ □

Remark 1:

It is interesting to observe what happens in (4.28) with constant A, C, K, R_2, when $t_0 \to -\infty$. If (A - KC) has all its eigenvalues in the left half plane, is stable, then P approaches zero, but if A - KC has an eigenvalue $\lambda_i > 0$ then $P \nrightarrow 0$, and such that $(A - KC - PC^T R_2^{-1} C)$ has the eigenvalue $-\lambda_i$ instead (cf. [5]).

Remark 2:

If $R_0 = 0$, then $P(t) \equiv 0$ and

$$\hat{x}(t|t_1) = \hat{x}(t|t)$$

given by

$$\begin{cases} d\hat{x}(t|t) = A\hat{x}(t|t)dt + K\big(dy - C\hat{x}(t|t)dt\big) \\ \hat{x}(t_0|t_0) = 0 \end{cases} \tag{4.31}$$

which is equal to the state equation (4.26) (in mean square sense) with dv solved from the observation equation.

Note also that (4.28) is not continuous in the initial value. Even a very small change in R_0 may cause a considerable change in P(t) for unstable (A - KC).

Remark 3:

The innovations representation for (4.1) described in Remark 2 after Theorem 5 is of the form (4.26) with $x(t_0) = 0$, $Ed\tilde{y}d\tilde{y}^T = R_2 dt$, $K = PC^T R_2^{-1}$ and with stable (A - KC). The state in (4.26) is (in mean square sense) equal to the filter estimate of the state in (4.1).

The filter estimate and the smoothing estimate of the state in (4.26) are equal and in mean square sense equal to the state itself, since they are defined by the same differential equation.

5. OPTIMIZATION.

5.1. Introduction.

Next application of the triangularization lemma is the optimization problem. The well-known case with linear dynamics and quadratic loss is worked out both with unrestricted and restricted end state vector. The end time is considered to be fix.

Like in the filter case half of the problem vanishes with the free end state.

When the end point is somewhat restricted it results in a two point boundary value problem and the full effect of the triangularization becomes obvious. It is also showed how the controllability is needed for the posed problem to make sense.

5.2. Free End Point Problem.

Let the performance of a system

$$\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \end{cases} \qquad (5.1)$$

be described by the functional

$$V = \frac{1}{2}\int_{t_0}^{t_1} (x^T Q_1 x + u^T Q_2 u)dt + \frac{1}{2}\, x^T(t_1) Q_0 x(t_1) \qquad (5.2)$$

Q_1 and Q_0 are nonnegative definite and Q_2 positive definite.

The problem is now to find the best input signal $u = u(t)$, $t \in (t_0, t_1)$, to the system (5.1) so that the measure V is minimized.

Using the nomenclature of Chapter 3:

Minimize the quadratic functional expression

$$2V = \{x, x(t_1)\} \cdot \{Q_1 x,\ Q_0 x(t_1)\} + u \cdot Q_2 u \qquad (5.2')$$

under the linear constraint

$$x = LBu + gx_0 \qquad (5.1')$$

or

$$\{x, x(t_1)\} = \bar{L}(Bu, x_0) \qquad (5.1'')$$

Define also a new operator Q on $L_2[t_0, t_1] x R^n$ by

$$Q\{x, x(t_1)\} = \{Q_1 x, Q_0 x(t_1)\}$$

Thus

$$2V = \bar{L}(Bu, x_0) \cdot Q\bar{L}(Bu, x_0) + u \cdot Q_2 u$$

and using adjoint operators:

$$2V = (Bu, x_0) \cdot \bar{L}^{*} Q\bar{L}(Bu, x_0) + u \cdot Q_2 u$$

$\bar{L}^{*}$ was computed in Section 3.

Thus

$$\bar{L}^{*}Q\bar{L} = \begin{bmatrix} L^{*}Q_1L + hQ_0T_1L & L^{*}Q_1g + hQ_0T_1g \\ T_0\left\{L^{*}Q_1L + hQ_0T_1L\right\} & T_0\left\{L^{*}Q_1g + hQ_0T_1g\right\} \end{bmatrix} =$$

$$= \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \tag{5.3}$$

and

$$V = \frac{1}{2}\, u \cdot (Q_2 + B^TM_1B)u + \frac{1}{2}\, u \cdot B^TM_2x_0 +$$

$$+ \frac{1}{2}\, x_0 \cdot M_3Bu + \frac{1}{2}\, x_0 \cdot M_4x_0$$

The adjoint of

$$M_3B = T_0L^{*}Q_1LB + T_0hQ_0T_1LB$$

is equal to

$$B^TM_2 = B^TL^{*}Q_1g + B^ThQ_0T_1g$$

since according to Chapter 3

$$g = (T_0L^{*})^{*}$$

and

$$h = (T_1L)^{*}$$

as well as

$$T_1 g = (T_0 h)^*$$

This can also be seen from the fact that $\bar{L}^* Q \bar{L}$ is self-adjoint. Consequently

$$V = \frac{1}{2}\, u \cdot (Q_2 + B^T M_1 B) u + u \cdot B^T M_2 x_0 + \frac{1}{2}\, x_0 \cdot M_4 x_0 \tag{5.4}$$

Now remains the unconstrained minimization of a quadratic functional for which the following well-known theorem [1] is directly applicable.

<u>Theorem 5.1</u>:

Consider the quadratic functional

$$V = x \cdot Px + 2r \cdot x + b \tag{5.5}$$

where P is a self-adjoint, positive definite linear operator. Then the minimizing x is the solution of the linear equation

$$- Px = r \tag{5.6}$$

The minimizing x is unique.

The proof is done by "completing squares". □

Inserting from (5.4), (5.3)

$$P = Q_2 + B^T (L^* Q_1 + h Q_0 T_1) LB$$

and

$$r = B^T(L^* Q_1 + hQ_0T_1)gx_0$$

gives $u = u_{min}$ from

$$\left(Q_2 + B^T(L^* Q_1 + hQ_0T_1)LB\right)u = - B^T(L^* Q_1 + hQ_0T_1)gx_0 \qquad (5.7)$$

The left hand operator contains Fredholm operators and the solution is hard to obtain. The "triangularization" of Chapter 2 can, however, be used just as in Chapter 4.

Regard $S = S(t)$ as the unique matrix solution of

$$\begin{cases} -\dot{S} = A^TS + SA + Q_1 - SBQ_2^{-1}B^TS \\ S(t_1) = Q_0 \end{cases} \qquad (5.8)$$

and also regard S as a linear operator

$$-\frac{d}{dt}S + S\frac{d}{dt} = A^TS + SA + Q_1 - SBQ_2^{-1}B^TS$$

Since from Chapter 3

$$L^*\left(-\frac{d}{dt} - A^T\right) = I - hT_1$$

and $\left(\frac{d}{dt} - A\right)L = I$ as well as $\left(\frac{d}{dt} - A\right)g = 0$, this means that

$$SL + L^*S + L^*SBQ_2^{-1}B^TSL = L^*Q_1L + hT_1Q_0L$$

and

$$Sg + L^{*}SBQ_2^{-1}B^TSg = L^{*}Q_1g + hT_1Q_0g$$

Thus

$$\left[Q_2 + B^T(L^{*}Q_1 + hT_1Q_0)LB\right] = (Q_2 + B^TL^{*}SB)Q_2^{-1}(Q_2 + B^TSLB)$$

and

$$B^T(L^{*}Q_1 + hT_1Q_0)g = (Q_2 + B^TL^{*}SB)Q_2^{-1}B^TSg$$

In Chapter 3 it was proven that the dynamic systems

$(Q_2 + B^TL^{*}SB)$ and $(Q_2 + B^TSLB)$

are invertible, provided that Q_2 is nonsingular.
Consequently from (5.7)

$$\begin{aligned}(Q_2 + B^TSLB)u &= - Q_2(Q_2 + B^TL^{*}SB)^{-1}(Q_2 + B^TL^{*}SB) \cdot \\ &\quad \cdot Q_2^{-1}B^TSgx_0 \\ &= - B^TSgx_0\end{aligned}$$

has a unique solution $u = u_{min}$.

But

$$x = LBu + gx_0$$

thus

$$Q_2u = - B^TS(LBu + gx_0) = - B^TSx$$

or

$$u = - Q_2^{-1} B^T S x \tag{5.9}$$

The closed loop performance

$$x = - LBQ_2^{-1} B^T S x + g x_0$$

can be written

$$\begin{cases} T_0 x = x_0 \\ (L^{-1} + BQ_2^{-1} B^T S) x = 0 \end{cases}$$

or

$$\begin{cases} T_0 x = x_0 \\ \frac{d}{dt} x(t) = (A - BQ_2^{-1} B^T S) x(t) \end{cases} \tag{5.10}$$

which is summarized in the theorem:

Theorem 5.2:

The function u defined by (5.9) uniquely minimizes the criterion (5.2) subject to the constraint (5.1) giving the closed loop system (5.10). □

Remark 1:

Since u(t) according to (5.9) only requires x(t) the calculation of the control law can be made off line.

Remark 2:

Like in the filter case the backward integration vanishes and the necessary calculation is considerably simplified.

Remark 3:

The minimum value of V is

$$V = \frac{1}{2} x_o \cdot T_o S g x_o = \frac{1}{2} x_o \cdot S(t_o) x_o$$

which is obtained using (5.3), (5.4), (5.7) and the triangularizations.

5.3. Some Elements of the State Vector Fixed at t_1.

An extension of the foregoing problem is to require that the state vector at the end point lies in some prescribed hyperplane, parametrized as

$$x(t_1) = b(z)$$

where z is of lower dimension, say q, than x.

It is of no principal importance to assume that only the q last components of $x(t_1)$ are free while the first n-q components are fixed. It is then also natural to have

$$Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & Q_{02} \end{bmatrix}$$

where only Q_{02} (of order $q \times q$) is different from zero.

The problem is now to minimize

$$V = \frac{1}{2}\int_{t_0}^{t_1} x^T Q_1 x dt + \frac{1}{2} x^T(t_1) Q_0 x(t_1) + \frac{1}{2}\int_{t_0}^{t_1} u^T Q_2 u dt$$

with respect to u, under the restriction

$$\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = a \\ x(t_1) = b(z) = \begin{bmatrix} c \\ z \end{bmatrix}, \quad z \text{ arbitrary} \end{cases}$$

and rewritten in operator notation:

$$V = \frac{1}{2}(x,b) \cdot Q(x,b) + \frac{1}{2} u \cdot Q_2 u \qquad (5.11)$$

$$(x,b) - \bar{L}(Bu,a) = 0 \qquad (5.12)$$

The route used in the previous section cannot be applied directly. The equation (5.4) is the same but there is the additional linear constraint:

$$c = [I \quad 0]T_1 LBu + [I \quad 0]T_1 ga \qquad (5.13)$$

which is not solved. Theorem 5.1 is therefore not directly applicable.

However, completing squares are useful, and use can thus be taken of the solution of the unconstrained minimization in the foregoing section.

Let u_f be that solution:

$$u_f = - \left\{ Q_2 + B^T(L^* Q_1 + hQ_0T_1)LB \right\}^{-1} B^T(L^* Q_1 + hQ_0T_1)ga$$

and using the Riccati equation triangularization:

$$u_f = - (Q_2 + B^TSLB)^{-1}B^TSga =$$

$$= - Q_2^{-1}B^TS(I + LBQ_2^{-1}B^TS)^{-1}ga$$

The problem (5.11) - (5.13) can now be reformulated using the previous results and the definition of u_f:

Minimize

$$V = \frac{1}{2}(u - u_f) \cdot \left\{(Q_2 + B^TL^*SB)Q_2^{-1}(Q_2 + B^TSLB)\right\}^{-1}(u-u_f) +$$

$$+ \frac{1}{2}a \cdot S(t_o)a \qquad (5.14)$$

Under the constraint

$$[I \quad 0]T_1LB(u - u_f) = c - [I \quad 0]T_1ga + [I \quad 0]T_1LBQ_2^{-1}B^TS \cdot$$

$$\cdot (I + LBQ_2^{-1}B^TS)^{-1}ga$$

$$= c - [I \quad 0]T_1(I + LBQ_2^{-1}B^TS)^{-1}ga \qquad (5.15)$$

Defining

$$v = Q_2^{-1}(Q_2 + B^TSLB)(u - u_f) \qquad (5.16)$$

$$U = [I \quad 0]T_1 LB(Q_2 + B^T SLB)^{-1} Q_2 =$$

$$= [I \quad 0]T_1 (L^{-1} + BQ_2^{-1} B^T S)^{-1} B \qquad (5.17)$$

$$y = c - [I \quad 0]T_1 (I + LBQ_2^{-1} B^T S)^{-1} ga \qquad (5.18)$$

then (5.14) and (5.15) are shown equivalent to

$$V = \frac{1}{2} v \cdot Q_2 v + \frac{1}{2} a \cdot S(t_o) a \qquad (5.19)$$

and

$$Uv = y \qquad (5.20)$$

This means that after having solved the unconstrained minimization the constrained minimization was much simpler.

Theorem 5.3:

The minimization problem (5.11) - (5.12) is equivalent to:

Minimize

$$V = \frac{1}{2} v \cdot Q_2 v + \frac{1}{2} a \cdot S(t_o) a \qquad (5.19)$$

Under the restriction

$$c = [I \quad 0]T_1 x \qquad (5.20')$$

where x satisfies

$$\begin{cases} \frac{d}{dt} x = (A - BQ_2^{-1}B^T S)x + Bv \\ T_0 x = a \end{cases} \tag{5.21}$$

Proof:

(5.16) gives $V = u + Q_2^{-1}B^T SLBu + Q_2^{-1}B^T Sga = u + Q_2^{-1}B^T Sx$ and (5.21) is verified. (5.20') follows then from (5.17), (5.18), (5.20). □

Define for convenience the fundamental matrix to (5.21) to be $\Psi(t,s)$, which is thus the kernel of the operator

$$(L^{-1} + BQ_2^{-1}B^T S)^{-1}$$

It still remains to determine v, which can be done using the formulation of Theorem 5.3 and a well-known theorem, see for instance [4]:

Theorem 5.4:

Let P be a positive definite self adjoint linear operator, and U a not necessarily invertible operator. If the compound operator $UP^{-1}U^*$ is invertible then

$$Ux = y$$

has the solution

$$x_0 = P^{-1}U^*(UP^{-1}U^*)^{-1}y$$

which also minimizes

$$J = x \cdot Px$$

under the linear constraint. □

According to this

$$v = Q_2^{-1}U^*(UQ_2^{-1}U^*)^{-1}y$$

provided that the inverse exists.

Rewrite:

$$UQ_2^{-1}U^* = [I \quad 0]T_1(L^{-1} + BQ_2^{-1}B^TS)^{-1}BQ_2^{-1}B^T \cdot$$

$$\cdot (I + L^*SBQ_2^{-1}B^T)^{-1}h\begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$= [I \quad 0]\int_{t_0}^{t_1} \Psi(t_1,s)BQ_2^{-1}B^T\Psi^T(t_1,s)ds\begin{bmatrix} I \\ 0 \end{bmatrix} = W(t_0) \qquad (5.22)$$

where the integral is the controllability gramian for the closed loop system (5.21).

$$y = c - [I \quad 0]T_1x_f = c - [I \quad 0]\Psi(t_1,t_0)a \qquad (5.23)$$

if x_f is the state obtained with $u = u_f$, that is if no care is taken of the constraint.

Thus

$$v = Q_2^{-1}B^T(I + L^*SBQ_2^{-1}B^T)^{-1}h\begin{bmatrix} I \\ 0 \end{bmatrix}W^{-1}(t_0)y$$

or

$$v = Q_2^{-1} B^T \xi$$

with the backward integration

$$\begin{cases} -\dfrac{d}{dt}\,\xi = (A - BQ_2^{-1}B^T S)^T \xi \\ T_1 \xi = \begin{bmatrix} I \\ 0 \end{bmatrix} W^{-1}(t_0) y \end{cases} \tag{5.24}$$

Summarize in the theorem

Theorem 5.5:

(5.11) restricted by (5.12) is minimized by the input

$$u = -Q_2^{-1} B^T Sx + Q_2^{-1} B^T \xi \tag{5.25}$$

with x given by (5.21) and ξ by (5.24), (5.23) and (5.22), and

$$V_{min} = \frac{1}{2}\, a \cdot T_o S g a + \frac{1}{2}\, \xi \cdot BQ_2^{-1} B^T \xi =$$

$$= \frac{1}{2}\, a^T S(t_0) a + \frac{1}{2}\, y^T W^{-1}(t_0) y \qquad \square$$

Remark 1:

It follows from (5.25) and (5.23) that t_o can be variable as long as $W(t_o)$ has an inverse.

5.4. Optimization under Other Types of Constraints.

Some other types of constraints are easily handled with the operator technique. Assume, for instance, that there is an absolute limit on the input term

$$u \cdot Q_2 u \leq C$$

instead of being included in the cost function.

$$V = x \cdot Q_1 x + T_1 x \cdot Q_0 T_1 x; \quad x = LBu + gx_0$$

Inbedding this in the problem

$$V_e = V + eu \cdot Q_2 u$$

having a minimum for u_e, which can be obtained as in 5.2, it is possible to show that

$$u_e \cdot Q_2 u_e$$

is strictly decreasing with e and either has a finite limit, when $e \to 0$, which is less than C, or it equals C for only one e. This u_e also minimizes V under the constraint $u \cdot Q_2 u \leq u_e \cdot Q_2 u_e = C$ since it minimizes V_e.

A corresponding analysis can be done for another problem with integral constraint:

$$\begin{cases} \text{Minimize } u \cdot Q_2 u \text{ under the constraint} \\ x \cdot Q_1 x \leq C \end{cases}$$

Compare Bellman [2].

6. SEPARATION THEOREM.

6.1. Introduction.

The results of Chapters 4 and 5 can be combined and the separation theorem is possible to prove using the operator notation.

Assume that the loss

$$V = \frac{1}{2} E\left\{\int_{t_0}^{t_1} x^T(t)Q_1x(t)dt + x^T(t_1)Q_0x(t_1) + \right.$$

$$\left. + \int_{t_0}^{t_1} u^T(t)Q_2u(t)dt\right\} \tag{6.1}$$

should be minimized when x defined by the stochastic differential

$$\begin{aligned} dx &= Axdt + Budt + dv \\ dy &= Cxdt + de \end{aligned} \tag{6.2}$$

and $x(t_0) = a$ has mean value m and covariance R_0. Like in Chapter 4 v and e are independent Wiener processes with incremental covariance R_1dt and R_2dt respectively. They are also independent of $x(t_0)$.

Two problems are formulated dependent of what information is available for choosing u(t)

1) $\{x(s) \quad s \leq t\}$

2) $\{y(s) \quad s \leq t\}$

6.2. Reformulation of the Minimization.

In operator formulation (6.2) gives

$$x = LBu + ga + L \frac{dv}{dt} \tag{6.3}$$

where the last term must be interpreted as a stochastic integral and special care must be taken in the calculus. (6.1) can be reformulated like

$$V = \frac{1}{2} E\left\{x \cdot Q_1 x + T_1 x \cdot Q_0 T_1 x + u \cdot Q_2 u\right\} \tag{6.4}$$

Now following Section 5.2 I insert (6.3) into (6.4) and use the triangularization of $Q_2 + B^T L^* Q_1 L^* B$ by the Riccati equation

$$\begin{cases} -\dot{S} = A^T S + SA + Q_1 - SBQ_2^{-1} B^T S \\ T_1 S = Q_0 \end{cases} \tag{6.5}$$

with

$$\begin{cases} L^* Q_1 L + hT_1 Q_0 L = SL + L^* S + L^* SBQ_2^{-1} B^T SL \\ L^* Q_1 g + hT_1 Q_0 g = Sg + L^* SBQ_2^{-1} B^T Sg \end{cases} \tag{6.6}$$

leading to

$$V = \frac{1}{2} E\Big\{u \cdot (I + B^T L^* SBQ_2^{-1}) Q_2 (I + Q_2^{-1} B^T SLB) u +$$

$$+ 2u \cdot \left(B^T L^* S \frac{dv}{dt} + (I + B^T L^* SBQ_2^{-1}) B^T S (L \frac{dv}{dt} + ga)\right) +$$

$$+ \frac{dv}{dt} \cdot (SL + L^{*}S + L^{*}SBQ_2^{-1}B^{T}SL)\frac{dv}{dt} +$$

$$+ 2 \frac{dv}{dt} \cdot (Sg + L^{*}SBQ_2^{-1}B^{T}Sg)a +$$

$$+ a \cdot T_0(Sg + L^{*}SBQ_2^{-1}B^{T}Sg)a\Big\}$$

Note that all realistic controls u(t) can be functions of the noise dv only up to time t. Since $L^{*}\frac{dv}{dt}$ only depends of dv after that time and since v is a Wiener process

$$Eu \cdot B^{T}L^{*}S \frac{dv}{dt} = 0 \tag{6.7}$$

Completing squares and introducing

$$u_f = - (I + Q_2^{-1}B^{T}SLB)^{-1}Q_2^{-1}B^{T}S\left(L \frac{dv}{dt} + ga\right)$$

gives

$$V = \frac{1}{2} E\Big\{(I + Q_2^{-1}B^{T}SLB)(u - u_f) \cdot$$

$$\cdot Q_2(I + Q_2^{-1}B^{T}SLB)(u - u_f) +$$

$$+ a \cdot T_0Sga + \frac{dv}{dt} \cdot (SL + L^{*}S)\frac{dv}{dt} +$$

$$+ 2 \frac{dv}{dt} \cdot Sga\Big\} \tag{6.8}$$

But a and v are independent and thus

$$\frac{dv}{dt} \cdot Sga = \int_{t_0}^{t_1} a^T \phi^T(t,t_0) S(t) dv(t) = 0$$

and since all differentials are Ito differentials

$$E \frac{dv}{dt} \cdot L^* S \frac{dv}{dt} = E \int_{t_0}^{t_1} dv^T(t) \int_{t}^{t_1} \phi^T(s,t) S(s) dv(s) = 0$$

$$E \frac{dv}{dt} \cdot SL \frac{dv}{dt} = E \int_{t_0}^{t_1} dv^T(t) S(t) \int_{t_0}^{t} \phi(t,s) dv(s) =$$

$$= E \int_{t_0}^{t_1} \mathrm{tr}\, S(t) \int_{t_0}^{t} \phi(t,s) dv(s) dv^T(t) =$$

$$= \int_{t_0}^{t_1} \mathrm{tr}\, S(t) R_1(t) dt \qquad (6.9)$$

Note also that

$$(I + Q_2^{-1} B^T SLB)(u - u_f) =$$

$$= u + Q_2^{-1} B^T SLBu + Q_2^{-1} B^T S \left(L \frac{dv}{dt} + ga \right) =$$

$$= u + Q_2^{-1} B^T Sx \qquad (6.10)$$

which can be summed up in

Theorem 6.1:

Assume that x follows the state equation (6.3) and that S is defined by (6.5) then

$$V = \frac{1}{2} E\left\{x \cdot Q_1 x + T_1 x \cdot Q_0 T_1 x + u \cdot Q_2 u\right\} =$$

$$= \frac{1}{2} E\left\{(u + Q_2^{-1} B^T Sx) \cdot Q_2 (u + Q_2^{-1} B^T Sx)\right\} +$$

$$+ m^T S(t_0) m + \text{tr } S(t_0) R_0 +$$

$$+ \int_{t_0}^{t_1} \text{tr } S(t) R_1(t) dt \qquad (6.11)$$

If $u = - Q_2^{-1} B^T Sx$ is an admissible strategy, then this certainly minimizes V. □

Remark 1:

Since the operator $Q_2^{-1} B^T S$ only implies multiplication with the corresponding matrices, best u is a function only of the current state. This is owing to the concept of state, i.e. minimum information about the system status, described by (6.2) or (6.3). But it also accounts for the structure of the loss (6.1) and (6.4). If Q_1 and Q_2 were not "diagonal" as operators, best input would not be given by a "diagonal" operation on the state.

Remark 2:

Note that one major assumption is done. It is eq. (6.7) saying that no dependence is possible between the control u(t) and dv(s) for s > t. If v were for instance a known forcing function instead of unpredictable noise, (6.11) would not be valid. Best input should be a function not only of the state but also of this forcing function. The concept state is somewhat dubious in this case.

6.3. The Separation Theorem.

In problem 2) of the introduction 6.1.

$$u = - Q_2^{-1} B^T S x$$

is not an admissible control.

If u(t) is restricted to linear functionals of y(s) up to time t the first term of (6.11) can be rewritten:

$$\frac{1}{2} E(u + Q_2^{-1} B^T S x) \cdot Q_2 (u + Q_2^{-1} B^T S x) =$$

$$= \frac{1}{2} E(u + Q_2^{-1} B^T S \hat{x}_f) \cdot Q(u + Q_2^{-1} B^T S \hat{x}_f) +$$

$$+ Eu \cdot Q_2 Q_2^{-1} B^T S \tilde{x}_f + \frac{1}{2} E Q_2^{-1} B^T S \tilde{x}_f \cdot Q_2 Q_2^{-1} B^T S \tilde{x}_f =$$

$$= \frac{1}{2} E(u + Q_2^{-1} B^T S \hat{x}_f) \cdot Q_2 (u + Q_2^{-1} B^T S \hat{x}_f) +$$

$$+ \frac{1}{2} \int_{t_0}^{t_1} \operatorname{tr} PSBQ_2^{-1} B^T S dt \qquad (6.12)$$

with $\hat{x}_f(t)$ being the "linear projection" of x(t) onto $\mathcal{Y}_t = \{y(s), s \leq t\}$ and $\tilde{x}_f(t) = x(t) - \hat{x}_f(t)$ being independent of $\mathcal{Y}_t$ and thus also u(t). $\tilde{x}_f(t)$ has the covariance P(t) defined in Chapter 4.

Regarding this

$$u = - Q_2^{-1} B^T S \hat{x}_f$$

certainly minimizes V for the problem 2).

<u>Theorem 6.2:</u>

The loss function (6.1) for the dynamic system (6.2) is uniquely minimized under the restriction that u(t) be a linear functional of y up to time t by

$$u = - Q_2^{-1} B^T S \hat{x}_f \tag{6.13}$$

and the minimum loss is

$$V_{min} = \frac{1}{2}\left\{ m^T S(t_0) m + \text{tr } S(t_0) R_0 + \int_{t_0}^{t_1} \text{tr } S(t) R_1(t) dt + \int_{t_0}^{t_1} \text{tr } PSBQ_2^{-1} B^T S dt \right\} \tag{6.14}$$

where $\hat{x}_f$ is defined by

$$\begin{cases} d\hat{x}_f = A\hat{x}_f dt + Budt + PC^T R_2^{-1}(dy - C\hat{x}_f dt) \\ \hat{x}(t_0) = m \end{cases} \tag{6.15}$$

and S by (6.5) and P by

$$\begin{cases} \dot{P} = AP + PA^T + R_1 - PC^T R_2^{-1} CP \\ P(t_0) = R_0 \end{cases} \tag{6.16}$$

□

Remark 1:

If the linearity assumption on u is omitted the theorem holds only for normal processes. However, if $\hat{x}_f(t)$ means the conditional expectation of x(t) given $\mathcal{Y}_t$ and P(t) the conditional variance (6.13) and (6.14) still hold, but $\hat{x}$ is a nonlinear functional of $\mathcal{Y}_t$ when normality is not assumed.

Remark 2:

If u(t) is a linear functional of $\mathcal{Y}_{t-T}$, $\hat{x}_f(t)$ should be changed to $\hat{x}(t|t-T)$, the best linear predictor.

u(t) being a functional of $\mathcal{Y}_{t+T}$ violates (6.7) (see Remark 2 after Theorem 1).

Remark 3:

The minimization can also be performed for other types of information processes giving another $\mathcal{Y}_t$. The observations can for instance be performed at discrete instants during the interval.

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