LUND UNIVERSITY

Stabilisation of Uncertain Systems

Molander, Per

1979

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

PER MOLANDER Stabilisation of Uncertain Systems

CODEN: LUTFD2/(TFRT-1020)/1-111/(1979)

Stabilisation of Uncertain Systems

PER MOLANDER

Department of Automatic Control • Lund Institute of Technolog

Stabilisation of Uncertain Systems

av

Per Molander

FK,CI

Akademisk avhandling som för avläggande av teknisk doktorsexamen vid Tekniska Fakulteten vid Universitetet i Lund kommer att offentligen försvaras i sal M:E, Maskinhuset, Lunds Tekniska Högskola, fredagen den 21 september 1979 kl 10.15.

AV-centralen i Lund 1979

Dokumentutgivare
Lund Institute of Technology
Handläggare Dept of Automatic Control
K J Åström
Författare
Per Molander

Dokumentnamn
REPORT
Dokumentbeteckning
LUTFD2/(TFRT-1020)/1-111/(1979)
Utgivningsdatum
August 1979
Ärendebeteckning

Dokumenttitel och undertitel

Stabilisation of Uncertain Systems

Referat (sammandrag)

The possibility of stabilising a nonlinear, time-varying system by means of a linear, constant feedback regulator is investigated. Existence theorems for such robust regulators are given for the case of large disturbances. Both state feedback and output feedback are considered.

The results are compared to known results on random-parameter systems. The dependence on the stochastic convergence mode chosen is illustrated by examples.

Referat skrivet av
Author

Förslag till ytterligare nyckelord
Geometric control theory; linear-quadratic optimal control; stability; stochastic stability.

Klassifikationssystem och -klass(er)

Indextermer (ange källa)
Automatic control; feedback control; nonlinear systems; stability; stochastic processes. (Thesaurus of Engineering and Scientific Terms, Engineers Joint Council, NY, USA).

Omfång
111 pages

Övriga bibliografiska uppgifter

Språk
English

Sekretessuppgifter

ISSN

ISBN

Dokumentet kan erhållas från
Department of Automatic Control
Lund Institute of Technology
Box 725, S-220 07 Lund 7, Sweden

Mottagarens uppgifter

Pris

DOKUMENTDATABLAD enligt SIS 62 10 12

SIS-DB 1

Blankett LU 11:25 1976-07

PREFACE

A few words on the genesis of this work may be necessary to explain some of its idiosyncrasies. The germ of the problems discussed was provided by Karl Johan Åström, who suggested an investigation of Kalman-Bucy-type filters for nonlinear systems. This eventually lead to the results presented in Part I of the thesis.

When this work was completed, my interest was attracted to the area of stochastic stability. I found that problems related to my own had been studied within this framework. It was natural to compare the results obtained using different approaches, more precisely the role of the modelling in the design. This is the reason why the second part contains relatively few results that are new to readers acquainted with the field.

The text is intended for readers who have a general background in control theory and know the basics of deterministic stability theory (Lyapunov stability, the circle criterion etc.). For Part II, a general familiarity with stochastic convergence concepts belong to the prerequisites, but no knowledge of stochastic stability is assumed. In some instances, mathematical rigour has been sacrificed for the sake of readability.

Valuable criticism on the manuscript was given by Bo Egardt, Per-Olof Gutman, Jan Sternby and Karl Johan Åström. To these and other friends and colleagues at the department in Lund, from whom and together with whom I have learned what I now know about control theory, I wish to express my sincere gratitude for the years past.

Part of the work was done while I was with Jan C Willems at the Dept of Mathematics of the University of Groningen.

The hospitality of this department is gratefully acknowledged.

Last but not least I wish to thank Birgitta Tell, who drew the figures, and Eva Dagnegård, who produced a magnificently typed version of a barely decipherable hand-written manuscript.

Lund, in June, 1979

Per Molander

CONTENTS

PART II - STOCHASTIC MODELS

INTRODUCTION

The problem of uncertainty

The modelling of any physical system is subject to uncertainty. Such uncertainty may be due to nonmeasurable disturbances and unknown or only partly known system parameters.

Since any systematic design procedure is based on a model of the system to be controlled, the synthesis is liable to the same uncertainty as the modelling. In many cases, the regulator has an inherent robustness against disturbances and parameter variations. This is generally true for feedback regulators, which is in fact one of the main reasons for using feedback.

However, there are numerous cases in which the influence of the uncertainty on the system performance is so large that it has to be accounted for in the design. A typical example of large parameter variations caused by exogeneous variables is the variation of aircraft dynamics with the dynamic pressure and the Mach number. A servo motor with an unknown nonlinear characteristic is another example. In these situations, standard linear design techniques are inadequate.

Adaptive versus robust design

There are two basically different ways of tackling the design problem for uncertain systems. One is to measure and/or estimate changes in the system parameters, and to base the synthesis on the current estimates. If the parameter variations are caused by exogeneous variables which can be measured with sufficient accuracy, it is often sufficient to precompute a finite number of regulators and to choose the one that best suits the current parameter values. This is referred to as *gain scheduling*. Alternatively, it is possible to estimate the unknown parameters using some on-line identification method, and then to modify the controller according to the current estimates. This yields an *adaptive* regulator.

An entirely different method of tackling the problem of unknown or varying parameters is to design the regulator apriori with the object of quenching the effect of the uncertainties. This approach, producing a *robust* regulator, is the main topic of the present work.

Previous work

The problem of designing for low sensitivity is an old one in control theory, and the literature on this topic is extensive. An elaborate method based on classical frequency domain techniques has been developed by Horowitz and Sidi ([Hor 1], [Hor 2]). Lead-lag compensators are used to twist the Nichols plot of the open-loop transfer function in order to satisfy given time-domain specifications. These techniques are somewhat troublesome to extend to multivariable systems.

Also in modern control theory much interest has been devoted to the problem of low-sensitivity design; see for instance the book by Cruz ([Cru]) for an overview of the subject. For small variations around the nominal plant, sensitivity derivatives have been used extensively. Various methods to reduce the eigenvector sensitivity, trajectory sensitivity, or the sensitivity of some performance index have been proposed. Rigorously speaking, these methods are efficient only for local variations. For large parameter fluctuations, the design has mostly been based on minimax-strategies, i.e. basically a trial-and-error approach.

Previous work which is more in the vein of the present one is a long series of papers by Gilman and Rhodes (see [Gil], which contains further references). In these papers the influence on the performance index of substantial nonlinear terms in the plant equation is studied. However, no guidelines for the design are given.

Synopsis

The problem posed in the present work is: when can a nonlinear and/or timevarying system be stabilised by means of a linear feedback regulator? The requirement on stability is of course far from sufficient for practical purposes, but it permits the formulation of precise theorems. Further, the results can be modified to cover the case when the closed-loop poles are required to lie in some prescribed subset of the complex plane.

There is of course a danger in isolating one aspect of the control problem (in this case stability), namely that other relevant aspects are left aside. Therefore, the theoretical results presented are to serve more as a moral support for the designer. The trade-off between the various specifications in a given problem remains to be done.

In order to impose some structure on the problem, a model for the disturbances must be chosen. The standard dichotomy of modern control theory - deterministic versus stochastic models - appears also in this context. Part I discusses the problem in a deterministic setting. The disturbances are modelled as cone-bounded nonlinearities. Chapter 2 contains the main existence theorems for a stabilising feedback as the confining sector of the nonlinearities increases without bound. Output feedback is also discussed. To our knowledge, most of these results are new, but some known results (due to Haussmann [Hau]) have been included for the sake of completeness.

Chapter 3 gives the corresponding theorems for sampled-data systems, with an emphasis on the differences between the discrete-time and continuous-time results.

The analysis is based on modern frequency-domain stability criteria. These are known to yield conservative results. This is natural, since very little is assumed about the nonlinearity. In some applications, it is reasonable to assume that the parameter variations are independent of the state, and that the average behaviour over long time intervals is known, even though the variations may be large locally in time. This motivates the use of stochastic models.

The introduction of a stochastic element poses some new problems. Firstly, the intuitive idea of noisy parameters must be given a rigorous formulation. This is more than a mathematical game. The stability conditions obtained depend critically on the stochastic integral concept chosen. Secondly, the many different stochastic convergence concepts offer a variety of possibilities as the Lyapunov stability definitions are to be rephrased in stochastic terminology. These topics are discussed in some detail in Chapter 4.

Chapter 5 finally contains a study of the stabilisability problem for random-parameter systems, and can be regarded as the stochastic counterpart of Chapter 2. The mean-square stabilisation problem was treated by Willems and Willems in [Wil] using Wonham's solution of the linear-quadratic optimal control problem for white-noise parameter systems. These results are compared to what was obtained in Chapter 2. The main object is thus to study the influence of the modelling on the regulator obtained. This final chapter also includes a discussion of the relevance of the various stability definitions. It is shown that the two problems of stabilising a control system in the mean square and almost surely, respectively, display qualitative differences.

NOTATION

The notation used is standard.Capital letters (A, B, M, N) are used for matrices and small letters (x, m, n) for vectors or column matrices.

Im(A) denotes the subspace spanned by the columns of A, and Ker(A) the kernel of A considered as a mapping. $\lambda(A)$ is used for the eigenvalues of A, and $A^{-1}(\cdot)$ for the inverse image (notice that the matrix inverse of A need not exist).

$\mathcal{V}^{\perp}$ is the orthogonal complement of the subspace . $\sum_{i} \mathcal{V}_{i}$ denotes the span of the subspaces $\mathcal{V}_{i}$, and $< A \mid B > =$
$= \sum_{i=0}^{n-1} A_{i}\, \mathrm{Im}(B)$.

In the stochastic section, $E\{\cdot\}$ is an expectation value, and $E\{\cdot \mid \Sigma\}$ means the expected value conditioned with respect to some σ-algebra Σ.

PART I - DETERMINISTIC MODELS

CHAPTER 1. PRELIMINARIES

The basic definitions and main theorems to be used in the sequel are given here. The stability definitions are specialised to motions described by ordinary differential equations, since that will be the main concern in the present work. They are easily translated to the case of difference equations and more general motions. Further, the unperturbed motion (i.e. the motion whose stability is being studied) is assumed to be the solution which is identically zero. It will be called the *equilibrium*, the *null solution*, or the *trivial solution*.

1.1 Basic definitions

Consider the system of ordinary differential equations in $x = (x_1, x_2, \ldots, x_n)$:

$$\begin{cases} \dfrac{dx_1}{dt}(t) = f_1(x_1(t), x_2(t), \ldots, x_n(t), t) & \\ \dfrac{dx_2}{dt}(t) = f_2(x_1(t), x_2(t), \ldots, x_n(t), t) & t > t_0 \\ \vdots & \\ \dfrac{dx_n}{dt}(t) = f_n(x_1(t), x_2(t), \ldots, x_n(t), t) & \end{cases} \qquad (1.1)$$

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{pmatrix} \triangleq x_0 .$$

The functions f_i are assumed to satisfy

$$f_i(0, 0, \ldots, 0, t) = 0, \qquad i = 1, 2, \ldots, n,$$

so that $x(t) \equiv 0$ is a solution of equation (1.1). It is further assumed that the f_i's are regular enough for the general existence and uniqueness conditions to be satisfied over the entire half-axis $t \geqslant t_0$ (see e.g. [Cod]).

Definition 1.1 The equilibrium is said to be *stable* iff for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\| x_0 \| \leqslant \delta$$

implies

$$\| x(t) \| \leqslant \varepsilon$$

for all $t \geqslant t_0$. □

Definition 1.2 The equilibrium is said to be *attractive* iff there exists a $\delta > 0$ such that

$$\| x_0 \| \leqslant \delta$$

implies

$$\lim_{t \to \infty} x(t) = 0. \qquad (*)$$

The set of x_0's such that (*) holds form the *domain of attraction*. □

Definition 1.3 The equilibrium is said to be *asymptotically stable* iff it is stable and attractive. It is *uniformly asymptotically stable* iff further δ in the above definitions can be chosen independently of t_0. It is *uniformly* *asymptotically stable in the large* if further the domain of attraction is the whole of $\mathbb{R}^n$. □

In the sequel, the only concern will be with asymptotic stability, and "stability" and "asymptotic stability" will be used indiscriminately.

The above stability definitions, which are rephrased versions of Lyapunov's definitions in his pioneering work, refer to perturbations of the initial state. In many practical situations, other forms of perturbations may be equally or more relevant. Assume for instance that a small (in some suitably chosen norm) term is added to the right-hand side of equation (1.1). An investigation of the behaviour of the solutions $x(t)$, $t > t_0$, leads naturally to the concept of *stability under persistent disturbances* (or *total stability*). It can be shown ([Hah]) that uniform asymptotic stability in the sense of Lyapunov implies total stability, which makes it relevant to consider Lyapunov stability even if the ultimate interest is in additive disturbances.

Alternatively, the f_i's may be perturbed by functions g_i, which are not necessarily small, but which satisfy

$$g_i(0, 0, \ldots, 0, t) = 0.$$

Such perturbations arise as models of parameter fluctuations or nonlinearities which are not accounted for in the modelling. The problem of investigating stability (in the Lyapunov sense) under such disturbances is the one envisaged in the present work.

1.2 Frequency domain stability criteria

An early attempt in this direction was the formulation of the absolute stability problem by Lure and Postnikov ([Lur]). The configuration, which has become a standard one in modern stability theory, consists of a linear, time--invariant link in the forward path and a time-invariant nonlinear function in the feedback loop.

Lure and Postnikov proposed to find conditions on G(s) that would ensure stability of the feedback system under the sole condition that

$$\sigma\Phi(\sigma) \geq 0 \quad \text{for all } \sigma.$$

The breakthrough came with Popov ([Pop]), who proved a criterion based on the frequency response function $G(i\omega)$. The condition turned out to be a positive-realness one.

Definition 1.4 Let G(s) be a square transfer matrix whose entries are analytic in the open right half-plane. Then G(s) is said to be *positive real* if

$$(G(s) + G^T(-s)) \geq 0$$

for all s with $Re(s) > 0$.
G(s) is termed *strictly positive real* if $G(s-\varepsilon)$ is positive real for some $\varepsilon > 0$. □

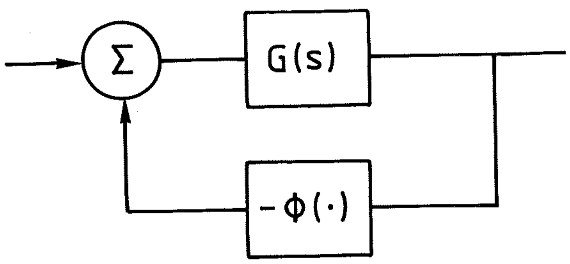

Fig. 1.1 - The standard configuration.

The synthesis problem is thus reduced to one of making a certain transfer function positive real.

The only concern in this work is with time-varying nonlinearities, and the Popov criterion is therefore not applicable. The appropriate stability theorem to use is instead the circle criterion. Early versions were given by Rozenvasser ([Roz]) and Narendra and Goldwyn ([Nar 1]). Their results were generalised by Sandberg and Zames using functional analysis methods. Two forms of the multivariable circle criterion will be used here. They are in fact equivalent, and one can be obtained from the other by means of a simple loop transformation (see [Nar 2], which also contains a proof of the theorem).

<u>Circle criterion</u>. Consider the configuration of Fig. 1.1, where $\Phi(\cdot,\cdot)$ is a mapping from $\mathbb{R}^q \times \mathbb{R}$ into $\mathbb{R}^q$ satisfying

$$\Phi(\sigma,t)^T(\Phi(\sigma,t) - \kappa\cdot\sigma) \leq 0$$

for some positive scalar κ and all $\sigma \in \mathbb{R}^q$, $t \geq t_0$.

Then the trivial solution of the ordinary differential equation corresponding to the given feedback system is uniformly asymptotically stable in the large if $\kappa\cdot G(s) + I$ is strictly positive real. □

<u>Circle criterion, small-gain form</u>. Consider the same configuration as above but with $\Phi(\cdot,\cdot)$ subject to

$$\|\Phi(\sigma,t)\| \leq \kappa\cdot\|\sigma\|$$

for some positive scalar κ and all $\sigma \in \mathbb{R}^q$, $t \geq t_0$.

Then stability holds if

$$\sup_{\omega\in\mathbb{R}} \max_i |\lambda_i(G^T(-i\omega)\ G(i\omega))| < \frac{1}{\kappa^2}. \quad \square$$

1.3 Geometric control theory concepts

A brief review of some fundamental concepts from Wonham's geometrical control theory will be given here. For a detailed exposé including proofs, the reader is referred to [Won 4].

An (A,B)-*invariant subspace* $\mathcal{V}$ of $\mathbb{R}^n$ is a subspace that satisfies

$$A\mathcal{V} \subseteq \mathcal{V} + \mathrm{Im}(B).$$

Equivalently, there exists an L such that

$$(A - BL^T)\ \mathcal{V} \subseteq \mathcal{V}.$$

Every subspace $\mathcal{W}$ of $\mathbb{R}^n$ contains a *maximal* (A,B)-*invariant subspace*. The maximal (A,B)-invariant subspace of $\mathcal{W}$, denoted by $\mathcal{V}^*$, can be generated by the algorithm

$$\begin{cases} \mathcal{V}^{(0)} = \mathcal{W} \\ \mathcal{V}^{(\mu)} = \mathcal{W} \cap A^{-1}(\mathcal{V}^{(\mu-1)} + \mathrm{Im}(B)) \end{cases}$$

which converges in a finite number of steps to $\mathcal{V}^*$. (A^{-1} denotes inverse image and does not imply that the indicated matrix inverse exists.)

A *controllability subspace* $\mathcal{R}$ of $\mathbb{R}^n$ is a subspace for which there exists an L such that

$$\mathcal{R} = \langle A - BL^T \mid (\mathrm{Im}(B) \cap \mathcal{R}) \rangle.$$

Every subspace $\mathcal{W}$ of $\mathbb{R}^n$ contains a *maximal controllability subspace*. The maximal controllability subspace in $\mathcal{W}$, denoted by $\mathcal{R}^*$, is a subspace of $\mathcal{V}^*$ and can be generated by the algorithm

$$\begin{cases} \mathcal{R}^{(0)} = 0 \\ \mathcal{R}^{(\mu)} = \mathcal{V}^* \cap (A\mathcal{R}^{(\mu-1)} + \mathrm{Im}(B)) \end{cases}$$

which converges in a finite number of steps to $\mathcal{R}^*$.

The following important pole assignment property holds ([Won 4], Prop. 4.1 and Cor. 5.2):

if $\mathcal{V}^*$ is a maximal (A,B)-invariant subspace of $\mathcal{W} \subseteq \mathbb{R}^n$ and $\mathcal{R}^*$ the corresponding controllability subspace, the spectrum of $(A-BL^T)$ is freely assignable on $\mathcal{R}^* + (\mathbb{R}^n - \mathcal{V}^*)$ with L subject to

$$(A - BL^T)\ \mathcal{V}^* \subseteq \mathcal{V}^*.$$

Finally, let L be such that

$$(A - BL^T)\ \mathcal{V}^* \subseteq \mathcal{V}^*.$$

Consider the canonical projection $P: \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathcal{R}^*$ and denote by A^* the mapping induced by $(A-BL^T)$ in $\mathbb{R}^n/\mathcal{R}^*$. Let p(s) be the minimal polynomial of A^*, restricted to $\mathcal{V}^*/\mathcal{R}^*$, and let $p_-(s)p_+(s)$ be a factorization of p(s), where $p_-(s)$ contains all the zeros of p(s) in the closed left halfplane. Then the space $\mathcal{V}^*_-$ is defined as

$$\mathcal{V}^*_- = P^{-1}(\mathrm{Ker}(p_-(A^*)) \cap \mathcal{V}^*/\mathcal{R}^*).$$

For readers who are not used to this geometric apparatus, some comments relating these definitions to more well-known concepts might be justified. Any linear subspace $\mathcal{W}$ of $\mathbb{R}^n$ can be written as the kernel of some linear mapping,

$$\mathcal{W} = \mathrm{Ker}(M^T).$$

Assume first that

$$\mathrm{rank}(M) = \mathrm{rank}(B).$$

Then it can be shown (this is done in Chapter 2) that an L which realises

$$(A - BL^T)\ \mathcal{V}^* \subset \mathcal{V}^*$$

places poles in all the zeros of the transfer matrix

$$H(s) = M^T(sI - A)^{-1}\ B.$$

The condition for $\mathcal{V}^*$ to equal $\mathcal{W}$ is consequently that H(s) have a maximal number of zeros (= n - rank(M)), or, equivalently, that M^TB be nonsingular ([Kou]). Further, $\mathcal{V}_*$ will equal $\mathcal{V}^*$ if all the zeros are located in the closed left half-plane. Notice that in general the spectrum on $\mathcal{V}^*$ will be fixed in this case, so $\mathcal{R}^* = 0$.

If

$$\text{rank}(B) > \text{rank}(M),$$

H(s) will in general have no zeros, and it can be shown (this is also done in the following chapter) that if this is the case, it is possible to satisfy

$$(A - BL^T)\ \mathcal{V}^* \subset \mathcal{V}^*$$

and still to place the closed-loop poles arbitrarily, so, in general,

$$\mathcal{V} = \mathcal{V}^* = \mathcal{R}^*$$

in this case.

Finally, if

$$\text{rank}(B) < \text{rank}(M),$$

$$\mathcal{V}^* = \mathcal{R}^* = 0$$

in general.

The physical interpretation of the subspace $\mathcal{V}^*$ is that a motion starting in it can be kept there by means of a suitably chosen feedback. The same holds for $\mathcal{R}^*$, with the additional requirement that the dynamics of the motion can be chosen arbitrarily. $\mathcal{V}_*$ finally contains the modes that are "naturally" stable plus the ones whose spectrum is freely assignable subject to the constraint that

$$(A - BL^T)\ \mathcal{V}_* \subseteq \mathcal{V}_*.$$

In summary: generically, any linear subspace of codimension equal to the number of inputs is an (A,B)-invariant subspace, and any subspace of codimension less than the number of inputs is a controllability subspace.

CHAPTER 2. STABILISATION OF SYSTEMS WITH CONE-BOUNDED NONLINEARITIES

This chapter contains the main theorems on robust regulators for systems given by equations perturbed by time-varying nonlinearities. The most important contributions are Theorem 2.2 and Corollary 2.1, which deal with plant uncertainty. These results also provide some extensions (Thm. 2.3) of a theorem by Kwakernaak and Sivan ([Kwa 2]) on limiting forms of linear-quadratic optimal control.

For input-channel disturbances, some easy extensions of the standard results on the robustness of optimal regulators are given in § 2.3. Haussmann's theorem on stabilisability under control-dependent disturbances (Thm. 2.4) has been included for later reference.

The above solutions assume that the state is accessible for measurement. The more general problem of output feedback is discussed in the last two sections. As in the standard linear case, the solution is based on a reconstruction of the state. This consequently calls for the construction of an observer for the nonlinear system in question. A convergence result for the basic observer structure, which is a straightforward extension of the linear Kalman-Bucy filter, is proved in § 2.4 (Thm 2.5, Cor. 2.2) by simply dualising Theorem 2.2 and its corollary. Overall stability of the system and the observer follows trivially from a perturbation theorem (Thm. 2.6). For practical purposes, however, Theorem 2.7 on reduced observers is believed to be much more important.

2.1 Formulation of the robust regulator problem (RRP)

Consider a nonlinear, time-varying system given by

$$\frac{dx}{dt}(t) = Ax(t) - N\Phi(M^T x(t), t) + Bu(t) + T\Theta(u(t), t). \tag{2.1}$$

Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, A is $(n\times n)$, M and N $(n\times q)$, and B and T are $(n\times p)$ matrices. $\Phi(\cdot,\cdot)$ and $\Theta(\cdot,\cdot)$ are nonlinear mappings from $\mathbb{R}^q \times \mathbb{R}$ into $\mathbb{R}^q$ and $\mathbb{R}^p \times \mathbb{R}$ into $\mathbb{R}^p$, respectively, which satisfy the conicity conditions

$$\begin{aligned} \|\Phi(\sigma,t)\| &\leq \kappa\cdot\|\sigma\| \quad \text{for all} \quad \sigma \in \mathbb{R}^q,\ t \geq t_0. \\ \|\Theta(\tau,t)\| &\leq \kappa\cdot\|\tau\| \quad \text{for all} \quad \tau \in \mathbb{R}^p,\ t \geq t_0. \end{aligned} \tag{2.2}$$

(There is no restriction in assuming the same sector for Φ and Θ, since this may always be accomplished by a re-scaling of N and T.)

Further, (A,B) is assumed to be a controllable pair. Specifically, this implies that if T is zero (no control--dependent disturbances), A can be assumed to be stable without loss of generality.

Equation (2.1) is quite general, although it has been written in a rather special form to make an application of the circle criterion easy. The minus sign is chosen for conventional reasons.

Relatively to Equations (2.1), (2.2) the following problem (the *robust**) *regulator problem*, RRP) may be posed:

i) Given A,B,M,N, and T, when does there exist, for any given κ, a linear, constant feedback law

$$u(t) = -L^T x(t) \tag{2.3}$$

---

*) Notice that some authors (e.g. Davison) use the word "robust" in a different sense.

such that the null solution of (2.1) - (2.3) is uniformly asymptotically stable in the large?

ii) If such an L does not exist for any given κ, can apriori bound on the maximally permissible sector radius κ be given?

Definition 2.1 *Perfect robustness* (PR) of (A,B) with respect to (M,N,T) is said to be achievable if a stabilising L exists for any given κ. □

A study of simultaneous perturbations in the plant and in the input channels is difficult to carry through. These two problems will therefore be considered separately.

2.2 Plant uncertainty

In this section, the following special case of equation (2.1) is considered:

$$\frac{dx}{dt}(t) = Ax(t) - N\Phi(M^T x(t), t) + Bu(t). \qquad (2.1)'$$

2.2.1 Solution of the RRP

Since Φ is a nonlinear time-varying function, the multivariable circle criterion is the appropriate stability theorem to use. Recall (§ 1.2) that, for a given κ, stability holds if

$$\sup_{\omega \in \mathbb{R}} \max_i |\lambda_i(G^T(-i\omega)\; G(i\omega))| < \frac{1}{\kappa^2},$$

where in this case

$$G(s) = M^T(sI - A + BL^T)^{-1} N. \qquad (2.4)$$

The problem of the transfer matrix (2.4) is that, unless

$Im(N) \subseteq Im(B)$, not only the poles but also the zeros depend on L. This indicates that a straightforward high-gain design need not be successful.

In any case, it might be tempting to let all eigenvalues of the matrix $(A-BL^T)$ tend to negative infinity along some properly chosen half-rays in the left half-plane, which are symmetric with respect to the real axis.

To motivate the definition below, a simple special case will be discussed. Let

$$A = \begin{pmatrix} * & * & \dots & * & * \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

be the controllable canonical form of the linear part of the system (asterisks denoting possibly non-zero entries). It is fairly easy to see that disturbances of the form $N\Phi(M^T x(t), t)$ can be quenched by the regulator for the above choice of L if

$$Im(N) \subseteq Im(B).$$

Now, the ones of the A-matrix denote the integrators by which the input can reach successively the subspaces AB, A^2B etc. It seems reasonable that N could belong to the subspace spanned by B and AB provided that the one in the (2,1) position is not touched by the disturbances, i.e. provided that

$$M^T B = 0.$$

Generalising this idea leads to the following definition.

<u>Definition 2.2</u> Let the sequence $S^{(\mu)}$ be defined by

$$\begin{cases} S^{(0)} = Im(B) \\ S^{(\mu)} = S^{(\mu-1)} + A(Ker(M^T) \cap S^{(\mu-1)}), \end{cases}$$

and let S^* be the first $S^{(\mu)}$ satisfying

$$S^{(\mu)} = S^{(\mu-1)}. \quad \square$$

Theorem 2.1 PR of (A,B) with respect to (M,N) is achievable if

$$\mathrm{Im}(N) \subseteq S^*. \quad \square$$

Proof. Consider first the single-input case. If $\mathrm{Im}(N) \subseteq \mathrm{Im}(B)$, G(s) may be written as q(s)/p(s), where q(s) is independent of L and p(s) is the closed-loop characteristic polynomial. If the closed-loop poles p_i, $i = 1, 2, \ldots, n$ are chosen as $t \cdot p_{i0}$, where t is real and the p_{i0}'s are in the left half-plane,

$$p(s) = \prod_{i=1}^{n} (s - t \cdot p_{0i}) = t^n \prod_{i=1}^{n} (\frac{s}{t} - p_{0i}).$$

Writing q(s) in the same way shows that G(s) will indeed tend to zero uniformly on the imaginary axis as t tends to infinity, since the degree of the numerator is at most (n-1).

For the general single-input case $\mathrm{Im}(N) \subseteq S^*$, write

$$M^T(sI - A + BL^T)^{-1} AB =$$
$$= M^T(sI - A + BL^T)^{-1}(-sI + A - BL^T) B +$$
$$+ M^T(sI - A + BL^T)^{-1} (sI + BL^T) B =$$
$$= - M^TB + M^T(sI - A + BL^T)^{-1} B \cdot (s + L^TB).$$

Continuing this way gives

$$M^T(sI - A + BL^T)^{-1} A^{\mu}B =$$
$$= - (M^TA^{(\mu-1)}B + sM^TA^{(\mu-2)}B + \ldots + s^{(\mu-1)}M^TB) +$$
$$+ M^T(sI - A + BL^T)^{-1}B(s^{\mu} + s^{(\mu-1)}L^TB + \ldots + L^TA^{(\mu-1)}B).$$

If $\mathrm{Im}(N) \subseteq S^*$, the first term above will give a zero contribution to the expansion of $G(s) = M^T(sI-A+BL^T)^{-1}N$. Notice also that the second term is strictly proper, since the last factor of this term is compensated for by a greater pole excess of $G(s)$ (cf. [Kou]).

With the same pole placement as above,

$$L^T A^\nu B = O(t^{\nu+1}).$$

This implies that the coefficients of the factor

$$(s^\mu + s^{(\mu-1)} L^T B + \ldots + L^T A^{(\mu-1)} B) \tag{2.5}$$

remain bounded as the complex variable is changed from s to s/t. By the same argument as above, it can be inferred that the modulus of G(s) can be made arbitrarily small uniformly over the imaginary axis by choosing t large enough.

For the general multi-input case, choose a basis b_i, Ab_i etc. in S^* and complete this to a basis for $\mathbb{R}^n$. In this basis, A and B have the following structure (asterisks denoting possibly non-zero entries):

$$A = \left(\begin{array}{ccc|ccc|c|ccc}
\multicolumn{3}{c|}{\longleftarrow \theta_1 \longrightarrow} & \multicolumn{3}{c|}{\longleftarrow \theta_2 \longrightarrow} & & \multicolumn{3}{c}{\longleftarrow \theta_p \longrightarrow} \\
 & 0 & * & & & * & \cdots & & & * \\
1 & & \vdots & & 0 & \vdots & & & 0 & \vdots \\
 & \ddots & \vdots & & & \vdots & & & & \vdots \\
0 & 1 & * & & & * & & & & * \\
\hline
 & & * & & 0 & * & & & & * \\
 & 0 & \vdots & 1 & & \vdots & & & 0 & \vdots \\
 & & \vdots & & \ddots & \vdots & & & & \vdots \\
 & & * & 0 & 1 & * & & & & * \\
\hline
 & \vdots & & & \vdots & & & & \vdots & \\
\hline
 & & * & & & * & & & 0 & * \\
 & 0 & \vdots & & 0 & \vdots & & 1 & & \vdots \\
 & & \vdots & & & \vdots & & & \ddots & \vdots \\
 & & * & & & * & & 0 & 1 & *
\end{array}\right); \quad B = \left(\begin{array}{cccc}
\multicolumn{4}{c}{\longleftarrow p \longrightarrow} \\
1 & & & \\
0 & & 0 & \\
\vdots & & & \\
0 & & & \\
\hline
0 & 1 & & \\
\vdots & 0 & 0 & \\
\vdots & \vdots & & \\
0 & 0 & & \\
\hline
 & \vdots & & \\
\hline
 & & & 1 \\
 & & & 0 \\
 & 0 & & \vdots \\
 & & & 0
\end{array}\right).$$

After a standard change of basis, this can be brought into

$$
A = \left(\begin{array}{ccc|ccc|c|ccc}
\multicolumn{3}{c|}{\leftarrow \theta_1 \rightarrow} & \multicolumn{3}{c|}{\leftarrow \theta_2 \rightarrow} & & \multicolumn{3}{c}{\leftarrow \theta_p \rightarrow} \\
* & \cdots & * & * & \cdots & * & \cdots & * & \cdots & * \\
1 & & 0 & & & & & & & \\
 & \ddots & & & 0 & & & & 0 & \\
0 & & 1 & & & & & & & \\
\hline
* & \cdots & * & * & \cdots & * & & * & \cdots & * \\
 & & & 1 & & 0 & & & & \\
 & 0 & & & \ddots & & & & 0 & \\
 & & & 0 & & 1 & & & & \\
\hline
 & \vdots & & & \vdots & & & & \vdots & \\
\hline
* & \cdots & * & * & \cdots & * & & * & \cdots & * \\
 & & & & & & & 1 & & 0 \\
 & 0 & & & 0 & & & & \ddots & \\
 & & & & & & & 0 & & 1
\end{array}\right) ; \quad B \text{ as before.}
$$

The feedback matrix L can be chosen so that all entries outside the diagonal blocks are zero. If this is done, the closed-loop poles will be given by the eigenvalues of diagonal blocks. (Using this structure, it may not be possible to generate all closed-loop pole configurations, but this is irrelevant for the conclusion.) If L is chosen so that the closed-loop poles tend to infinity as indicated above,

$$
\ell_{ij} = \begin{cases} O(t^{j-i+1}) & \text{in the diagonal blocks} \\ O(1) & \text{otherwise} \end{cases} , \quad t \to \infty
$$

Consequently, $L^T A^\nu B = O(t^{\nu+1})$, and it follows that the elements of $G(s)$ can be made arbitrarily small uniformly over the imaginary axis if t is chosen large enough. □

<u>Remark 1</u>. An examination of the above calculations shows that nothing is gained by letting *all* poles tend to infinity. In fact, it is sufficient to place μ of them, μ being the dimension of S^*, far out in the left half-plane.

This observation will be important in the sequel. □

Remark 2. The space S^* is in fact the orthogonal complement of the maximal (A^T,M)-invariant subspace of $\mathrm{Ker}(B^T)$ of the dual system (B^T,A^T,M). This (not very intuitive) interpretation will be exploited later. □

The condition for the high-gain design to be successful is a fairly restrictive one. However, there are other, more efficient ways of reducing the effects of the disturbances. The idea is to make the system maximally unobservable from M^T. This may fix some of the closed-loop poles. The freedom that is left is used to let the remaining poles tend to negative infinity along the lines of the preceding theorem.

Let V^* be the maximal (A,B)-invariant subspace of $\mathrm{Ker}(M^T)$, and let R^* be the corresponding maximal controllability subspace. Finally, let $\underline{V^*}$ be the subspace of V^* that was defined in § 1.3. The following theorem then holds.

Theorem 2.2 PR of (A,B) with respect to (M,N) is achievable if

$$\mathrm{Im}(N) \subseteq S^* + \underline{V^*}. \quad \square$$

Proof. Choose an L such that

$$(A-BL^T)\ V^* \subseteq V^*.$$

Then $\underline{V^*}$ consists of the modes in $\mathrm{Ker}(M^T)$ with eigenvalues fixed in the left half-plane plus the ones whose spectrum is freely assignable. These modes do not contribute to G(s) and can thus be discarded as long as internal stability is ensured. There may be eigenvalues on the imaginary axis for the chosen L, but if this is the case, L may be perturbed slightly in order to achieve asymptotic stability without destroying the PR property. Outside V^*, where the spectrum is freely assignable, the poles may be placed arbitrarily far out in the left half-plane. If $\mathcal{V}^* \cap \mathcal{S}^* \neq 0$, it follows from a theorem by Bengtsson ([Ben], Thm. 4.1) that $\mathcal{V}^* \cap \mathcal{S}^* \subseteq \mathcal{R}^*$, so there is no conflict between these two strategies. □

The following corollary gives a somewhat more concrete picture of the situation.

<u>Corollary 2.1</u> i) If $p=q$, PR of (A,B) with respect to (M,N) is acheivable if

$$H(s) = M^T(sI-A)^{-1}B$$

is invertible and has no zeros in the open right half-plane.

ii) If $p > q$, PR of (A,B) with respect to (M,N) is achievable if H(s) is right invertible and has no zeros in the open right half-plane.

iii) If $p < q$, PR of (A,B) with respect to (M,N) is in general not achievable (using these methods; cf., however, Remark 1 following the proof). □

<u>Proof</u>. i) $(p=q)$ The proof consists of two parts. First the invertibility assumption will be shown to imply that

$$\mathcal{S}^* + \mathcal{V}^* = \mathbb{R}^n.$$

The claim then follows once it has been proved that $\mathcal{V}^*_- = \mathcal{V}^*$ if H(s) is minimum-phase.

Notice first that the algorithm generating $\mathcal{V}^*$ can be modified slightly to look like

$$\begin{cases} \mathcal{V}^{(0)} = \mathrm{Ker}(M^T) \\ \mathcal{V}^{(\mu)} = \mathcal{V}^{(\mu-1)} \cap A^{-1}(\mathcal{V}^{(\mu-1)} + \mathrm{Im}(B)). \end{cases}$$

Compare this with the $\mathcal{S}^*$ algorithm:

$$\begin{cases} \mathcal{S}^{(0)} = \mathrm{Im}(B) \\ \mathcal{S}^{(\mu)} = \mathcal{S}^{(\mu-1)} + A(\mathrm{Ker}(M^T) \cap \mathcal{S}^{(\mu-1)}). \end{cases}$$

A moment of reflection in the right direction shows that

$$(S^{(\mu)})^{\perp} = V_d^{(\mu)},$$

$V_d^{(\mu)}$ being the sequence generating V_d^*, the maximal (A^T,M)--invariant subspace of $Ker(B^T)$ in the dual system (B^T,A^T,M). Further, invertibility of H(s) is equivalent to ([Sil])

$$V^* \cap Im(B) = 0,$$

which in turn implies that $R^* = 0$. According to the previously mentioned theorem by Bengtsson ([Ben], Thm. 4.1),

$$R^* = V^* \cap (V_d^*)^{\perp},$$

which completes the first part of the proof.

To conclude, it will be shown that any L satisfying

$$(A-BL^T)\ V^* \subseteq V^*$$

places closed-loop poles at the zero locations (this was even proposed as a definition in [Ben]). The zeros of the invertible transfer matrix

$$H(s) = M^T(sI-A)^{-1}B$$

can be defined as the complex members s for which the matrix

$$\begin{pmatrix} sI-A & | & B \\ - - - & | & - - \\ M^T & | & 0 \end{pmatrix}$$

loses rank ([Ros 2]). Further, the poles corresponding to unobservable modes of the closed-loop system $(M^T, A-BL^T, B)$ are the points where

$$\begin{pmatrix} sI-A+BL^T \\ - - - - - \\ M^T \end{pmatrix} \qquad (2.6)$$

loses rank ([Ros 1], Ch. 2). From this it is clear that an eigenvalue corresponding to an unobservable mode is also a zero. Now assume that s_0 is a zero, and that

$$v = \begin{bmatrix} v_1 \\ --- \\ v_2 \end{bmatrix}$$

is an (n+p)-vector satisfying

$$\begin{cases} (s_0 I - A)\ v_1 + Bv_2 = 0 \\ M^T v_1 = 0. \end{cases} \qquad (2.7)$$

Clearly, $v_1 \neq 0$, whence there exists an (n×p)-matrix L satisfying

$$L^T v_1 = v_2.$$

Thus (2.7) can be written as

$$\begin{cases} (s_0 I - A + BL^T)\ v_1 = 0 \\ M^T v_1 = 0\ , \end{cases}$$

which on comparison with (2.6) shows that v_1 is an unobservable mode and that s_0 is the corresponding eigenvalue. Summarising, all zeros will be eigenvalues for any L satisfying

$$(A-BL^T)\ \mathcal{V}^* \subseteq \mathcal{V}^*.$$

The minimum-phase condition then ensures that

$$\mathcal{V}^* = \mathcal{V}^*_- \ .$$

ii) $(p > q)$ That zeros of H(s) (if any) are also closed--loop poles for any L subject to

$$(A-BL^T)\ \mathcal{V}^* \subseteq \mathcal{V}^*$$

follows in the same way as before. It will be shown that, in the absence of zeros,

$$\mathcal{R}^* = \mathcal{V}^*.$$

To simplify matters, assume that

$$\mathrm{Ker}(M^T) = \mathcal{V}^*.$$

Recall that $\mathrm{Ker}(M^T)$ is a controllability subspace iff

$$\mathrm{Ker}(M^T) = \sum_{i=0}^{n-1} (A-BL^T)^i (\mathrm{Ker}(M^T) \cap \mathrm{Im}(B)) \tag{2.8}$$

for any L such that

$$(A-BL^T)\ \mathcal{V}^* \subseteq \mathcal{V}^*.$$

Assume that

$$\mathrm{Ker}(M^T) \cap \mathrm{Im}(B) = \mathrm{Im}(B_1),$$

where

$$B = \left[B_1 \mid B_2 \right].$$

The definition of zeros given for the square case above is equally valid for nonsquare systems ([Dav]). In a suitable basis, this matrix takes the form

$$\begin{array}{c} \\ q \\ (n-q) \\ q \end{array} \left(\begin{array}{c|c|c|c} \overset{\leftarrow q \rightarrow}{sI_q - \tilde{A}_{11}} & \overset{\leftarrow (n-q) \rightarrow}{0_{q,(n-q)}} & \overset{\leftarrow (p-q) \rightarrow}{0_{q,(p-q)}} & \overset{\leftarrow q \rightarrow}{B_{21}} \\ \hline \tilde{A}_{12} & sI_{(n-q)} - \tilde{A}_{22} & B_{12} & B_{22} \\ \hline I_q & 0_{q,(n-q)} & 0_{q,(p-q)} & 0_{q,q} \end{array} \right) \tag{2.9}$$

where $\tilde{A} = (A-BL^T)$. If (2.8) does not hold, there is an s_0 such that the $(n-q)\times(n+p-2q)$-block in the centre of (2.9) has rank $< (n-q)$. But then the full matrix has rank $< (n+q)$ for $s = s_0$, which was to be proved.

iii) $(p < q)$ $\dim(\mathcal{V}^*) + \dim(\mathcal{S}^*) \leqslant (n-1)$. □

Remark 1. The corollary is weaker than the theorem, since N does not enter. For instance, PR is achievable with respect to any number of nonlinearities in Im(B). But as a condition on M, A, and B only, the minimum-phase condition is necessary. For, dualising the previous results, it is easy to see how N should be chosen in order to make the system maximally uncontrollable from B. Picking such an N and letting $\Phi(\cdot,\cdot) = I_q$ implies that system will contain uncontrollable, unstable modes irrespectably of the choice of L. Whether the minimum-phase condition is necessary for a fixed N is a considerably more complex problem. □

Remark 2. Nonsquare systems have zeros only in "exceptional" cases. PR is thus generically achievable in case $p > q$. □

According to iii) of the corollary, PR will in general not be achievable in the case $p < q$. There will have to be a trade-off between the maximally permissible sectors for the different nonlinearities. The theory used so far gives no indication of how this problem should be tackled. A method for solving the RRP in this case, which is perhaps the most common in practical applications, is proposed in the following section. A slightly different approach is suggested in Chapter 5.

For this case $(p < q)$ and cases $p = q$ when PR is not achievable, it would be desirable to produce bounds for the maximal sector of the nonlinearities, based on A, B, M, and N. This is difficult, however, already for the simpler problem in which the nonlinearity is replaced by a constant, unknown gain, ranging over some finite interval. Obviously, the critical values of the gain (if any) for which the system will contain uncontrollable, unstable modes, are bounds for the permissible sector. That the interval in which the gain can take its values contain no such critical value is, however, not a sufficient condition for stabilisability, as is shown by the following counterexample.

Example 2.1 Consider the linear, time-invariant system given by

$$\frac{dx}{dt}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) - \kappa \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} (1 \; -10 \;\; 20)\, x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t).$$

It is readily verified that this system is controllable for all positive values of κ. Applying the feedback

$$u = - L^T x = - \sum_{i=1}^{3} \ell_i x_i$$

yields the closed-loop characteristic polynomial

$$p_\kappa(s) = s^3 + s^2(\ell_1+5\kappa) + s(-10\kappa + \ell_2(1-2\kappa) - \ell_3\kappa) + \\ + \ell_1 \cdot 40\kappa + \ell_2 \cdot 20\kappa + \ell_3(1+8\kappa).$$

For $\kappa = 0$,

$$p_\kappa(s) = s^3 + \ell_1 s^2 + \ell_2 s + \ell_3,$$

and necessary conditions for stability are

$$\ell_i > 0, \quad i = 1, 2, 3. \tag{2.10}$$

For $\kappa = 1$,

$$p_\kappa(s) = s^3 + s^2(\ell_1+5) + s(-10-\ell_2-\ell_3) + 40\ell_1 + 20\ell_2 + 9\ell_3,$$

and a necessary condition for stability is

$$10 + \ell_2 + \ell_3 < 0. \tag{2.11}$$

Clearly, (2.10) and (2.11) are not compatible, whence there exists no fixed, linear, constant feedback law that stabilises the system for all values of κ in an interval that contains [0,1]. □

2.2.2 Solution of the RRP using linear-quadratic optimal control

The object of this section is to show that the feedback law derived previously is easily generated within the framework of linear-quadratic optimal control (LQOC) theory.

Consider the problem of minimising, with respect to $u(\cdot)$, the performance index

$$J_\rho = \int_0^\infty (x(s)^T MM^T x(s) + \rho\, u(s)^T Ru(s))\, ds, \qquad (2.12)$$

where $x(\cdot)$ solves

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Bu(t) \\ x(0) = x_0. \end{cases} \qquad (2.13)$$

It is well known that the solution is given by

$$u(t) = -\rho^{-1}R^{-1}B^T P_\rho x(t),$$

where P_ρ is the largest solution of the algebraic Riccati equation

$$A^T P_\rho + P_\rho A + MM^T - \rho^{-1} P_\rho BR^{-1}B^T P_\rho = 0. \qquad (2.14)$$

The minimum value of J_ρ is given by $x_0^T P_\rho x_0$.

The asymptotic behaviour of the solution P_ρ and the resulting closed-loop system was investigated by Kwakernaak and Sivan in [Kwa 2]. Define

$$P_0 = \lim_{\rho \to 0} P_\rho$$

(the limit always exists) and let

$$\begin{cases} p = \operatorname{rank}(B) \\ q = \operatorname{rank}(M). \end{cases}$$

Then the results of [Kwa 2] may be summarised as follows:

i) if $p < q$, $P_0 \neq 0$.

ii) if $p = q$, $P_0 = 0$ if and only if $H(s) = M^T(sI-A)^{-1}B$ has no zeros in the open right half-plane, provided that $H(s)$ is invertible.

iii) if $p > q$ a sufficient but not necessary condition for P_0 to be zero is that there exists a $(p\times q)$-matrix D such that $\tilde{H}(s) = M^T(sI-A)^{-1}BD$ satisfies the condition of ii).

Consider first the case $p = q$. Clearly, P_0 cannot be equal to zero if the closed-loop system has finite poles, whose corresponding modes are observable from M^T. This implies that the solution of the LQOC problem (2.12),(2.13) asymptotically reproduces the feedback gain obtained in § 2.2.1 by entirely different methods, provided that $H(s)$ is minimum-phase. If, on the contrary, $H(s)$ has right half-plane zeros, some closed-loop poles tend to the mirror images of these zeros with respect to the imaginary axis as ρ tends to zero. The corresponding modes then remain observable from M^T also in the limit, and $P_0 \neq 0$. The modes discussed here correspond to $\mathcal{V}^*$ in the earlier notation; that $\mathcal{S}^*$ belongs to $\mathrm{Ker}(P_0)$ can be inferred directly from the Riccati equation.

Consider now the case $q < p$. It will be shown that the minimum-phase condition is valid also in this case. Section 2.2.1 exhibits a feedback law that makes the closed-loop system unobservable from M^T and places the poles outside $\mathcal{V}^*$ far out in the left half-plane, all this provided that $H(s) = M^T(sI-A)^{-1}B$ is minimum-phase. A straightforward calculation shows that this suboptimal u makes J_ρ zero in the limit, whence P_0 must be zero.

Further, if $H(s)$ has a zero, then this zero will asymptotically be a pole of the closed-loop system generated from (2.12),(2.13). To see this, consider the standard frequency domain equality obtained from the Riccati equation (2.14) (cf. [Kwa 1], § 3.8):

$$p_{CL}(s)p_{CL}(-s) = p(s)p(-s)\det(I_p + \frac{1}{\rho} R^{-1}H^T(-s)H(s)). \tag{2.15}$$

($p_{CL}(s)$ and $p(s)$ are the closed-loop and open-loop characteristic polynomials, respectively.) If $H(s)$ is invertible, $\det(H(s)) \not\equiv 0$, and it follows from (2.15) that the finite closed-loop poles approach the zeros of $H(s)$ as ρ tends to zero. However, if $q < p$, $\det H(s) \equiv 0$, and the coefficient of ρ^{-p} is zero. The first non-vanishing coefficient is that of ρ^{-q} ([Gan], p. 70), which equals the sum of all $(q\times q)$-minors of $H^T(-s)H(s)$. If $H(s)$ has a zero s_0, $(s-s_0)$ is a factor of all these minors of $H(s)$ ([Kon]), and it follows that s_0 (or its mirror image in case it is a right-half-plane zero) is indeed a closed-loop pole in the limit. In the nonminimum-phase case, the closed-loop system will have finite poles, whose modes remain observable from M^T as ρ tends to zero, and P_0 will be different from zero.

Summarising, the minimum-phase criterion is valid also in the case $q < p$, and it is further generically satisfied. The following extension of Kwakernaak-Sivan's result has thus been established.

<u>Theorem 2.3</u> Let P_ρ be the largest solution of the algebraic Riccati equation (2.14), and assume that

i) $\text{rank}(M) = \text{rank}(B)$ and

$H(s) = M^T(sI-A)^{-1}B$ is invertible,

or that

ii) $\text{rank}(M) < \text{rank}(B)$ and

$H(s)$ is right invertible.

Then the rank of

$$P_0 = \lim_{\rho\to 0} P_\rho$$

equals the number of right-half-plane zeros of H(s), and

$$\mathrm{Ker}(P_0) = S^* + \mathcal{V}^*_- . \quad \square$$

Remark 1. It is not clear from the proof in [Kwa 2] whether the assumption on invertibility is essential. That this condition is in fact necessary can be concluded from the following counterexample.

Example 2.2 Consider (2.12),(2.13) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad M^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$H(s) = M^T(sI-A)^{-1}B = \begin{pmatrix} 1/s^2 & 1/s \\ 1/s^3 & 1/s^2 \end{pmatrix}$$

has no zeros, but

$$P_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \square$$

Remark 2. Although Thms. 2.2 and 2.3 are closely related, they are independent. To see the difference, let

$$\tilde{A} = A - \rho^{-1}BR^{-1}B^TP_\rho,$$

and pick any vector $x_0 \in \mathrm{Im}(N)$. Then

$$\lim_{\rho\to 0} x_0^T P_\rho x_0 =$$

$$= \lim_{\rho\to 0} x_0^T\left(\int_0^\infty \exp(\tilde{A}^T s)(MM^T + \frac{1}{\rho} P_\rho BR^{-1}B^TP_\rho)\exp(\tilde{A}s)ds\right)x_0 \geq$$

$$\geq \lim_{\rho\to 0} x_0^T\left(\int_0^\infty \exp(\tilde{A}^T s)\; MM^T \exp(\tilde{A}s)ds\right)x_0 =$$

$$= \lim_{\rho\to 0} x_0^T\left(\int_{-\infty}^\infty (-i\omega I-\tilde{A}^T)^{-1}\; MM^T(i\omega I-\tilde{A})^{-1}d\omega\right)x_0$$

so that, with $G(s) = M^T(sI-\tilde{A})^{-1}N$,

$$\mathrm{Im}(N) \subseteq \mathrm{Ker}(P_0)$$

implies

$$\lim_{\rho\to 0} \int_{-\infty}^{\infty} G^T(-i\omega)\, G(i\omega)\, d\omega = 0.$$

However, for the circle criterion an estimate of

$$\sup_{\omega\in\mathbb{R}} \max_{i} |\lambda_i(G^T(-i\omega)G(i\omega))|$$

is asked for, and in general there are no implications between L_1-convergence and L_∞-convergence. □

The reason why the LQOC apparatus has been introduced is twofold:

i) It provides what is believed to be a sane way of tackling the case $q > p$. PR is in general not achievable in this case, and the maximally permissible sectors for the nonlinearities are consequently interdependent. It is then possible to choose the penalty on x as a weighted sum:

$$J_\lambda = \int_0^\infty (x(s)^T Q_\lambda x(s) + u(s)^T R u(s))\, ds,$$

$$Q_\lambda = \sum_{i=1}^{q} \lambda_i m_i m_i^T,$$

where a λ_i should be chosen large if the corresponding sector radius is large and vice versa.

ii) So far the only concern has been to produce a feedback gain matrix that guarantees stability under certain perturbations. In practice, stability will not be the only requirement on the design. The LQOC framework permits one to make a suitable trade-off between robustness and other specifications, such as the transient behaviour of the nominal plant. This is basically a trial-and-error procedure, as is synthesis via LQOC in the standard case.

2.3 Input channel uncertainty

The case of control-dependent disturbances will now be tackled. The special form of equation (2.1) to be studied is consequently

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) + T\Theta(u(t),t). \qquad (2.1)''$$

Section 2.3.1 considers the case when the implementation of a desired linear, constant feedback law is corrupted by timevarying nonlinearities. The results can be expressed in classical terms such as gain and phase margins. In § 2.3.2 more general control-dependent disturbances are considered.

2.3.1 Gain and phase margins for high-gain regulators

Since the discovery by Kalman of the frequency domain inequality satisfied by linear-quadratic optimal control regulators, an abundance of literature has been produced on the robustness of these regulators (see e.g. [Kwa 1], § 3.9, and [Saf]). The basic results are as follows. Let

$$G(s) = L^T(sI-A+BL^T)^{-1}B,$$

where L is the optimal gain derived from the Riccati equation:

$$\begin{cases} L^T = R^{-1}B^TP \\ A^TP + PA + Q - PBR^{-1}B^TP = 0. \end{cases}$$

Elementary manipulations of this equation lead to the equality

$$(I_p-G(-s))^T\, R(I_p-G(s)) =$$
$$= R - B^T((-sI_n-A+BL^T)^T)^{-1}\, Q(sI_n-A+BL^T)^{-1}B. \qquad (2.16)$$

(Notice that the above equation is given for the closed--loop transfer function and that it is thus not quite standard.) Suppose for simplicity that $R = I_p$. If Q is at least positive semidefinite, the second term on the right-hand side is nonpositive on the imaginary axis. A straightforward application of the circle criterion then shows that system stability will be retained if the nominal input is perturbed by any timevarying nonlinearity $\Phi(\cdot,\cdot)$ satisfying the sector condition

$$\sigma^T\Phi(\sigma,t) \geq \sigma^T\sigma(-\frac{1}{2}+\varepsilon) \quad \text{for some } \varepsilon > 0. \tag{2.17}$$

This is illustrated in Fig. 2.1. This covers gain drops by fifty per cent, and also any dynamical perturbations (for instance neglected actuator dynamics) subject to the above sector condition. The only requirement is that Q be positive-semidefinite, and no restriction is laid on the plant. It can be shown by examples that (2.17) is tight, i.e. cannot be improved without further information about A and/or Q.

Various extensions of the above result have appeared in the literature, for instance in [Won 2]. In this paper, the problem of maximising the permissible sector for the nonlinearity Φ was also posed. It will be shown that this problem has a trivial solution and that, as such, it is not very meaningful.

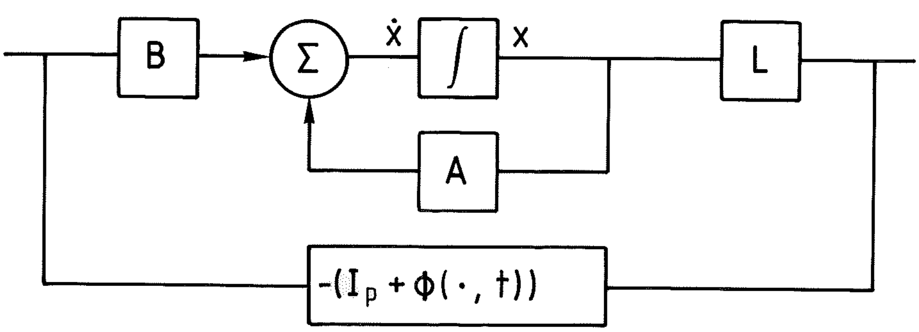

Fig. 2.1 - Illustrating the robustness result of (2.17).

Obviously, a necessary sector condition for open-loop unstable plants is

$$\sigma^T\Phi(\sigma,t) \geq \sigma^T\sigma(-1+\varepsilon), \qquad \varepsilon > 0. \qquad (2.18)$$

For certain feedback gains, this is also sufficient.

Proposition 2.1 For any positive ε, there exists a feedback gain L that guarantees stability of the closed-loop system for any perturbation from the nominal gain subject to inequality (2.18). □

Proof. Choose any feedback gain L_0 designed via linear--quadratic optimal control with a $Q \geq 0$ and use

$$L = \frac{1}{\varepsilon} \cdot L_0. \quad \square$$

Maximal robustness in the above sense can thus be achieved by choosing a large nominal gain. An alternative means to produce such a gain is to let the penalty ρ on the input tend to zero in the optimal control problem along the lines of § 2.2.2. This is not quite equivalent, however. A necessary condition for improved robustness as ρ tends to zero is that the function $B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B$ have no zeros on the imaginary axis.

In classical terminology, stability under the condition (2.18) implies infinite gain margin, and asymptotically 100 per cent gain reduction tolerance and 90 degrees phase margin.

A few words of caution are necessary here. Using high gains in order to achieve a larger gain reduction tolerance and a better phase margin may not be so successful as the above results would seem to indicate. For instance, if the gain drop is due to saturation, there is obviously no point in increasing the nominal gain, since this will only make the regulator saturate more often.

Secondly, high gains means high bandwidth, which in turn implies that dynamics neglected in the modelling phase may become important. This implies that what has been gained in phase margin may very well be consumed by no longer negligible actuator dynamics, for instance.

These arguments indicate that the problem of maximising the robustness of the controller in the above sense is not well-posed. For the solution to be practically meaningful, robustness must be achieved without recourse to high gains. These matters will be discussed elsewhere ([Mol]).

It is necessary to point out that the gain and phase margins hold separately. A simultaneous gain drop and phase shift in the implementation may well result in instability even though they are in the permissible interval when considered separately.

Finally, it should be stressed that the above results assume that the state is accessible for measurement. Some errors on this point have occurred in the literature. If the state is not measurable, and the synthesis is based on an observer, two cases may occur. If the disturbance $\Phi(\cdot,\cdot)$ is known, it can be included in the observer, and the robustness result is still valid. If it is not known, *there is no generally valid robustness result whatsoever* ([And], § 9.1, [Doy]).

2.3.2 General disturbances

The purely geometric problem of selecting linear combinations of the inputs in order to achieve complete disturbance isolation was solved by Haussmann in [Hau]. Haussmann discusses the stochastic problem of stabilising, in the L_2-sense, a linear system with white-noise input-dependent disturbances. The results are geometric in nature, however, and are equally valid for the problem to be treated here. Only a brief sketch of the arguments will be given here. For the details, as well as for the necessity part of the proof, the reader is referred to [Hau].

The system equation is repeated for convenience:

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) + T\Theta(u(t),t). \qquad (2.1)''$$

The problem is to find conditions under which (2.1)" can be stabilised using linear state feedback as the confining sector of the nonlinearity $\Theta(\cdot,\cdot)$ increases without bound. The condition rank(T) = p, which was introduced in the formulation of the RRP, can be dropped in this context.

The basic idea is given in the following simple proposition.

<u>Proposition 2.2</u> Let X_- be the subspace of $\mathbb{R}^n$ spanned by the stable modes of A. Assume that (A,B) is a stabilisable pair, and that

$$\mathrm{Im}(T) \subseteq X_- . \qquad (2.19)$$

Then the system given by Equation (2.1)" is stabilisable for arbitrarily large disturbances $\Theta(\cdot,\cdot)$. □

The proof is omitted.

In the single-input case, this exhausts all possibilities. In the multivariable case, however, it may be possible to proceed under the additional assumption that the disturbances enter linearly. In this case, the system equation can be written as

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) + T(u)\kappa(t),$$

where

$$T(u) = \sum_{i=1}^{p} T_i u_i,$$

$$\kappa(t) = (\kappa_1(t), \kappa_2(t), \ldots, \kappa_q(t)),$$

and the T_i´s are $(n \times q)$ matrices. The idea is to split the state space into a sequence of subspaces, where each subsystem is stabilised by a control producing disturbances only in the subsystems that have already been stabilised.

Theorem 2.4 Define

$$M_T(X) = \{Bu;\ \mathrm{Im}(T(u)) \in X\}.$$

Let the sequence $\mathcal{T}^{(\mu)}$ be generated according to

$$\begin{cases} \mathcal{T}^{(0)} = X_- \\ \mathcal{T}^{(\mu)} = \mathcal{T}^{(\mu-1)} + < A \mid M_T(\mathcal{T}^{(\mu-1)}) > , \end{cases}$$

and let $\mathcal{T}^*$ be the first $\mathcal{T}^{(\mu)}$ satisfying

$$\mathcal{T}^{(\mu)} = \mathcal{T}^{(\mu-1)}.$$

Then the system given by equation (2.1)" is stabilisable for arbitrarily large disturbances if

$$\mathcal{T}^* = \mathbb{R}^n. \quad \square$$

It should be stressed that the condition of the theorem is a restrictive one.

2.4 Observers for nonlinear systems

The solution of the RRP presented above assumes that the state is accessible for measurement. If this is not the case, some sort of state reconstruction will be necessary, which poses the problem of constructing an observer for the nonlinear system in question. Great efforts have been made in the area of nonlinear filtering, but the solutions which are optimal in some sense are generally of limited value due to their complexity. This justifies a search for simpler, suboptimal solutions, which can be proved to converge. Although the problem of reconstruction will be discussed in a deterministic setting here, it should be stressed that the question whether there exists a fixed-gain observer of the type to be discussed below is of theoretical interest in itself for the corresponding stochastic problem (cf. [Tar]).

2.4.1 Basic observer structure

Consider the nonlinear time-varying system given by

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) - N\Phi(M^T x(t), t) + F(u(t)) \\ y(t) = C^T x(t). \end{cases} \qquad (2.20)$$

$F(\cdot)$ is any nonlinear, n-vector-valued function, which includes inputs and known disturbances. As usual, M and N are $(n\times q)$-matrices, and C is an $(n\times m)$-matrix. The function $\Phi(\cdot,\cdot)$ is assumed to be known. Notice that the assumption on linear observations is not so restrictive as it may seem, since a nonlinearity in the output may often be transferred into a nonlinearity in the plant via a change of state variables.

A simple alternative for reconstructing the state of (2.20) is obviously to use an ordinary Kalman-Bucy-type filter with a fixed, linear gain. Postulating thus the observer structure

$$\frac{d\hat{x}}{dt}(t) = A\hat{x}(t) - N\Phi(M^T\hat{x}(t), t) + F(u(t)) + K(y - C^T\hat{x}(t)) \qquad (2.21)$$

yields the following equation for the reconstruction error $e = x - \hat{x}$:

$$\frac{de}{dt}(t) = (A-KC^T)\ e(t) - N\Phi(M^Tx(t),t) + N\Phi(M^T\hat{x}(t),t) \triangleq$$
$$\triangleq (A-KC^T)\ e(t) - N\Psi(M^Te(t),t). \qquad (2.22)$$

Notice that, since $x(t)$ is not known, $\Psi(\cdot,t)$ is an unknown function of e, but that it is confined to lie within the incremental sector of $\Phi(\cdot,t)$. The initial values of (2.22) range over the whole of $\mathbb{R}^n$, and a K is therefore sought which will guarantee asymptotic stability in the large of the null solution.

Two important differences between the regulator and the observer cases should be noticed. In the observer case, $\Phi(\cdot,\cdot)$ must be a known function, since otherwise the estimate will be biased. Further, the sector of $\Psi(\cdot,t)$ appearing in equation (2.22) is the *incremental* sector of $\Phi(\cdot,t)$, which is always larger than the confining sector of $\Phi(\cdot,t)$.

The following theorem and its corollary are proved by simply dualising the results of § 2.2. The involved spaces S_d^*, V_d^*, and V_{-d}^* are now subspaces of the dual of $\mathbb{R}^n$.

Theorem 2.5 There exists a K such that the null solution of equation (2.22) is globally uniformly asymptotically stable for $\Phi(\cdot,t)$ incrementally in any finite sector if

$$\text{Im}(M^T) \subseteq S_d^* + V_{-d}^*. \quad \square$$

Corollary 2.2 i) If $m = q$, a K that ensures consistency of the estimate exists if

$$H(s) = C^T(sI-A)^{-1}N$$

is invertible and has no zeros in the open right half--plane.

ii) If $m > q$, a K that ensures consistency exists if $H(s)$ is left invertible and has no zeros in the open right half-plane.

iii) If $m < q$, a K ensuring convergence for all incremental sectors if $\Phi(\cdot,t)$ can in general not be found (using these methods; cf., however, Remark 1 following Cor. 2.1). □

2.4.2 An application: state estimation in autopilots for large tankers

Consider the motion of the ship shown in Fig. 2.2. u and v are the forward and lateral velocities, respectively, ψ is the course, $r = d\psi/dt$ is the yaw rate, and δ is the rudder angle. u may often be considered as constant in the analysis.

The ship motion can, with sufficient accuracy, be described by a third order system (this is the model currently used in autopilots, cf. [Käl]), which linearised around the

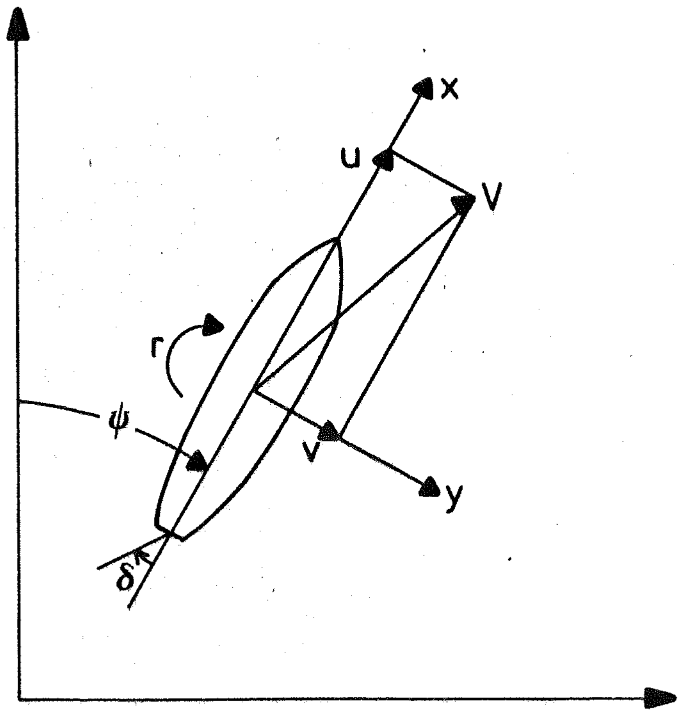

Fig. 2.2 - Coordinates and variables used for the equation of motion.

origin, is given by the equations

$$\begin{pmatrix} \frac{dv}{dt}(t) \\ \frac{dr}{dt}(t) \\ \frac{d\psi}{dt}(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v(t) \\ r(t) \\ \psi(t) \end{pmatrix} + \begin{pmatrix} b_{11} \\ b_{12} \\ 0 \end{pmatrix} \delta(t).$$

If the length unit is chosen as the length of the ship, and the time unit the time it takes to travel a length unit (given the forward velocity u), the coefficients a_{ij} and b_{ij} turn out to be remarkably invariant for different ships. They depend heavily on the load conditions, however. If the wind and wave disturbances are negligible, a_{13} and a_{23} can be set to zero, and the system in this case contains a pure integrator.

Although the above linear model is sufficiently accurate when controlling the ship for constant course, it is inadequate when the course of the ship is being changed. Nonlinear effects become important even at moderate yaw rates.

The synthesis problem is the following. Suppose that the course ψ is measured. Is it possible to find a constant--gain state estimator of the type (2.21) that converges in presence of the nonlinearities?

Example 2.3 A typical set-up for a tanker, when nonlinear effects are taken into account, is

$$\begin{cases} \frac{dv}{dt}(t) = a_{11}\cdot v(t) - c\cdot v(t)\cdot |v(t)| + a_{12}\cdot r(t) + b_{11}\cdot\delta(t) \\ \frac{dr}{dt}(t) = a_{21}\cdot v(t) + a_{22}\cdot r(t) + b_{12}\cdot\delta(t) \\ \frac{d\psi}{dt}(t) = r(t) \end{cases}$$

with

$$\begin{cases} a_{11} = -0.94 \\ a_{12} = -0.36 \\ a_{21} = -1.75 \\ a_{22} = -1.19 \end{cases}$$

$$\begin{cases} b_{11} = 0.23 \\ b_{12} = -1.24 \end{cases}$$

and

$$c = 2.00.$$

Theoretically, the sector of the nonlinearity $\Phi(v) = v \cdot |v|$ covers the whole of the first and third quadrants, but for physical reasons, the sway rate v is bounded apriori. A conservative estimate for the relevant interval is [-0.5, 0.5]. Without changing the behaviour of the estimator in the physically relevant interval, $\Phi(\sigma)$ may consequently be replaced by

$$\tilde{\Phi}(\sigma) = \begin{cases} \sigma + 0.25 & \sigma \leqslant -0.5 \\ \Phi(\sigma) & -0.5 < \sigma \leqslant 0.5 \\ \sigma - 0.25 & \sigma > 0.5. \end{cases}$$

$\Phi(\sigma)$ and $\tilde{\Phi}(\sigma)$ are shown in Fig. 2.3. The incremental sector of $\tilde{\Phi}(\sigma)$ is [0, 1].

In the notation of § 2.1,

$$A = \begin{pmatrix} -0.94 & -0.36 & 0 \\ -1.75 & -1.19 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad M = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};$$

$$N = \begin{pmatrix} 2.00 \\ 0 \\ 0 \end{pmatrix}.$$

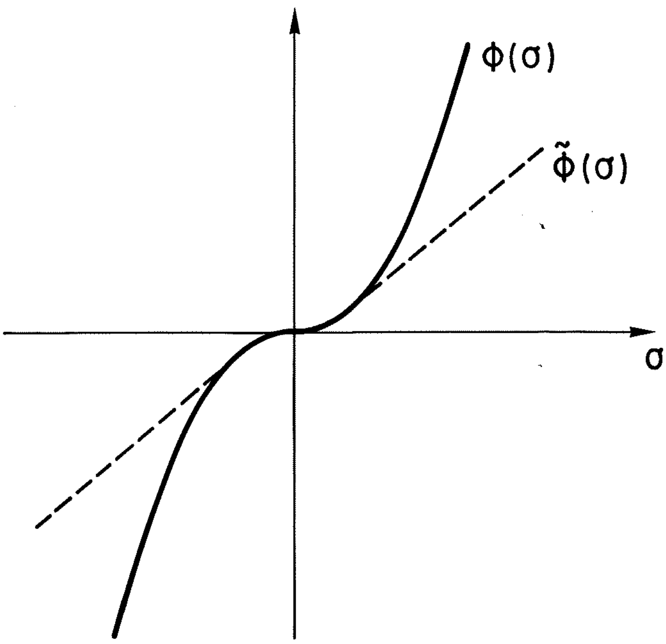

Fig. 2.3 - Modifying the nonlinearity $\Phi(\sigma)$.

An inspection shows that N is orthogonal to C^T and C^TA, whence the condition of Thm. 2.1 (or rather its dual) is satisfied. In fact, the condition on

$$G(s) = M^T(sI - A + KC^T)^{-1}N$$

is rather weak; the circle criterion guarantees (physically relevant) global convergence of the estimator (2.21) if $(G(s)+1)$ is strictly positive real. The eigenvalues of A are 0, -0.27, and -1.87, and if the eigenvalues of $(A-KC^T)$ are chosen for instance as -3, -4, and -5, it can be shown that this condition is indeed satisfied. This pole placement also gives a bandwidth which is reasonable with respect to the measurement disturbances. □

Example 2.4 The second example is a fully loaded tanker of the same size as the previous one. The state equations are

$$\begin{cases} \frac{dv}{dt}(t) = a_{11}\cdot v(t) - c_{11}\cdot v(t)\cdot|v(t)| + a_{12}\cdot r(t) + b_{11}\cdot\delta(t) \\ \frac{dr}{dt}(t) = a_{21}\cdot v(t) + a_{22}\cdot r(t) - c_{22}\cdot r(t)\cdot|r(t)| + b_{12}\cdot\delta(t) \\ \frac{d\psi}{dt}(t) = r(t) \end{cases}$$

with

$$\begin{cases} a_{11} = -0.39 \\ a_{12} = -0.45 \\ a_{21} = -3.40 \\ a_{22} = -1.58, \end{cases}$$

$$\begin{cases} b_{11} = 0.097 \\ b_{12} = -0.81, \end{cases}$$

and

$$\begin{cases} c_{11} = 1.18 \\ c_{22} = 1.25. \end{cases}$$

(The open-loop system is unstable, but this is irrelevant for the analysis.)

Since the number of nonlinearities exceeds the number of outputs, this is a case for the LQOC design. With

$$A = \begin{pmatrix} -0.39 & -0.45 & 0 \\ -3.40 & -1.58 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$N = \begin{pmatrix} 1.18 & 0 \\ 0 & 1.25 \\ 0 & 0 \end{pmatrix},$$

the condition for global convergence is that

$$G(s) = M^T(sI - A + KC^T)^{-1}N$$

satisfy

$G(s) + I$ strictly positive real.

A K was found by solving, for different values of λ, the Riccati equation

$$AP_\lambda + P_\lambda A^T + Q_\lambda - P_\lambda CC^T P_\lambda = 0, \qquad (2.23)$$

where

$$Q_\lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\lambda = 10$ turned out to be sufficient, yielding the gain matrix

$$K = \begin{pmatrix} -3.49 \\ -5.65 \\ 3.51 \end{pmatrix}.$$

The positive realness condition may of course be checked by means of the Yakubovič-Kalman lemma; alternatively the P_λ solving equation (2.23) can be shown to work as a Lyapunov function for the nonlinear system in question. □

2.5 Stabilisation using output feedback

The results from the preceding sections will now be combined to provide a solution of the stabilisation problem, when the full state is not accessible for measurement. In § 2.5.1, a simple separation theorem is proved under the same assumption as in § 2.4, namely that $\Phi(\cdot,\cdot)$ is a known function. This assumption is somewhat awkward, for there may be more efficient ways of designing a regulator than those presented above, if $\Phi(\cdot,\cdot)$ is known. Section 2.5.2 presents a simplified observer structure which works without this unrealistic assumption, and for which stability conditions can be expressed in known quantities.

2.5.1 A separation theorem

It is natural to ask whether, as in linear systems, the overall stability of the system, controlled by feedback from the observer, can be guaranteed from considerations of stability via state feedback and consistency of the estimate separately. Under very general assumptions, the answer is indeed affirmative.

Theorem 2.6 Consider the system of ordinary differential equations

$$\begin{cases} \frac{dx}{dt}(t) = f(x(t), e(t), t) & (2.24) \\ \frac{de}{dt}(t) = g(e(t), t) & (2.25) \end{cases}$$

where the null solution of (2.24) is uniformly asymptotically stable in the large for $e(\cdot) \equiv 0$ and the null solution of (2.25) is asymptotically stable in the large. Assume that $f(x, \cdot, t)$ is continuous at the origin, uniformly in x and t. Then the equilibrium of (2.24),(2.25) is globally asymptotically stable. □

Proof. Due to the continuity assumption, (2.24) can be written as

$$\frac{dx}{dt}(t) = f(x(t), 0, t) + h(t),$$

where

$$h(t) = f(x(t), e(t), t) - f(x(t), 0, t)$$

tends to zero as $t \to \infty$. The assertion then follows from a general perturbation theorem ([Hah], Thm. 68.2). □

Remark. In the case treated here, the uniform asymptotic stability is guaranteed by the use of the circle criterion (which ensures even exponential stability). □

2.5.2 Simplified observer structure

The observer structure (2.21) assumes that the exact shape of the nonlinearities is known. This is of course unrealistic, particularly in the time-varying case. The estimate will therefore in general be biased. It is a trivial matter to verify that the synthesis presented above will produce a small (in any reasonable sense of the word) bias, if no or false information on $\Phi(.,.)$ is used.

This can be exploited further. Suppose that the observer is part of a regulator whose task it is to stabilise the given system. If the regulator works, the state will be close to zero, and the bias will be small. Since the nonlinearities are assumed to be contained in symmetric sectors, an obvious way of reducing the complexity of the observer is to omit the nonlinearity from the right-hand side of equation (2.21). For the resulting closed-loop system, the following robustness result holds.

Theorem 2.7 Consider the nonlinear, time-varying system

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) - N\Phi(M^T x(t), t) + Bu(t) \\ y(t) = C^T x(t) \end{cases} \tag{2.26}$$

together with the observer

$$\frac{d\hat{x}}{dt}(t) = Ax(t) + Bu(t) + K(y(t) - C^T\hat{x}(t)) \tag{2.27}$$

and the feedback

$$u(t) = - L^T\hat{x}(t). \tag{2.28}$$

Assume that the condition of Thm. 2.2 and its dual is satisfied for (A,B) and (C^T,A) with respect to (M,N). Then the trivial solution of equations (2.26) - (2.28) is asymptotically stable in the large for a proper choice of K and L. □

Proof. The equations for $x(\cdot)$ and the reconstruction error $e(\cdot)$ are

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) - N\Phi(M^T x(t), t) - BL^T(x(t) - e(t)) \\ \frac{de}{dt}(t) = (A-KC^T)\ e(t) - N\Phi(M^T x(t), t). \end{cases}$$

This system can be analysed as a nonlinear feedback system with

$$G(s) = \begin{pmatrix} M^T & 0 \end{pmatrix} \begin{pmatrix} sI-A+BL^T & BL^T \\ 0 & sI-A+KC^T \end{pmatrix}^{-1} \begin{pmatrix} N \\ N \end{pmatrix}$$

in the forward path and $\Phi(\cdot,\cdot)$ in the feedback loop.

Expanding G(s) gives

$$G(s) = M^T(sI-A+BL^T)^{-1}N + M^T(sI-A+BL^T)^{-1}BL^T(sI-A+KC^T)^{-1}N$$
$$\triangleq G_1(s) + G_2(s)\cdot G_3(s).$$

Consider the second term in the expression for G(s). According to the PR assumption on the pair (A,B), the factor $G_2(s)$ can be made arbitrarily small by a proper choice of L. More specifically,

$$\sup_{\omega\in\mathbb{R}} \max_i |\lambda_i(G_2(-i\omega)^T G_2(i\omega))| = O(t^{-1})$$

if

$$L(t) = O(t), \quad t \to \infty,$$

t being the scalar parameter in the proof of Thm. 2.1. This estimate together with the dual assumption on (C^T,A) show that the term $G_2(s)G_3(s)$ can be made arbitrarily small. The same goes for $G_1(s)$, which proves the claim. □

In case PR is not achievable, the above expression for G(s) still provides an estimate for maximally permissible sector of $\Phi(\cdot,t)$. Notice that the overall stability is not necessarily enhanced by the inclusion of the nonlinear term in the observer.

CHAPTER 3. DISCRETE-TIME PROBLEMS

The natural framework for high-gain controllers is continuous time. The obvious reason is that the closed-loop poles of a sampled-data system have to lie within the unit circle, and consequently there are very precise limits on the magnitude of the feedback gains once the sampling interval has been fixed.

The conditions for perfect robustness to be achievable will consequently be more restrictive in sampled-data systems if the same methods are used. As in the continuous-time case, the space $\mathbb{R}^n$ is split into two subspaces, which are complementary under certain conditions. In one of these, V^* in the previous terminology, the disturbances are completely decoupled. In the other, S^*, only finite-sector nonlinearities can be coped with.

No complete theory will be given. The theorems merely serve to show some resemblances and differences between the continuous-time and discrete-time problems. In § 3.2 it is shown that, under certain conditions generically satisfied, the achievable robustness of a sampled-data system approaches that of the corresponding continuous control system as the sampling interval tends to zero.

3.1 The RRP for sampled-data control systems

3.1.1 Plant disturbances

Consider the system described by the set of difference equations

$$x(t+1) = Ax(t) - N\Phi(M^T x(t), t) + Bu(t). \tag{3.1}$$

As usual, x belongs to $\mathbb{R}^n$, u is a p-vector, and $\Phi(\cdot,\cdot)$ is a nonlinear time-varying function from $\mathbb{R}^q \times \mathbb{R}$ into $\mathbb{R}^q$ satisfying the conicity condition

$$\|\Phi(\sigma,t)\| \leq \kappa \cdot \|\sigma\| \quad \text{for all} \quad \sigma \in \mathbb{R}^q. \tag{3.2}$$

The control function $u(\cdot)$ is assumed to be a linear, constant feedback law

$$u(t) = - L^T x(t). \tag{3.3}$$

With respect to (3.1) - (3.3) the same questions may be asked as in the corresponding continuous-time problem. A sufficient condition for perfect robustness to be achievable is given in the following theorem. The notation is identical to that of Chapter 2.

Theorem 3.1 Perfect robustness of (A,B) with respect to (M,N) is achievable if

$$\mathrm{Im}(N) \subseteq \mathcal{V}^*_- . \quad \square$$

Proof. Since the theory of disturbance decoupling is identical for continuous-time and discrete-time problems, the proof of the corresponding result in Chapter 2 is unaltered. $\square$

For more general perturbations, the following finite-sector condition is sufficient. Only the square case $q = p$ is considered.

Theorem 3.2 Consider (3.1) - (3.3) with $p = q$. Assume that the matrix (M^TB) is invertible, and let the component of N along $\mathcal{S}^*$ be denoted by $N_{\mathcal{S}^*}$. Then a stabilizing L exists for (3.1), (3.2) if

i) $\mathrm{Im}(N) \subseteq \mathcal{S}^* + \mathcal{V}^*_-$

and

ii) $\kappa < [\max_i \lambda_i((M^TN_{\mathcal{S}^*})^T(M^TN_{\mathcal{S}^*}))]^{-1/2}$. $\square$

Proof. The proof is very much like the one in the continuous-time case. Notice first that the invertibility of $(M^T B)$ implies

$$\mathcal{S}^* = \mathrm{Im}(B)$$

and

$$\mathcal{S}^* \oplus \mathcal{V}^* = \mathbb{R}^n,$$

so that N can be uniquely decomposed along $\mathcal{S}^*$ and $\mathcal{V}^*$. Further, the component along $\mathcal{V}^*$ does not contribute to the transfer function, if L is chosen properly. On $\mathcal{S}^*$ (= Im(B)), the poles can be placed at the origin, yielding the closed--loop transfer function

$$G(z) = z^{-1}(M^T N_{\mathcal{S}^*}).$$

The condition then follows from a straightforward application of the circle criterion. □

Remark. In case $\mathcal{S}^*$ and Im(B) do not coincide, the situation becomes more complicated. □

In analogy with what was done for continuous-time systems, approximate versions of this geometrically designed feedback gain may easily be generated from a linear-quadratic optimal control problem. The discrete-time version of "cheap control" is known as the (output) dead-beat regulator. The behaviour of the closed-loop poles as the penalty on the input tends to zero is analogous to that in the continuous--time case (see [Kwa 1], Section 6.4).

3.1.2 Input channel disturbances

Sampled-data optimal controllers do not enjoy the same robustness properties as their continuous-time counterparts. This fact seems to have attracted little attention in the literature, and a quick derivation of the frequency domain inequality for these controllers will therefore be given.

Consider thus the problem of minimizing, with respect to $u(\cdot)$, the performance index

$$J = \sum_{s=0}^{\infty} (x(s)^T Qx(s) + u(s)^T Ru(s)),$$

where $x(\cdot)$ is governed by the difference equation

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ x(0) = x_0. \end{cases}$$

As is well known, the solution is given by

$$u(t) = -(R+B^T PB)^{-1} B^T PAx(t),$$

P being the solution of the algebraic Riccati-type equation

$$A^T PA - P + Q - A^T PB(R+B^T PB)^{-1} B^T PA = 0. \tag{3.4}$$

If the term

$$z^{-1} PA + zA^T P - A^T PA$$

is added to both members, (3.4) can be rewritten as

$$(z^{-1} I_n - A^T) PA + A^T P(zI_n - A) + A^T PB(R+B^T PB)^{-1} B^T PA =$$

$$= -(z^{-1} I_n - A^T) P(zI_n - A) + Q.$$

Multiplying from the left and the right by $B^T (z^{-1} I_n - A^T)^{-1}$ and $(zI_n - A)^{-1} B$ respectively yields

$$B^T PA(zI_n - A)^{-1} B + B^T (z^{-1} I_n - A^T)^{-1} A^T PB +$$

$$+ B^T (z^{-1} I_n - A^T)^{-1} A^T PB(R+B^T PB)^{-1} B^T PA(zI_n - A)^{-1} B =$$

$$= -B^T PB + B^T (z^{-1} I_n - A^T)^{-1} Q(zI_n - A)^{-1} B.$$

Introducing

$$G(z) = (R+B^T PB)^{-1} B^T PA(zI_n - A)^{-1} B,$$

this can be written as

$$(I_p + G^T(z^{-1}))\ (R + B^T PB)\ (I_p + G(z)) =$$
$$= R + B^T(z^{-1}I_n - A^T)^{-1}\ Q(zI_n - A)^{-1}\ B.$$

If Q is positive semidefinite, the second term on the right-hand side is nonnegative on the unit circle, and the following inequality results:

$$(I_p + G^T(z^{-1}))(R+B^T PB)(I_p + G(z)) \geq R \quad \text{for } |z| = 1. \qquad (3.5)$$

This result is weaker than the corresponding inequality for continuous-time systems ([Kwa 1], § 3.9). Further, the inequality depends on P, the solution of the Riccati equation (3.4), whence no universally valid robustness result can be stated.

It was pointed out in Chapter 2 that the phase margin and gain reduction tolerance could be increased by reducing the penalty on u(·) in the performance index. For the dead-beat regulator there is no corresponding result, and conditions must be imposed on the plant in order to produce robustness results. An example is given below.

Example 3.1 Consider the single-input, single-output system described by the difference equation

$$A(q^{-1})\ y(t) = q^{-1}\ B(q^{-1})\ u(t),$$

where

$$\begin{cases} A(q^{-1}) = \sum_{i=0}^{n} a_i\, q^{-i}, & a_0 \neq 0, \\ B(q^{-1}) = \sum_{i=0}^{n-1} b_i\, q^{-i}, & b_0 \neq 0. \end{cases}$$

A(·) and B(·) are assumed to have all their zeros outside the unit circle. Assume that the implementation of the dead-beat regulator is corrupted by the timevarying nonlinearity $\Phi(\cdot,\cdot)$ satisfying the conicity condition

$$0 \leq \sigma\Phi(\sigma,t) \leq \sigma^2 .$$

A sufficient condition for stability is then that $A(q^{-1})$ be strictly positive real.

The proof is based on the polynomial identity generating the dead-beat controller. If $G(q^{-1})$ solves

$$1 = A(q^{-1}) + q^{-1}G(q^{-1}), \tag{3.6}$$

then the dead-beat regulator is given by

$$u(t) = -\frac{G(q^{-1})}{B(q^{-1})}y(t) + u_r(t),$$

where u_r denotes the reference value. The controller configuration is shown in Fig. 3.1.

The transfer function of the linear part is

$$H(q^{-1}) = \frac{q^{-1}B(q^{-1})}{A(q^{-1})} \cdot \frac{G(q^{-1})}{B(q^{-1})} = \frac{q^{-1}G(q^{-1})}{A(q^{-1})} .$$

Using the sector condition on $\Phi(\cdot,t)$, the circle criterion guarantees stability of the closed-loop system if $(H(q^{-1})+1)$ is strictly positive real. The claim now follows from the identity (3.6).

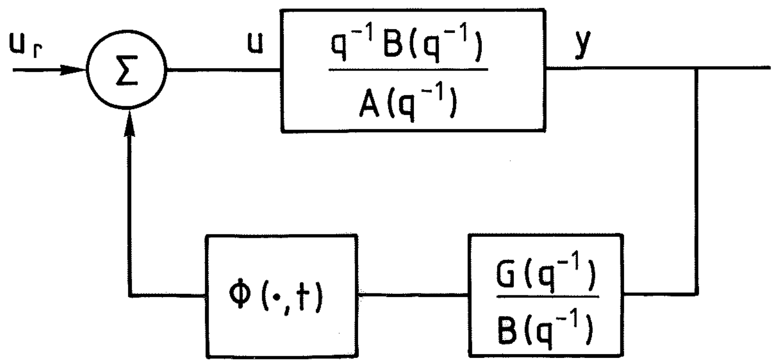

Fig. 3.1 - The configuration of Example 3.1.

If the transfer function of the plant contains more time delays, i.e.

$$A(q^{-1})\ y(t) = q^{-k}\ B(q^{-1})\ u(t), \qquad k > 1,$$

the identity (3.6) is changed into

$$1 = A(q^{-1})\ F(q^{-1}) + q^{-k}\ G(q^{-1}),$$

where $F(\cdot)$ and $G(\cdot)$ have degrees $(k-1)$ and $(n-1)$ respectively. The condition for stability is then that $A(q^{-1})F(q^{-1})$ be strictly positive real. □

To close this section, a few words should be said about the more general control-dependent disturbances treated for the continuous-time case in § 2.3.2. Clearly, Haussmann's results are purely geometric in nature and consequently do not depend on the structure of the set where the time variable takes its values. The statements and the proofs thus remain valid without changes for the discrete-time case.

3.2 Effects of fast sampling

The discussion so far assumes that the sampling interval is given. It is natural from the designer's point of view to ask what is the effect of the sampling interval on the achievable robustness. In particular, it seems intuitively clear that the robustness results for continuous-time systems should be recovered as the length of the sampling interval tends to zero. This need not be the case, however.

The reason is that the minimum-phase quality of the continuous-time system may be lost even if the sampling interval is chosen short. Some additional assumption is needed in order order to prevent this from happening.

<u>Lemma 3.1</u> Consider a continuous-time system with the square transfer matrix G(s). If G(s) has a maximal number of zeros and is strictly minimum-phase, then the same holds for the corresponding sampled-data system if the sampling interval is short enough. □

<u>Proof</u>. A simple computation shows that, under the assumption of the lemma, the zeros of the sampled-data system $z_i^{(s)}$ can be expressed as

$$z_i^{(s)} = 1 + z_i^{(c)} \cdot T + O(T^2), \qquad T \to 0$$

where $z_i^{(c)}$ are the zeros of the continuous-time system. □

Consider now the system given by the set of ordinary differential equations

$$\frac{dx}{dt}(t) = Ax(t) - N\Phi(M^T x(t), t) + Bu(t) \tag{3.7}$$

together with its sampled-data counterpart

$$\bar{x}(t+T) = \bar{A}\bar{x}(t) - \bar{N}\Phi(M^T\bar{x}(t), t) + \bar{B}u(t). \tag{3.8}$$

Only the case rank(M) = rank(B) will be discussed.

Notice first that, under the assumption of the lemma, the system $(M^T, \bar{A}, \bar{B})$ will be minimum-phase if (M^T, A, B) is. This means that the decoupling part of the feedback carries over from the continuous-time system to the sampled version.

If Equation (3.7) is written as an integral equation, the following estimates are easily derived:

$$\begin{cases} A = I + A \cdot T + O(T^2) \\ N = N \cdot T + O(T^2) \\ B = B \cdot T + O(T^2) \end{cases} \qquad T \to 0$$

Further,

$$\| \Phi(\sigma, t) \| \leq \kappa \cdot \| \sigma \|$$

implies

$$\| \overline{\Phi}(\sigma,t) \| \leq \kappa \cdot (1 + O(T)) \cdot \| \sigma \|$$

For plant disturbances, the estimate on $\overline{N}$ shows that the κ of Theorem 3.2 increases to infinity as T tends to zero.

For input channel disturbances, the estimate on $\overline{B}$ together with the fact that the solution P of equation (3.4) decreases with T implies that inequality (3.5) asymptotically reproduces the result from continuous-time optimal system.

PART II - STOCHASTIC MODELS

CHAPTER 4. PRELIMINARIES

The stabilisation problem will now be studied within a stochastic framework. The introduction of stochastic assumptions requires some care both in the modelling and the analysis. The problem of giving the white-noise assumption a rigorous formulation by means of an Itô equation is discussed in § 4.1. The following section presents the basic stability definitions to be used in the sequel together with a necessary and sufficient condition for the almost-sure stability of linear random-parameter equations. It should be stressed that these two sections are included not for the sheer mathematical pleasure, but because the stability results depend critically on the model and the convergence mode chosen.

Section 4.3 contains a brief review of stochastic Lyapunov functions. Relations between the existence of Lyapunov functions and moment stability are also given.

4.1 Modelling

Consider the Langevin equation

$$\frac{dx}{dt}(t) = f(x(t), t) + \sigma(x(t), t)\ \dot{w}(t) \quad (4.1)$$

where x, f, and $\sigma \in \mathbb{R}^n$, and $\dot{w}$ is scalar "white noise". The meaning of Equation (4.1) is that the drift term $f(x(t), t)$ is perturbed by a disturbance $\sigma(x(t), t)\dot{w}$, where $\dot{w}$ has a spectral density which is constant up to frequencies which are high compared to the time constants of the noise-free system. The equation was proposed by Langevin in [Lan] as a model for the Brownian motion.

For analysis purposes, a rigorous version of Equation (4.1) is needed. The following Itô equation was originally proposed:

$$dx(t) = f(x(t), t)dt + \sigma(x(t), t)dw(t). \quad (4.2)$$

In [Won 1], Wong and Zakai analysed the relationship between Itô integrals and ordinary integrals obtained from Equation (4.1), when $\dot{w}$ is a process with large but finite bandwidth. It was shown that a correction term must be introduced if Equation (4.1) is to be modelled by an Itô equation. The modified equation corresponding to (4.1) is

$$dx(t) = f(x(t), t) + \frac{1}{2}\ \frac{d\sigma}{dx}(x(t), t)\ \sigma(x(t), t) +$$

$$+ \sigma(x(t), t)\ dw(t),$$

$d\sigma/dx$ being the Jacobian of σ with respect to x. The same fact led Stratonovič ([Str]) to propose an alternative definition of the stochastic integral, such that the above correction term (frequently called the "Stratonovič correction") is zero.

In the sequel, only Itô equations will be considered, since the transition between the various integral definitions is in most cases trivial.

4.2 Stochastic stability

There are several modes of convergence to choose between when adapting the Lyapunov stability concepts to a stochastic framework. Naively speaking, a practically oriented analyst should of course be interested primarily in the physically observable behaviour of the solution process. What can be observed is the sample paths, and one is thus led naturally to the concept of stability with probability one. However, also moment stability may be relevant in some applications, since it measures in some sense the frequency and magnitude of excursions made by the process from the equilibrium. In particular, the popularity of quadratic loss functions motivates an interest in second-moment stability.

4.2.1 Basic definitions

In the definitions given below $x(t;x_0,t_0)$ is a stochastic process which is the solution of some differential or difference equation. The initial value at t_0 is x_0. The equilibrium solution studied is the null solution.

Let $\|\cdot\|_p$ denote the $\mathbb{R}^n$-norm defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \qquad p > 0.$$

$\|\cdot\|$ will be used when the value of p is irrelevant.

Definition 4.1 The null solution is said to be *almost surely stable* (or *stable with probability one*) if for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists a $\delta > 0$ such that

$$P\left(\sup_{\|x_0\| \leq \delta} \sup_{t \geq t_0} \|x(t;x_0,t_0)\| > \varepsilon_1\right) < \varepsilon_2. \quad \square$$

Alternatively, Definition 4.1 may be expressed as ([Koz 1])

$$P\left(\lim_{\|x_0\| \to 0} \sup_{t \geq t_0} \|x(t;x_0,t_0)\| = 0 \right) = 1.$$

It is thus clear that the deterministic stability definition holds for almost all sample paths.

Definition 4.2 The equilibrium is said to be *almost surely attractive* (or *attractive w.p. 1*) if there exists a $\delta > 0$ such that $\|x_0\| \leq \delta$ implies

$$P\left(\lim_{t\to\infty} \|x(t;x_0,t_0)\| = 0 \right) = 1. \quad \square$$

Definition 4.3 The equilibrium is said to be *almost surely asymptotically stable* (or *asymptotically stable w. p. 1*) if it is stable and attractive w.p. 1.

It is *globally asymptotically stable w.p. 1* if further δ in Definition 4.2 can be taken arbitrarily large. □

The following definitions describe the time evaluation of the moments of the process.

Definition 4.4 The null solution is said to be *p-th mean asymptotically stable* if the function

$$E \left\{ \|x(t;x_0,t_0)\|_p^{\,p} \right\}$$

is asymptotically stable.

It is *p-th mean exponentially stable* if there is a $C > 0$ and an $\alpha > 0$ such that

$$E \left\{ \|x(t;x_0,t_0)\|_p^{\,p} \right\} \leq C \, \|x_0\|_p^{\,p} \cdot \exp(-\alpha\,(t-t_0)). \quad \square$$

Notice that $E\{\|x(t;x_0,t_0)\|_p^{\,p}\}$ is a deterministic function. In some cases it is possible to derive a differential (difference) equation for this function, which can be studied using deterministic Lyapunov theory. A stronger version of Definition 4.4 uses

$$E \left\{ \sup_{t \geq t_0} \|x(t;x_0,t_0)\|_p{}^p \right\},$$

but this will not be necessary for our purposes.

P-th-mean stability implies q-th-mean stability for $q < p$. There are also implications between moment stability and almost sure stability. For instance, it has been shown by Kozin and Sugimoto ([Koz 3]) that for linear Itô equations, almost-sure stability is equivalent to p-th-moment stability as p tends to zero.

4.2.2 A necessary and sufficient condition for almost-sure stability

Consider the linear system given by

$$\begin{cases} x(t+1) = A(t)\ x(t) \\ x(0) = x_0 \ , \end{cases} \tag{4.3}$$

where x is an n-vector and

$$\{A(t)\}_{t=0}^{\infty}$$

is a sequence of independent, identically distributed random matrices with given distribution function. The problem is to give conditions on this distribution which ensure stability with probability one of the trivial solution of (4.3). To this end, let

$$\Theta = x / \|x\| ,$$

where Θ is thus the coordinate vector on the unit sphere. Obviously,

$$\log(\|x(t)\|) = \log(\|x_0\|) + \sum_{s=0}^{t-1} \log(\|A(s)\Theta(s)\|) . \tag{4.4}$$

Now, $\Theta(t)$ defines a Markov process on the unit sphere which is ergodic under general assumptions. Assume that the corresponding invariant measure (i.e. the "steady--state" measure defined by the process) can be calculated.

If conditions for the validity of the strong law of large numbers are satisfied, the sum in (4.4) will behave roughly like its mean value, or more precisely

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} \log(\| A(s)\Theta(s)\|) = +\infty \quad \text{or} \quad -\infty$$

according as

$$E\{\log(\|A\Theta\|)\} > 0 \quad \text{or} \quad < 0. \tag{4.5}$$

The expectation is taken with respect to the original distribution of A and the invariant measure of Θ. Stability will consequently hold almost surely if the expectation value in (4.5) is negative.

The relevance of (4.5) to the stability problem was established by Khasminskii ([Kha]). Unfortunately, the invariant measure can be computed analytically only in exceptional cases. One such case is the linear Itô equation

$$dx(t) = Ax(t)dt + Bx(t)dw(t). \tag{4.6}$$

As in the discrete-time case, the key step is the projection of the process on the unit sphere by the introduction of $\Theta = x / \|x\|$. The stability limit is determined by the expected value of $L(\log(\|x\|))$ with respect to the invariant measure of the diffusion on the unit sphere (L is the Kolmogorov backward operator pertaining to equation (4.6)). This expectation value can be obtained in closed form. The reason is that the variable Θ itself satisfies an Itô equation, from which the invariant measure can be determined. If equation (4.6) is two-dimensional, the Θ-process is scalar, and Feller's theory for one-dimensional diffusions can be applied (see [Itô 1] (in Russian), [Itô 2];

cf. also [Åst], [Koz 2], which contain illustrative examples). The picture may be complicated by singular points, i.e. points where the diffusion term of equation (4.6) vanishes. In such cases, Feller's classification of singular points must be invoked.

4.3 Stochastic Lyapunov functions

The possibility of using Lyapunov functions in studying the stability of random-parameter systems was discovered by Bertram and Sarachik ([Ber]) and Kats and Krasovskii ([Kat]). Recall that a Lyapunov function for a deterministic motion is a positive definite function which is non--increasing along the trajectories of the motion. The existence of such a function is sufficient to guarantee the stability of the system. If the Lyapunov function is decreasing along the trajectories, asymptotic stability is ensured. In the stochastic theory, the change of the Lyapunov function is replaced by the expected change, conditioned with respect to the present state. If this is negative, stability can be inferred from a supermartingale convergence theorem. Only the most important theorem and no proof will be given here. An extensive reference is the book by Kushner ([Kus 2]).

Only homogeneous Markov processes will be considered, i.e. processes whose transition probabilities are independent of the initial time. This simplifies the analysis somewhat, since the Lyapunov function candidates can be chosen time-independent.

4.3.1 A stochastic Lyapunov theorem

The continuous-parameter stochastic version of Lyapunov's theorem on stability will now be stated. The discrete-time case is simpler, since the technicalities of the Itô calculus do not enter. Only a brief sketch of the conditions are given. For a detailed account, the reader is referred to [Kus 2].

Definition 4.5 The function $\varphi(\cdot)$ is said to be in the domain of the *weak infinitesimal operator* L of the process x(t) if

i) $$\lim_{\delta\to 0} \frac{E\{\varphi(x(t+\delta))|x(t)=x\} - \varphi(x(t))}{\delta} \triangleq L(\varphi) \text{ exists } = \psi(x)$$

and

ii) $$\lim_{\delta\to 0} E\{\psi(x(t+\delta))|x(t)=x\} = \psi(x). \quad \square$$

Example 4.1 For Itô processes, the weak infinitesimal operator L is given by the Kolmogorov backward operator $\mathcal{L}$. Explicitly, let the Itô equation

$$dx(t) = f(x(t))dt + \sigma(x(t))dw(t)$$

be given. Then, if $\mathcal{L}(\varphi)$ is defined,

$$L(\varphi) = \mathcal{L}(\varphi) = \sum_{i=1}^{n} f_i \cdot \frac{\partial\varphi}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i\sigma_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}. \quad \square$$

Theorem 4.1 ([Kus 2]) Let x(t) be a continuous-parameter Markov process, and let V(x) be a positive function in the domain of L. Define

$$\Omega_M = \{x;\ V(x) \leqslant M\}.$$

If, for all $x \in \Omega_M$,

$$L(V)(x) \leqslant -W(x) \leqslant 0,$$

then

i) $P\left(\sup_{t_0 \leq t < \infty} V(x(t)) \geq M \mid x(t_0) = x_0 \right) \leq V(x_0)/M,$

and

ii) $\lim_{t \to \infty} W(x(t)) = 0$ with probability $\geq (1 - V(x_0)/M)$. □

4.3.2 Relations to moment stability

A general existence theorem for stochastic Lyapunov functions was proved by Kushner ([Kus 1]) by rephrasing Massera's proof for the deterministic problem in stochastic terminology. The theorem to follow has a rather special form, which gives a connection between Lyapunov functions and moment stability. Special cases have appeared before in the literature.

Theorem 4.2 Let $x(t;x_0,t_0)$ be a continuous parameter Markov process which is homogeneous in time, and let L be its weak infinitesimal operator. Let $x(t;0,t_0) \equiv 0$ be the equilibrium.

i) If the equilibrium is p-th mean exponentially stable for some $p > 0$, there exists a time-invariant function V solving

$$L(V)(x) = -q(x) \tag{4.7}$$

with V and q subject to

$$\alpha_1 \|x\|_p^p \leq V(x) \leq \alpha_2 \|x\|_p^p \tag{4.8}$$

$$\alpha_3 \|x\|_p^p \leq q(x) \leq \alpha_4 \|x\|_p^p, \tag{4.9}$$

all $\alpha_i > 0$.

In particular, p-th mean exponential stability for some $p > 0$ implies stability with probability one.

ii) Assume that equation (4.7) has a solution V for some q with V and q satisfying (4.8), (4.9). Then the equilibrium is p-th mean exponentially stable. □

Proof. i) Consider

$$V(x,t,\tau) = E\left\{ \int_t^\tau q(x(s))ds \mid x(t)=x \right\}.$$

If x(t) is p-th mean exponentially stable,

$$\lim_{\tau\to\infty} E\left\{ \int_t^\tau q(x(s))ds \mid x(t)=x \right\} < \infty ,$$

and

$$\lim_{\tau\to\infty} V(x,t,\tau) = V(x,t)$$

exists, independent of time due to the homogeneity. Equation (4.7) then follows from the definition of V and Dynkin's formula ([Kus 2], p. 10).

ii) From Dynkin's formula one obtains

$$E\{V(x(t))|x(t_0)=x_0\} =$$

$$= V(x_0) + E\left\{ \int_{t_0}^t -q(x(s))ds | x(t_0)=x_0 \right\}.$$

Using the inequalities (4.8), (4.9) this implies

$$E\{V(x(t))|x(t_0) = x_0\} \leq$$

$$\leq V(x_0) - E\left\{ \int_{t_0}^t \alpha_3 \|x(s)\|_p^p ds | x(t_0)=x_0 \right\} \leq$$

$$\leq V(x_0) - E\left\{ \int_{t_0}^t \frac{\alpha_3}{\alpha_2} V(x(s))ds | x(t_0)=x_0 \right\} =$$

$$= V(x_0) - \frac{\alpha_3}{\alpha_2} \int_{t_0}^t E\left\{ V(x(s)) | x(t_0)=x_0 \right\} ds,$$

whence

$$E\left\{V(x(t))\,|\,x(t_0)=x_0\right\} \leq V(x_0)\; e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)} .$$

Again, using (4.8),

$$E\left\{\,||x(t)||_p^{\,p}\,\middle|\,x(t_0)=x_0\right\} \leq \frac{\alpha_2}{\alpha_1}\,||x_0||_p^{\,p}\; e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)} ,$$

which proves the claim. □

Remark. Suppose that the interest is in asymptotic stability rather than exponential stability, and that the stability region in some parameter space is to be determined. If the situation is such that Khasminskii's result can be applied, the relevant stability condition is expressed as the negativity of the inequality (4.5). Assuming that the expectation value (4.5) depends continuously on the parameters in question, there will then be room for an ε to make the system exponentially stable. Under these conditions, there will consequently be no gap between asymptotic and exponential stability. □

CHAPTER 5. STABILISATION OF RANDOM PARAMETER SYSTEMS

The problem of stabilising - in different possible senses of the word - a random-parameter system will now be tackled. In contrast to what was done in Chapter 2, it will be formulated as an optimization problem. If the optimal solution exists, it can, at least formally, be obtained via the functional equation of dynamic programming. Minimisation over an infinite time-horizon yields a time-independent solution of the functional equation, which can be used as a Lyapunov function for the closed-loop system to prove stability.

As usual, the case of linear systems and quadratic loss functions is exceptional in the sense that it permits a fairly explicit solution of the dynamic-programming equation. Known results for this solution are compared to what was derived for the corresponding deterministic problem in Chapter 2.

The almost-sure stabilisation problem is also discussed. The optimal solution cannot in general be obtained in closed form. It is therefore natural to consider suboptimal solutions, which can be shown to stabilise the system. Since the goal formulated at the outset of this work was to investigate what can be achieved with linear regulators, the study is confined to this class. No general answers are available here, but examples are given which show that there are indeed qualitative differences between the almost-sure and the mean-square stabilisation problems.

The main stress is on continuous-time systems.

5.1 An optimisation problem

Consider a motion given by the continuous-parameter stochastic process x(t), which is assumed to be homogeneous in time. The process is controlled from the input u. The object is to minimise, over some class of permissible control functions, the performance index

$$J_{\tau} = E\left\{ \int_{t}^{\tau} q(x(s),u(s))ds \mid x(t) = x \right\} \qquad (5.1)$$

where $q(\cdot,\cdot)$ is some nonnegative function.

Assume that the minimum exists within the given class of controls, and define

$$V(x,t,\tau) = \min_{u(\cdot)} J_{\tau}$$

Using dynamic programming ([Won 3]), the following functional equation can be derived for u and V:

$$\min_{u} [L_u(V)(x,t) + q(x,u)] = 0. \qquad (5.2)$$

Here, L_u denotes the infinitesimal generator obtained for a given control u. Equation (5.2) is a version of the Hamilton-Jacobi-Bellman equation. Assume that there exists a unique u* minimising the left member of (5.2). Inserting this u gives an equation for V only. If this equation can be solved, u* can be expressed as a feedback,

$$u^*(t) = - \ell^*(x(t)).$$

This results in a new equation for V:

$$L_{\ell^*}(V)(x,t) + q(x,\ell^*(x)) = 0. \qquad (5.3)$$

If the optimisation over an infinite horizon is meaningful, the V solving this equation will be time-independent. Comparing this with Theorem 4.2 shows that a positive V solving the time-invariant version of Equation (5.3) will work as a Lyapunov function for the closed-loop system.

5.2 Stabilisation in the mean-square

5.2.1 Solution of the dynamic-programming equation

Consider the linear Itô equation

$$dx(t) = Ax(t)dt + \left(\sum_{i=1}^{k} d\kappa_i A_i\right) x(t) + Bu(t)dt + + \left(\sum_{i=1}^{\ell} d\lambda_i B_i\right) u(t) \tag{5.4}$$

together with the performance criterion

$$J_\tau = E\left\{\int_t^\tau (x(s)^T Qx(s) + u(s)^T Ru(s))\, ds \mid x(t) = x\right\}. \tag{5.5}$$

As usual, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, A, A_i, B, and B_i are matrices of appropriate dimensions, and the κ_i's and the λ_i's are independent unit Wiener processes. Q is assumed to be positive semidefinite and R positive definite (although the latter condition may be dispensed with under certain conditions).

The first to treat this problem in any generality were Wonham ([Won 3]) and independently, in a special case, Kleinman ([Kle]). It turns out that the solution of the Hamilton-Jacobi-Bellman equation (5.3) can be obtained as a quadratic form,

$$V(x,t) = x^T P(t) x,$$

where P(t) satisfies the Riccati-type equation

$$\begin{cases} -\dfrac{dP}{dt}(t) = A^T P(t) + P(t)A + Q + \displaystyle\sum_{i=1}^{k} A_i^T P(t) A_i - \\ \qquad - P(t)\, B\left(R + \displaystyle\sum_{i=1}^{\ell} B_i^T P(t) B_i\right)^{-1} B^T P(t) \\ P(\tau) = 0. \end{cases} \tag{5.6}$$

The optimal input u* is a linear state feedback,

$$u^*(t) = -\left(R + \sum_{i=1}^{\ell} B_i^T P(t) B_i\right)^{-1} B^T P(t) x(t).$$

If the system is mean-square stabilisable,

$$\lim_{T\to\infty} J_T = J$$

will exist, and equation (5.3) will have a time-invariant solution. Equivalently, the solution P(t) of (5.6) will converge to a constant matrix P, which satisfies the algebraic equation

$$A^T P + PA + Q + \sum_{i=1}^{k} A_i^T PA_i - PB\left(R + \sum_{i=1}^{\ell} B_i^T PB_i\right)^{-1} B^T P = 0. \tag{5.7}$$

The term

$$\sum_{i=1}^{k} A_i^T PA_i$$

has the effect of increasing the penalty on x and tends to produce large gains, whereas the term

$$\sum_{i=1}^{\ell} B_i^T PB_i$$

increases the penalty on u and thus has the opposite effect. These tendencies are conflicting and there may indeed be situations where no positive solution of equation (5.7) exists.

5.2.2 Consequences for the mean-square stabilisability

The solvability of equation (5.7) with a positive P and its implications for the problem of stabilising, in the mean-square sense, the null solution of equation (5.4) was examined by Willems and Willems in [Wil]. A brief account of these results will be given here, with an emphasis on the case of large noise intensities.

State-dependent noise

Consider first the case of state-dependent noise only, and assume for simplicity that all perturbation matrices A_i have unit rank. Let the A_i's be factored according to

$$A_i = \sigma_i \cdot n_i m_i^T \qquad i = 1, 2, \ldots, k. \tag{5.8}$$

Here, σ_i are variance parameters introduced for convenience. Let $\overline{m}_i$, $i=1,2,\ldots,\overline{k}$, be a basis for the subspace spanned by the m_i's, and form the matrix

$$\overline{M} = \mathrm{col}(\overline{m}_i).$$

Consider further the algebraic Riccati equation in P_ρ:

$$A^T P_\rho + P_\rho A + \overline{M}\overline{M}^T - \rho^{-1} P_\rho B R^{-1} B^T P_\rho = 0. \tag{5.9}$$

Let P_0 be the limiting solution of Equation (5.9) as the scalar parameter ρ tends to zero (this limit exists). The following result is proved in [Wil]:

> The null solution of Equation (5.4) is mean-sqaure stabilisable for all noise intensities σ_i if and only if
>
> i) the deterministic system (corresponding to $\sigma_i=0$, all i) is stabilisable in the usual sense,
>
> and
>
> ii) $\sum_{i=1}^{k} \mathrm{Im}(n_i) \subseteq \mathrm{Ker}(P_0)$. □

This result should be compared to Thms. 2.2 and 2.3. It turns out that the conditions for L_2-stabilisability of the system (5.4) for arbitrarily large noise intensities and stabilisability in the deterministic sense of the system given by the equations (2.1)', (2.2), (2.3) for an arbitrarily large sector radius κ are the same.

The resemblance goes somewhat further. Using the notation (5.8), Equation (5.7) takes the form

$$A^T P_\sigma + P_\sigma A + Q + \sum_{i=1}^{k} \sigma_i^2 \cdot n_i^T P_\sigma n_i \cdot m_i m_i^T - P_\sigma B R^{-1} B^T P_\sigma = 0, \tag{5.10}$$

where the notation P_σ is used to show the dependence of P upon the variance parameters σ_i. Consider first the case $k=1$. Then it is clear from (5.10) that the asymptotic closed-loop pole configuration obtained as the σ_i's tend to infinity is the same as produced from the cheap-control used in § 2.2.2.

For $k > 1$, Equation (5.10) is more interesting. Recall that an equation of the type (5.9) was suggested in § 2.2.2 for solving the robust regulator problem in the case when perfect robustness is not achievable. A drawback of that approach is that the information about the n-vectors is not used. An attractive feature about Equation (5.10) is that it provides an automatic scaling of the relative weights of the matrices $m_i m_i^T$, in the sense that an n_i with a large component outside $\mathrm{Ker}(P_0)$ (as defined from Equation (5.9)) will tend to penalise the corresponding $m_i m_i^T$ more. This suggests the use of Equation (5.10) in the design for robustness also in systems where Equation (5.4) is not a good model of the disturbances. After all, the object of the design is to produce a good regulator rather than a good model. The price paid for the automatic scaling of the weights is the somewhat greater computational complexity of Equation (5.10) (or Equation (5.6), if straightforward integration is used to find the steady--state solution) as compared to the ordinary Riccati equation.

Control-dependent noise

The case of control-dependent disturbances only will now be examined. Also this problem is related to a limiting form of the ordinary deterministic LQOC problem, namely the minimum-energy control. More precisely, consider the algebraic Riccati equation

$$A^TS_\zeta + S_\zeta A + \zeta Q - S_\zeta BR^{-1}B^TS_\zeta = 0. \qquad (5.11)$$

S_ζ is a decreasing function of ζ and thus

$$S_0 = \lim_{\zeta\to 0} S_\zeta$$

exists, in fact independent of Q. In [Wil] the condition for mean-square stabilisability is expressed using equation (5.11) as follows:

i) The null solution of Equation (5.4) is L_2-stabilisable if and only if there exists an R such that

$$\sum_{i=1}^{\ell} B_i^T S_0 B_i < R.$$

ii) With $B_i = \sigma_i \cdot B_{i0}$, stabilisability for arbitrarily large σ_i's holds if and only if the conditions of Thm. 2.4 are satisfied. □

The above result is possible to motivate from intuitive reasoning. Since the control action introduces noise in the system, it should be minimised in order to minimise the noise level of the system. However, the conclusion is somewhat different from what was obtained for some of the deterministic input channel disturbances in Chapter 2. For instance, it was shown that high integrity against gain drops is achieved by high-gain controllers.

Conclusion

The moral lesson contained in these results is that whereas the modelling of plant disturbances has a minor influence in the design for robustness, it is highly important to specify the character of the disturbances in the input channels.

5.2.3 Discrete-time mean-square stabilisability

It is an easy matter to translate the above results into their discrete-time counterparts. In fact, the proofs become somewhat simpler, since the technicalities of the Itô calculus do not enter.

Consider the system given by the difference equation

$$x(t+1) = \left(A + \sum_{i=1}^{k} \kappa_i(t)A_i\right)x(t) + \left(B + \sum_{i=1}^{\ell} \lambda_i(t)B_i\right) u(t). \tag{5.12}$$

Here, the κ_i's and the λ_i's are assumed to be independent white-noise sequences. The object is to minimise, with respect to $u(\cdot)$, the performance index

$$J_\tau = E\left\{ \sum_{s=t}^{\tau} (x(s)^T Qx(s) + u(s)^T Ru(s)) \,|\, x(t) = x\right\},$$

where Q is positive semidefinite and R is positive definite. Using dynamic programming, it can be shown that the optimal input is a linear feedback from the state, and that the optimal loss is given by

$$J_\tau = x^T P(t)x,$$

where P(t) solves the difference equation

$$\begin{cases} P(t) = A^T P(t+1)A + Q + \left(\sum_{i=1}^{k} A_i^T P(t+1)A_i\right) - \\ \quad - A^T P(t+1)B\left[R + B^T P(t+1)B + \right. \\ \quad \left. + \left(\sum_{i=1}^{\ell} B_i^T P(t+1)B_i\right)\right]^{-1} B^T P(t+1)\,A \\ P(\tau) = 0. \end{cases} \tag{5.13}$$

Under the condition of mean-square stabilisability, P(t) converges to a constant matrix P, which solves the algebraic matrix equation

$$A^T PA - P + Q + \left(\sum_{i=1}^{k} A_i^T PA_i \right) -$$

$$- A^T PB \left[R + B^T PB + \left(\sum_{i=1}^{\ell} B_i^T PB_i \right) \right]^{-1} B^T PA = 0. \qquad (5.14)$$

Conversely, the existence of a positive solution of equation (5.14) guarantees stabilisability in the mean square. This requires stabilisability of the noise-free system and is further a condition on the noise intensities involved.

5.2.4 The "uncertainty threshold"

It was pointed out earlier that the algebraic Riccati-type equations arising from the dynamic-programming equation (Equations (5.7) and (5.14)) may not have a positive solution if the noise levels are sufficiently high. This implies that the optimisation of the quadratic performance over an infinite horizon has no solution, or, equivalently, that the system in question is not mean-square stabilisable. Based on a first-order scalar equation, Athans, Ku, and Gershwin in [Ath] introduced the term "uncertainty threshold" for the values of the variance parameters for which a solution ceases to exist. It is claimed in [Ath] that "this result has several implications in engineering and socio-economic systems, since it points out that there is a quantifiable boundary between our ability of making optimal decisions or not as a function of the modeling uncertainty".

There are at least two reasons why such a basic concept as an uncertainty threshold should *not* be tied to equations (5.7), (5.14):

i) For continuous-time systems, the first reason lies in the modelling problem. Having the discussion of § 4.1 in mind, it is clear that, for most applications, the Stratonovič interpretation of the stochastic integral is the appropriate one. This means that the Stratonovič correction must be taken into account before the Itô calculus is used. However, in presence of control-dependent noise, this correction depends on the yet-to-choose feedback law. There may be ways to overcome this difficulty, for instance using iterative methods, but in any case Equation (5.7) will have to be modified.

ii) The introduction of a stochastic framework requires some care in the choice of an appropriate stability concept. The bulk of the literature is concerned with L_2-stability, but the reason for this seems to be relative computational simplicity rather than intrinsic importance.

What then should be the decisive criterion in the choice between the various available convergence concepts? For the practising engineer, it seems reasonable to adopt the relevance to the actual physical behaviour of the process as the touch-stone by which the convergence concepts should be judged. This implies that almost-sure convergence rather than L_p-convergence is the logical basis for the definition of an uncertainty threshold.

It can be argued that knowledge of the ultimate behaviour of the moments yields some insight into how large excursions the process will make from the equilibrium. On the other hand, there is no reason to single out the second moment in this context. In some applications, interest in higher moments is in fact justified (see [San] for a discussion of such problems in a deterministic setting).

This is not to say that Equations (5.7), (5.14) are uninteresting. On the contrary, based on the arguments in § 5.2.2, they are believed to provide an effective means to design low-sensitivity regulators. But the role attributed to them in [Ath] seems difficult to justify.

5.3 Stabilisability in the almost-sure sense

An investigation of the almost-sure stabilisability of a control system should concentrate on the dynamic-programming equation (5.2), where the penalty function q contains only p-moments for p close to zero. Unfortunately, this equation becomes untractable already for second-order systems. But since the only concern in this work is with linear regulators, only such regulators will be considered.

The first example is a scalar first-order equation, which despite its simplicity already illustrates some of the points of the preceding discussion.

Example 5.1 Consider the Itô equation

$$dx = ax\,dt + bu\,dt + \sigma u\,du \tag{5.15}$$

where a is assumed to be greater than zero. The problem is to determine for what noise intensities the system can be stabilised.

It is readily verified that Equation (5.7) yields

$$\sigma^2 < \frac{b^2}{2a}$$

as a necessary and sufficient condition for mean-square stabilisability. The almost-sure stabilisability problem is trivial in this case, since stability is enhanced relatively to the noise-free system by the introduction of the term $\sigma u\,du$. The system is stabilisable almost surely for arbitrarily large noise intensities, and no uncertainty threshold exists. □

The second example shows that the minimum-phase condition that turned out to be crucial for stabilisability in both the deterministic and the mean-square sense, is no longer relevant for the almost-sure stabilisability problem.

Example 5.2 Consider the control system described by the linear Itô equation

$$\begin{cases} dx(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t)\, dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)\, dt + \\ \qquad + \sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & a \end{pmatrix} x(t)\, dw(t) & (5.16) \\ du(t) = - \ell^T dx(t) = - \ell_1\, dx_1(t) - \ell_2\, dx_2(t) & (5.17) \end{cases}$$

The problem is to investigate the stabilisability almost surely of the trivial solution of (5.16) using the linear feedback (5.17) as a function of the variance parameter σ.

Notice first that if a is positive, the minimum-phase condition is satisfied, so the system is stabilisable in the mean square sense, thus a fortiori with probability one, for arbitrarily large noise intensities. This is no longer true if a is negative, and the almost-sure stabilisability must be examined separately. The analysis will be based on Khasminskii's condition for almost-sure stability. Recall (§ 4.2.2) that a necessary and sufficient condition is that the expected value of $L(\log(||x||))$ with respect to the invariant measure of the diffusion process, projected on the unit sphere, be negative. The process (5.16) is singular, i.e. the diffusion term vanishes for certain x's. To determine the invariant measure, the singular points of the diffusion must be studied.

On the unit circle, the generator L takes the form

$$L = \frac{\sigma^2}{2} (\cos \Theta + a \sin \Theta)^2 \cos^2 \Theta \frac{d^2}{d\Theta^2} +$$

$$+ \left(- \sigma^2 (\cos \Theta + a \sin \Theta)^2 \cos \Theta \sin \Theta + \cos^2 \Theta + \right.$$

$$\left. + \ell_1 \cos \Theta \sin \Theta + \ell_2 \sin^2 \Theta \right) \cdot \frac{d}{d\Theta} \triangleq$$

$$\triangleq a(\Theta) \frac{d^2}{d\Theta^2} + b(\Theta) \frac{d}{d\Theta} .$$

It follows that the singular points of the process are $\Theta = \pm \pi/2$, $\Theta = \Theta_0 = \arctan(- 1/a)$ and $\Theta = \Theta_0 + \pi$.

Introducing Feller's canonical speed $s(\Theta)$ and canonical measure $m(\Theta)$ ([Itô 1]), one obtains

$$ds(\Theta) = \exp \left(- \int^{\Theta} \frac{b(\varphi)}{a(\varphi)} d\varphi\right) d\Theta =$$

$$= \frac{1}{\cos^2 \Theta} \cdot \exp \left[- \frac{2}{\sigma^2} \left(\frac{\ell_2}{a^2} \cdot \tan \Theta + \right.\right.$$

$$+ \frac{\ell_1 a - 2\ell_2}{a^3} \cdot \ln(|1 + a \cdot \tan \Theta|) +$$

$$\left.\left. + \frac{\ell_1 a - \ell_2 - a^2}{a^3} \cdot \frac{1}{1 + a \cdot \tan \Theta}\right)\right] d\Theta, \tag{5.18}$$

and

$$dm(\Theta) = \frac{1}{a(\Theta)} \cdot \exp \left(\int^{\Theta} \frac{b(\varphi)}{a(\varphi)} d\varphi\right) d\Theta =$$

$$= \frac{2}{\sigma^2 (\cos \Theta + a \sin \Theta)^2} \cdot \exp \left[\frac{2}{\sigma^2} \left(\frac{\ell_2}{a^2} \cdot \tan \Theta + \right.\right.$$

$$+ \frac{\ell_1 a - 2\ell_2}{a^3} \cdot \ln (|1 + a \cdot \tan \Theta|) +$$

$$\left.\left. + \frac{\ell_1 a - \ell_2 - a^2}{a^3} \cdot \frac{1}{1 + a . \tan \Theta}\right)\right] d\Theta. \tag{5.19}$$

Consider first the interval $[- \frac{\pi}{2}, \Theta_0]$. It follows from (5.18), (5.19) that for this interval, $- \frac{\pi}{2}$ is an entrance point. Θ_0 is an exit point provided that $a < 0$ and

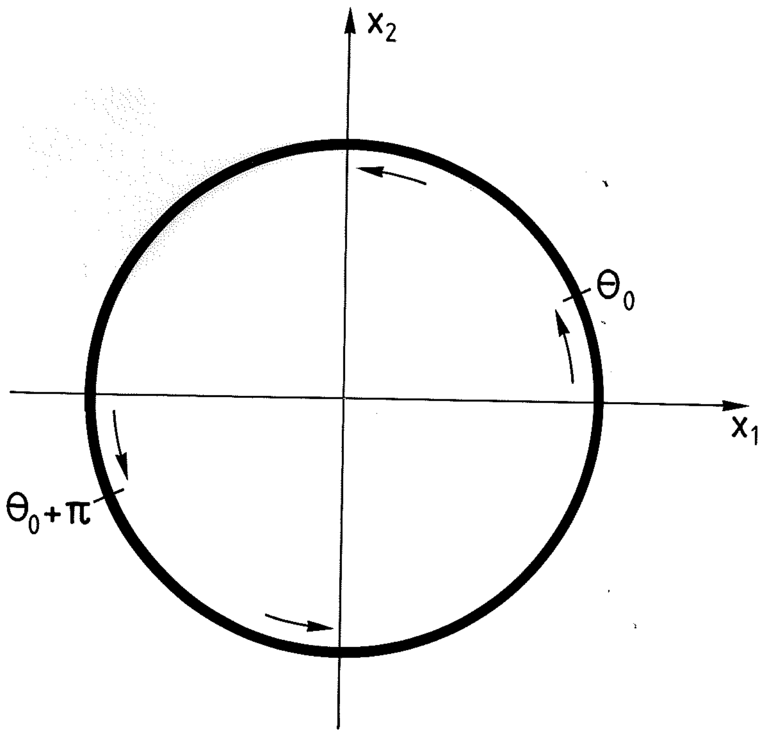

Fig. 5.1 - Diffusion picture of Eqns. (5.16), (5.17).

ℓ_1, $\ell_2 > 0$. For the interval $[\Theta_0, \frac{\pi}{2}]$, one obtains that Θ_0 is an entrance point (assuming that the above inequalities hold), whereas $\frac{\pi}{2}$ is an exit point. Further, the diffusion is symmetric with respect to the origin.

It follows that the Θ-process will diffuse around the unit circle in the counterclockwise direction without being trapped at any point. It is ergodic over the entire unit circle.

The invariant measure of the process can be obtained in closed form using its factorisation in the speed and scale measures, but no explicit form will be needed. Instead, consider the integrand

$$L(\log(\|x\|)) = \frac{\sigma^2}{2}(\cos\Theta + a\sin\Theta)^2\cos 2\Theta - \ell_1\cos^2\Theta - $$
$$- (\ell_2 - 1)\sin\Theta\cos\Theta. \tag{5.20}$$

An inspection of the three terms forming (5.20) shows that the integrand can be made negative on the entire unit circle for any given σ if ℓ_1 is chosen large enough. Necessarily, the expected value is then negative, which ensures stability. □

CONCLUSION

The results presented in this work, although limited in their scope, are none the less believed to provide some general guidelines in the design of robust regulators. A few final remarks will be made here on the major issues discussed.

Modelling

One of the major contributions is perhaps the study of the effect of the modelling on the synthesis. In summary, *the modelling of large plant disturbances is less critical than that of input-channel disturbances.* The design for robustness against the former class of perturbations leads to some combination of high gains and disturbance decoupling, irrespectively of their fine-structure. In fact, it follows from standard stability theorems that robustness is ensured against any form of deterministic dynamic disturbances using the synthesis of Chapter 2.

In contrast, synthesis for robustness with respect to input-channel disturbances requires a fairly accurate description of these disturbances. In some cases a high-gain regulator results, in others more cautious forms.

From the theoretical point of view it may be interesting to note that the stochastically phrased problem permits the formulation of necessary and sufficient conditions for stabilisability, whereas only sufficient conditions are given in Part I of this work (at least using all the data of the problem).

What then should be considered as the "best" way of modelling a disturbance, about which only little information is available? It is known that the circle-criterion-type theorems generally produce conservative results, the reason for this being that the assumptions on the nonlinearity are very weak. On the synthesis level, this may imply that too much energy is devoted to the stability problem, which may deteriorate the performance of the regulator in other respects. The assumption on ergodicity of the time-varying parameters, which is easy to justify in many applications, enhances the stability considerably (at least in the almost-sure sense).

Further, and this may be more important in practical design problems, the L_2-design based on Equation (5.7) captures some interesting quantitative features of the design problem, which are closely related to classical concepts. In fact, the cautious character of the regulator obtained from equation (5.7) reflects the classical trade--off between accuracy and fast response (high gains) on one hand and stability and low noise sensitivity (low gains) on the other (generally referred to as the bandwidth compromise). This is an appealing feature of the L_2--design, which, combined with its relative computational simplicity, makes it attractive from the user's point of view.

Adaptivity versus robustness

An adaptive regulator can in principle handle arbitrarily large parameter variations, provided that these variations are slow compared to the time constants of the controlled system (see for instance [Ste]). The constant linear feedback regulator, on the contrary, has a satisfactory performance only in certain regions of the parameter space. Bounds for these regions are given by the manifolds (if any) where the system will contain unstable, uncontrollable modes. On the other hand, there are no bounds on the rate of variation of the parameters.

The adaptive regulator is fairly complex in structure, and requires a rather deep knowledge of estimation theory (stochastic or deterministic). The much simpler structure of the robust linear feedback regulator is compensated for by the use of (sometimes) large control signals, something that is bound to create problems in certain applications.

In summary, the choice between these two design principles should not be too difficult to make in practical cases.

The limits of regulation

The concept of "threshold of uncertainty" introduced by Athans and co-workers, which was discussed in § 5.2.4, is believed to be an important subject for theoretical analysis. However, for the reasons given above, it is felt that convergence almost surely rather than in the mean square is the sound basis for such a definition. This discussion is of course of academic interest only, since a control system with noise levels anywhere in the near of the stability limit (be it mean-square or almost-sure stability) is useless for practical purposes.

At the heart of this problem lies a yet-to-define concept of stochastic controllability.

REFERENCES

[And]
Anderson, B.D.O. and Moore, J.B.(1971): *Linear Optimal Control*. Prentice-Hall, Englewood Cliffs, N.J.

[Åst]
Åström, K.J.(1965): On a first-order stochastic differential equation. *Int. J. Control* 1, No. 4, 301-326.

[Ath]
Athans, M., Ku, R., and Gershwin, R.G.(1977): The uncertainty threshold principle: Some fundamental limitations of optimal decision making under dynamic uncertainty. *IEEE Trans. Aut. Control* 22, 491-495.

[Ben]
Bengtsson, G.(1973): A theory for control of linear multivariable systems. Report TFRT-1006, Dept. of Automatic Control, Lund Institute of Technology, Lund, Sweden.

[Ber]
Bertram, J.E. and Sarachik, P.E.(1959): Stability of circuits with randomly time-varying parameters. *Trans. IRE-PGIT* 5, 260.

[Cod]
Coddington, E.A. and Levinson, N.(1955): *Theory of Ordinary Differential Equations*. McGraw-Hill, New York--Toronto-London.

[Cru]
Cruz, J.B.(Ed.)(1973): *System Sensitivity Analysis*. Dowden, Hutchinson, and Ross, Stroudsburg, Penn.

[Dav]
Davison, E.J. and Wang, S.H.(1974): Properties and calculation of transmission zeros of linear multivariable systems. *Automatica* 10, 643-658.

[Doy]
Doyle, J.C.(1978): Guaranteed margins for LQG regulators. *IEEE Trans. Aut. Control* 23, No. 4, 756-757.

[Gan]
Gantmacher, F.R.(1959): *The Theory of Matrices, Vol. 1*. Chelsea, New York.

[Gil]
Gilman, A.S. and Rhodes, I.B.(1976): Cone-bounded nonlinearities and mean-square bounds - quadratic regulation bounds. *IEEE Trans. Aut. Control* 21, 472-483.

[Hah]
Hahn, W.(1967): *Stability of Motion*. Springer-Verlag, Berlin-Heidelberg-New York.

[Hau]
Haussmann, U.G.(1973): Stability of linear systems with control dependent noise. *SIAM J. Control* 11, 382-394.

[Hor 1]
Horowitz, I.M. and Sidi, M.(1972): Synthesis of feedback systems with large plant ignorance for prescribed time-domain tolerances. *Int. J. Control* 16, No. 2, 287-309.

[Hor 2]
Horowitz, I.M. and Sidi, M.(1973): Synthesis of cascaded multi-loop feedback systems with large plant parameter ignorance. *Automatica* 9, 589-600.

[Itô 1]
Itô, K.(1963): *Veroyatnostnye Protsessy. Vypusk II.* Izd. Inostr. Lit., Moscow.

[Itô 2]
Itô, K. and McKean, H.P.(1965): *Diffusion Processes and their Sample Paths*. Springer-Verlag, Berlin-Heidelberg-New York.

[Käl]
Källström, C.G., Åström, K.J., Thorell, N.E., Eriksson, J., and Sten, L. (1978): Adaptive autopilots for large tankers. Preprints 7th IFAC World Congress, Helsinki, Finland, 477-484.

[Kat]
Kats, I.I. and Krasovskii, N.N.(1960): On the stability of systems with random parameters. *Prikl. Mat. i Mekh.* 24, 809-823.

[Kha]
Khasminskii, R.Z.(1967): Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems. *Th. Prob. Appls.* 1, 144-147.

[Kle]
Kleinmann, D.(1969): Optimal stationary control of linear systems with control-dependent noise. *IEEE Trans. Aut. Control* 14, 673-677.

[Kon]
Kontakos, T.(1973): PhD Thesis, Univ. of Manchester, as quoted in Kouvaritakis, B. and MacFarlane, A.G.J.(1976): Geometric approach to analysis and synthesis of system zeros. Part 2. Non-square systems. *Int. J. Control* 23, No. 2, 167-181.

[Kou]
Kouvaritakis, B. and Shaked, U.(1976): Asymptotic behaviour of root-loci of linear multivariable systems. *Int. J. Control* 23, No. 3, 297-340.

[Koz 1]
Kozin, F.(1969): A survey of stability of stochastic systems. *Automatica* 5, 95-112.

[Koz 2]
Kozin, F. and Prodromou, S.(1971): Necessary and sufficient conditions for almost sure sample stability of linear Itô equations. *SIAM J. Appl. Math.* 21, 413-424.

[Koz 3]
Kozin, F. and Sugimoto, S.(1977): Relations between sample and moment stability for linear stochastic differential equations. In *Proc. Conf. Stoch. Diff. Eqns. and Appls.* (Ed. J.D. Mason). Academic Press, New York--San Francisco-London.

[Kus 1]
Kushner, H.(1967): Converse theorems for stochastic Lyapunov functions. *SIAM J. Control* 5, No. 2, 228-233.

[Kus 2]
Kushner, H.J.(1967): *Stochastic Stability and Control*. Academic Press, New York-London.

[Kwa 1]
Kwakernaak, H. and Sivan, R.(1972): *Linear Optimal Control Systems*. J. Wiley, New York.

[Kwa 2]
Kwakernaak, H. and Sivan, R.(1972): The maximally achievable accuracy of linear optimal regulators and linear optimal filters. *IEEE Trans. Aut. Control* 17, 79-86.

[Lan]
Langevin, P.(1908): Sur la théorie du mouvement brownien. *C. R. Acad. Sci. Paris* 146, 530-533.

[Lur]
Lure, A.I. and Postnikov, V.N.(1944): On the theory of stability of control systems. *Prikl. Mat. i Mekh.* 8, 246-248.

[Mol]
Molander, P. and Willems, J.C. (to appear): Design of feedback controllers with specified gain and phase margins.

[Nar 1]
Narendra, K.S. and Goldwyn, R.M.(1964): A geometrical criterion for the stability of certain nonlinear non-autonomous systems. *IEEE Trans. Circuit Theory* 11, 406-407.

[Nar 2]
Narendra, K.S. and Taylor, J.H.(1973): *Frequency Domain Criteria for Absolute Stability*. Academic Press, New York-London.

[Pop]
Popov, V.M.(1960): Nouveaux criteriums de stabilité pour les systèmes automatiques non-linéaires. *Revue d'Electrotechn. et d'Energetique, Acad. de la Rép. Pop. Roumaine* 5, No. 1.

[Ros 1]
Rosenbrock, H.H.(1970): *State-space and Multivariable Theory*. Nelson, London.

[Ros 2]
Rosenbrock, H.H.(1973): The zeros of a system. *Int. J. Control* 18, 297-299; Correction to 'The zeros of a system'. *Int. J. Control* 20, 525-527.

[Roz]
Rozenvasser, E.N.(1963): The absolute stability of non-linear systems. *Avtom. i Telemekh.* 24, No. 5.

[Saf]
Safonov, M.G. and Athans, M.(1977): Gain and phase margin for multiloop LQG regulators. *IEEE Trans. Aut. Control* 22, 173-179.

[San]
Sandor, J. and Williamson, D.(1977): Design of non-linear regulators for linear plants. *IEEE Trans. Aut. Control* 22, 47-50.

[Sil]
Silverman, L.M. and Payne, H.J.(1971): Input-output structure of linear systems with application to the decoupling problem. *SIAM J. Control* 9, 199-233.

[Ste]
Sternby, J.(1977): Topics in dual control. Dept. of Automatic Control, Lund Inst. of Technology, Lund, Sweden. CODEN: LUTFD2/(TFRT-1012)/1-135/(1977).

[Str]
Stratonovič, R.L.(1966): A new representation for stochastic integrals and equations. *SIAM J. Control* 4, 362-371.

[Tar]
Tarn, T.-J. and Rasis, Y.(1976): Observers for nonlinear stochastic systems. *IEEE Trans. Aut. Control* 21, 441-448.

[Wil]
Willems, J.L. and Willems, J.C.(1976): Feedback stabilisability for stochastic systems with state and control dependent noise. *Automatica* 12, 277-283.

[Won 1]
Wong, E. and Zakai, M.(1965): On the convergence of ordinary integrals to stochastic integrals. *Ann. Math. Stat.* 36, 1560-1564.

[Won 2]
Wong, P.-K., Stein, G., and Athans, M.(1978): Structural reliability and robustness properties of optimal linear-quadratic multivariable regulators. Preprints 7th IFAC World Congress, Helsinki, Finland, pp. 1797-1805.

[Won 3]
Wonham, W.M.(1970): Random differential equations in control theory. In *Prob. Meth. in Appl. Math., Vol 2* (Ed. A.T. Bharucha-Reid). Academic Press, New York-London.

[Won 4]
Wonham, W.M.(1974): *Linear Multivariable Control. A Geometric Approach*. Springer-Verlag (Lecture Notes in Economics and Mathematical Systems 101), Berlin-Heidelberg-New York.