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1. Introduction

Most of the existing control theory assumes that models for the processes to be controlled and their environments are known. In many practical situations such assumptions are highly unrealistic because the desired models are often not available. The problem can be avoided in two different ways, using modelling and system identification methods to obtain the models or by introducing more sophisticated control algorithms which do not require knowledge of the process models. In principle, it is possible to generate such algorithms by considering the problem of controlling a system with constant but unknown parameters. The algorithms obtained in this way are optimal but they cannot be determined with a reasonable computational effort. It is, however, possible to determine suboptimal control strategies. One strategy of this type called STUREI was analysed in Astrom-Wittenmark (1973). It was shown that under reasonable conditions the regulator has the *self-tuning property*. This means that if the parameters of the regulator converge, the regulator obtained in the limit will be identical to the optimal regulator that could have been designed if the parameters had been known.

There are many ways to generate suboptimal strategies. One possibility is to make the assumption that the problem of controlling a system with unknown parameters can be separated into an identification problem and a control problem. This was proposed, for example, by Kalman (1958). Algorithms of this type are also investigated in Peterka (1970) and Wieslander-Wittenmark (1970).

Analysis of convergence of a gneeral class of stochastic control algorithms, which includes several self-tuning regulators is given in

Ljung (1974). A detailed discussion of the application of the general convergence theorems for the self-tuning regulator STUREI is given in Ljung-Wittenmark (1974).

The self-tuning property of the algorithm STUREI depends on a delicate balance of the bias in the least squares estimate and the fact that the control law is determined from an erroneous model. A consequence of this is that the algorithm STUREI has several limitations. It will only converge to the optimal regulator when the criterion is such that there is no penalty on the control. The algorithm STUREI is not a dual regulator in the sense of Feldbaum. This means that the regulator does not attempt to introduce control energy for the explicit purpose of reducing parameter uncertainty. The algorithm cannot easily be modified to give a dual control law because any attempt to introduce perturbation signals will upset the balance between the bias and the modelling error.

In this report the notion of self-tuning regulators is extended. It is shown that it is possible to obtain a self-tuning estimator for both process and noise parameters based on least squares calculations only. Since both process and noise parameters are estimated, it is then possible to obtain self-tuning regulators where control actions are penalised and also regulators with dual control features.

The report is organised as follows. The recursive estimation algorithm is given in Section 2. The algorithm is identical to the one proposed by Young (1970) and closely related to one proposed by Panuska (1968). A first attempt at analysis is given in Section 3 where it is shown that certain covariances must vanish. A

characterisation of possible limit points is given in Section 4 where it is shown that the true parameter values form a limit point. In special cases it is also shown that this is the only limit point. Section 6 deals with convergence conditions. The analysis is based on the differential equations introduced by Ljung (1974). The relation to other methods like recursive maximum likelihood and model reference methods are briefly covered in Sections 7 and 8.

2. The Algorithm

A system with one input and one output is considered. It is desired to obtain a mathematical model relating the output y to the input u. The algorithm can be described as follows:

ALGORITHM

At each time N the parameters of the prediction model

$$\hat{y}(t) = -A(q^{-1})y(t)+B(q^{-1})u(t)+C(q^{-1})\varepsilon(t), \quad t=N,N-1,... \tag{2.1}$$

are determined in such a way that the prediction error defined by

$$\varepsilon(t) = y(t)-\hat{y}(t) \tag{2.2}$$

is as small as possible in the sense of least squares, that is:

$$V = \sum_{t=1}^{N} \varepsilon^2(t) \quad \text{minimic} \tag{2.3}$$

In equation (2.1) $\{u(t), t=N-1,N-2,...\}$ denotes the sequence

of inputs that are applied to the system, $\{y(t),\ t=N,N-1,\ldots\}$ the sequence of observed outputs and $\{\varepsilon(t),\ t=N-1,N-2,\ldots\}$ the prediction errors obtained in the previous steps. Furthermore $A(x)$, $B(x)$ and $C(x)$ denote polynomials

$$A(x) = \alpha_1 x + \ldots + \alpha_n x^k$$

$$B(x) = \beta x + \ldots + \beta_\ell x^\ell$$

$$C(x) = \gamma_1 x + \ldots + \gamma_m x^m$$

whose coefficients are the unknown parameters and q^{-1} is the backward shift operator. Notice that there is no constant term in either of the polynomials which means that (2.1) is a true prediction model. #

In order to write the equations for the least squares estimates of the parameters in a compact form the following notation is introduced

$$\Theta = [\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_\ell,\gamma_1,\ldots,\gamma_m]^T \tag{2.4}$$

$$\phi(t-1) = [-y(t-1),\ldots,-y(t-m),u(t-1),\ldots,u(t-\ell),\varepsilon(t-1),\ldots,\varepsilon(t-m)] \tag{2.5}$$

The prediction $\hat{y}$ given by (2.1) can then be written as

$$\hat{y}(t) = \phi(t-1)\Theta \tag{2.1}'$$

The prediction error defined by (2.2) becomes

$$\varepsilon(t) = y(t)-\phi(t-1)\Theta \tag{2.2}'$$

and the criterion (2.3) can be written as

$$V_N(\theta) = \sum_{t-1}^{N} [y(t)-\phi(t-1)\theta]^2 \qquad (2.3)'$$

The least squares estimate is the value of the parameter which minimises (2.3)'. It is easy to show that the estimate is given by the normal equations

$$\sum_{t=1}^{N} \phi^T(t-1)[y(t)-\phi(t-1)\theta(N)] = 0 \qquad (2.5)$$

which can also be written as

$$[\sum_{t=1}^{N} \phi^T(t-1)\phi(t-1)]\theta(N) = \sum_{t=1}^{N} \phi^T(t-1)y(t)$$

It is well known that the estimate satisfies the following recursive equations

$$\theta(t+1) = \theta(t)+P(t+1)\phi^T(t)[y(t+1)-\phi(t)\theta(t)] \qquad (2.6)$$

where

$$P(t+1) = P(t)-P(t)\phi^T(t)[1+\phi(t)P(t)\phi^T(t)]^{-1}\phi(t)P(t) \qquad (2.7)$$

Notice that $\phi(t)$ is a function of the previous estimates $\theta(t)$, $\theta(t-1)$, ... because the residuals $\varepsilon(t)$, $\varepsilon(t-1)$, ... depend on these estimates. Compare equation (2.5).

Remark

It follows from equations (2.1) and (2.2) that

$$\hat{y}(t) = -A(q^{-1})\hat{y}(t)+B(q^{-1})u(t)+[C(q^{-1})-A(q^{-1})]\varepsilon(t)$$

$$= -A(q^{-1})\hat{y}(t)+B(q^{-1})u(t)+C'(q^{-1})\varepsilon(t) \qquad (2.8)$$

The algorithm is therefore in essence equivalent to fitting the parameters of the polynomials A, B and C' of the model (2.8) by least squares. This is the approach taken in Wittenmark (1974).

3. Limiting Properties

The properties of the algorithm described in the previous section will now be analysed. It will first be assumed that the estimated parameters converge as N goes to infinity and that $||\phi(t)||$ is bounded. Equations (2.2)' and (2.5) give

$$\frac{1}{N}\sum_{t=1}^{N}\phi^T(t-1)\varepsilon(t)+\frac{1}{N}\sum_{t=1}^{N}\phi^T(t-1)\phi(t-1)[\theta(t-1)-\theta(N)] = 0 \qquad (3.1)$$

If the parameter estimates converge, that is

$$\theta(t) \to \theta_0 \quad \text{as} \quad t \to \infty$$

then the second term in the equation can be bounded by

$$\left|\frac{1}{N}\sum_{t=1}^{N}\phi^T(t-1)\phi(t-1)[\theta(t)-\theta(N)]\right|$$

$$\leq \frac{1}{N}\{N_1 \sup_{1\leq t\leq N_1}\phi^T(t-1)\phi(t-1)\left|\theta(t)-\theta(N)\right|\}$$

$$+\frac{1}{N}\sum_{t=N_1}^{N}\phi^T(t-1)\phi(t-1)\sup_{t\geq N_1}\left|\theta(t)-\theta(N)\right| \leq \frac{k_1}{N}+k_2\cdot\varepsilon_N$$

where k_1 and k_2 are constants and ε_N can be made arbitrarily small by choosing N_1 sufficiently large. If it is assumed that $\phi(t)$ is bounded it is thus found that the second term in (3.1) will go to zero as $N \to \infty$. The following result can now be obtained.

Theorem 1

Assume that the parameter estimates of the algorithm converge to such values that the polynomial $1+C^0(z)$ has all its zeros outside the unit disc. Let the sequences $\{y(t)\}$ and $\{u(t)\}$ be bounded and assume that the limits

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} u(t+\tau)u(t)$$

exist. Then the limiting values of the parameter estimates are such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon^0(t+\tau)y(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t+\tau)y(t) = 0, \tau=1,2,\ldots,k$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon^0(t+\tau)u(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t+\tau)u(t) = 0, \tau = 1,2,\ldots,\ell$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon^0(t+\tau)\varepsilon^0(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t+\tau)\varepsilon(t) = 0, \tau = 1,2,\ldots,m \tag{3.2}$$

where $\varepsilon^0(t)$ are the prediction errors computed using the limiting estimates.

Proof

The second equalities are already proven. They follow directly from the normal equations for the least squares estimate. To complete the proof the following lemma will first be shown.

Lemma 1

Consider the sequences

$$y(t) = \sum_{s=1}^{t} h(t,s)u(s), \quad z(t) = \sum_{s=1}^{t} k(t,s)u(s)$$

where the sequence $\{u(t)\}$ is bounded and the weighting functions h and k have the property

$$|h(t,s)-h_0(t-s)| < C_t e^{-\alpha|t-s|}$$

where $C_t \to 0$ as $t \to \infty$. Then

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} y(t)z(t) = \lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} y_0(t)z_0(t)$$

where

$$y_0(t) = \sum_{s=1}^{t} h_0(t-s)u(s)$$

$$z_0(t) = \sum_{s=1}^{t} k_0(t-s)u(s)$$

if the limit of the right hand side exists.

Proof of Lemma 1

First consider

$$|y(t)-y_0(t)| = \left| \sum_{s=1}^{t} [h(t,s)-h_0(t-s)]u(s) \right|$$

$$\leq \frac{C_t}{1-e^{-\alpha}} \sup_{1\leq s\leq t} |u(s)|$$

Since C_t goes to zero as $t \to \infty$, $y(t)$ will thus converge to $y_0(t)$.

Now consider

$$\frac{1}{t}\sum_{s=1}^{t}[y(s)z(s)-y_0(s)z_0(s)] =$$

$$= \frac{1}{t}\sum_{s=1}^{t}[y(s)z(s)-y_0(s)z(s)+y_0(s)z(s)-y_0(s)z_0(s)]$$

$$= \frac{1}{t}\sum_{s=1}^{t}[y(s)-y_0(s)]z(s)+\frac{1}{t}\sum_{s=1}^{t}y_0(s)[z(s)-z_0(s)]$$

But

$$\frac{1}{t}\sum_{s=1}^{t}\left|[y(s)-y_0(s)]z(s)\right| \leq$$

$$\leq \frac{1}{t}\{n \sup_{1\leq s\leq n}\left|(y(s)-y_0(s))z(s)\right|+(t-n)\sup_{n\leq s\leq t}\left|(y(s)-y_0(s))\cdot z(s)\right|\}$$

$$\leq \frac{An}{t} + \varepsilon(n)$$

where $\varepsilon(n)$ can be made arbitrarily small by choosing n sufficiently large, because z is bounded and $y(s) \to y_0(s)$ as $s \to \infty$. The statement of the lemma is thus proven. #

Under the assumptions of the theorem the sequences $\{u(t)\}$ and $\{y(t)\}$ are bounded. It follows from equations (2.1) and (2.2) that

$$\varepsilon(t) = y(t)-\hat{y}(t) = y(t)+\hat{A}(q^{-1})y(t)-\hat{B}(q^{-1})u(t)-\hat{C}(q^{-1})\varepsilon(t)$$

Hence

$$[1+\hat{C}(q^{-1})]\varepsilon(t) = [1+\hat{A}(q^{-1})]y(t)-\hat{B}(q^{-1})u(y)$$

The coefficients of this difference equation will depend on time since the estimates are time dependent. For sufficiently large t

the equation will, however, be arbitrarily close to the zeros of the polynomial $1+C^0(x)$. The assumptions of the lemma thus hold and the theorem is proven.

4. Possible Limiting Points

In the previous section a characterisation was given of the properties of the estimates under the strong assumption that the estimates converge. The characterisation did not require specific assumptions on the input output relation. In this section it will be assumed that the input output data used in the experiment are generated by the system

$$A(q^{-1})y(t) = B(q^{-1})u(t)+C(q^{-1})e(t) \tag{4.1}$$

where $A(x)$, $B(x)$ and $C(x)$ are polynomials

$$A(x) = 1+a_1x+\ldots+a_nx^n$$

$$B(x) = b_1x\ldots+b_nx^n$$

$$C(x) = 1+c_1x\ldots+c_nx^n$$

and q^{-1} is the backwards shift operator. It is shown in Astrom (1970) that the one step predictor for the process (4.1) can be written as

$$\hat{y}(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t) + \frac{C(q^{-1})-A(q^{-1})}{A(q^{-1})}e(t)$$

or

$$\hat{y}(t) = [1-A(q^{-1})]\hat{y}(t)+B(q^{-1})u(t)+[C(q^{-1})-A(q^{-1})]e(t)$$

Observing that

$$y(t) = \hat{y}(t)+e(t)$$

it is easy to show that the equation above can be rewritten as

$$\hat{y}(t) = [1-A(q^{-1})]y(t)+B(q^{-1})u(t)+[C(q^{-1})-1]e(t) \qquad (4.2)$$

It thus follows that

$$\begin{aligned} &E\ e(t+\tau)y(t) = 0,\ \tau = 1,\ldots,n,\ldots \\ &E\ e(t+\tau)u(t) = 0,\ \tau = 1,\ldots,n,\ldots \\ &E\ e(t+\tau)e(t) = 0,\ \tau = 1,\ldots,n,\ldots \end{aligned} \qquad (4.3)$$

Summing up the following result is obtained.

Lemma 2

Let the algorithm of Section 2 be applied to a system described by (4.1) and let $k \geq n$, $\ell \geq n$ and $m \geq n$. Assume that the parameter estimates converge to values such that the polynomial $1+C(x)$ has all its zeros outside the unit circle. Then a possible limit point is given by

$$\alpha_i = a_i \quad i = 1,2,\ldots,n$$

$$\beta_i = b_i \quad i = 1,2,\ldots,n \tag{4.4}$$

$$\gamma_i = c_i \quad i = 1,2,\ldots,n$$

It will now be investigated whether there can possibly be any other limit points. It is again assumed that the parameter estimates converge. The limiting properties will now be investigated. Equations (2.1) and (2.2) give

$$[1+C(q^{-1})]\hat{y}(t) = [C(q^{-1})-A(q^{-1})]y(t)+B(q^{-1})u(t)$$

or

$$\hat{y}(t) = \frac{C-A}{1+C}\,y(t)+\frac{B}{1+C}\,u(t) \tag{4.5}$$

where (q^{-1}) has been deleted in the polynomial notation $A(q^{-1})$ in order to save writing. Furthermore equation (2.2) gives

$$\varepsilon(t) = y(t)-\hat{y}(t) = \frac{1+A}{1+C}\,y(t)-\frac{B}{1+C}\,u(t) \tag{4.6}$$

If the input-output sequencewere generated from the system (4.1) then

$$y(t) = \frac{B}{A}\,u(t)+\frac{C}{A}\,e(t)$$

and the equation above can be written as

$$\varepsilon(t) = \frac{(1+\hat{A})C}{(1+\hat{C})A}e(t) - \frac{A\hat{B}+(1+\hat{A})B}{A(1+\hat{C})}u(t) \tag{4.7}$$

The conditions of Theorem 1 now give $k + \ell + m$ conditions to determine the $k + \ell + m$ parameters of the polynomials. The conditions are obtained in terms of a set of nonlinear algebraic equations. It is in general not trivial to analyse these equations. Some specific examples will therefore be investigated.

Example 4.1

Consider the system

$$y(t+1)+ay(t) = bu(t)+e(t+1)+ce(t) \tag{4.8}$$

where the input u is assumed to be a sequence of uncorrelated random variables which are also uncorrelated with $\{e(t)\}$. Let $|a| < 1$ and $|c| < 1$ and assume that the algorithm of Section 2 is applied using the prediction model

$$\hat{y}(t+1) = \alpha y(t)+\beta u(t)+\gamma\varepsilon(t) \tag{4.9}$$

Furthermore assume that the parameter estimates converge. Since $\{u(t)\}$, $\{y(t)\}$ and $\{\varepsilon^0(t)\}$ are stationary stochastic processes, the time-averages appearing in Theorem 1 will be ensemble averages. The conditions of Theorem 1 can then be written as

$$\begin{aligned} r_{\varepsilon_0 y}(1) &= 0 \\ r_{\varepsilon_0 u}(1) &= 0 \\ r_{\varepsilon_0 \varepsilon_0}(1) &= 0 \end{aligned} \tag{4.10}$$

where $\{\varepsilon_0(t)\}$ denotes the residuals calculated using the limiting values of the parameter estimates. The subscript in ε_0 will be suppressed in the following in order to avoid messy notation.

It follows from equation (2.2) that

$$\text{cov}\,[\varepsilon(t+1),u(t)] = \text{cov}\,[y(t+1)-\hat{y}(t+1),u(t)]$$

$$= \text{cov}\,[y(t+1),u(t)] - \text{cov}\,[\hat{y}(t+1),u(t)]$$

Since u(t) and u(s) are uncorrelated if $t \neq s$ and since u(t) is uncorrelated with e(s) $\forall$ s it follows that

$$\text{cov}\,[y(t+1),u(t)] = \text{cov}\,[-ay(t)+bu(t)+e(t+1)+ce(t),u(t)]$$

$$= b\ \text{cov}\,[u(t),u(t)]$$

and

$$\text{cov}\,[\hat{y}(t+1),u(t)] = \text{cov}\,[-\alpha y(t)+\beta u(t)+\gamma\varepsilon(t),u(t)]$$

$$= \beta\ \text{cov}\,[u(t),u(t)]$$

The condition (3.2) then gives

$$r_{\varepsilon u}(1) = (b-\beta)\ \text{cov}\,[u(t),u(t)] = 0$$

Hence

$$\beta = b \qquad (4.11)$$

Equations (4.8) and (4.9) give

$$\text{cov}\,[\varepsilon(t+1),\hat{y}(t+1)] = -\alpha r_{\varepsilon y}(1)+\beta r_{\varepsilon u}(1)+\gamma r_{\varepsilon\varepsilon}(1) = 0$$

Furthermore it follows from equation (4.10) that

$$\text{cov}\,[\varepsilon(t+1),\hat{y}(t)] = \text{cov}\,[\varepsilon(t+1),y(t)-\varepsilon(t)] = 0$$

But

$$\text{cov}\,[\varepsilon(t+1),\hat{y}(t)] =$$

$$\text{cov}\,[y(t+1),\hat{y}(t)] - \text{cov}\,[\hat{y}(t+1),\hat{y}(t)]$$

Since

$$\text{cov}\,[y(t+1),\hat{y}(t)] =$$

$$\text{cov}\,[-ay(t)+bu(t)+e(t+1)+ce(t),\hat{y}(t)] =$$

$$-a\ \text{cov}\,[y(t),\hat{y}(t)]$$

and

$$\text{cov}\,[\hat{y}(t+1),\hat{y}(t)] =$$

$$\text{cov}\,[-\alpha y(t)+\beta u(t)+\gamma\varepsilon(t),\hat{y}(t)] =$$

$$-\alpha\ \text{cov}\,[y(t),\hat{y}(t)]$$

it follows from equation (4.10) that

$$0 = (\alpha-a)\ \mathrm{cov}\,[y(t),\hat{y}(t)] = (\alpha-a)\ \mathrm{cov}\,[\hat{y}(t),\hat{y}(t)]$$

Hence

$$\alpha = a \tag{4.12}$$

Subtracting equations (4.8) and (4.9) and using (4.11) and (4.12) gives

$$\varepsilon(t+1)+\gamma\varepsilon(t) = e(t+1)+ce(t) \tag{4.13}$$

The covariance of the process $\{\varepsilon(t)\}$ at lag 1 is given by

$$r_{\varepsilon}(1) = \frac{(c-\gamma)(1-\gamma c)}{1-\gamma^2}$$

The condition (4.10) gives

$$\gamma = c$$

or

$$\gamma = 1/c$$

It is thus found that the conditions (4.10) imply that α and β are given uniquely and that there are two possible values for γ, $\gamma = c$ and $\gamma = 1/c$ respectively. Since $|c|<1$ the value $\gamma=1/c$ corresponds to a polynomial $1+C(x) = 1+x/c$ with zeros inside the unit disc. The equation (4.13) is thus unstable for this value of γ and the conditions of Theorem 1 are violated. It is thus found that there is only one set of parameters which satisfy the conditions of Theorem 1.

Example 4.2

Consider the system

$$A(q^{-1})y(t) = C(q^{-1})e(t) \tag{4.14}$$

where $A(q^{-1})$ and $C(q^{-1})$ are n^{th} order polynomials in the backward shift operator q^{-1}. It is assumed that the polynomials $A(x)$ and $C(x)$ have all their zeros outside the unit disc. Assume that the algorithm of Section 2 is applied using the prediction model

$$\hat{y}(t) = -\hat{A}(q^{-1})y(t)+C(q^{-1})\varepsilon(t) \tag{4.15}$$

Assuming that the parameter estimates converge, it follows from Theorem 1 and Lemma 2 that one possible stationary point is given by

$$\hat{A}(q^{-1}) = A(q^{-1})-1$$

$$C(q^{-1}) = C(q^{-1})-1$$

It will now be investigated whether this is the only stationary point. To do so it is observed that the residuals are given by equation (4.7) which in this particular case reduces to

$$\varepsilon(t) = \frac{[1+\hat{A}(q^{-1})]C(q^{-1})}{[1+C(q^{-1})]A(q^{-1})}\, e(t) \tag{4.16}$$

Introduce the stochastic process $\{v(t)\}$ defined by

$$v(t) = \frac{C(q^{-1})}{[1+C(q^{-1})]A(q^{-1})}\, e(t)$$

Then

$$\varepsilon(t) = [1+A(q^{-1})]u(t)$$

$$y(t) = [1+C(q^{-1})]u(t) \qquad (4.17)$$

Furthermore

$$r_{\varepsilon y}(\tau) = \text{cov}\,\{\varepsilon(t+\tau), [1+C(q^{-1})]u(t)\}$$

$$= [1+C(q)]r_{\varepsilon u}(\tau)$$

and

$$r_{\varepsilon\varepsilon}(\tau) = \text{cov}\,\{\varepsilon(t+\tau), [1+A(q^{-1})]u(t)\}$$

$$= [1+A(q)]r_{\varepsilon u}(\tau)$$

The conditions (3.2) thus imply

$$r_{\varepsilon u}(1)+\alpha_1 r_{\varepsilon u}(2)+\ldots+\alpha_k r_{\varepsilon u}(k+1) = 0$$

$$r_{\varepsilon u}(2)+\alpha_1 r_{\varepsilon u}(3)+\ldots+\alpha_k r_{\varepsilon u}(k+2) = 0$$

$$\vdots$$

$$r_{\varepsilon u}(m)+\alpha_1 r_{\varepsilon u}(m+1)+\ldots+\alpha_k r_{\varepsilon u}(k+m) = 0$$

$$r_{\varepsilon u}(1)+\gamma_1 r_{\varepsilon u}(2)+\ldots+\gamma_m r_{\varepsilon u}(m+1) = 0$$

$$r_{\varepsilon u}(2)+\gamma_1 r_{\varepsilon u}(3)+\ldots+\gamma_m r_{\varepsilon u}(m+2) = 0$$

$$\vdots$$

$$r_{\varepsilon u}(k)+\gamma_1 r_{\varepsilon u}(k+1)+\ldots+\gamma_m r_{\varepsilon u}(m+k) = 0$$

These equations can be written as

$$\begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_n & & & \\ & 1 & \alpha_1 & & \alpha_n & & \\ & & \ddots & & & & \\ & & & 1 & \alpha_1 & \cdots & \alpha_k \\ 1 & \gamma_1 & \cdots & \gamma_m & & & \\ & 1 & \gamma_1 & & \gamma_m & & \\ & & \ddots & & & & \\ & & & 1 & \gamma_1 & & \gamma_m \end{bmatrix} \begin{bmatrix} r_{\varepsilon U}(1) \\ r_{\varepsilon U}(2) \\ \\ \\ \\ \\ \\ r_{\varepsilon U}(k+m) \end{bmatrix} = 0 \qquad (4.18)$$

Since $r_{\varepsilon U}(\tau)$ is also a function of the parameters α_i and γ_i the equations are nonlinear. Notice, however, that the determinant of the matrix on the left is a resultant. It then follows that if the polynomials $1+A(x)$ and $1+C(x)$ are relatively prime then the matrix is nonsingular and the equation (4.16) implies that

$$r_{\varepsilon U}(\tau) = 0, \quad \tau = 1,2,\ldots,k+m \qquad (4.19)$$

Since ε and u are outputs at a dynamical system of order $n+m$ it follows from (4-18) and the Yule-Walker equation that

$$r_{\varepsilon U}(\tau) = 0 \ \forall \ \tau$$

if $k \geq n$. The equation (4.17) then implies that

$$r_\varepsilon(\tau) = 0 \quad \tau \neq 0 \tag{4.20}$$

where ε is given by (4.16). It then follows that

$$[1+A(x)][x^n+A^*(x)]C(x)C^*(x) =$$

$$[1+C(x)][x^n+C^*(x)]A(x)A^*(x)$$

where A* is the reciprocial polynomial, that is

$$A^*(x) = x^n A(x^{-1})$$

Since it was postulated that $1 + C(x)$ and $C(x)$ have all zeros outside the unit disc and since $A(x)$ and $C(x)$ are relatively prime it follows from (4.21) that

$$\hat{C}(x) = C(x)-1 \tag{4.21}$$

Since the residuals are linear in the coefficients of the polynomial $\hat{A}$, it then follows that the solution of the least squares problem is unique and it is found that

$$\hat{A}(x) = A(x)-1 \tag{4.22}$$

It is thus found that if the algorithm is applied to a stationary time series then there is only one stationary point which satisfies the conditions of Theorem 1.

Remark

The analysis of this example is in essence equivalent to the proof of Theorem 2 in Åström-Wittenmark (1973).

Example 4.3

Consider the system

$$A(q^{-1})y(t) = B(q^{-1})u(t)+e(t)$$

where $A(q^{-1})$ and $B(q^{-1})$ are polynomials in the backward shift operator q^{-1}. Assume that the algorithm of Section 2 is applied using the prediction model

$$\hat{y}(t) = -\hat{A}(q^{-1})y(t)+\hat{B}(q^{-1})u(t)$$

and that the parameter estimates converge. The residuals associated with the limits of the parameters

$$\varepsilon_0(t) = y(t)-\hat{y}_0(t) = [1+A_0(q^{-1})]y(t)-B_0(q^{-1})u(t)$$

are then linear in the parameters. The equations (3.2) reduce to

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^0(t+\tau)y(t) = 0,\ \tau = 1,\ldots,k$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^0(t+\tau)u(t) = 0,\ \tau = 1,\ldots,\ell$$

These linear equations can be written as

$$\left[\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \phi^T(t)\phi(t)\right](\theta-\theta_0) = 0$$

The matrix on the left hand side is positive definite if $k = \ell = m$ and if the input signal is persistently exciting of order $2n$. The solution to (3.2) given by Lemma 2 is then unique.

5. Convergence

The algorithm of Section 2 is a special case of the stochastic algorithms analysed by Ljung (1974). Convergence conditions can therefore be obtained by the methods developed by Ljung. To apply these methods it is first assumed that the input $u(t)$ is such that the sums

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u(t+\tau)u(t) \quad \text{exist for all } \tau \tag{5.1}$$

This assumption is necessary because the results in Ljung (1974) are based on stationarity assumptions. It is also necessary to impose a stability condition. This condition can be formulated as

> Assume that the polynomial $1 + C(x)$ has its zeros outside the unit disc infinitely often or that the algorithm is modified in such a way that the polynomial $1 + C(x)$ will always have its zeros outside the unit disc. (5.2)

Introduce the functions f and G defined by

$$f(\theta) = E\varepsilon(t;\theta)\phi(t;\theta) \tag{5.3}$$

$$G(\theta) = E\phi^T(t;\theta)\phi(t;\theta) \tag{5.4}$$

where the domain is taken as the subset of $R^{k+\ell+m}$ on which the polynomial $1 + C(x)$ has all the zeros outside the unit disc. The scalar $\varepsilon(t;\theta)$ is defined by equations (2.1) and (2.2) and the vector $\phi(t;\theta)$ is defined by equation (2.5).

It was shown in the previous section that the function f vanishes for the true parameters, that is

$$\theta = \theta_0 = [-a_1,\ldots,-a_n,b_1,\ldots b_n,c_1,\ldots,c_n]$$

Examples were also given when $\theta = \theta_0$ is the only zero of f. It follows from the results of Ljung (1974) that the estimate will converge to θ^* if (θ^*,S^*) is a globally asymptotically stable solution of the differential equations

$$\frac{d\theta(\tau)}{dt} = S(\tau)f(\theta(\tau)) \tag{5.5}$$

$$\frac{dS(\tau)}{dt} = S(\tau)-S(\tau)G(\theta(\tau))S(\tau) \tag{5.6}$$

such that S^* is positive definite.

Example 5.1

Consider the first order moving average process

$$y(t) = e(t)+ce(t-1)$$

The equation (4.16) gives

$$\varepsilon(t) = \frac{1+cq^{-1}}{1+\gamma q^{-1}}\, e(t)$$

Hence

$$f(\gamma) = r_\varepsilon(1) = \frac{(c-\gamma)(1-\gamma c)}{1-\gamma^2}$$

$$G(\gamma) = r_\varepsilon(0) = \frac{1-2\gamma c+c^2}{1-\gamma^2}$$

We have

$$f'(c) = -1$$

$$G(\gamma) = 1 + \frac{(\gamma-c)^2}{1-\gamma^2} > 0 \qquad |\gamma| \leq 1$$

The solution of $\gamma = c$ is therefore an asymptotically stable solution to the equation (5.5).

6. Relations to Other Recursive Estimation Schemes

The algorithm discussed in Section 2 is closely related to several other recursive estimation schemes. The relations to stochastic approximations and recursive maximum likelihood estimation and the model reference method are discussed briefly in this section.

Stochastic Approximations

The stochastic approximation scheme is obtained by replacing the matrix P in equation (2.6) by a scalar which goes to zero as 1/t with increasing t. The algorithm then becomes

$$\theta(t+1) = \theta(t)+\gamma(t)\phi^T(t)[y(t+1)-\phi(t)\theta(t)] \qquad (6.1)$$

This gives a significant reduction in the number of computations required because it is no longer necessary to iterate the matrix equation (2.7). The algorithm (6.1) is identical to the algorithm proposed by Panuska (1968). A suitable normalisation of the coefficients γ is

$$\gamma(t) = A \Big/ \sum_{k=1}^{t} \phi(k)\phi^T(k) \tag{6.2}$$

If the estimates given by equation (6.1) with $\gamma(t) \sim 1/t$ converge they will converge to values such that the equations (3.2) hold. In this respect the stochastic approximation algorithm is thus similar to the algorithm of Section 2. The convergence conditions are, however, different.

Recursive Maximum Likelihood

The algorithm of Section 2 is also closely related to recursive maximum likelihood estimators for the model (4.1). It is shown in Söderström (1973) that a recursive maximum likelihood estimator for the parameters of the model (4.1) can be written as

$$\theta(t+1) = \theta(t) - P(t+1)\psi(t+1)\varepsilon(t+1) \tag{6.3}$$

where

$$\theta = \text{col}[\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n, \hat{c}_1, \ldots, \hat{c}_n] \tag{6.4}$$

$$\hat{C}(q^{-1})\varepsilon(t) = \hat{A}(q^{-1})y(t) - \hat{B}(q^{-1})u(t) \tag{6.5}$$

$$\psi(t) = \text{col}\left[\frac{\partial\varepsilon(t)}{\partial\hat{a}_1}, \ldots, \frac{\partial\varepsilon(t)}{\partial\hat{a}_n}, \frac{\partial\varepsilon(t)}{\partial\hat{b}_1}, \ldots, \frac{\partial\varepsilon(t)}{\partial\hat{b}_n}, \frac{\partial\varepsilon(t)}{\partial\hat{c}_1}, \ldots, \frac{\partial\varepsilon(t)}{\partial\hat{c}_n}\right] \tag{6.6}$$

$$\hat{C}(q^{-1}) \frac{\partial \varepsilon(t)}{\partial \hat{a}_1} = y(t-i) \tag{6.7}$$

$$\hat{C}(q^{-1}) \frac{\partial \varepsilon(t)}{\partial \hat{b}_1} = u(t-i) \tag{6.8}$$

$$\hat{C}(q^{-1}) \frac{\partial \varepsilon(t)}{\partial \hat{c}_1} = -\varepsilon(t-i) \tag{6.9}$$

$$P(t) = \left[\sum_{k=1}^{t} \Psi(t)\Psi^T(t) \right]^{-1} \tag{6.10}$$

This algorithm reduces to the algorithm of Section 2 if it is observed that equation (6.5) implies that

$$\varepsilon(t) = y(t) - \phi(t)\theta$$

where ϕ is given by (2.5) and if the equations (6.7), (6.8) and (6.9) are approximated by

$$\frac{\partial \varepsilon(t)}{\partial \hat{a}_1} \approx y(t-i)$$

$$\frac{\partial \varepsilon(t)}{\partial \hat{b}_1} \approx -u(t-i)$$

$$\frac{\partial \varepsilon(t)}{\partial \hat{c}_1} \approx -\varepsilon(t-i)$$

Then

$$\Psi(t) = -\phi(t-1)$$

and the equation (6.3) becomes identical to (2.6).

If the recursive ML algorithm converges the limiting values of the parameters are such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_0(t+\tau) C_0^{-1}(q^{-1}) y(t) = 0 \quad \tau = 1,2,\ldots,n$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_0(t+\tau) C_0^{-1}(q^{-1}) u(t) = 0 \quad \tau = 1,2,\ldots,n \qquad (6.10)$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_0(t+\tau) C_0^{-1}(q^{-1}) \varepsilon(t) = 0 \quad \tau = 1,2,\ldots,n$$

Compare with the equations (3.2).

The Model Reference Method

The model reference method introduced by Whitaker (1959) is based on the following idea: to adjust the parameters of a model its output is compared with the output of the System. The model parameters are then adjusted in such a way that their rate of change is proportional to the gradient of the norm of the model error. See Figure 6.1.

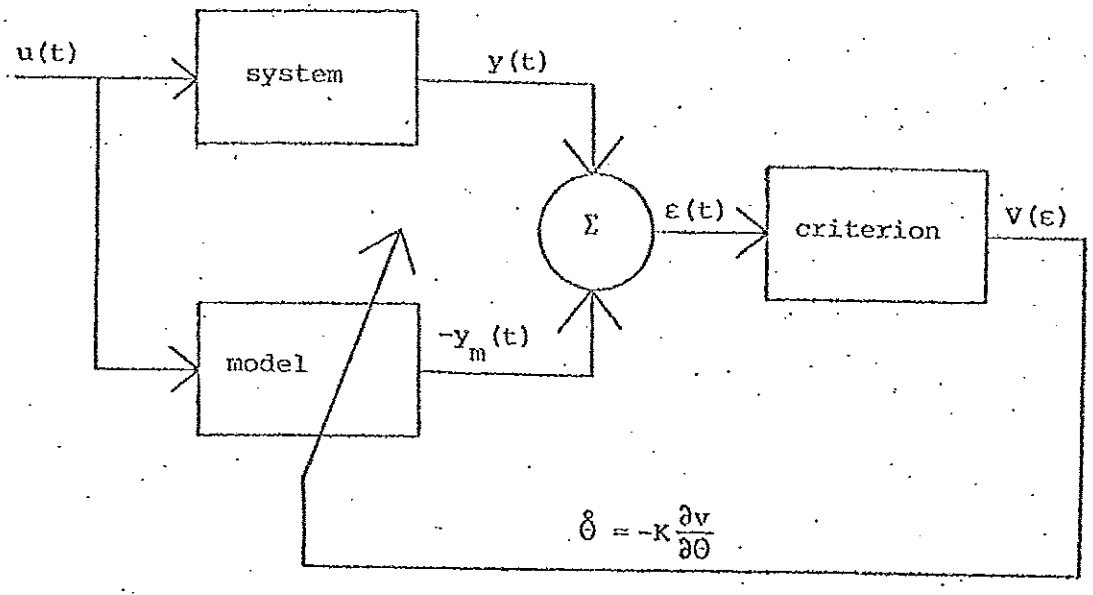

Fig. 6.1 Schematic Diagram of Whitaker's Model Reference Method

It will now be shown that the recursive estimation schemes can be interpreted as model reference arguments. To do so consider the system (4.1) and let the model in Figure 6.1 be a prediction model

$$\hat{C}(q^{-1})\varepsilon(t) = \hat{A}(q^{-1})y(t)-\hat{B}(q^{-1})u(t)$$

Let the criterion be

$$V_t(\varepsilon) = \varepsilon^2(t)$$

Whitaker's rule then gives

$$\hat{a}_i(t+1) = \hat{a}_i(t)-K\cdot\varepsilon(t)\frac{\partial\varepsilon(t)}{\partial\hat{a}_i}$$

An application of Whitakers rule thus leads to equations similar to those obtained from the recursive maximum likelihood method if the gain K is properly selected. Compare the above equation with equations (6.3) and (2.6).

7. Conclusions

It has been shown that it is possible to derive estimation algorithms based on least squares recursive estimation which have good asymptotic properties even for systems with correlated residuals. These algorithms can be exploited to construct predictors and self-tuning regulators which can be used in cases where the algorithm STUREI is not appropriate.

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