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Abstract

Relays are common in automatic control systems. It is well-known that a linear dynamical system under relay feedback can give complex oscillations. In this paper it is proved that several of these phenomena can actually be captured by third-order systems. It is shown that the existence of fast switches and sliding modes is completely characterized by the high-frequency asymptote of the Nyquist curve for the linear part of the system. A novel method for analyzing linear dynamical systems under relay feedback is also introduced. Trajectories for a class of third order systems are proved to converge in a certain sense.

Key words
Nonlinear control systems, Limit cycles, Relays, Sliding mode, Oscillation

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1. Introduction

Analysis of relay feedback systems is a classical topic in control theory. The early work was motivated by relays in electromechanical systems and simple models for dry friction. An important property of these systems are their tendency to oscillate. The design of simple relay controllers in aerospace applications described in Flügge-Lotz (1953) gave for instance inspiration to the development of the self-oscillating adaptive controller in the 1960's. Recently new interest of relay feedback appeared due to the idea of using relays for tuning of simple controllers, see Åström and Hägglund (1984). By simply replacing the controller by a relay, measure the amplitude and frequency of the possible oscillation, and out of these derive the controller parameters, a robust control design method is given. Even if this method is widely spread and accepted in practice, it has not been theoretically investigated in greater detail. The idea of putting the plant under relay feedback is also used in other applications. In Smith and Doyle (1993) perturbation bounds are estimated for robust control design, and in Lundh and Åström (1994) it is shown how initialization of adaptive controllers can be done. More historical comments on relays in control systems and their applications are given in Tsypkin (1984) and Åström (1993).

A linear system under relay feedback is shown in Figure 1. Analysis of this system is a nontrivial task. Restrictions of the linear dynamics have to be made. The monograph Andronov *et al.* (1965) is an early classical reference (first edition published in Russian in 1937) discussing oscillations in mostly second-order systems using phase-plane analysis. For some systems a fruitful approach to get approximate results is the describing function method, see Atherton (1975) and Mees (1981). In Yakubovich (1973) a frequency condition is used to give sufficient conditions for a certain type of oscillation. The major reference about relay control systems Tsypkin (1984) surveys a number of analysis methods. An intuitive stability condition therein, saying roughly that if a linear process is stable under arbitrary large proportional feedback it is also stable under relay feedback, is proved in Anosov (1959). Applicable stability results for a more general class of nonlinearities is given in Yakubovich (1964).

Even if relay feedback systems have been studied for a long time, they are far from fully understood. There is, for instance, little known about

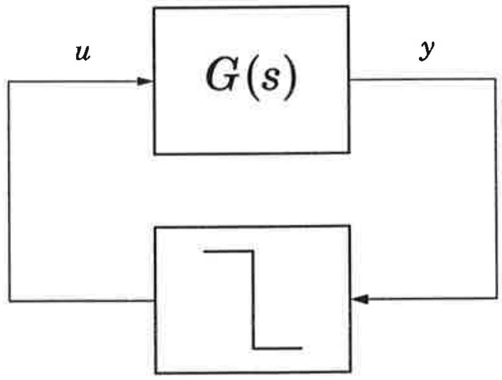

Figure 1. Relay feedback of linear system G.

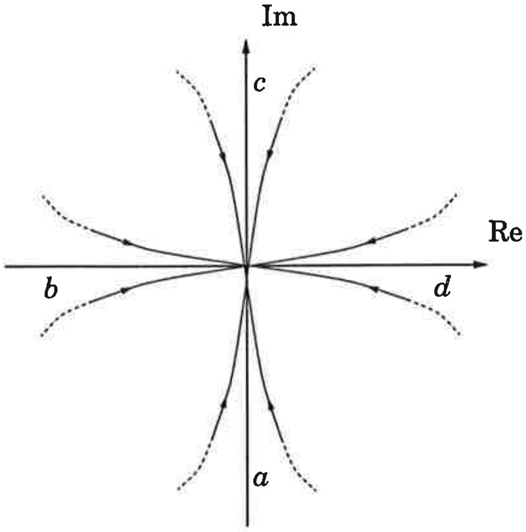

Figure 2. Sliding modes occur when the high-frequency asymptote of the Nyquist curve is the negative imaginary axis and fast switches if it is the negative real axis.

globally attractive limit cycles. For second-order processes analysis can be done in the phase-plane. Stable second-order nonminimum phase processes can in this way be shown to have a globally attractive limit cycle. In Megretski (1996) it is proved that this also holds for processes having an impulse response sufficiently close in a certain sense to a second-order nonminimum phase process.

The main contribution of our work is to analyze third-order linear systems under relay feedback. It is shown that despite the low dimension, these systems have a rich structure. Sliding modes as well as a type of fast relay switches can appear. The phenomena can be detected from the high-frequency asymptote of the Nyquist curve for the linear part of the system, see Figure 2. If positive steady-state gain is assumed, the high-frequency asymptote a gives sliding modes, the asymptote b gives fast switches, while c and d do not. Furthermore, a method for analysing relay feedback systems based on convergence of switch plane intersections is also introduced in this paper.

The outline is as follows. Some notations and the definition of a limit cycle are given in Section 2. Section 3 discusses sliding modes in third-order relay feedback systems and Section 4 considers fast switches. A method for global analysis of limit cycles is introduced in Section 5, followed by a new result on convergence in Section 6. Finally, concluding remarks are given in Section 7. Some of the proofs are collected in an appendix. A short version of this paper is published as Johansson and Rantzer (1996).

2. Preliminaries

Consider the relay feedback system in Figure 1. The process G is a stable and strictly proper linear transfer function with scalar input u and scalar output y. In state-space form G is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and A is a Hurwitz matrix, that is, all eigenvalues of A lie in the open left half plane. The relay feedback is defined by

$$u = -\operatorname{sgn} y = \begin{cases} -1 & \text{if} \quad y > 0 \\ 1 & \text{if} \quad y < 0 \end{cases} \tag{2}$$

The *switch plane* $\mathcal{S}$ is the hyperplane of dimension $n-1$ where the output vanishes, that is, $\mathcal{S} := \{x : Cx = 0\}$. On either side of $\mathcal{S}$ the feedback system is linear. If $Cx > 0$ the dynamics are given by $\dot{x} = Ax - B$, and if $Cx < 0$ we have $\dot{x} = Ax + B$. We also introduce the notation $\mathcal{S}_+ := \{x \in \mathcal{S} : CAx + CB > 0\}$. If nothing else is mentioned, we assume the process G to have positive steady-state gain. Since the linear dynamics on each side of $\mathcal{S}$ have fixed points equal to $\pm A^{-1}B$, positive steady-state gain guarantees that the trajectories do not tend to any of these two fixed points, and thus ensures a relay switch to occur. The differential equation (1)–(2) is only valid outside the switch plane. By letting $u \in [-1, 1]$ for $x \in \mathcal{S}$, the solution can still be a continuous function which satisfies (1)–(2) everywhere, see Filippov (1988) and Yakubovich (1973).

Let the Poincaré map $g = g(x) : \mathcal{S}_+ \to \mathcal{S}_+$ map a point x to next switch plane intersection of the trajectory starting at x and reflect the intersection in the origin. We have

$$g(x) := -e^{Ah(x)}x + (e^{Ah(x)} - I)A^{-1}B \tag{3}$$

where $h(x)$ is the *switch time*, that is, the unique time it takes between the two intersections x and $g(x)$. Recall that $CB = 0$ if and only if the relative degree of G is greater than one. If the steady-state gain $G(0) = -CA^{-1}B$ is positive, then $CB < 0$ if and only if the relative degree is one and G has an odd number of zeros in the right half plane.

3. Sliding Modes

It is well-known that *sliding modes* (or *Filippov solutions*) can occur in relay feedback systems, see Filippov (1988). This can easily be understood from studying $\dot{y} = CAx \pm CB$ close to $\mathcal{S}$. We see that depending on the value of CB a classification of the directions of the trajectories divide the switch plane into two or three regions.

EXAMPLE 1
Consider the process

$$G(s) = \frac{\beta s + 1}{(s+1)(s+2)}$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ \beta \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

Then $\mathcal{S}$ equals the x_1-axis, see Figure 3. The points $p_{1,2}$, where the trajectories change directions, are given by the solutions of

$$CAx \pm CB = 0$$

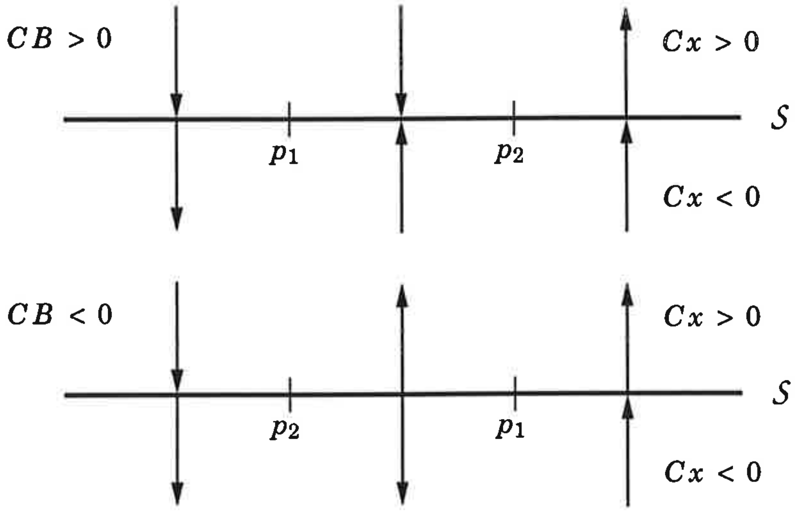

Figure 3. The switch plane $\mathcal{S}$ and the trajectories for the system in Example 1. If $CB > 0$ there exist sliding modes.

that is, $p_1 = (-\beta, 0)$ and $p_2 = (\beta, 0)$. For $CB = \beta > 0$ there exist sliding modes, while for $CB < 0$ the region between p_1 and p_2 is repelling. The region vanishes if $CB = 0$.

Applying nonsmooth Lyapunov stability theory, it is shown in an example in Shevitz and Paden (1994) that all solutions converge to the origin if $CB > 0$. □

The condition in the example for existence of sliding modes directly generalizes to processes of order $n > 2$. Then p_1 and p_2 denotes hyperplane of order $n-2$, still separating the switch plane into two or three regions. It follows that a sliding mode can occur if and only if $CB > 0$.

It is well-known that oscillations can occur in mechanical control systems due to friction. The oscillation may include a stick-slip motion, that is, the mechanical device is moving only a part of each period. A fifth-order system is reported to behave this way in Atherton *et al.* (1985), but the following example shows that third-order dynamics are sufficient.

EXAMPLE 2
Consider

$$G(s) = \frac{s(s-\zeta)}{(s+1)(s^2+s+1)} \tag{4}$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -\zeta \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

Figure 4 shows the (clockwise) limit cycle is for $\zeta = 1$. The fixed points of $\dot{x} = Ax - B$ and $\dot{x} = Ax + B$ are marked with asterisks and lie on

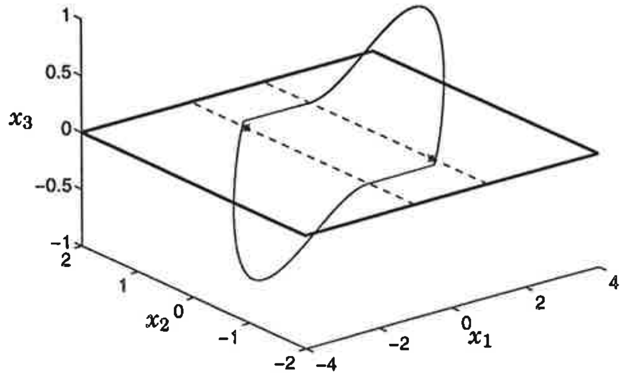

Figure 4. Limit cycle with sliding mode.

the boundary of the sliding mode region $\{x \in \mathcal{S} : |CAx| < CB\}$ (dashed lines). The sliding mode part of each period decreases with increasing ζ, so that $\zeta = 10$ gives a smooth limit cycle. The intersection of the Nyquist curve of (4) with the negative real axis does not correspond to the true period of the oscillation. Hence, the first-order harmonic balance solution (the describing function method) gives an erroneous estimate of the limit cycle period for this example, see Mees (1981). □

4. Fast Switches

In this section a necessary and sufficient condition for the occurrence of fast relay switches similar to sliding modes is proved. From previous section it is obvious that if $CB > 0$ there exist sliding modes and if $CB < 0$ there cannot exist any arbitrarily fast relay switches. Therefore consider a third-order system with $CB = 0$. Figure 5 shows trajectories close to $\{x \in \mathcal{S} : CAx = 0\}$ for examples with $CAB > 0$ and $CAB < 0$. The tick marks indicate $CA^2\chi_1 - CAB = 0$ and $CA^2\chi_2 + CAB = 0$, that is, where $\ddot{y} = 0$ on $\{x \in \mathcal{S} : CAx = 0\}$. Solid trajectories are above the switch plane ($Cx > 0$) and dashed under. The figure suggests that the switch times $h(\cdot)$ can be arbitrarily short if $CAB > 0$. A proof will be given next.

THEOREM 1
Consider the relay feedback system (1)–(2) with $n = 3$, $CB = 0$, and $CAB \neq 0$. Then $\inf_{x \in \mathcal{S}_+} h(x) + h(g(x)) = 0$ if and only if $CAB > 0$.

Proof: Let $\phi_-(t, x)$ denote the trajectory of $\dot{x} = Ax - B$ at time t starting in x at time $t = 0$. Consider $x_0 \in \mathcal{S}$ such that $CAx_0 = 0$ and $CA^2x_0 - CAB < 0$. It follows directly that

$$C\phi_-(t, x_0) = (CA^2x_0 - CAB)\frac{t^2}{2} + O(t^3) \qquad (5)$$

Hence, $C\phi_-(t_0, x_0) < 0$ for t_0 sufficiently small. Furthermore, for this t_0 we have $C\phi_-(t_0, x) < 0$ for $x \in \mathcal{S}_+$ with $|x - x_0|$ sufficiently small. It follows for $x \in \mathcal{S}_+$ that $h(x) \to 0$ as $x \to x_0$, and also that $g(x) \to -x_0$.

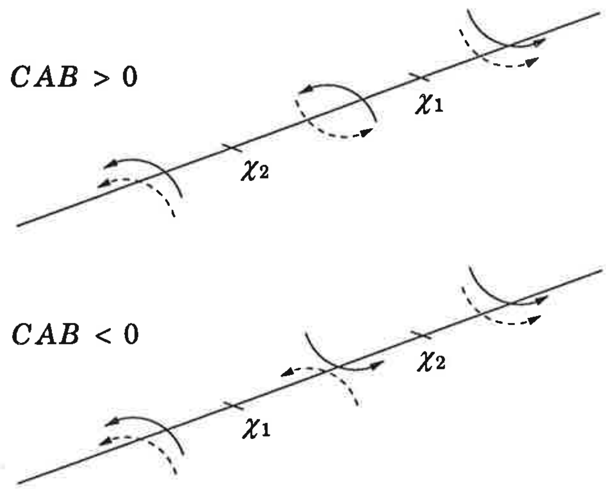

Figure 5. Illustration of Theorem 1. Fast relay switches occur if $CAB > 0$.

A symmetric argument with $g(x)$ gives $h(g(x)) \to 0$ as $x \to x_0$ if x_0 also satisfies $CA^2x_0 + CAB > 0$. Hence, sufficiency follows, since for $CAB > 0$ there exists $x \in \mathcal{S}_+$ such that $|CA^2x| < CAB$.

If

$$CA^2x_0 - CAB > 0 \tag{6}$$

for $x_0 \in \mathcal{S}$ and $CAx_0 = 0$, then (5) gives that $C\phi_-(t_0, x_0) > 0$ for t_0 sufficiently small. Hence, $C\phi_-(t_0, x) > 0$ for $x \in \mathcal{S}_+$ and $|x - x_0|$ sufficiently small, and thus $h(x) > t_0$. If $CA^2x_0 - CAB < 0$ and

$$CA^2x_0 + CAB < 0 \tag{7}$$

then in the same way there exists $t_1 > 0$ such that $h(g(x)) > t_1$ for $|x - x_0|$ sufficiently small. If $CAB < 0$ one of the two inequalities (6) and (7) most hold for $x_0 \in \mathcal{S}$ satisfying $CAx_0 = 0$ and thus necessity follows. $\square$

EXAMPLE 3
Consider the third-order process

$$G(s) = \frac{\zeta - s}{\zeta(s+1)^3}$$

with state-space representation

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 + 1/\zeta \\ -1 - 2/\zeta \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

The limit cycle period $2h$ is in Figure 6 shown as a function of the zero ζ. The dashed line corresponds to the limit cycle for the process $1/(s+1)^3$. The relay feedback system is stable if $\zeta \in (-3, 0)$. The convergence to limit cycle is quite different depending on if $\zeta < -3$ or $\zeta > 0$. This illustration

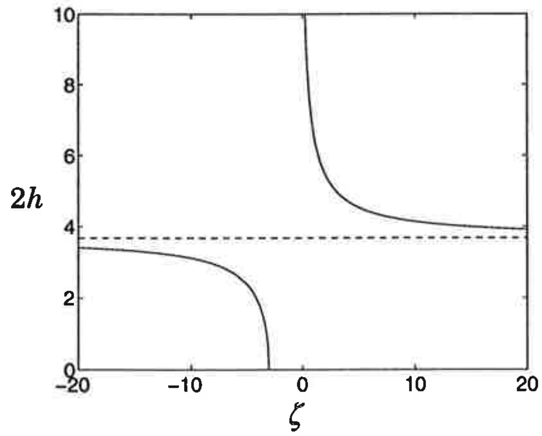

Figure 6. The limit cycle period as a function of zero location in Example 3.

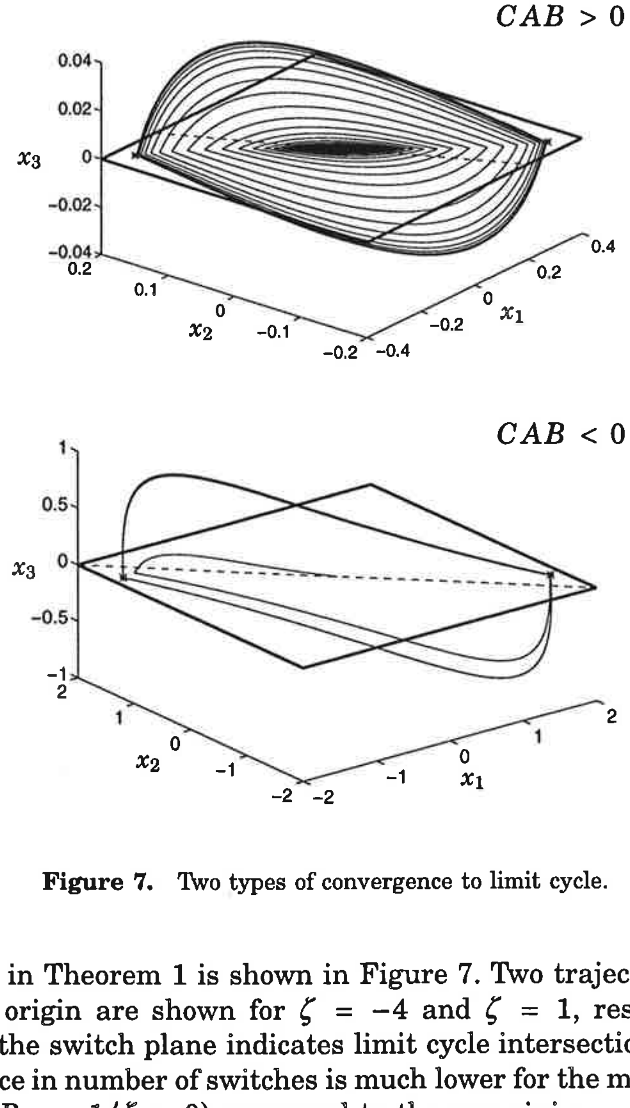

Figure 7. Two types of convergence to limit cycle.

of the result in Theorem 1 is shown in Figure 7. Two trajectories starting close to the origin are shown for $\zeta = -4$ and $\zeta = 1$, respectively. The asterisks in the switch plane indicates limit cycle intersections. The speed of convergence in number of switches is much lower for the minimum phase system ($CAB = -1/\zeta > 0$) compared to the nonminimum phase system ($CAB < 0$). □

5. Global Analysis of Limit Cycles

We now derive a method for analyzing relay feedback system using the Poincaré map g in (3). Let $\phi(t, x_0)$ denote the trajectory of (1)–(2) that starts at x_0. A *closed orbit* is a trajectory such that $\phi(t_1, x_0) = \phi(t_2, x_0)$ for some $t_1 < t_2$. A point p is said to be a *limit point* of the trajectory if there exists a sequence $\{t_k\}$, with $t_k \to \infty$ as $k \to \infty$, such that $\phi(t_k, x_0) \to p$ as $k \to \infty$. The set of all limit points is the *limit set* of the trajectory and is denoted $\mathcal{L}$. A limit set that is a closed orbit is called a *limit cycle*. The limit cycle is called *simple* if it has exactly two intersections with the switch plane $\mathcal{S}$. It is said to be *globally attractive* if it is the limit set of all possible trajectories.

An obvious question is if it exists relay feedback systems not having a unique stable limit cycle. For higher order systems, the answer is yes as shown by the following example.

EXAMPLE 4
Let

$$G(s) = \frac{(s+1)^2}{(s+0.1)^3(s+7)^2}$$

Depending on the initial conditions, the relay feedback system tends to either a slow or a fast limit cycle. In Figure 8 the relay output u is shown for the two cases after the initial transient has disappeared. Analysis shows that the limit cycles are locally stable. □

Denote k successive mappings by $g^k(x)$. If $\phi^*(t, x_0)$ is part of a stable simple limit cycle, and thus $\phi^*(t, x_0) \in \mathcal{L}$ for all $t \geq 0$, then the intersections with $\mathcal{S}$ equals $\pm x^* \in \mathcal{L}$, where x^* is a fixed point of g: $x^* = g(x^*)$. Hence, solving the equation $x = g(x)$ gives candidates for simple limit cycle intersections with $\mathcal{S}_+$. The solution is given by

$$x = (e^{Ah} + I)^{-1}(e^{Ah} - I)A^{-1}B$$

The following proposition gives necessary conditions for existence of a simple stable limit cycle, see Tsypkin (1984) and Åström (1993).

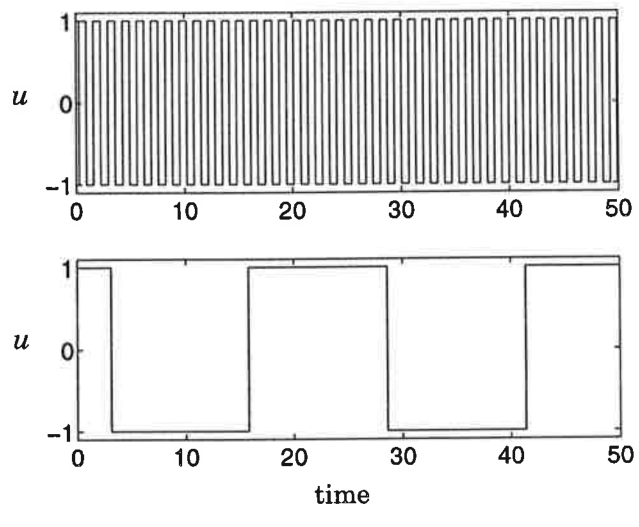

Figure 8. Two stable limit cycles for the system in Example 4.

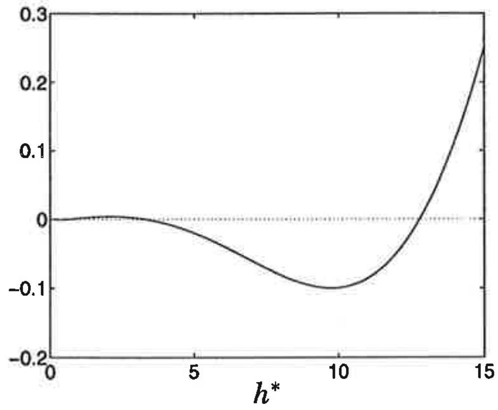

Figure 9. The left-hand side of (8) as a function of h^* in Example 5.

PROPOSITION 1
Consider the relay feedback system (1)–(2). If there exists a simple limit cycle with switch plane intersections $\pm x^*$ and period time $2h^*$, then

$$C(e^{Ah^*} + I)^{-1}(e^{Ah^*} - I)A^{-1}B = 0 \tag{8}$$

The limit cycle is stable if all eigenvalues of

$$\frac{dg}{dx}(x^*) = (I - \frac{(Ax^* + B)C}{C(Ax^* + B)})e^{Ah^*} \tag{9}$$

are in the open unit disc. □

Notice that the trivial solution $h^* = 0$ always satisfies (8). It is easy to show that this is the only solution for first-order and second-order stable processes with no zeros. Hence, these processes exhibit no simple limit cycles under relay feedback.

EXAMPLE 5
Consider the relay feedback system in Example 4. Figure 9 shows the left-hand side of (8) as a function of h^*. The zero-crossings are at 0.66, 3.32, and 12.80. The maximum eigenvalues of (9) for the three cases are 0.60, 1.42, and 0.64, respectively. Only the first and third zero-crossings thus come from a locally stable limit cycle. Notice that we cannot draw any conclusions about convergence. □

A numerical method for analyzing convergence to limit cycles is given by letting

$$X_0 = \{x \in \mathcal{S} : CAx = 0\} \cup \{x \in \mathcal{S}_+ : |x| = R\}$$

for a sufficiently large R and studying the set recursion $X_k = g(X_{k-1})$. Remark that since $|u| \le 1$ and A is Hurwitz, the existence of a globally attractive and invariant ball $\{x : |x| \le R\}$ is trivial, compare Hsu (1990). For systems of order three and less the method is easy to visualize.

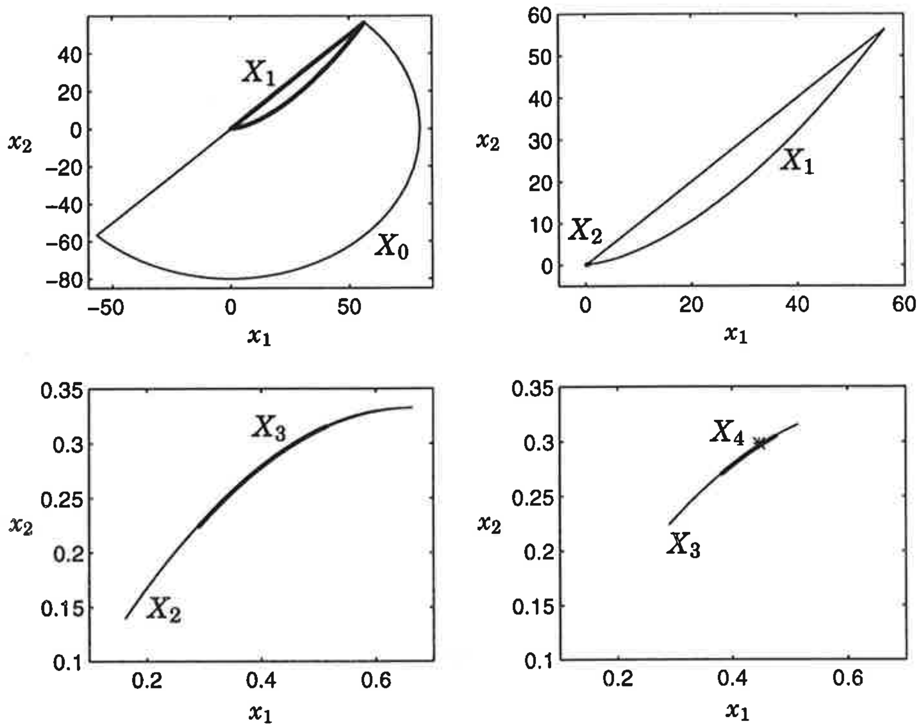

Figure 10. Convergence to limit cycle illustrated with a numerical method.

EXAMPLE 6
Consider the process

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 3 & -3 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

Hence, $\mathcal{S} = \{x : x_3 = 0\}$ and $\mathcal{S}_+ = \{x \in \mathcal{S} : x_1 > x_2\}$. Let X_0 be a semicircle disc with radius 80. Figure 10 shows the set recursion under four iterations. The first diagram shows X_0 together with X_1 (drawn with thicker lines), the second X_1 and X_2, etc. In the last diagram the fixed point $x^* = (0.45, 0.30, 0)$ is marked by an asterisk. The contraction is remarkably fast, in particular during the first two iterations. This agrees with the behavior noted also when using relay feedback in practice, see Åström and Hägglund (1984). □

By applying local stability analysis around x^* for a given system, it is possible to prove global convergence to a stable limit cycle with the method described above.

6. Area Contraction

For a class of processes we are able to prove global area contraction.

DEFINITION 1
A subset $\mathcal{V} \subset \mathcal{S}_+$ is called *globally attractive*, if for all $x \in \mathcal{S}_+$ there exists k such that $g^k(x) \in \mathcal{V}$. If $g(x) \in \mathcal{V}$ for all $x \in \mathcal{V}$, then $\mathcal{V}$ is *invariant*.

The area of a measurable set X is denoted by $\mathcal{A}(X) = \int dX$. The map $g = g(x) : \mathcal{S}_+ \to \mathcal{S}_+$ is called *area contractive*, if there exist constants $c \in \mathbb{R}$ and $\rho \in [0, 1)$ such that

$$\mathcal{A}(g^k(X)) \le c\rho^k, \qquad k \ge 0$$

for all compact sets $X \subset \mathcal{S}_+$. □

Area contraction is weaker than ordinary contraction. For instance, the existence of a unique fixed point for g is not guaranteed. Still it improves the understanding of the behavior of some relay feedback systems. It is possible to show area contraction for a class of third-order processes including the one in Example 6. The following theorem states that the switch plane intersections of all solutions of (1)–(2) converge to a region with vanishing area.

THEOREM 2
Let A, B, C satisfy

$$C(sI - A)^{-1}B = \frac{K}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}, \qquad K > 0, \quad \lambda_3 > \lambda_2 > \lambda_1 > 0 \tag{10}$$

and consider the set recursion $X_k = g(X_{k-1})$ with g as in (3) and X_0 a compact subset of $\mathcal{S}_+$. Then, for $\{X_k\}$ there exist $c \in \mathbb{R}$ and $\rho \in (0, 1)$ such that

$$\mathcal{A}(X_k) \le c\rho^k, \qquad k \ge 0$$

□

The proof of Theorem 2 is divided into two lemmas, which are proved in Appendix. Lemma 1 gives area contraction a geometric interpretation.

LEMMA 1
Assume $A + A^T < 0$. If $CB = 0$, then g is area contractive in every invariant compact subset of

$$\mathcal{U} := \{x \in \mathcal{S}_+ : B^T Ax \le 0\}$$

□

LEMMA 2
For A, B, C as in (10), there exist $\varepsilon, R > 0$ such that

$$g^k(x) \in \{x \in \mathcal{S}_+ : B^T Ax \le 0, CAx > \varepsilon, |x| \le R\} =: \mathcal{V} \tag{11}$$

for all $x \in \mathcal{S}_+$ and $k \ge 3$. □

The compact set $\mathcal{V}$ is illustrated in Figure 11.

Proof of Theorem 2: For each compact set $X_0 \subset \mathcal{S}_+$, Lemma 2 gives that there exists a globally attractive and invariant compact set $\mathcal{V}$. This follows from considering $g^k(x)$, $k \ge 3$, since $g^3(x) \in \mathcal{V}$ and for each $g^k(x) \in \mathcal{V}$ it holds that $g^{k+1}(x) \in \mathcal{V}$, $k \ge 3$. The proof is completed by applying Lemma 1 and the definition of area contraction. □

The top left diagram in Figure 10 shows that $g(X_0)$ is a subset of $\mathcal{U}$ in Example 6. Notice that $B^T Ax = 0$ corresponds to $x_2 = -x_1$.

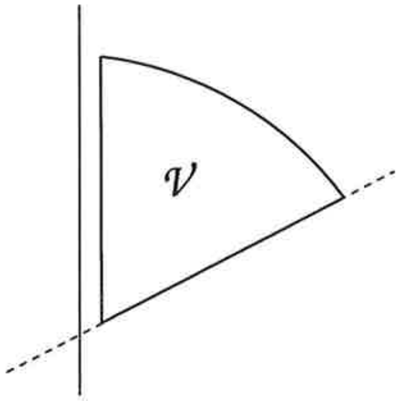

Figure 11. The compact set $\mathcal{V}$. On the solid line $CAx = 0$ and on the dashed line $B^T Ax = 0$.

7. Conclusions

The problem of oscillations in linear systems under relay feedback has been addressed. Some heuristics were given, and it was shown that also for third-order systems several important phenomena can arise. It was shown that the existence of arbitrarily fast switches and sliding modes can be detected from the high-frequency asymptote of the Nyquist curve. The second part of the paper introduced a method for global analysis of relay feedback systems based on the Poincaré map between switch plane intersections. In particular, for a certain class of processes it was proved that the intersection points converge to a region with vanishing area.

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9. Appendix

In this section we prove Lemmas 1 and 2.

Proof of Lemma 1: Consider the switch plane intersection x in a compact invariant set $\mathcal{X} \subset \mathcal{U}$ and denote the surrounding disc

$$\mathcal{B}_\varepsilon(x) := \{z \in \mathcal{X} : |z - x| \leq \varepsilon\}$$

Let $\Phi_-(t, \mathcal{B})$ be the set $\mathcal{B}$ after time t following the dynamics $\dot{x} = Ax - B$. The trajectories intersecting $\mathcal{B}_\varepsilon(x)$ pass through the hyperplane $\mathcal{N}_v(x) := \{z : v^T(z - x) = 0\}$. In particular, define

$$H_-(x, \mathcal{B}_\varepsilon(x)) := \{\mathcal{N}_{Ax-B}(x) \cap \Phi_-(t, \mathcal{B}_\varepsilon(x)), t \in \mathbb{R}\}_x$$

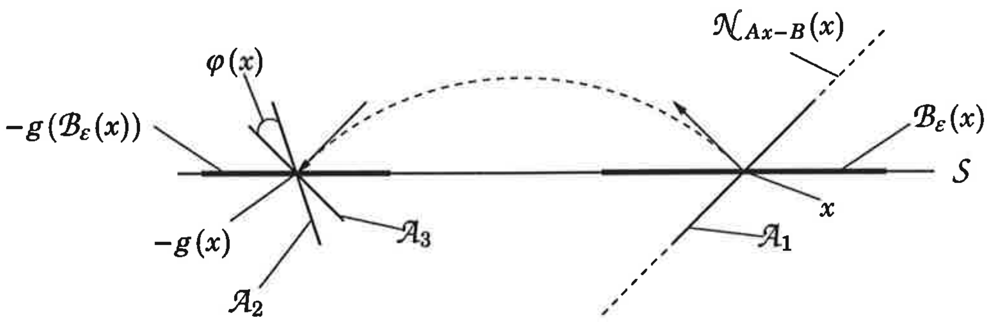

Figure 12. Illustration of area contraction in the proof of Lemma 1.

see Figure 12. The notation $\{\cdot\}_x$ means that the set should be restricted to the connected component including x. Hence, ε small implies that t in the set above belongs to a small interval around zero. Introduce the projection matrix $P_v := I - vv^T/(v^Tv)$. Then,

$$\mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x))) = \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x))(1 + O(\varepsilon))$$

Furthermore,

$$\begin{aligned} \mathcal{A}_1 := \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x)) &= \mathcal{A}(\mathcal{B}_\varepsilon(x))\cos\theta(x) \\ \mathcal{A}(P_{Ax+B}\mathcal{B}_\varepsilon(x)) &= \mathcal{A}(\mathcal{B}_\varepsilon(x))\cos\alpha(x) \end{aligned}$$

and

$$\frac{\cos\theta(x)}{\cos\alpha(x)} = \frac{|Ax + B|}{|Ax - B|}$$

where $\theta(x)$ is the angle of refraction and $\alpha(x)$ the angle of incidence for a trajectory passing through $\mathcal{S}$ at x. Introduce

$$\mathcal{A}_2 := \mathcal{A}(\Phi_-(h(x), H_-(x, \mathcal{B}_\varepsilon(x)))) \qquad \mathcal{A}_3 := \mathcal{A}(H_-(-g(x), -g(\mathcal{B}_\varepsilon(x))))$$

and let φ be the angle between the surfaces defined by $\mathcal{A}_2$ and $\mathcal{A}_3$ as in Figure 12, so that $\mathcal{A}_2 \geq \mathcal{A}_2\cos\varphi(x) = \mathcal{A}_3(1 + O(\varepsilon))$. Then, for $\varepsilon > 0$ sufficiently small

$$\begin{aligned} \mathcal{A}_2 &= \mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x)))\det e^{Ah(x)} \leq \mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x))) \\ &= \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x))(1 + O(\varepsilon)) \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{A}(P_{Ag(x)+B}\,g(\mathcal{B}_\varepsilon(x))) = \mathcal{A}(P_{A(-g(x))-B}(-g(\mathcal{B}_\varepsilon(x)))) &= \mathcal{A}_3(1 + O(\varepsilon)) \\ &\leq \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x))(1 + O(\varepsilon)) \end{aligned}$$

Since $\mathcal{X}$ is an invariant compact subset of $\mathcal{U}$, there exists $\kappa \in (0,1)$ such that $\cos\theta(x)/\cos\alpha(x) < \kappa$ for all $x \in \mathcal{X}$. Hence, there exists $\bar{\rho} \in (0,1)$ independent of x and $\varepsilon_x > 0$ depending on x so that

$$\mathcal{A}(P_{Ag(x)+B}\,g(\mathcal{B}_\varepsilon(x))) \leq \bar{\rho}\mathcal{A}(P_{Ax+B}\mathcal{B}_\varepsilon(x)), \quad \text{for all } x \in \mathcal{X}, \varepsilon < \varepsilon_x$$

For k mappings thus

$$\mathcal{A}(g^k(\mathcal{B}_\varepsilon(x)))\cos\theta(g^k(x)) = \mathcal{A}(P_{Ag^k(x)+B}\,g^k(\mathcal{B}_\varepsilon(x))) \leq \bar{\rho}^k\mathcal{A}(P_{Ax+B}\mathcal{B}_\varepsilon(x))$$

which gives that there exists $c \in \mathbb{R}$ such that

$$\mathcal{A}(g^k(\mathcal{B}_\varepsilon(x))) \leq c\bar{\rho}^k$$

The proof is completed by noting that it is possible to select a finite number of these discs $\mathcal{B}_\varepsilon$ to cover any compact set. □

The following three lemmas are used in the proof of Lemma 2. It is assumed that A, B, C satisfy (10) and without restriction $A = -\operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. We use the notation $\phi_-(t,x)$ for the trajectory of $\dot{x} = Ax - B$ at time t starting in x at time $t = 0$.

LEMMA 3

$$B^T A g(x) < 0, \qquad \text{for all } x \in \mathcal{S}_+$$

Proof: We show that for all $t > 0$

$$\phi_-(t, x_0) \notin \{x : Cx > 0, CAPx < 0, B^T APx < 0\} =: \mathcal{W}$$

where the projection matrix $P := P_{C^T} = I - C^T C/(CC^T)$. The set $\mathcal{W}$ is hence a cone in the state-space. Notice that $\{x \in \mathcal{S} : B^T Ax < 0, CAx < 0\}$ is a subset of $\mathcal{W}$. Cauchy-Schwartz' inequality on C^T and AC^T gives

$$CC^T \cdot CA^2 C^T > (CAC^T)^2$$

and on $(-A)^{1/2}C^T$ and $(-A)^{-1/2}C^T$ gives

$$CAC^T \cdot CA^{-1}C^T > (CC^T)^2$$

Thus, $CAPAC^T > 0$ and $CAPA^{-1}C^T < 0$. There exists $\tau_i, \sigma_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$CAPA = \tau_1 C + \tau_2 CAP + \tau_3 B^T AP \tag{12}$$

$$B^T APA = \sigma_1 C + \sigma_2 CAP + \sigma_3 B^T AP \tag{13}$$

hold and $\tau_1, \sigma_1 > 0$ and $\tau_3, \sigma_2 < 0$. This follows from multiplying (12) from right by C^T and B and (13) by C^T and $A^{-1}C^T$:

$$\begin{aligned} 0 &< CAPAC^T = \tau_1 CC^T \\ 0 &< CA^2B = \tau_3 B^T AB \\ 0 &< CA^2B = \sigma_1 CC^T \\ 0 &= \sigma_1 CA^{-1}C^T + \sigma_2 CAPA^{-1}C^T \end{aligned}$$

The existence of τ_i, σ_i implies that for all x

$$\begin{cases} Cx > 0 \\ CAPx = 0 \\ B^T APx < 0 \end{cases} \Rightarrow \quad CAP\dot{x} \geq CAPAx > 0$$

and

$$\begin{cases} Cx > 0 \\ CAPx < 0 \\ B^T APx = 0 \end{cases} \Rightarrow \quad B^T AP\dot{x} > B^T APAx > 0$$

It thus holds that no trajectories enter $\mathcal{W}$ through neither the hyperplane $\{x : CAPx = 0\}$ nor $\{x : B^T APx = 0\}$. □

LEMMA 4
There exists $\varepsilon > 0$ such that

$$|g^2(x)| > \varepsilon, \qquad \text{for all } x \in \mathcal{S}_+$$

Proof: Given $\varepsilon_1 > 0$, assume that $|g(x)| < \varepsilon_1$. Lemma 3 gives that for all $x \in \mathcal{S}_+$ there exists $\tau \in \mathbb{R}$ such that $\tau CAg(x) + CA^2g(x) \geq 0$. It follows from Taylor expansion that

$$0 = C\phi_-(t, g(x)) = tCAg(x) + \frac{t^2}{2}CA^2g(x) + O(t^3)$$

or

$$0 = CAg(x) + \frac{t}{2}CA^2g(x) + O(t^2) \geq \left(1 - \frac{t\tau}{2}\right)CAg(x) + O(t^2)$$

For sufficiently small t this contradicts the inequality $CAg(x) > 0$. Hence, there exists a $t_0 > 0$ such that $h(g(x)) > t_0$ for all $g(x)$ satisfying $|g(x)| < \varepsilon_1$. Furthermore, since $A < -\lambda_1 I$, it follows that

$$\begin{aligned}|g^2(x)| &= |\phi_-(h(g(x)), g(x))| \geq |A^{-1}B| - |e^{Ah(g(x))}(g(x) - A^{-1}B)| \\ &\geq |A^{-1}B|(1 - e^{-\lambda_1 t_0}) - |g(x)| > \varepsilon_1\end{aligned}$$

for $\varepsilon_1 = |A^{-1}B|(1 - e^{-\lambda_1 t_0})/2$. Hence, $|g^2(x)| > \varepsilon_1$ if $|g(x)| < \varepsilon_1$. If instead $|g(x)| \geq \varepsilon_1$, assume $g^2(x) = 0$. Then

$$Cg(x) = C(I - e^{-Ah(g(x))})A^{-1}B = \int_0^{h(g(x))} Ce^{-At}B\,dt \tag{14}$$

and for $t > 0$

$$\begin{aligned}Ce^{-At}B &= \mathcal{L}^{-1}\left\{K \cdot \frac{1}{s - \lambda_1} \cdot \frac{1}{s - \lambda_2} \cdot \frac{1}{s - \lambda_3}\right\} \\ &= Ke^{\lambda_1 t} * e^{\lambda_2 t} * e^{\lambda_3 t} > 0\end{aligned}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform and $*$ convolution. The right hand side of (14) is thus a strictly increasing function in $h(g(x))$, so we have a contradiction. By continuity we get that there exists $\varepsilon_2 > 0$ such that $|g(x)| \geq \varepsilon_1$ implies $|g^2(x)| \geq \varepsilon_2$. The proof is completed by choosing $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. $\square$

LEMMA 5
There exists $\delta > 0$ such that

$$CAg^3(x) > \delta, \qquad \text{for all } x \in \mathcal{S}_+$$

Proof: Consider a point $g_0 \in \mathcal{S}$ such that $CAg_0 = 0$ and $CA^2g_0 > 0$, and thus $B^TAg_0 < 0$. Then, for $-g_0$ it holds that

$$C\phi_-(t, -g_0) = -\frac{t^2}{2}CA^2g_0 + O(t^3)$$

so that $C\phi_-(-t_0, -g_0) < 0$ for $t_0 > 0$ sufficiently small. For such t_0, we have $C\phi_-(-t_0, -g(z)) < 0$ and $h(z) < t_0$ for $z \in \mathcal{S}_+$ with $|g(z) - g_0|$ sufficiently small. Hence, $h(z) \to 0$ and $B^T Az \to -B^T Ag_0 > 0$ as $|g(z) - g_0| \to 0$. In particular, there is a disc $\mathcal{D} \subset \mathcal{S}$ around g_0 such that $B^T Az > 0$ if $g(z) \in \mathcal{D}$. Moreover, for all $\varepsilon > 0$ and $R > \varepsilon$, it is possible to cover the line

$$\{x \in \mathcal{S} : CAx = 0, B^T Ax < 0, \varepsilon \le |x| \le R\}$$

with a finite number of such discs $\mathcal{D}_k$, $k = 1, \ldots, N$. From Lemma 4 we know that there exists $\varepsilon > 0$ so that $|g^2(x)| > \varepsilon$. Now assume that there exists a k such that $g^3(x) \in \mathcal{D}_k$. Then, $B^T Ag^2(x) > 0$. Lemma 3 says however that this cannot be true. We have a contradiction and the proof is complete. □

Proof of Lemma 2: The existence of R is trivial and the existence of ε is an immediate consequence of Lemmas 3–5. □