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A Convergence Proof for Relay Feedback Systems

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Abstract

Lemmas 1 and 2 in Johansson *et al.* (1997) are proved.

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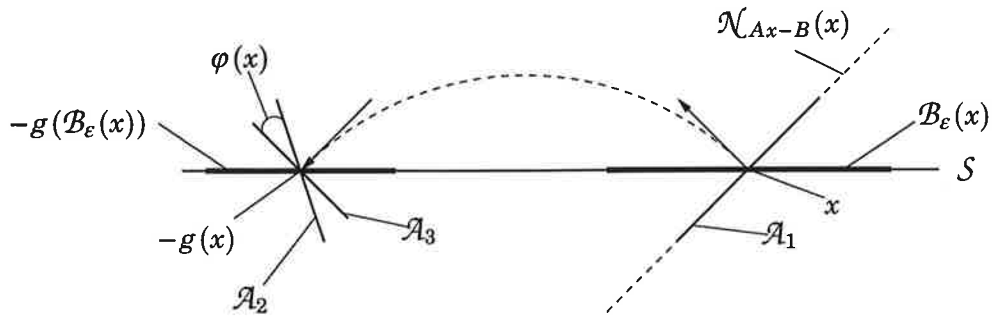

Figure 1 Illustration of area contraction in the proof of Lemma 1.

Lemmas 1 and 2 stated in Johansson *et al.* (1997) (and below) are proved in this report. These proofs complete the proof of Theorem 2 in Johansson *et al.* (1997) on trajectory convergence for relay feedback systems. Recall the paper for notations.

LEMMA 1
Assume A is stable and $CB = 0$. Then, g is area contractive in every invariant compact subset of

$$\mathcal{U} := \{x \in \mathcal{S}_+ : B^T Ax \leq 0\}$$

Proof: Consider the switch plane intersection x in a compact invariant set $\mathcal{X} \subset \mathcal{U}$ and denote the surrounding ball

$$\mathcal{B}_\varepsilon(x) := \{z \in \mathcal{X} : |z - x| \leq \varepsilon\}$$

Let $\Phi_-(t, \mathcal{B})$ be the set $\mathcal{B}$ after time t following the dynamics $\dot{x} = Ax - B$. The trajectories intersecting $\mathcal{B}_\varepsilon(x)$ pass through a hyperplane $\mathcal{N}_v(x) := \{z : v^T(z - x) = 0\}$. In particular, define

$$H_-(x, \mathcal{B}_\varepsilon(x)) := \{\mathcal{N}_{Ax-B}(x) \cap \Phi_-(t, \mathcal{B}_\varepsilon(x)), t \in \mathbb{R}\}_x$$

see Figure 1. The notation $\{\cdot\}_x$ means that the set should be restricted to the connected component including x. Hence, ε small implies that t in the set above belongs to a small interval around zero. Introduce the projection matrix $P_v := I - vv^T/(v^T v)$. Then,

$$\mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x))) = \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x))(1 + O(\varepsilon))$$

Furthermore,

$$\begin{aligned} \mathcal{A}_1 := \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x)) &= \mathcal{A}(\mathcal{B}_\varepsilon(x)) \cos\theta(x) \\ \mathcal{A}(P_{Ax+B}\mathcal{B}_\varepsilon(x)) &= \mathcal{A}(\mathcal{B}_\varepsilon(x)) \cos\alpha(x) \end{aligned} \tag{1}$$

and

$$\frac{\cos\theta(x)}{\cos\alpha(x)} = \frac{|Ax + B|}{|Ax - B|}$$

where $\theta(x)$ is the angle of refraction and $\alpha(x)$ the angle of incidence for a trajectory passing through $\mathcal{S}_+$ at x. Then, $\cos\theta(x)/\cos\alpha(x) < 1$ for all $x \in \mathcal{U}$ since

$$\frac{\cos^2\theta(x)}{\cos^2\alpha(x)} = \frac{x^T A^T Ax + B^T B + 2B^T Ax}{x^T A^T Ax + B^T B - 2B^T Ax} < 1$$

Introduce

$$\mathcal{A}_2 := \mathcal{A}(\Phi_-(h(x), H_-(x, \mathcal{B}_\varepsilon(x)))) \qquad \mathcal{A}_3 := \mathcal{A}(H_-(-g(x), -g(\mathcal{B}_\varepsilon(x))))$$

and let φ be the angle between the surfaces defined by $\mathcal{A}_2$ and $\mathcal{A}_3$ as in Figure 1, so that $\mathcal{A}_2 \geq \mathcal{A}_2 \cos\varphi(x) = \mathcal{A}_3(1 + O(\varepsilon))$. Then, for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}\mathcal{A}_2 &= \mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x))) \det e^{Ah(x)} \leq \mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x))) \\ &= \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x))(1 + O(\varepsilon))\end{aligned}$$

where $|\det\exp(Ah(x))| < 1$ since A is a Hurwitz matrix. Hence,

$$\begin{aligned}\mathcal{A}(P_{Ag(x)+B}g(\mathcal{B}_\varepsilon(x))) = \mathcal{A}(P_{A(-g(x))-B}(-g(\mathcal{B}_\varepsilon(x)))) &= \mathcal{A}_3(1 + O(\varepsilon)) \\ \leq \mathcal{A}(P_{Ax-B}\mathcal{B}_\varepsilon(x))(1 + O(\varepsilon))\end{aligned} \tag{2}$$

Since $\mathcal{X}$ is an invariant compact subset of $\mathcal{U}$, there exists $\kappa \in (0, 1)$ such that $\cos\theta(x)/\cos\alpha(x) < \kappa$ for all $x \in \mathcal{X}$. Hence, from (1) and (2) we have that there exists $\bar{\rho} \in (0, 1)$ independent of x and $\varepsilon_x > 0$ depending on x so that

$$\mathcal{A}(P_{Ag(x)+B}g(\mathcal{B}_\varepsilon(x))) \leq \bar{\rho}\mathcal{A}(P_{Ax+B}\mathcal{B}_\varepsilon(x)), \quad \text{for all } x \in \mathcal{X}, \varepsilon < \varepsilon_x$$

For k mappings thus

$$\mathcal{A}(g^k(\mathcal{B}_\varepsilon(x)))\cos\theta(g^k(x)) = \mathcal{A}(P_{Ag^k(x)+B}g^k(\mathcal{B}_\varepsilon(x))) \leq \bar{\rho}^k\mathcal{A}(P_{Ax+B}\mathcal{B}_\varepsilon(x))$$

which gives that there exists $\bar{c} > 0$ such that

$$\mathcal{A}(g^k(\mathcal{B}_\varepsilon(x))) \leq \bar{c}\bar{\rho}^k$$

The proof is completed by noting that it is possible to select a finite number of these discs $\mathcal{B}_\varepsilon$ to cover any compact set. □

The following three lemmas are used in the proof of Lemma 1. It is assumed that A, B, C satisfy

$$C(sI - A)^{-1}B = \frac{K}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)}, \qquad K > 0, \quad \lambda_3 > \lambda_2 > \lambda_1 > 0$$

and without restriction $A = -\operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. We use the notation $\phi_-(t, x)$ for the trajectory of $\dot{x} = Ax - B$ at time t starting in x at time $t = 0$.

LEMMA 3
$$B^TAg(x) < 0, \qquad \text{for all } x \in \mathcal{S}_+$$

Proof: We show that for all $t > 0$,

$$\phi_-(t, x_0) \notin \{x : Cx > 0, CAPx < 0, B^TAPx < 0\} =: \mathcal{W}$$

where the projection matrix $P := P_{C^T} = I - C^TC/(CC^T)$. The set $\mathcal{W}$ is hence an $\mathbb{R}^2$ cone in $\mathbb{R}^3$. Notice that $\{x \in \mathcal{S} : B^TAx < 0, CAx < 0\}$ is a subset of $\mathcal{W}$. Cauchy-Schwartz' inequality on C^T and AC^T gives

$$CC^T \cdot CA^2C^T > (CAC^T)^2$$

and on $(-A)^{1/2}C^T$ and $(-A)^{-1/2}C^T$ gives

$$CAC^T \cdot CA^{-1}C^T > (CC^T)^2$$

Thus, $CAPAC^T > 0$ and $CAPA^{-1}C^T < 0$. There exist $\tau_i, \sigma_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$CAPA = \tau_1 C + \tau_2 CAP + \tau_3 B^T AP \tag{3}$$

$$B^T APA = \sigma_1 C + \sigma_2 CAP + \sigma_3 B^T AP \tag{4}$$

hold and $\tau_1, \sigma_1 > 0$ and $\tau_3, \sigma_2 < 0$. This follows from multiplying (3) from right by C^T and B and (4) by C^T and $A^{-1}C^T$:

$$\begin{aligned} 0 &< CAPAC^T = \tau_1 CC^T \\ 0 &< CA^2B = \tau_3 B^T AB \\ 0 &< CA^2B = \sigma_1 CC^T \\ 0 &= \sigma_1 CA^{-1}C^T + \sigma_2 CAPA^{-1}C^T \end{aligned}$$

The existence of τ_i implies that for all x,

$$\left\{ \begin{array}{l} Cx > 0 \\ CAPx = 0 \\ B^T APx < 0 \end{array} \right. \quad \Rightarrow \quad CAP\dot{x} \geq CAPAx > 0$$

and the existence of σ_i implies that for all x,

$$\left\{ \begin{array}{l} Cx > 0 \\ CAPx < 0 \\ B^T APx = 0 \end{array} \right. \quad \Rightarrow \quad B^T AP\dot{x} > B^T APAx > 0$$

It thus holds that no trajectories enter $\mathcal{W}$ through neither the hyperplane $\{x : CAPx = 0\}$ nor $\{x : B^T APx = 0\}$. □

The following lemma follows from Anosov (1959) and is stated without proof. Notice that for states close to the origin, $\{A, B, C\}$ is approximately equal to a triple integrator. In Section 5 in Johansson *et al.* (1997), it was shown that the origin is unstable for a triple integrator under relay feedback.

LEMMA 4
There exists $\varepsilon > 0$ such that if $|x| < \varepsilon$ with $x \in \mathcal{S}_+$, then $|g(x)| > \varepsilon$. □

Next, we prove that there exist no arbitrarily fast relay switches in the region $\{x \in \mathcal{S}_+ : B^T Ax < 0, |x| > \varepsilon\}$.

LEMMA 5
There exists $\delta > 0$ such that

$$CAg^2(x) > \delta, \qquad \text{for all } x \in \mathcal{S}_+$$

Proof: Consider a point $g_0 \in \mathcal{S}$ such that $CAg_0 = 0$ and $CA^2g_0 > 0$, and thus $B^TAg_0 < 0$. Then, for $-g_0$ it holds that

$$C\phi_-(t, -g_0) = -\frac{t^2}{2}CA^2g_0 + O(t^3)$$

so that $C\phi_-(-t_0, -g_0) < 0$ for $t_0 > 0$ sufficiently small. For a fixed such t_0, we have $C\phi_-(-t_0, -g(z)) < 0$ and $h(z) < t_0$ for $z \in \mathcal{S}_+$ with $|g(z) - g_0|$ sufficiently small. Hence, $h(z) \to 0$ and $B^TAz \to -B^TAg_0 > 0$ as $|g(z) - g_0| \to 0$. In particular, there is a disc $\mathcal{D} \subset \mathcal{S}$ around g_0 such that $B^TAz > 0$ if $g(z) \in \mathcal{D}$. Moreover, for all $\varepsilon > 0$ and $R > \varepsilon$, it is possible to cover the line

$$\{x \in \mathcal{S} : CAx = 0, B^TAx < 0, \varepsilon \le |x| \le R\}$$

with a finite number of such discs $\mathcal{D}_k$, $k = 1, \dots, N$. From Lemma 4 we know that there exists $\varepsilon > 0$ so that $|g(x)| > \varepsilon$. Now assume that there exists a k such that $g^2(x) \in \mathcal{D}_k$. Then, $B^TAg(x) > 0$. Lemma 3 says, however, that this cannot be true. We have a contradiction and the proof is complete. □

The proof of Lemma 2 now follows.

LEMMA 2
For A, B, C as in

$$C(sI - A)^{-1}B = \frac{K}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)}, \qquad K > 0, \quad \lambda_3 > \lambda_2 > \lambda_1 > 0 \tag{5}$$

there exist $\varepsilon, R > 0$ such that

$$g^k(x) \in \{x \in \mathcal{S}_+ : B^TAx \le 0, CAx > \varepsilon, |x| \le R\} =: \mathcal{V} \tag{6}$$

for all $x \in \mathcal{S}_+$ and $k \ge 2$.

Proof: The existence of R follows from that A is stable. The existence of ε is an immediate consequence of Lemmas 3 and 5. □

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