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ON CAUSAL AND STABLE SOLUTIONS TO RATIONAL MATRIX EQUATIONS WITH APPLICATIONS TO CONTROL THEORY

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ON CAUSAL AND STABLE SOLUTIONS TO RATIONAL MATRIX EQUATIONS WITH APPLICATIONS TO CONTROL THEORY*)

by

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1. INTRODUCTION

Linear matrix equations of the type

$$T(s)K(s) = H(s) \tag{1.1}$$

where $T(s)$, $K(s)$ and $H(s)$ are matrices with elements in $R(s)$, the field of rational functions in s with real coefficients, arise in several important control problems. In (1.1), it is assumed that $T(s)$ and $H(s)$ are given and that $K(s)$ is unknown. In fact, (1.1) represents the transfer matrix formulation of such problems as the general noninteraction problem, the model matching problem and the disturbance localization problem. Equations of the type (1.1) also appear in observer problems. This fact is demonstrated in detail in Section 6, where we show that these problems can be posed and solved as a problem of the type (1.1). Therefore, if we are able to characterize the admissible solution set (defined below) for (1.1) we also have a common characterization of the set of all solutions to these control problems. This type of question is of specific importance in synthesis, where possible non-uniqueness of the solution can be exploited to satisfy various additional design criteria which may be awkward to formalize in advance.

In control applications $T(s)$, $K(s)$ and $H(s)$ represent the external descriptions of the open-loop system, the controller and the closed-loop system respectively. Since we are only dealing with causal systems we assume that $T(s)$ and $H(s)$ are strictly proper and insist that $K(s)$ be proper. We see from (1.1) that this is no restriction with regard to $T(s)$ and $H(s)$ since (1.1) can be divided by s^k, with k large enough to make both $T(s)$ and $H(s)$ strictly proper.

Moreover, $K(s)$ is the external (feedforward) description of the control and should be implementable as an internally stable feedback control. It is shown in [1] that this restriction is essentially the same as requiring that $K(s)$ be a stable rational matrix, i.e. having all its poles within a specified region of the complex plane. The causality and stability restrictions on $K(s)$ make the problem unconventional from a mathematical point of view. Some partial results for this type of problem have been given in [4] and [10].

In this paper we are concerned with the existence, construction and characterization of solutions to (1.1) of this restricted type. Besides being able to give general answers to these questions, we will also relate the solvability of (1.1) to basic state space concepts such as supremal invariant and controllability subspaces [11] and the associated state feedback transformations. In this way we obtain new insight into these concepts in a frequency domain setting.

Our results will be stated in terms of a version of (1.1) of the form

$$C(s-A)^{-1}\left(BK(s) + E\right) = 0 \tag{1.2}$$

where A, B, C and E are real matrices. That (1.2) is equivalent to (1.1) is easily seen in the following way: rewrite (1.1) as

$$\begin{bmatrix} T(s) & -H(s) \end{bmatrix} \begin{pmatrix} K(s) \\ I \end{pmatrix} = 0 \tag{1.3}$$

and let $(C,A,\tilde{B})$ be a state realization of $\tilde{T}(s) = \begin{bmatrix} T(s) & -H(s) \end{bmatrix}$. Partition $\tilde{B} = \begin{bmatrix} B & E \end{bmatrix}$ compatibly with the blocks in $\tilde{T}(s)$. Substitution of

$$\tilde{T}(s) = C(s-A)^{-1}\begin{bmatrix} B & E \end{bmatrix}$$

into (1.3) yields the equivalence.

The paper is organized as follows. Technical preliminaries are given in Sec. 2. The solvability question is resolved in Sec.3. A characterization of the entire stable and proper solution set is given in Sec. 4. Some control applications are presented in Sec. 5.

Notations.

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$ denote the fields of real numbers, complex numbers and rational functions with real coefficients respectively. Script letters $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are used for real vector spaces of finite dimensions and capital Roman letters A, B, C denote linear maps (real matrices). The image of A is written Im A or sometimes $\mathcal{A}$. The kernel of A is written Ker A.

Let $A:\mathcal{X} \to \mathcal{X}$ and $B:\mathcal{U} \to \mathcal{X}$ be a pair of linear maps. The controllable subspace for (A,B) is $\langle A|\mathcal{B}\rangle = \mathcal{B} + \ldots + A^{n-1}\mathcal{B}$. A subspace $\mathcal{V}$ is said to be (A,B)-invariant if there is a linear map F such that $(A+BF)\mathcal{V} \subset \mathcal{V}$. A subspace $\mathcal{R}$ is said to be an (A,B)-controllability subspace if $\mathcal{R} = \langle A+BF|\mathcal{B}\cap\mathcal{R}\rangle$ for some linear map F. The family of all maps F such that $(A+BF)\mathcal{V} \subset \mathcal{V}$ is written $\underline{F}(\mathcal{V})$. The families of subspaces $\mathcal{I}(A,B,\mathcal{D}) = \{\mathcal{V} | \mathcal{V} \subset \mathcal{D}$ and $(A+BF)\mathcal{V} \subset \mathcal{V}$ some $F\}$ and $\mathcal{C}(A,B,\mathcal{D}) = \{\mathcal{R} | \mathcal{R} \subset \mathcal{D}$ and $\mathcal{R} = \langle A+BF | \mathcal{B} \cap \mathcal{R}\rangle$ some $F\}$ have unique supremal elements denoted sup $\mathcal{I}(A,B,\mathcal{D})$ and sup $\mathcal{C}(A,B,\mathcal{D})$ respectively [11].

Let $A:\mathcal{X} \to \mathcal{X}$ be a linear map, $\alpha(s)$ its minimal polynomial and

$$\mathbb{C} = \mathbb{C}^{+} \cup \mathbb{C}^{-} \tag{1.4}$$

an arbitrary disjoint partition of the complex plane such that $\mathbb{C}^{-}$ is symmetric with respect to the real axis and contains at least one real point. Then we define

$$\mathcal{X}^{\pm}(A) = \operatorname{Ker} \alpha^{\pm}(A)$$

where $\alpha = \alpha^{+}\alpha^{-}$ is a factorization of α corresponding to the partition (1.4). A rational function $r(s)$ is said to be proper if $r(\infty) < \infty$ and strictly proper if $r(\infty) = 0$. It is stable if $r(s) = q(s)/p(s)$, where $q(s)$ and $p(s)$ are relatively prime, and $p(s)$ has all its zeros within $\mathbb{C}^{-}$. A rational matrix $R(s)$ is said to be proper, strictly proper or stable respectively, if each element of $R(s)$ has the stated property.
A more detailed description of concepts and basic results in the geometric state space theory can be found e.g. in [11].

2. PRELIMINARIES

2.1. Feedback Realizations.

In (1.2), K(s) is regarded as the external feedforward description of a feedback control. To define the feedback control internally, let the system Σ be

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx \end{aligned} \tag{2.1}$$

where $x \in \mathcal{X}(\approx \mathbb{R}^n)$ is the state vector, $u \in \mathcal{U}(\approx \mathbb{R}^r)$ is the vector of control inputs, $y \in \mathcal{Y}(\approx \mathbb{R}^q)$ the output vector, and $w \in \mathcal{W}(\approx \mathbb{R}^m)$ is the vector of external inputs. Here, A, B, C and E are linear maps (matrices) between the appropriate vector spaces. We assume that B is of full column rank and that (A,B) is stabilizable. For this system, we define a dynamic feedback control in the following way. Let

$$\dot{x}_a = B_a u_a, \qquad x_a \in \mathcal{X}_a, \qquad u_a \in \mathcal{U}_a \tag{2.2}$$

be a set of integrators adjoined to (2.1), with B_a an isomorphism. Introducing the extended state vector $x_e := (x, x_a)$ and the extended input vector $u_e := (u, u_a)$, we may write (2.1) and (2.2) together as

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u_e + E_e w \\ y_e &= C_e x_e \end{aligned} \tag{2.3a}$$

Here

$$A_e = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; \qquad B_e = \begin{bmatrix} B & 0 \\ 0 & B_a \end{bmatrix}; \qquad E_e = \begin{bmatrix} E \\ 0 \end{bmatrix};$$

$$C_e = [C \quad 0]; \qquad x_e \in \mathcal{X}_e := \mathcal{X} \oplus \mathcal{X}_a; \tag{2.3b}$$

$$u_e \in \mathcal{U}_e := \mathcal{U} \oplus \mathcal{U}_a; \; n_a := \dim \mathcal{X}_a$$

By a dynamic feedback control for (2.1) we then mean a state feedback control

$$u_e = F_e x_e + G_e w \tag{2.4}$$

for the extended system (2.2). This is a standard dynamic extension technique used e.g. in the geometric state space theory [11]. The input/output transfer matrix relating u to w is readily computed from (2.3) - (2.4) as

$$K(s) = QF_e(s-A_e-B_eF_e)^{-1}(B_eG_e+E_e) + QG_e \tag{2.5}$$

where $Q: \mathcal{U}_e \to \mathcal{U}$ is the projection on $\mathcal{U}$ along $\mathcal{U}_a$. Conversely, if (2.1) and K(s) are given, any triple (F_e, G_e, n_a), related to (2.1) as in (2.3b) and (2.4) and satisfying (2.5), is said to be a <u>feedback realization</u> of $\{\Sigma, K(s)\}$. We see that different feedback realizations merely correspond to state coordinatizations of

$$u(s) = K(s)w(s) \tag{2.6}$$

as a feedback control to (2.1). The internal stability of the closed loop system is determined by the spectrum of the map $A_e^* = A_e + B_eF_e$. A feedback realization (F_e, G_e, n_a) which is such that A_e^* has its spectrum within the specified region $\mathbb{C}^-$ of the complex plane, cf. (1.1), is said to be <u>internally stable w.r.t. $\mathbb{C}^-$</u>. The relationship between K(s) and the corresponding feedback realizations (F_e, G_e, n_a) is explained in [1], from which we take the following result.

<u>Theorem 2.1. There exists an internally stable feedback realization of</u> $\{\Sigma, K(s)\}$ <u>iff</u>

(i) K(s) <u>is proper</u>

(ii) $P(s) := (s-A)^{-1}\{BK(s) + E\}$ <u>is stable w.r.t.</u> $\mathbb{C}^-$. □

If we insist on feedback realizations of the form $(F_e, 0, n_a)$, which are appropriate if $w(\cdot)$ is assumed not accessible to measurement, Th. 2.1 still holds if "proper" is replaced by "strictly proper". Also, if only output feedback is allowed from, say, $z = Hx$ with (H,A) an observable pair, Th. 2.1 holds unchanged. Moreover, the stability of $P(s)$ is related to the closed-loop transfer matrix in the following way [1].

Theorem 2.2. Assume (C,A) is observable. Then $P(s)$ is stable w.r.t. $\mathbb{C}^-$ iff $K(s)$ and $T_*(s)$ are both stable w.r.t. $\mathbb{C}^-$, where $T_*(s)$ is the closed loop transfer matrix $T_*(s) := C(s-A)^{-1} \cdot \cdot \left(BK(s) + E\right)$.

The usefulness of these results resides in the fact that internal stability can be viewed as a property of $K(s)$. For more results in this direction, and on constructions of feedback realizations, see [1].

2.2. Restatement of the Problem.

In view of the discussion above we accept solutions $K(s)$ to (1.2) which have an internally stable feedback implementation, i.e. we require

(a) $P(s) := (s-A)^{-1}\left(BK(s) + E\right)$ is a stable rational matrix w.r.t. $\mathbb{C}^-$. (2.7)

If in addition (C,A) is observable (a) is equivalent to

(a') $K(s)$ is a stable rational matrix w.r.t. $\mathbb{C}^-$ (2.8)

since $T_*(s)$ introduced in Theorem 2.2 is zero in this case (1.2). Let us call a solution $K(s)$ satisfying (a) or (a') internally stable.

To summarize, our problem is: to find a proper (strictly proper) and internally stable solution K(s) to (1.2).

In view of the discussion on feedback realizations, we may view a solution either as a rational matrix K(s) or as a triple (F_e, G_e, n_a) which is a feedback realization of K(s). The problem can easily be restated in terms of feedback realizations using the following lemma.

Lemma 2.1. Let (C,A,B) denote a system with transfer matrix $T(s) := C(s-A)^{-1}B$. Then

(i) $T(s) = 0$ iff $\langle A|\mathcal{B}\rangle \subset \operatorname{Ker} C$

(ii) T(s) is stable w.r.t. $\mathbb{C}^-$ iff $\langle A|\mathcal{B}\rangle \cap X^+(A) \subset \operatorname{Ker} C$.

Proof. Follows from Kalman's structure theorem [5]. □

The properties of K(s) can now be expressed as properties of an arbitrary feedback realization. For convenience, introduce

$$
\begin{aligned}
A_e^* &:= A_e + B_e F_e \\
B_e^* &:= B_e G_e + E_e
\end{aligned}
\tag{2.9}
$$

which are obtained as the closed loop system matrices when (2.4) is substituted in (2.3a).

Proposition 2.1. Let K(s) be a proper rational matrix and (F_e, G_e, n_a) an arbitrary feedback realization of $(\Sigma, K(s))$. Then

(i) K(s) <u>solves</u> (1.2) <u>iff</u>

$$\mathcal{R}_e := \langle A_e^* | \mathcal{B}_e^* \rangle \subset \operatorname{Ker} C_e \tag{2.10}$$

(ii) K(s) <u>is stable iff</u>

$$\mathcal{R}_e^+ := \mathcal{R}_e \cap \mathcal{X}_e^+(A_e^*) \subset \operatorname{Ker} C_e \tag{2.11}$$

(iii) K(s) <u>is strictly proper iff</u>

$$QG_e = 0 \tag{2.12}$$

<u>where Q is as in</u> (2.5).

<u>Proof</u>. Here, (i) follows from Lemma 2.1(i) since

$$0 = C(s-A)^{-1}\left(BK(s) + E\right) = C_e(s-A_e^*)^{-1}B_e^*$$

where the second equality follows from (2.3-5). The statement (ii) follows from Lemma 2.1(ii) since

$$P(s) = (s-A)^{-1}\left(BK(s) + E\right) = P_x(s-A_e^*)^{-1}B_e^*$$

where $P_x:\mathcal{X}_e \to \mathcal{X}$ is the projection on $\mathcal{X}$ along $\mathcal{X}_a$. Finally, (iii) is immediate from (2.5). □

3. GENERAL SOLVABILITY

In the statements of the theorems below, we need just a few algebraic concepts associated with Σ. Introduce the following subspaces

$$\mathcal{W}^* := \bigcap_{i=1}^{n} \operatorname{Ker} CA^{i-1} \left(= \sup \mathcal{I}(A,0,\operatorname{Ker} C)\right)$$

$$\mathcal{V}^* := \sup \mathcal{I}(A,B,\operatorname{Ker} C)$$

$$\mathcal{R}^* := \sup \mathcal{C}(A,B,\operatorname{Ker} C) \tag{3.1}$$

$$\mathcal{B} := \operatorname{Im} B; \quad \mathcal{E} = \operatorname{Im} E$$

These subspaces have the following systematic interpretation. The subspace $\mathcal{W}^*$ is the unobservable subspace for Σ. The subspace $\mathcal{V}^*$ is the supremal unobservable subspace obtained by the transformation $(C,A) \overset{F}{\to} (C,A+BF)$. The subspace $\mathcal{R}^*$ is the reachable set of states in Σ for inputs $u(\cdot)$ which keep the output zero, [9]. Note that $\mathcal{V}^*$ and $\mathcal{R}^*$ are easily constructed, cf. e.g. [11].

3.1. Nondynamic Solutions.

Let us first treat the case with nondynamic or constant $K(s)$, i.e. $K(s) = K$ where K is a real matrix. In view of Proposition 2.1, an equivalent problem is to find a triple (F_e, G_e, n_a) with $F_e = 0$, $G_e = K$ and $n_a = 0$ so that (2.10) and (2.11) hold.

<u>Theorem 3.1.</u> <u>There is a constant, internally stable solution</u> K <u>to</u> (1.2) <u>iff</u>

$$\mathcal{W}^- + \mathcal{B} \supset \mathcal{E}$$

<u>where</u>

$\mathcal{W}^- := \mathcal{X}^-(A) \cap \mathcal{W}^*$

The following lemma is needed.

<u>Lemma 3.1. There is a linear map</u> K <u>such that</u> $\mathrm{Im}(BK+E) \subset \mathcal{V}$ <u>iff</u> $\mathcal{V} + \mathcal{B} \supset \mathcal{E}$.

<u>Proof</u>. (If) Let w_i, $i \in \underline{d} := (1, 2, \ldots, d)$, be a basis for $\mathcal{W}$. There are $v_i \in \mathcal{V}$ and $u_i \in \mathcal{U}$ such that $Ew_i = v_i + Bu_i$. Define K such that $Kw_i = -u_i$. K exists since w_i is a basis. Then, $(BK+E)w_i = v_i$ and therefore $\mathrm{Im}(BK+E) \subset \mathcal{V}$.

(Only if) Let w_i, $i \in \underline{d}$, be the same basis as above. There are $v_i \in \mathcal{V}$ such that $(BK+E)w_i = v_i$, $i \in \underline{d}$; hence, $Ew_i = -BKw_i + v_i \in$ $\in \mathcal{V} + \mathcal{B}$. □

<u>Proof of Theorem 3.1</u>. (If) By Lemma 3.1 there is K such that $\mathrm{Im}(BK+E) \subset \mathcal{W}^-$. Then K solves the problem since, with the same notation as in Proposition 2.1, $\mathcal{R}_e = \langle A|\mathrm{Im}(BK+E)\rangle \subset \langle A|\mathcal{W}^-\rangle \subset$ $\subset \mathrm{Ker}\, C$. Also, $\mathcal{R}_e \cap \mathcal{X}^+(A) \subset \mathcal{W}^- \cap \mathcal{X}^+(A) = 0$, i.e. both (2.10) and (2.11) are satisfied.

(Only if) Assume K is a solution. Then $\mathcal{R}_e = \langle A|\mathrm{Im}(BK+E)\rangle$ and hence $\mathcal{R}_e \supset \mathrm{Im}(BK+E)$ and therefore $\mathcal{B} + \mathcal{R}_e \supset \mathcal{E}$ by Lemma 3.1. Moreover, $A\mathcal{R}_e \subset \mathcal{R}_e$ and $\mathcal{R}_e \subset \mathrm{Ker}\, C$, so $\mathcal{R}_e \subset \mathcal{W}^*$; we also have $\mathcal{R}_e \cap \mathcal{X}^+(A) = 0$, i.e. $\mathcal{R}_e \subset \mathcal{X}^-(A)$. Hence, $\mathcal{R}_e \subset \mathcal{X}^-(A) \cap \mathcal{W}^*$ and there follows $\mathcal{W}^- + \mathcal{B} \supset \mathcal{R}_e + \mathcal{B} \supset \mathcal{E}$. □

If we do not insist on stability with respect to $\mathbb{C}^-$, we may simply replace $\mathbb{C}^-$ by $\mathbb{C}$. From Theorem 3.1 there follows immediately (since $\mathcal{W}^- = \mathcal{W}^*$ in this case):

Corollary 3.1. There is a constant solution to (1.2) iff $\mathcal{W}^* + \mathcal{B} \supset \mathcal{E}$.

3.2. Dynamic Solutions.

The dynamic problem is to find a solution K(s) which is proper (or strictly proper) and internally stable. An equivalent statement is, by Prop. 2.1, to find a triple (F_e, G_e, n_a) such that (2.10) and (2.11) hold. If K(s) is restricted to be strictly proper, in addition (2.12) must hold with Q defined as in (2.5). The solvability question is resolved in

Theorem 3.2. There is a solution K(s) to (1.2) which is internally stable w.r.t. $\mathbb{C}^-$ iff

(i) $\quad \mathcal{V}^- + \mathcal{B} \supset \mathcal{E}$ if K(s) is restricted to be proper,

(ii) $\quad \mathcal{V}^- \supset \mathcal{E}$ if K(s) is restricted to be strictly proper,

where $\mathcal{V}^- := \mathcal{V}^* \cap \mathcal{X}^-(A+BF) + \mathcal{R}^*$. Here the choice of $F \in \underline{\mathbf{F}}(\mathcal{V}^*)$ is arbitrary.

Proof. (If) Take $F \in \underline{F}(\mathcal{V}^*)$ such that the spectrum of $(A+BF)|\mathcal{R}^*$ is assigned to $\mathbb{C}^-$. This is always possible (see [11], Thm. 5.1). Then $\mathcal{R}^* \subset \mathcal{X}^-(A+BF)$, and therefore

$$\begin{aligned} &\mathcal{V}^- = \mathcal{X}^-(A+BF) \cap \mathcal{V}^* \\ &(A+BF)\mathcal{V}^- \subset \mathcal{V}^- \end{aligned} \tag{3.2}$$

We treat separately the case when K(s) is strictly proper or merely proper.

(K(s) strictly proper): We claim that (F,0,0) is a solution, i.e. satisfies (2.10-12). Certainly, (2.12) is satisfied.

Since $\mathcal{V}^- \supset \mathcal{E}$, we also have $\mathcal{R}_e = \langle A+BF \mid \mathcal{E}\rangle \subset \langle A+BF \mid \mathcal{V}^-\rangle = \mathcal{V}^- \subset$ $\subset \operatorname{Ker} C_e$, and therefore (2.10) holds; also, $\mathcal{R}_e \cap X^+(A+BF) \subset$ $\subset \mathcal{V}^- \cap X^+(A+BF) = 0$, so (2.11) holds.

(K(s) proper): Since $\mathcal{V}^* = \sup \mathcal{I}(A+BF,0,\operatorname{Ker} C)$, there follows from Thm. 3.1 by the substitution $A \to A+BF$ that there is a G, such that (2.10) and (2.11) hold. We have now constructed a solution (F,G,0). K(s) is then obtained from (2.5).

(Only if) Let (F_e,G_e,n_a) be a solution and let $\mathcal{R}_e$ be as in Prop. 2.1; also let $\mathcal{R}_e^{\pm} := \mathcal{X}_e^{\pm}(A_e+B_eF_e)$. Moreover, let $P: \mathcal{X}_e \to \mathcal{X}$ be the projection on $\mathcal{X}$ along $\mathcal{X}_a$. By inspection of (2.2) it follows that P has the property

$$PA_e = AP; \qquad PB_e = BQ; \qquad PE_e = E; \qquad CP = C_e \tag{3.3}$$

where Q is defined as in (2.5). Let $\mathcal{V}_e^- := \mathcal{X}_e^-(A_e+B_e\tilde{F}_e) \cap \mathcal{V}_e^* + \mathcal{R}_e^*$ with $\tilde{F}_e \in \underline{F}(\mathcal{V}_e^*)$; the subscript indicates that (3.1) is computed for (A_e,B_e,C_e). Then

$$P\mathcal{V}_e^- = \mathcal{V}^- \tag{3.4}$$

which follows from [11], Lemma 10.6. Moreover, for any subspace $\mathcal{S} = \mathcal{V}_e \cap \mathcal{X}_e^-(A_e+B_eF_e)$ with $F_e \in \underline{F}(\mathcal{V}_e)$, we have $\mathcal{S} \subset \mathcal{V}_e^-$, (see [11] Lemma 5.8). The remainder of the proof is now straightforward.

(K(s) proper): We have $\mathcal{R}_e = \mathcal{R}_e^+ \oplus \mathcal{R}_e^- \supset \operatorname{Im}(B_eG_e+E_e)$ and therefore by Lemma 3.1, $\mathcal{R}_e^- + \mathcal{R}_e^+ + \mathcal{B}_e \supset \mathcal{E}_e$. Hence, $\mathcal{V}_e^- + \mathcal{R}_e^+ + \mathcal{B}_e \supset \mathcal{E}_e$. Applying P to both sides and noting that $\operatorname{Ker} \mathcal{R}_e^+ \subset \mathcal{X}_a = \operatorname{Ker} P$ by (2.11), gives $\mathcal{V}^- + \mathcal{B} \supset \mathcal{E}$. The strictly proper case can be treated similarly. □

We note that if stability is not an issue we simply replace $\mathbb{C}^-$ by $\mathbb{C}$ in Theorem 3.2 and the result holds with $\mathcal{V}^-$ replaced by $\mathcal{V}^*$. We also see from the proof above that, if a solution exists, it can always be taken to be of the form (F,G,0) or (F,0,0). Hence, we have

<u>Corollary 3.2.</u> <u>If an internally stable solution of (1.2) (in the sense of Theorem 3.2) exists, it can always be taken to be of the form</u>

(i) $(F,G,0)$, <u>i.e.</u> $K(s) = F(s-A-BF)^{-1}(BG+E) + G$, <u>with real matrices</u> F <u>and</u> G, <u>if</u> K(s) <u>is restricted to be proper</u>

(ii) $(F,0,0)$, <u>i.e.</u> $K(s) = F(s-A-BF)^{-1}E$, <u>with a real matrix</u> F, <u>if</u> K(s) <u>is restricted to be strictly proper</u>.

By Corollary 3.2 we conclude that there exists an internally stable and proper solution to (1.2) [or (1.1)] iff there are real matrices F and G such that

$$C(s-A-BF)^{-1}(BG+E) = 0 \tag{3.5}$$

and such that the controllable spectrum of (A+BF,BG+E)+) is stable. Similarly, there exists an internally stable and strictly proper solution to (1.2) [or (1.1)] iff there is a real matrix F such that

$$C(s-A-BF)^{-1}E = 0 \tag{3.6}$$

and such that the controllable spectrum of (A+BF,E) is stable. The corresponding K(s) is given by (2.5). Solving a matrix equation of the form (1.2) [or (1.1)] is essentially the same problem as constructing a supremal (A,B)-invariant subspace with the associated feedback matrix. If in addition, stable solutions are desired we are also faced with a pole assignment problem, i.e. finding $F \in \underline{F}(V^*)$ such that $(A+BF)|V^*$ is stable. There are easy constructions for these purposes, see e.g. [11].

+) i.e. the spectrum of $(A+BF)|\langle A+BF|\mathrm{Im}(BG+E)\rangle$.

4. CHARACTERIZATION OF THE SOLUTION SET.

If $T(s)$ is a $q \times r$ rational matrix, the set of all rational vectors $t(s)$ such that

$$T(s)t(s) = 0 \tag{4.1}$$

is a subspace of $\mathbb{R}^r(s)$. In other words, the set of vectors $t(s)$ satisfying (4.1) is generated by a basis $\pi_i(s)$, $i \in \underline{d}$, where $d = r - \text{rank}\ T(s)$ and r is the number of columns in $T(s)$. Let

$$T(s) = C(s-A)^{-1}B \tag{4.2}$$

with A, B and C as in (1.2). Also let $\pi(s) = [\pi_1(s) \ \ldots \ \pi_d(s)]$. An arbitrary solution $K(s)$ to (1.2) can then be written as

$$K(s) = K_0(s) + \pi(s)Q(s) \tag{4.3}$$

where $K_0(s)$ is a specific solution and $Q(s)$ an arbitrary rational matrix. The characterization (4.3) is valid for arbitrary solutions. In our case, we are interested only in proper (or strictly proper) and stable solutions. It does not follow directly from standard algebra that this type of solution can be characterized in the form (4.3). However, by appropriate choices of $\pi(s)$ and $Q(s)$ we will in fact show that this is the case.

Let us first describe an appropriate basis $\pi(s)$.

<u>Theorem 4.1. Let</u> $\mathcal{V}^*$ <u>be as in</u> (3.1) <u>and take any real matrices</u> F <u>and</u> N, <u>where</u> N <u>is of full column rank, such that</u>

$$F \in \underline{F}(\mathcal{V}^*) \ ; \qquad \text{Im}(BN) = \mathcal{B} \cap \mathcal{V}^* \tag{4.4}$$

<u>Then the columns of</u>

$$\pi(s) := F(s-A-BF)^{-1}BN + N \qquad (4.5)$$

<u>form a basis for the solution set to</u> (4.1), <u>i.e. for</u> Ker T(s) <u>with</u> T(s) <u>regarded as a mapping</u> $\mathbb{R}^r(s) \to \mathbb{R}^q(s)$.

In order to prove this, we need

<u>Lemma 4.1</u>. <u>With</u> T(s) <u>as in</u> (4.2), <u>we have</u> rank $T(s) \geq r - \dim(\mathcal{B} \cap \mathcal{V}^*)$.

<u>Proof</u>. It is known that T(s) is left invertible, i.e. has full column rank, iff $\mathcal{V}^* \cap \mathcal{B} = 0$ and Ker B = 0, see [9]. The latter condition is satisfied by hypothesis. Let $d := \dim(\mathcal{B} \cap \mathcal{V}^*)$ and let $\mathcal{B}_1$ be a complement such that $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B} \cap \mathcal{V}^*$. Then $\dim \mathcal{B}_1 = r - d$. Take an $r \times (r-d)$ matrix K such that $\mathrm{Im}(BK) = \mathcal{B}_1$ and set $B_1 := BK$. It follows that B_1 has full column rank.

Also, it is easily verified that

$$\mathcal{V}^* = \sup \mathcal{I}(A, B_1, \mathrm{Ker}\, C)$$

$$\mathcal{V}^* \cap \mathcal{B}_1 = 0$$

and therefore T(s)K has full column rank. Hence, rank $T(s) \geq$ rank $T(s)K = r - d$ as desired. □

<u>Proof of Theorem 4.1</u>. First, $\mathrm{Im}(BN) = \mathcal{V}^* \cap \mathcal{B}$ and therefore $\langle A+BF \mid \mathrm{Im}(BN)\rangle = \mathcal{R}^* \subset \mathrm{Ker}\, C$ (see [11] Thm. 5.5). Hence,

$$C(s-A)^{-1}B\pi(s) = C(s-A-BF)^{-1}BN = 0$$

and therefore the columns of $\pi(s)$ belong to Ker T(s). Now,

$$\mathrm{rank}\, T(s) = r - \dim\big(\mathrm{Ker}\, T(s)\big)$$

and using Lemma 4.1, there follows

$\dim(\mathrm{Ker}\ T(s)) \leq \dim(\mathcal{B} \cap \mathcal{V}^*)$

Since $\pi(s)$ has $\dim(\mathcal{B} \cap \mathcal{V}^*)$ columns by construction, it suffices to show that they are linearly independent. Let $\alpha(s)$ be a rational vector such that $\pi(s)\alpha(s) = 0$, i.e.

$$(I + F(s-A-BF)^{-1}B)N\alpha(s) = 0$$

Since $F(s-A-BF)^{-1}B$ is strictly proper, it follows that $(I + F(s-A-BF)^{-1}B)$ is of full column rank. Hence, $N\alpha(s) = 0$, and since N is of full column rank, $\alpha(s) = 0$, i.e. we have shown that the columns of $\pi(s)$ are linearly independent. □

The basis $\pi(s)$ constructed in the theorem has a very special form which can be used to find a simple characterization of our solution set. Since $F \in \underline{F}(\mathcal{V}^*)$ can be chosen arbitrarily, we can always take F such that

$$(A+BF)\,|\,\mathcal{R}^* \tag{4.6}$$

has its spectrum within $\mathbb{C}^-$. Since in addition

$$\mathcal{R}^* = \langle A+BF\,|\,\mathrm{Im}(BN)\rangle \tag{4.7}$$

with N as in Thm. 4.1, it follows that with this choice of F, $\pi(s)$ is a stable rational matrix. Let us call a basis constructed in this way a <u>stable basis</u> for Ker T(s). From (4.7) we see that there is a direct correspondence between $\mathcal{R}^*$ and the kernel of T(s). In fact, $\pi(s)$ represents the input/output map of the control $u = Fx + Nw$.

In the following theorem we will give a characterization of the set of proper (strictly proper) and stable solutions of (1.2) [or (1.1)].

Theorem 4.2. Let $K_0(s)$ be any (strictly) proper and internally stable solution to (1.2). Also let $\pi(s)$ be a stable basis for Ker $T(s)$ of the form (4.5). The set of all (strictly) proper and internally stable solutions to (1.2) is then all $K(s)$ such that

$$K(s) = K_0(s) + \pi(s)Q(s) \tag{4.8}$$

for some (strictly) proper and stable rational matrix $Q(s)$.

Proof. An arbitrary solution to (1.2) is given by (4.3) for some rational matrix $Q(s)$. In our case, we are interested only in (strictly) proper and internally stable solutions. Since $K_0(s)$ is (strictly) proper, the set of all (strictly) proper solutions is generated by all $Q(s)$ such that $\pi(s)Q(s)$ is (strictly) proper in (4.3). In this case

$$\pi(s)Q(s) = \left(F(s-A-BF)^{-1}B + I\right)NQ(s)$$

Since N is of full column rank, it follows that $\pi(s)Q(s)$ is (strictly) proper iff $Q(s)$ is (strictly) proper. Hence, the set of all (strictly) proper solutions is generated by (4.8) with $Q(s)$ being (strictly) proper. It then remains to impose stability. Internal stability holds by definition iff

$$P(s) = (s-A)^{-1}\left[B\left(K_0(s) + \pi(s)Q(s)\right) + E\right] =$$

$$= (s-A)^{-1}\left(BK_0(s) + E\right) + (s-A)^{-1}\pi(s)Q(s) \tag{4.9}$$

is a stable rational matrix. Since $K_0(s)$ is internally stable by assumption, the first term in (4.9) is stable. The second term can also be written

$$(s-A-BF)^{-1}BNQ(s) \tag{4.10}$$

Since $\pi(s)$ is stable, i.e. F is chosen so that (4.6) is stable, the rational matrix $(s-A-BF)^{-1}BN$ is stable. Hence (4.10) is stable, iff $BNQ(s)$ is stable. However, BN is of full column rank, and therefore (4.10) is stable iff $Q(s)$ is stable. Hence, $P(s)$ is stable iff $Q(s)$ is stable which proves the result. □

Since $\mathbb{C}^-$ is quite arbitrary, the characterization in the theorem is valid also for proper (or strictly proper) solutions with no stability imposed (choose $\mathbb{C}^- = \mathbb{C}$). It is important to notice that the <u>internally stable</u> solutions to (1.2) are exactly those which can be implemented as an internally stable dynamic feedback control for (2.1). Since, as is demonstrated in the next section, several control problems can be formulated as an algebraic problem of the form (1.2), we have thus characterized the set of all internally stable solutions (feedback controls) for these problems. Such characterizations are of great importance in control synthesis, since we may be able to select among all solutions the one which best satisfies supplementary design criteria.

If the pair (C,A) in (1.2) is observable then, as has been noted before, $K(s)$ is internally stable iff $K(s)$ is a stable rational matrix. Thus, Theorem 4.2 can also be used to characterize the set of all stable and (strictly proper rational matrices $K(s)$ satisfying (1.1).

The characterization in the theorem is valid only for the specific choice of $\pi(s)$ in (4.8), i.e. (4.5). For constructions, we note that $\mathcal{V}^*$, F, N are already computed in the construction of a <u>specific</u> solution (cf. Sec. 3) so no further computations are necessary to characterize the whole solution set.

5. CONTROL APPLICATIONS

In this section, we show how certain control problems can be restated in the form (1.2) [or (1.1)] to which our results apply.

5.1. Model Matching.

In the model matching problem we are given the system Σ (2.1) with $E = 0$ and a model Σ_m:

$$\dot{x}_m = A_m x_m + B_m v_m$$
$$y_m = C_m x_m$$

i.e.

$$T_m(s) = C_m(s-A_m)^{-1}B_m$$

We wish to find a dynamic feedback control which makes the input/output behavior of Σ equal to $T_m(s)$. In addition, the controlled system must be internally stable w.r.t. $\mathbb{C}^-$. This problem has attracted much interest in the control literature, see e.g. [6, 7].

Let the control be described by its input/output map

$$u(s) = K(s)u_m(s)$$

Since the transfer matrix for the closed-loop system is to be $T_m(s)$, $K(s)$ must be chosen such that

$$C(s-A)^{-1}BK(s) = C_m(s-A_m)^{-1}B_m \tag{5.1}$$

For this problem, there exists an internally stable solution iff

$$(s-A)^{-1}BK(s) \tag{5.2}$$

is stable. Introducing

$$\bar{A} := \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}; \qquad \bar{B} := \begin{pmatrix} B \\ 0 \end{pmatrix};$$

$$\bar{E} := \begin{pmatrix} 0 \\ B_m \end{pmatrix}; \qquad \bar{C} := \begin{pmatrix} C & -C_m \end{pmatrix}$$

we see directly that (5.1) is equivalent to

$$\bar{C}(s-\bar{A})^{-1}(\bar{B}K(s) + \bar{E}) = 0 \tag{5.3}$$

and (5.2) is equivalent to the requirement

$$(s-\bar{A})^{-1}(\bar{B}K(s) + \bar{E}) \tag{5.4}$$

be stable w.r.t. $\mathbb{C}^-$. Hence, the problem is to find a proper and internally stable solution to (5.3). The solvability is given in Theorem 3.2 and the whole (internally stable) solution set is given in Theorem 4.2.

5.2. Disturbance Localization.

If w in (2.1) is interpreted as a disturbance, the problem (1.2) becomes the classical feedforward control of disturbances. Solvability for causal and internally stable feedforward controls is given in Theorem 3.2 and the solution set is characterized in Theorem 4.2. Results for this problem have earlier been given in [2, 3, 8].

5.3. Dynamic Decoupling.

Assume we are given the system

$$\dot{x} = Ax + Bu$$
$$y_i = C_i x \, ; \qquad i \in [1, 2, \ldots, d] \tag{5.4}$$

where $y_i \in \mathbb{R}^{q_i}$ are otuput vectors to be controlled independently. Take a dynamic feedback control with external inputs $w_i \in \mathbb{R}^{m_i}$, $i \in \underline{d}$. Represent this control by its input/output map

$$u = \sum_{i=1}^{d} K_i(s) w_i(s) \tag{5.5}$$

We wish to select $K_i(s)$ such that, when applied to (5.4), the transfer matrices relating w_i and y_j are zero for $i \neq j$, and nonzero (in the sense below) for $i = j$. Introduce

$$T_i(s) := C_i(s-A)^{-1}B$$

$$\hat{T}_i(s) := \hat{C}_i(s-A)^{-1}B; \qquad \hat{C}_i := \begin{bmatrix} C_1 \\ \vdots \\ C_{i-1} \\ C_{i+1} \\ \vdots \\ C_d \end{bmatrix}$$

Then we may express the <u>noninteraction</u> condition as

(a) $\quad \hat{T}_i(s) K_i(s) = 0; \qquad i \in \underline{d}$

As the <u>output controllability</u> condition we take

(b) $\quad p^T T_i(s) K_i(s) \neq 0; \qquad \forall\ i \in \underline{d}$ and $\forall\ p \in \mathbb{R}^{q_i},\ p \neq 0.$

Finally, we require that (5.5) have an internally stable implementation as a dynamic feedback control. By Thm. 2.1, this is equivalent to the requirement that the matrix

$$(s-A)^{-1}B[K_1(s) \ \dots \ K_d(s)]$$

be stable w.r.t. $\mathbb{C}^-$. Hence, the internal stability condition is

(c) $\quad (s-A)^{-1}BK_i(s)$ is stable w.r.t. $\mathbb{C}^-$; $i \in \underline{d}$.

The decoupling problem with internal stability can thus be formulated as: find integers m_i and proper rational $m \times m_i$ matrices $K_i(s)$, $i \in \underline{d}$, such that (a)-(c) hold. This formulation is in fact the external (transfer matrix) correspondence to the extended decoupling problem as formulated in [9]. Obviously, (c) can always be satisfied since for any specific $K_i(s)$ we can multiply $(s-A)^{-1}BK_i(s)$ by a scalar rational function to make it stable. Conditions (a) and (b) are satisfied iff

$$p^T T_i(s)\pi_i(s) \neq 0; \qquad i \in \underline{d};\ p \in \mathbb{R}^{q_i},\ p \neq 0. \tag{5.6}$$

where $\pi_i(s)$ is a basis for Ker $\hat{T}_i(s)$. Such a basis is described in Theorem 4.1. Taking F_i and N_i as in this theorem, i.e.

$$F_i \in \underline{F}(V_i^*); \qquad \mathrm{Im}(BN_i) = \mathcal{B} \cap V_i^*$$

$$V_i^* = \sup I(A,B,\mathrm{Ker}\ C_i) \qquad \pi_i(s) = F(s-A-BF)^{-1}BN_i + N_i$$

the condition (5.6) becomes

$$p^T C_i(s-A-BF_i)^{-1}BN_i \neq 0; \qquad \forall\ i \in \underline{d};\ \forall\ p \in R^{q_i},\ p \neq 0.$$

It is easily shown that this condition is equivalent to output pointwise controllability for $(C_i, A+BF_i, BN_i)$. The controllable subspace for the pair $(A+BF_i, BN_i)$ equals R_i^*, where

$$R_i^* = \sup C(A, B_i, \mathrm{Ker}\ C_i)$$

It follows that the necessary and sufficient condition for solvability of EDP is

$$C_i \mathcal{R}_i^* = \operatorname{Im} \hat{C}_i; \qquad i \in \underline{d}$$

This is the same condition as obtained in [9].

5.4. Observers for a Linear Transformation of the State.

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad\qquad x(t_0) = x_0 \\ y &= Cx \qquad z = Hx \end{aligned} \tag{5.7}$$

where y is the measured output and z the output to be reconstructed. Assume that the reconstructing device is allowed to be of the form

$$\begin{aligned} \dot{w} &= A_1 w + B_1 y \qquad\qquad w(0) = 0 \\ \hat{z} &= H_1 w + D_1 y \end{aligned} \tag{5.8}$$

i.e. only y is available (i.e. not u). For the reconstruction we require that $z(t) - \hat{z}(t) \to 0$ as $t \to \infty$ for all x_0, $u(\cdot)$. On taking Laplace transforms in (5.7) and (5.8), we get

$$z - \hat{z} = \big(H - K(s)C\big)(s-A)^{-1}Bu + K(s)C(s-A)^{-1}x_0$$

where $K(s)$ is the transfer matrix of the reconstruction (5.8) i.e. $K(s) = H_1(s-A_1)^{-1}B_1 + D_1$. The requirements on the reconstructor are satisfied if and only if

$$\begin{aligned} &\text{(i)} \quad \big(H - K(s)C\big)(s-A)^{-1}B = 0 \\ &\text{(ii)} \quad K(s)C(s-A)^{-1} \text{ is a stable rational matrix} \end{aligned} \tag{5.9}$$

Hence, the reconstruction problem is to find a proper rational matrix $K(s)$ which satisfies (i) and (ii), namely the dual of (1.2) (take transposes). The solvability is given by Theorem 3.2 (dualized) and the whole solution set by Theorem 4.2. If in (5.7), (A,B) is a controllable pair, the problem (5.9) is equivalent to

(i) the same as above

(ii') $K(s)$ is a stable rational matrix, (5.10)

c.f. the reasoning in Section 2.

Similar arguments apply to reconstructors of the form

$$\hat{z}(s) = K_1(s)y(s) + K_2(s)u(s)$$

i.e. both y and u are accessible for measurement.

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