Transmuted distributions and extrema of random number of variables

Kozubowski, Tomasz J.; Podgórski, Krzysztof

2016

Working Papers in Statistics
No 2016:6
Department of Statistics
School of Economics and Management
Lund University

Transmuted distributions and extrema of random number of variables

TOMASZ J. KOZUBOWSKI, UNIVERSITY OF NEVADA

KRZYSZTOF PODGÓRSKI, LUND UNIVERSITY

Transmuted distributions and extrema of random number of variables

Tomasz J. Kozubowski[a1] **and Krzysztof Podgórski**[b]

[a] *Department of Mathematics & Statistics, University of Nevada at Reno, USA*

[b] *Department of Statistics, Lund University, Sweden*

Abstract Recent years have seen an increased interest in the transmuted probability models, which arise from transforming a “base” distribution into its generalized counterpart. While many standard probability distributions were generalized throughout this simple construction, the concept lacked deeper theoretical interpretation. This note demonstrates that the scheme is more than just a simple trick to obtain a new cumulative distribution function. We show that the transmuted distributions can be viewed as the distribution of maxima (or minima) of a random number N of independent and identically distributed variables with the base distribution, where N has a Bernoulli distribution shifted up by one. Consequently, the transmuted models are, in fact, only a special case of extremal distributions defined through a more general N.

MSC: 60E05; 62E10

Keywords: Distribution theory; Extremes, Marschall-Olkin generalized distribution; Quadratic transmutation map; Random minimum; Random maximum; Sibuya distribution; Stochastic representation

1. Introduction

The idea of transforming of a cumulative distribution functions (CDF) through the *quadratic rank transmutation map* was introduced in an unpublished report Shaw and Buckley (2007) and applied to uniform, exponential and normal distributions. The map is defined by

$$u \to u + \alpha u(1-u), \;\; u \in [0,1], \alpha \in [-1,1], \tag{1}$$

and starting from a given *base* CDF F, new probability CDF's F_α are obtained through

$$F_\alpha(x) = (1+\alpha)F(x) - \alpha F^2(x), \;\; x \in \mathbb{R}, \tag{2}$$

[1]Corresponding author. Tel: + 1 775 784 6643; fax: + 1 775 784 6378.

E-mail address: tkozubow@unr.edu (T.J. Kozubowski).

Referred to as *transmuted distributions*, such extended models have been discussed in a large number of papers devoted to basic properties of theses distributions, including derivation their probability density functions (PDFs), quantile functions, moments, hazard rates, and maximum likelihood estimation. Notably, we find transmuted extensions for the log-logistic distribution in Granzotto and Louzada (2015), the extreme value distributions in Aryal and Tsokos (2009), generalized inverted exponential in Elbatal (2013), additive Weibull in Elbatal and Aryal (2013), generalized linear exponential in Tian et al. (2014) and Elbatal et al. (2013), generalized inverse Weibull in Khan and King (2013), Rayleigh in Merovci (2013), inverse Reyleigh in Khan and King (2015), generalized inverse Weibull in Merovci et al. (2014). Basic properties were also developed for other classes of distributions in some other recently published work, including inverse exponential, Gompertz, Fréchet, Marshall-Olkin Fréchet, exponentiated Fréchet, Weibull, complementary Weibull geometric, exponentiated modified Weibull, Weibull Lomax, Lomax and exponentiated Lomax, exponentiated gamma, Lindley, and Pareto distributions.

Despite their abundance, papers on the topic generally focus on rather elementary properties and do not provide any theoretical interpretation of the construction through transmuted mapping. Our goal is to fill this gap and show that all variables defined viz. (2) correspond to maxima (or minima) of a *random number* of independent and identically distributed (IID) components.

2. The main result

Let us consider an integer-valued random variable N_p having a Bernoulli distribution with parameter $p \in [0,1]$ shifted up by one, so it is taking on the values of 1 and 2 with probabilities $1-p$ and p, respectively. Since the probability generating function (PGF) of N_p is given by

$$G_p(s) = \mathbb{E}s^{N_p} = s(1-p+ps), \quad s \in [0,1], \tag{3}$$

it follows that, for $\alpha \in [-1,0]$, the CDF in (2) can be written as

$$F_\alpha(x) = G_{-\alpha}(F(x)), \quad x \in \mathbb{R}. \tag{4}$$

On the other hand, straightforward algebra shows that, in case $\alpha \in [0,1]$, the *survival function* (SF) $S_\alpha(x) = 1 - F_\alpha(x)$ admits the representation

$$S_\alpha(x) = G_\alpha(S(x)), \quad x \in \mathbb{R}, \tag{5}$$

where $S(x) = 1 - F(x)$ is the SF corresponding to the base CDF F. The CDF and the SF given by the right-hand-sides of (4)-(5) correspond, respectively, to the maximum and the minimum of N_p IID variables $\{X_i\}$ with the base CDF F, where N_p has the PGF G_p, and is independent of the $\{X_i\}$. This leads to the main result of this paper, stated below.

Proposition 2.1. *Let Y have a transmuted-F distribution, given by the CDF (2) with $\alpha \in [-1,1]$ and some base CDF F. Further, let X_1, X_2 be IID random variables with the CDF F, and let N_p be an integer-valued variable with the PGF (3), independent of the $\{X_i\}$. Then, for $\alpha \in [-1,0]$, we have*

$$Y \stackrel{d}{=} \bigvee_{j=1}^{N_p} X_j \tag{6}$$

with $p = -\alpha \in [0,1]$, while for $\alpha \in [0,1]$, we have

$$Y \stackrel{d}{=} \bigwedge_{j=1}^{N_p} X_j \tag{7}$$

with $p = \alpha \in [0,1]$.

Remark 2.1. Note that when $\alpha = 0$, the CDF F_α in (2) reduces to the base CDF F. In this case, we have $N_p = 1$ (a.s.) and both relations (6)-(7) hold simultaneously with their right-hand-sides reducing to X_1. Further, in the extreme cases $\alpha = -1$ and $\alpha = 1$, we have $N_p = 2$ (a.s.) and the relations (6) and (7) reduce to

$$Y \stackrel{d}{=} \max(X_1, X_2) \tag{8}$$

and

$$Y \stackrel{d}{=} \min(X_1, X_2), \tag{9}$$

respectively (cf., Shaw and Buckley, 2007).

Remark 2.2. The stochastic representations (6)-(7) can be alternatively stated as mixtures. For $\alpha \in [-1,0]$, the variable Y in (6) is either equal to $\max(X_1, X_2)$, with probability $p = -\alpha$, or to X_1, with probability $1-p$, and its CDF is the weighted average

$$F_\alpha(x) = (1-p)F(x) + pF^2(x), \quad x \in \mathbb{R}. \tag{10}$$

Similarly, for $\alpha \in [0,1]$, the variable Y in (7) is either equal to $\min(X_1, X_2)$, with probability $p = \alpha$, or to X_1, with probability $1-p$, and its SF is the weighted average

$$S_\alpha(x) = (1-p)S(x) + pS^2(x), \quad x \in \mathbb{R}. \tag{11}$$

Remark 2.3. Note that the *generalized transmuted distributions* generated by F, given by the CDF

$$G(x) = (1 + \sum_{j=1}^{k} \alpha_j) F(x) - \sum_{j=1}^{k} \alpha_j F^2(x), \;\; x \in \mathbb{R}, \tag{12}$$

and studied in Das and Barman (2015), are really the same as those given by (2).

Generalizations of transmuted classes of distributions can be obtained by taking more general N_p in the above set-up. For example, N_p can be chosen to have a binomial distribution with parameters $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $p \in [0, 1]$, shifted up by one. In this case the PGF (3) would be replaced by

$$G_p(s) = \mathbb{E} s^{N_p} = s(1 - p + ps)^n, \;\; s \in [0, 1], \tag{13}$$

with the generalized distribution having either the CDF (4) with $p = -\alpha$ if $\alpha \in [-1, 0]$ or the SF (5) with $p = \alpha$ if $\alpha \in [0, 1]$. Of course the PGF G_p can go beyond the polynomial function, leading to still more general classes of distributions connected with random maxima (6) or random minima (7) with the $\{X_i\}$ having the base PDF F. In case N_p is geometrically distributed, this scheme leads to the so called Marschall-Olkin generalized-F distributions, which have populated the literature since the appearance of such models (with exponential and Weibull base distributions) in Marshall and Olkin (1997). In closing, let us mention another two classes of quite popular generalizations of the base CDF F (and the corresponding survival function S), given by their power transformations. These also arise through random maxima and minima, with the N_p having the Sibuya distribution given by the PGF

$$G_p(s) = 1 - (1 - s)^p, \;\; s \in [0, 1]. \tag{14}$$

Here, the SF of the random maximum (6) becomes $[S(x)]^p$ while the CDF of the random minimum (7) is $[F(x)]^p$, where F and S are the CDF and the SF of the $\{X_i\}$ in (6)-(7). This class of distributions will be treated in further detail in future research.

References

Aryal, R. and Tsokos, P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods and Applications*, 71:1401–1407.

Das, K. and Barman, L. (2015). On some generalized transmuted distributions. *International Journal of Scientific & Engineering Research*, 6(2):1686–1691.

Elbatal, I. (2013). Transmuted generalized inverted exponential distribution. *Eco. Qual. Cont*, 28:125–133.

Elbatal, I. and Aryal, G. (2013). On the transmuted additive weibull distribution. *Austrian Journal of Statistics*, 42:117–132.

Elbatal, I., Diab, L., and Abdul-Alim, N. (2013). Transmuted generalized linear exponential distribution. *International Journal of Computer Applications*, 83(17):29–37.

Granzotto, D. and Louzada, F. (2015). The transmuted log-logistic distribution: Modeling, inference, and an application to a polled Tabapua race time up to first calving data. *Communications in Statistics-Theory and Methods*, 44:3387–3402.

Khan, M. and King, R. (2013). Transmuted generalized inverse Weibull distribution. *Journal of Applied Statistical Sciences*, 20(3):15–32.

Khan, M. and King, R. (2015). Transmuted modified inverse Rayleigh distribution. *Austrian Journal of Statistics*, 44:17–29.

Marshall, A. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3):641–652.

Merovci, F. (2013). Transmuted Rayleigh distribution. *Austrian Journal of Statistic*, 42:21–31.

Merovci, F., Elbatal, I., and Alaa, A. . T. t. g. i. W. d. (2014). The transmuted generalized inverse Weibull distribution. *Austrian Journal of Statistics*, 43:119–131.

Shaw, W. and Buckley, I. (2007). The alchemy of probability distributions: Beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. Research Report.

Tian, Y., Tian, M., and Zhu, Q. (2014). Transmuted linear exponential distribution: A new generalization of the linear exponential distribution. *Communications in Statistics-Simulation and Computation*, 43:2661–2677.

http://journals.lub.lu.se/stat

LUND UNIVERSITY
School of Economics and Management

Working Papers in Statistics 2016
LUND UNIVERSITY
SCHOOL OF ECONOMICS AND MANAGEMENT
Department of Statistics
Box 743
220 07 Lund, Sweden