LUND UNIVERSITY

Non-existence of three-dimensional travelling water waves with constant non-zero vorticity

Wahlén, Erik

Published in:
Journal of Fluid Mechanics

DOI:
10.1017/jfm.2014.131

2014

Link to publication

Citation for published version (APA):
Wahlén, E. (2014). Non-existence of three-dimensional travelling water waves with constant non-zero vorticity. *Journal of Fluid Mechanics*, *746*, R2. https://doi.org/10.1017/jfm.2014.131

Total number of authors:
1

General rights
Unless other specific re-use rights are stated the following general rights apply:
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.
• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: https://creativecommons.org/licenses/

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

Non-existence of three-dimensional travelling water waves with constant non-zero vorticity

E. Wahlén†

Centre for Mathematical Sciences, Lund University, PO Box 118, 22100 Lund, Sweden

(Received ?; revised ?; accepted ?. - To be entered by editorial office)

We prove that there are no three-dimensional bounded travelling gravity waves with constant non-zero vorticity on water of finite depth. The result also holds for gravity-capillary waves under a certain condition on the pressure at the surface, which is satisfied by sufficiently small waves. The proof relies on unique continuation arguments and Liouville type results for elliptic equations.

1. Introduction

While most of the theoretical studies of water waves throughout history have assumed irrotational flow, vorticity is important for describing the interaction of waves with non-uniform currents (Peregrine 1976). The mathematical theory of rotational water waves goes back to the beginning of the 19th century, when Gerstner (1809) constructed an explicit family of periodic travelling waves with non-zero vorticity using Lagrangian coordinates. Dubreil-Jacotin (1934) provided the first existence result for small-amplitude waves with a general distribution of vorticity. The research activity in this area has exploded in the last decade following work by Constantin & Strauss (2004), in which families of large-amplitude periodic gravity waves were constructed for a big class of vorticity distributions. See the book (Constantin 2011*a*) for a survey of recent results.

In all of the previous investigations of travelling waves with vorticity, the motion is assumed to be two-dimensional. In other words, the surface only varies in the direction of propagation and is homogeneous in the orthogonal direction, while the flow also varies in the vertical direction. In contrast, there is substantial literature on three-dimensional irrotational waves. Reeder & Shinbrot (1981) proved the first rigorous existence result for doubly periodic gravity-capillary waves, describing e.g. the oblique reflection of a two-dimensional wave at a wall. This was later followed by more general results for doubly periodic waves by Craig & Nicholls (2000) and Groves & Mielke (2001). By now a plethora of other types of solutions have been constructed, including waves which are localised in all horizontal directions, or localised in one-direction and periodic in another; see (Groves 2007) for an overview. The problem is substantially harder in the case of zero surface tension due to small-divisor problems. Nevertheless, Iooss & Plotnikov (2009, 2011) have recently succeeded in proving the existence of doubly periodic waves using Nash-Moser theory. It's currently unknown if there are other types of travelling gravity waves, although Craig (2002) proved a non-existence result for fully localised solitary waves.

In view of the above, it is natural to ask if there are three-dimensional water waves with non-zero vorticity. In order to understand why this is a difficult question, we note that the assumption of irrotational flow allows one to reduce Euler's equations to Laplace's equation for the velocity potential. This reduction is not available in the rotational case.

† Email address for correspondence: erik.wahlen@math.lu.se

Nevertheless, the problem simplifies significantly in two dimensions, since there is then a functional relationship between the vorticity and the stream function. In other words, $\omega = \gamma(\psi)$ for some function γ, where ω is the scalar vorticity and ψ the stream function. Using this observation, one can replace Euler's equations by a semilinear elliptic equation for the stream function and construct solutions for a large class of functions γ (see e.g. Constantin & Strauss 2004). In the case of constant vorticity, this elliptic equation is similar to Laplace's equation and it is possible to obtain a formulation in terms of a 'generalised velocity potential' which is very close to the irrotational one (see e.g. Constantin & Varvaruca 2011). The assumption of constant vorticity might seem like a natural starting point for a three-dimensional theory of rotational water waves, but as we shall demonstrate here this is not the case. Heuristically, the problem is that the vorticity has a direction which determines the fluid motion to a great extent. Due to the constancy of the vorticity, the flow is (locally) homogeneous in the direction of the vorticity vector (see eq. (2.11) below). We show that only a horizontal vorticity vector is compatible with a physical wave motion. We also use the fact that the assumption of constant vorticity implies that the velocity components are harmonic functions. This allows us to turn local arguments into global arguments using unique continuation principles.

Let us finally comment on the difference between our results and some recent related investigations. Constantin (2011*b*) and Constantin & Kartashova (2009) proved the two-dimensionality of constant-vorticity flow for gravity and capillary waves, respectively, under the assumption that the surface is a periodic travelling wave with no variation perpendicular to the direction of propagation, that is, $z = \eta(x - ct)$; see also (Stuhlmeier 2012) for similar results on solitary waves. In our results we only assume that $z = \eta(x - c_1 t, y - c_2 t)$ and show that, after a rotation, η does not depend on the second variable. Moreover, our methods work for all bounded solutions, including both solitary and periodic waves. On the other hand, Constantin and Kartashova do not assume that the flow is steady.

2. Preliminaries and main results

Let us recall the governing equations for three-dimensional travelling water waves (see Johnson 1997). The water is modelled as an incompressible, inviscid fluid of constant density $\varrho > 0$, bounded from below by an impermeable flat bottom and above by a free surface, which is assumed to be the graph of a function. The fluid is assumed to extend indefinitely in all horizontal directions. In a reference frame moving with the wave, the motion is described by a free surface $\eta(x, y)$ and a velocity field $\boldsymbol{u}(x, y, z)$ defined in the fluid domain

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \colon 0 < z < \eta(x, y)\}.$$

The velocity field $\boldsymbol{u} = (u, v, w)$ satisfies the steady Euler equations

$$uu_x + vu_y + wu_z = -\frac{1}{\varrho} P_x, \tag{2.1}$$

$$uv_x + vv_y + wv_z = -\frac{1}{\varrho} P_y, \tag{2.2}$$

$$uw_x + vw_y + ww_z = -\frac{1}{\varrho} P_z - g, \tag{2.3}$$

and the equation of mass conservation

$$u_x + v_y + w_z = 0 \tag{2.4}$$

in Ω. Here P is the pressure. Furthermore, the velocity field satisfies the kinematic boundary condition

$$w = 0 \tag{2.5}$$

on the bottom $\{z = 0\}$ and the kinematic and dynamic boundary conditions

$$w - u\eta_x - v\eta_y = 0, \tag{2.6}$$

$$P = \text{const.} \tag{2.7}$$

on the surface $\{z = \eta(x, y)\}$. By a solution of the water wave problem we mean a 4-tuple (η, u, v, w), with $\eta \in C^1(\mathbb{R}^2)$, satisfying $\inf_{(x,y)\in\mathbb{R}^2} \eta(x, y) > 0$, and $u, v, w \in C^1(\overline{\Omega})$, which solves the equations (2.1)–(2.7), meaning that there exists a C^1 function P such that the equations hold (see eq. (2.10) below). The pressure P is uniquely determined up to a constant by $\boldsymbol{u}$ through (2.1)–(2.3). In the presence of surface tension, equation (2.7) is replaced by

$$P + \sigma \frac{(1 + \eta_y^2)\eta_{xx} - 2\eta_x\eta_y\eta_{xy} + (1 + \eta_x^2)\eta_{yy}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} = \text{const.} \tag{2.8}$$

We then assume that $\eta \in C^2(\mathbb{R}^2)$.

The vorticity vector $\boldsymbol{\omega}$ is defined as the curl of $\boldsymbol{u}$, that is, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. In component form, we have that

$$(\omega_1, \omega_2, \omega_3) = (w_y - v_z, u_z - w_x, v_x - u_y). \tag{2.9}$$

Taking the curl of the Euler equations, one obtains the vorticity equation

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}, \tag{2.10}$$

which guarantees that the pressure can be recovered from the velocity field through (2.1)–(2.3). From now on we assume the following.

Assumption 1. *$\boldsymbol{\omega}$ is a non-zero constant vector and $\omega_1 = 0$.*

The second assumption can be imposed without loss of generality since the problem is invariant under rotations around the z-axis. When $\boldsymbol{\omega}$ is constant, the vorticity equation simplifies to

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}, \tag{2.11}$$

expressing that $\boldsymbol{u}$ is (locally) constant in the direction $\boldsymbol{\omega}$.

Before stating our main results, we introduce some notation. For a k-times differentiable vector-valued function $\boldsymbol{f}$ defined on the closure of an open subset U of $\mathbb{R}^n$ we denote by $\|\boldsymbol{f}\|_{k,\infty}$ the expression $\sup_{\boldsymbol{x}\in\overline{U}} \sum_{|\alpha|\leqslant k} \|\partial^\alpha \boldsymbol{f}(\boldsymbol{x})\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ and we have used multi-index notation. We let $\|\cdot\|_\infty := \|\cdot\|_{0,\infty}$.

Theorem 1 (Gravity waves). *Let $(\eta, \boldsymbol{u})$ be a solution of (2.1)–(2.7) satisfying Assumption 1 and $\|\boldsymbol{u}\|_\infty, \|\eta\|_\infty < \infty$. Then $\omega_3 = 0$, v is constant and the solution is independent of y.*

Theorem 2 (Gravity-capillary waves). *Let $(\eta, \boldsymbol{u})$ be a solution of (2.1)–(2.6), (2.8) satisfying Assumption 1, $\sup P_z|_{z=\eta} < 0$ and $\|\boldsymbol{u}\|_{1,\infty}, \|\eta\|_{2,\infty} < \infty$. Then the same conclusions as in Theorem 1 hold.*

Remark 1. *When the surface is flat, $\eta = \eta_0$, where η_0 is a constant, the dynamic boundary condition (2.8) reduces to (2.7), meaning that there's no difference between the*

gravity and gravity-capillary problems. Using Theorem 1, we find that $\boldsymbol{\omega} = (0, \omega_2, 0)$*,* v *is constant and* u*,* w *independent of* y*. Moreover,* $u_z - w_x = \omega_2$*,* $u_x + w_z = 0$*. The kinematic boundary condition at the surface shows that* $w|_{z=\eta_0} = 0$*. Since* w *is also harmonic and vanishes at the bottom, it must be identically* 0*. It then follows that* $u = \omega_2 z + u_0$*, where* u_0 *is an arbitrary constant.*

REMARK 2. *Note that* $P_z = -\varrho g$ *for the flat surface flow in the previous remark. If* $(\eta, \boldsymbol{u})$ *is sufficiently close to a solution with a flat surface in the* C^1*-norm it follows that the pressure condition* $\sup P_z|_{z=\eta} < 0$ *is satisfied. In order to compare the two solutions, we can extend the flat surface flow to the region* $\{z > \eta_0\}$ *using the formula* $\boldsymbol{u} = (\omega_2 z + u_0, v_0, 0)$*.*

3. Proofs of the main results

Combining (2.9) with (2.4) we find that that u, v and w are harmonic functions. In particular, they are real analytic, meaning that they can be expanded in (uniformly) convergent power series in a neighbourhood of any point in Ω (see Axler *et al.* 2001). It follows that the left hand sides of equations (2.1)–(2.3) also define real analytic functions. It is then easy to see by integrating equations (2.1)–(2.3) that the pressure can be locally expressed in the form of a power series, that is, it is real analytic.

The reader is reminded that Assumption 1 holds throughout this section. This implies that the vorticity vector has one of the following three forms, where $\omega_2, \omega_3 \neq 0$:

(a) $\boldsymbol{\omega} = (0, 0, \omega_3)$,
(b) $\boldsymbol{\omega} = (0, \omega_2, \omega_3)$,
(c) $\boldsymbol{\omega} = (0, \omega_2, 0)$.

The proofs consist of eliminating cases (a) and (b) and showing that (c) implies that the solution is independent of y.

3.1. *Proof of Theorem 1*

LEMMA 1. *There is no solution of* (2.1)–(2.7) *with* $\boldsymbol{\omega} = (0, 0, \omega_3)$ *and* $\|\boldsymbol{u}\|_\infty < \infty$*.*

Proof. If $\boldsymbol{\omega} = (0, 0, \omega_3)$, equation (2.11) yields that $u_z = v_z = w_z = 0$ and (2.9) then shows that $w_y = w_x = 0$, so that $\nabla w = 0$. We also find that $u = u(x, y)$, $v = v(x, y)$ with $u_x + v_y = 0$ and $v_x - u_y = \omega_3$. It follows that u and v are bounded harmonic functions in $\mathbb{R}^2$. Hence they are constant by Liouville's theorem for harmonic functions (Axler *et al.* 2001). But this contradicts the fact that $\omega_3 \neq 0$. □

LEMMA 2. *If* $\boldsymbol{\omega} = (0, \omega_2, \omega_3)$*, the solutions of* (2.1)–(2.7) *have the explicit form* $\boldsymbol{u} = (-\omega_3 y + \omega_2 z + u_0, 0, 0)$*,* $\eta = \eta_0$ *and* $P = -\varrho g z + P_0$ *where* u_0*,* $\eta_0 > 0$*,* P_0 *are arbitrary real numbers. In particular, there are no solutions with* $\|\boldsymbol{u}\|_\infty < \infty$*.*

Proof. We have from (2.11) that

$$\omega_2 w_y + \omega_3 w_z = 0,$$

which in conjunction with (2.5) gives $w = 0$ in a neighbourhood of the bottom. Hence w vanishes identically since it is real analytic and Ω is connected. It follows that $v_z = 0$ and $u_z = \omega_2$. We moreover have that $\omega_2 u_y + \omega_3 u_z = 0$, $\omega_2 v_y + \omega_3 v_z = 0$, $v_x - u_y = \omega_3$ and $u_x + v_y = 0$, so that $u_y = -\omega_3$ and $u_x = v_x = v_y = 0$. But this implies that $\nabla v = 0$ and $\nabla u = (0, -\omega_3, \omega_2)$, that is, $u = -\omega_3 y + \omega_2 z + u_0$, while $v = v_0$. From the Euler

equations, we see that

$$P_x = \varrho\omega_3 v_0, \quad P_y = 0 \quad \text{and} \quad P_z = -\varrho g,$$

whence

$$P = \varrho\omega_3 v_0 x - \varrho g z + P_0.$$

The dynamic boundary condition thus takes the form

$$\varrho\omega_3 v_0 x - \varrho g\eta = \text{const.}, \tag{3.1}$$

showing that η is independent of y and that $\eta_x = \omega_3 v_0/g$. The kinematic boundary condition now reduces to

$$0 = u\eta_x + v\eta_y = \frac{(-\omega_3 y + \omega_2\eta + u_0)\,\omega_3 v_0}{g}.$$

If $v_0 \neq 0$, we find that

$$\eta = -\frac{u_0}{\omega_2} + \frac{\omega_3}{\omega_2} y,$$

contradicting (3.1). It follows that $v_0 = 0$ and that η_x vanishes identically, so that $\eta = \eta_0$ is constant. This concludes the proof. □

LEMMA 3. *Assume that $\boldsymbol{\omega} = (0, \omega_2, 0)$. Then η, $\boldsymbol{u}$ and P are independent of y and v is constant.*

Proof. The vorticity equation (2.11) implies that $u_y = v_y = w_y = 0$, which in combination with (2.9) yields that $v_x = v_z = 0$. Hence, $\nabla v = 0$ and $P_y = 0$. It follows that v is constant throughout Ω.

We claim that P is independent of y. This is clear locally, but not globally since η might a priori depend on y. For z sufficiently close to 0, we have that $P(x, y_1, z) = P(x, y_2, z)$ since the points (x, y_1, z) and (x, y_2, z) can be joined by a line segment along which P_y vanishes. But then there is a maximal $z = z^*$ such that $P(x, y_1, z) = P(x, y_2, z)$ as long as $0 \leqslant z \leqslant z^*$. Suppose that z^* is strictly less than $\min\{\eta(x, y_1), \eta(x, y_2)\}$. By real analyticity of the function $z \mapsto P(x, y_1, z) - P(x, y_2, z)$ we obtain that $P(x, y_1, z) = P(x, y_2, z)$ for z in a neighbourhood of z^*, resulting in a contradiction. The same argument shows that u and w are independent of y.

The function $P = P(x, z)$ is defined on the projection

$$\Omega' = \{(x, z)\colon (x, y, z) \in \Omega \text{ for some } y \in \mathbb{R}\}$$

of Ω on the xz-plane. Note that

$$\Omega' = \{(x, z)\colon 0 < z < f(x)\}$$

where $f(x) = \sup_{y\in\mathbb{R}} \eta(x, y)$. Since η is continuous, it follows that $f\colon \mathbb{R} \to (0, +\infty]$ is lower semicontinuous (see Rudin 1987). Moreover, Ω' is an open, connected subset of $\mathbb{R}^2$. We claim that $\eta(x, y) = f(x)$. Suppose not. Then $z_0 = \eta(x_0, y_0) < f(x_0)$ for some x_0 and y_0, and $P(x_0, \eta(x_0, y_0)) = P_0$. By continuity, there is some $y(z)$ such that $z = \eta(x_0, y(z))$ for each z between z_0 and $f(x_0)$ and thus $P(x_0, z) = P_0$ for each such z. By lower semicontinuity we must also have that $\eta(x, y_0) < f(x)$ for x sufficiently close to x_0, which in combination with the previous argument implies that $P(x, z) = P_0$ on some open subset of Ω'. Since P is real analytic, this implies that P is constant throughout Ω'. But this yields a contradiction on the bottom since (2.3) gives $0 = -g$ there in view of the fact that $w(x, y, 0) = 0$. □

Theorem 1 follows by combining Lemmata 1–3.

3.2. *Proof of Theorem 2*

We next consider gravity-capillary waves, meaning that (2.7) is replaced by (2.8). Note that (2.7) was not used in the proof of Lemma 1, which therefore automatically holds for gravity-capillary waves as well.

In Lemma 2 we only used (2.7) to show that η is constant and $v = 0$. In particular, it is not needed in order to obtain the explicit form of u. Therefore, the conclusion that there are no solutions with $\|\boldsymbol{u}\|_\infty < \infty$ remains true.

It remains to consider the case $\boldsymbol{\omega} = (0, \omega_2, 0)$. In order to handle this case, we shall need a Liouville type theorem for elliptic equations of the form

$$a_{ij}\partial_i\partial_j f + b_i\partial_i f + cf = 0, \tag{3.2}$$

where we have used Einstein's summation convention. The following result is sufficient for our needs.

LEMMA 4. *Let a_{ij}, b_i and c, $i, j = 1, \dots, n$, be continuous functions on $\mathbb{R}^n$. Assume that the functions a_{ij} and b_i are bounded, that the matrix $(a_{ij}(\boldsymbol{x}))$ is symmetric and nonnegative for any $\boldsymbol{x} \in \mathbb{R}^n$ and that $\sup_{\boldsymbol{x}\in\mathbb{R}^n} c(\boldsymbol{x}) < 0$. Let $f \in C^2(\mathbb{R}^n)$ be a bounded solution of the elliptic equation* (3.2). *Then $f \equiv 0$.*

Lemma 4 is a special case of (Krylov 1996, Corollary 2.9.3), to which we refer for the proof.

LEMMA 5. *Assume that $\boldsymbol{\omega} = (0, \omega_2, 0)$, with $\omega_2 \neq 0$, and $\sup P_z|_{z=\eta} < 0$. Assume furthermore that $\|\boldsymbol{u}\|_{1,\infty}$ and $\|\eta\|_{2,\infty}$ are finite. Then η, $\boldsymbol{u}$ and P are independent of y.*

Proof. The proof is identical to the proof of Lemma 3 up to the point where the dynamic boundary condition is used. The boundary condition is now of the form

$$P(x, \eta(x,y)) + \sigma\frac{(1+\eta_y^2)\eta_{xx} - 2\eta_x\eta_y\eta_{xy} + (1+\eta_x^2)\eta_{yy}}{(1+\eta_x^2+\eta_y^2)^{3/2}} = \text{const.}$$

Differentiation with respect to y results in an equation of the form (3.2) for $f = \eta_y$, where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and

$$(a_{ij}) = \frac{\sigma}{(1+\eta_x^2+\eta_y^2)^{3/2}}\begin{pmatrix} 1+\eta_y^2 & -\eta_x\eta_y \\ -\eta_x\eta_y & 1+\eta_x^2 \end{pmatrix}.$$

It is easily verified that this matrix is nonnegative. The coefficient c is given by $c(x,y) = P_z(x, \eta(x,y))$. By assumption, all the coefficients are bounded and $\sup_{(x,y)\in\mathbb{R}^2} c(x,y) < 0$. An application of Lemma 4 therefore shows that η_y vanishes identically. □

Acknowledgements

This work was supported by the Swedish Research Council (grant no. 621-2012-3753). The author is grateful to the referees for several helpful suggestions.

REFERENCES

AXLER, S., BOURDON, P. & RAMEY, W. 2001 *Harmonic function theory*, 2nd edn., *Graduate Texts in Mathematics*, vol. 137. New York: Springer-Verlag.

Constantin, A. 2011*a* *Nonlinear water waves with applications to wave-current interactions and tsunamis*, *CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 81. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

Constantin, A. 2011*b* Two-dimensionality of gravity water flows of constant nonzero vorticity beneath a surface wave train. *Eur. J. Mech. B Fluids* **30** (1), 12–16.

Constantin, A. & Kartashova, E. 2009 Effect of non-zero constant vorticity on the nonlinear resonances of capillary water waves. *EPL* **86** (2).

Constantin, A. & Strauss, W. 2004 Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.* **57** (4), 481–527.

Constantin, A. & Varvaruca, E. 2011 Steady periodic water waves with constant vorticity: regularity and local bifurcation. *Arch. Ration. Mech. Anal.* **199** (1), 33–67.

Craig, W. 2002 Non-existence of solitary water waves in three dimensions. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **360** (1799), 2127–2135, recent developments in the mathematical theory of water waves (Oberwolfach, 2001).

Craig, W. & Nicholls, D. P. 2000 Travelling two and three dimensional capillary gravity water waves. *SIAM J. Math. Anal.* **32** (2), 323–359.

Dubreil-Jacotin, M. L. 1934 Sur la détermination rigoureuse des ondes permanentes périodiques d'ampleur finite. *J. Math. Pures Appl.* **13**, 217–291.

Gerstner, F. 1809 Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile. *Ann. Phys.* **32**, 412–445.

Groves, M. D. 2007 Three-dimensional travelling gravity-capillary water waves. *GAMM-Mitt.* **30** (1), 8–43.

Groves, M. D. & Mielke, A. 2001 A spatial dynamics approach to three-dimensional gravity-capillary steady water waves. *Proc. Roy. Soc. Edinburgh Sect. A* **131** (1), 83–136.

Iooss, G. & Plotnikov, P. 2011 Asymmetrical three-dimensional travelling gravity waves. *Arch. Ration. Mech. Anal.* **200** (3), 789–880.

Iooss, G. & Plotnikov, P. I. 2009 Small divisor problem in the theory of three-dimensional water gravity waves. *Mem. Amer. Math. Soc.* **200** (940), viii+128.

Johnson, R. S. 1997 *A modern introduction to the mathematical theory of water waves*. Cambridge: Cambridge University Press.

Krylov, N. V. 1996 *Lectures on elliptic and parabolic equations in Hölder spaces*, *Graduate Studies in Mathematics*, vol. 12. Providence, RI: American Mathematical Society.

Peregrine, D. H. 1976 Interaction of water waves and currents. *Advances in Applied Mechanics* **16**, 9–117.

Reeder, J. & Shinbrot, M. 1981 Three-dimensional, nonlinear wave interaction in water of constant depth. *Nonlinear Anal.* **5** (3), 303–323.

Rudin, W. 1987 *Real and complex analysis*, 3rd edn. New York: McGraw-Hill Book Co.

Stuhlmeier, R. 2012 On constant vorticity flows beneath two-dimensional surface solitary waves. *J. Nonlinear Math. Phys.* **19** (suppl. 1), 1240004, 9.