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Abstract

The formulae for linear quadratic control of a system
with uncertain parameters are derived in this note.
The results are of interest in order to take uncertainties
into account in computer aided design and also for
adaptive control algorithms.

Contents

1. INTRODUCTION

This note considers a linear quadratic problem where the
parameters of the system model are random variables.
The criterion is to minimize the expected loss where
the expectation is taken both with respect to process
and measurement disturbances and with respect to parameter
uncertainties. The solution of the problem is of interest
in order to evaluate the effects of uncertainties in
design of control laws for systems with uncertain
parameters and for adaptive control. Results of this
type was derived for single-input single-output systems
using transfer function representations in Wieslander
and Wittenmark (1971). The state-space formulas which
are simpler were used in Peterka and Åström (1973).

2. PROBLEM STATEMENT

Consider the system

$$x(t+1) = \Phi x(t) + \Gamma u(t) + e(t) \tag{1}$$

where $\{e(t)\}$ is a sequence of random vectors with zero mean values and covariances R_1. Assume that the parameters Φ and Γ are random variables with known mean values and known covariances. The elements of Φ and Γ are uncorrelated with $\{e(t)\}$. Assume that the state variable x can be measured without error. Determine a control strategy such that the criterion

$$E \left[\sum_{k=0}^{N} x^T(k)Q_1 x(k) + u^T(k)Q_2 u(k) \right] \tag{2}$$

is minimal. The matrices Q_1 and Q_2 are assumed known. The matrices Φ and Γ have mean values

$$E \Phi = \hat{\Phi}$$
$$E \Gamma = \hat{\Gamma} \tag{3}$$

To express the uncertainties of Φ and Γ. We introduce

$$\Phi = [\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n]$$
$$\Gamma = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n] \tag{4}$$

The uncertainties are defined by

$$R_{\varphi_i \varphi_j} = E [\varphi_i - \hat{\varphi}_i] [\varphi_j - \hat{\varphi}_j] = \text{cov} [\varphi_i, \varphi_j]$$

$$R_{\varphi_i \gamma_j} = E [\varphi_i - \overline{\varphi}_i] [\gamma_j - \hat{\gamma}_j] = \text{cov} [\varphi_i, \gamma_j] \tag{5}$$

$$R_{\gamma_i \gamma_j} = E [\gamma_i - \hat{\gamma}_i] [\gamma_j - \hat{\gamma}_j] = \text{cov} [\gamma_i, \gamma_j]$$

3. MAIN RESULT

To solve the problem we procede in the usual manner using Dynamic Programming. Assume that the minimum exists and introduce

$$V(x,t) = \min_{u} E\left[\sum_{k=t}^{N} x^T(k) Q_1 x(k) + u^T(k) Q_2 u(k)\right] \qquad (6)$$

where E denotes the conditional expectation given x(t). We have

$$V(x,N) = x^T Q_1 x \qquad (7)$$

It will now be shown by induction that V(x,t) is a quadratic form, i.e.

$$V(x,t) = x^T S(t)\psi + \alpha(t) \qquad (8)$$

This is apparently true for t = N with

$$\begin{cases} S(N) = Q_1 \\ \alpha(N) = 0 \end{cases}$$

Assuming that (8) holds for t + 1 it will be shown by induction that it holds also for t. We have

$$V(x,t) = \min_{u} E\left[x^T(t)Q_1 x(t) + u^T(t)Q_2 u(t) + V[x(t+1),t+1]\right] \qquad (9)$$

Drop the argument t in x(t) to simplify the writing, introduce (1) and (8) into (6). Then

$$V(x,t) = \min_{u} E\left[x^T Q_1 x + u^T Q_2 u + V(\Phi x + \Gamma u + e, t+1)\right] =$$

$$= \min_{u} E\left[x^T Q_1 x + u^T Q_2 u + (\Phi x + \Gamma u + e)^T S(t+1)(\Phi x + \Gamma u + e) + \alpha(t+1)\right] =$$

$$= \min_{u} \left[x^T Q_1 x + u^T Q_2 u + Ex^T \Phi^T S(t+1)\Phi x + Ex^T \Phi^T S(t+1)\Gamma u + \right.$$

$$\left. + Eu^T \Gamma^T S(t+1)\Phi x + Eu^T \Gamma^T S(t+1)\Gamma u + tr\ S(t+1)R_1 + \alpha(t+1)\right] \qquad (10)$$

To procede it is now necessary to evaluate

$$E \quad x^T \Phi^T S(t+1) \Phi x$$

$$E \quad x^T \Phi^T S(t+1) \Gamma u$$

$$E \quad u^T \Gamma^T S(t+1) \Gamma u$$

This is simple in principle but messy in detail unless we are careful about our notation. Equation (4) gives

$$\Phi^T S \Phi = \begin{pmatrix} \varphi_1^T \\ \varphi_2^T \\ \vdots \\ \varphi_n^T \end{pmatrix} S [\varphi_1 \quad \varphi_2 \ldots \varphi_n] = \begin{pmatrix} \varphi_1^T S \varphi_1 & \varphi_1^T S \varphi_2 & \ldots & \varphi_1^T S \varphi_n \\ \varphi_2^T S \varphi_1 & \varphi_2^T S \varphi_2 & \ldots & \varphi_2^T S \varphi_n \\ \vdots & & & \\ \varphi_n^T S \varphi_1 & \varphi_n^T S \varphi_2 & \ldots & \varphi_n^T S \varphi_n \end{pmatrix}$$

Taking mathematical expectations we find

$$E\Phi^T S \Phi = \hat{\Phi}^T S \hat{\Phi} + A_1(S) \tag{11}$$

$$A_1(S) = \begin{pmatrix} \text{tr } SR_{\varphi_1 \varphi_1} & \text{tr } SR_{\varphi_2 \varphi_1} & \ldots & \text{tr } SR_{\varphi_n \varphi_1} \\ \text{tr } SR_{\varphi_1 \varphi_2} & \text{tr } SR_{\varphi_2 \varphi_2} & \ldots & \text{tr } SR_{\varphi_n \varphi_2} \\ \vdots & & & \\ \text{tr } SR_{\varphi_1 \varphi_n} & \text{tr } SR_{\varphi_2 \varphi_n} & \ldots & \text{tr } SR_{\varphi_n \varphi_n} \end{pmatrix} \tag{12}$$

$$E \, \varphi_i^T S \varphi_j = E \, \varphi_j^T S \varphi_i = E \, \text{tr } \varphi_j^T S \varphi_i = E \, \text{tr } S \varphi_i \varphi_j^T = \text{tr } SR_{\varphi_i \varphi_j}$$

which implies

$$\text{tr } SR_{\varphi_j \varphi_i} = \text{tr } R_{\varphi_j \varphi_i} S = \text{tr } SR_{\varphi_i \varphi_j} = \text{tr } R_{\varphi_i \varphi_j} S \tag{14}$$

Similarly we find

$$E \, \Phi^T S \Gamma = \hat{\Phi}^T S \hat{\Gamma} + A_2(S) \tag{15}$$

$$A_2(S) = \begin{pmatrix} \text{tr } SR_{\Upsilon_1 \varphi_1} & \text{tr } SR_{\Upsilon_2 \varphi_1} & \cdots & \text{tr } SR_{\Upsilon_p \varphi_1} \\ \text{tr } SR_{\Upsilon_1 \varphi_2} & \text{tr } SR_{\Upsilon_2 \varphi_2} & \cdots & \text{tr } SR_{\Upsilon_p \varphi_2} \\ \vdots & & & \\ \text{tr } SR_{\Upsilon_1 \varphi_n} & \text{tr } SR_{\Upsilon_2 \varphi_n} & \cdots & \text{tr } SR_{\Upsilon_p \varphi_n} \end{pmatrix} \tag{16}$$

$$E \; \Gamma^T S \Gamma = \hat{\Gamma} S \hat{\Gamma} + A_3(S) \tag{17}$$

$$A_3(S) = \begin{pmatrix} \text{tr } SR_{\Upsilon_1 \Upsilon_1} & \text{tr } SR_{\Upsilon_1 \Upsilon_2} & \cdots & \text{tr } SR_{\Upsilon_1 \Upsilon_p} \\ \text{tr } SR_{\Upsilon_2 \Upsilon_1} & \text{tr } SR_{\Upsilon_2 \Upsilon_2} & \cdots & \text{tr } SR_{\Upsilon_2 \Upsilon_p} \\ \vdots & & & \\ \text{tr } SR_{\Upsilon_p \Upsilon_1} & \text{tr } SR_{\Upsilon_p \Upsilon_2} & \cdots & \text{tr } SR_{\Upsilon_p \Upsilon_p} \end{pmatrix} \tag{18}$$

Introducing (10), (15) and (17) into (8) we get

$$
\begin{aligned}
V(x,t) = \min_u \Big\{ & u^T[Q_2 + \hat{\Gamma}^T S(t+1)\hat{\Gamma} + A_3(S(t+1))] \, u \; + \\
& + u^T[\hat{\Gamma}^T S(t+1)\hat{\Phi} + A_2^T(S(t+1))] \, x \; + \\
& + x^T[\hat{\Phi}^T S(t+1)\hat{\Gamma} + A_2(S(t+1))] \, u \; + \\
& + x^T[Q_1 + \hat{\Phi}^T S(t+1)\hat{\Phi} + A_1(S(t+1))] \; + \\
& + \text{tr } S(t+1)R_1 + \alpha(t+1) \Big\}
\end{aligned}
$$

Summarizing the results we find

THEOREM

The optimal control strategy is given by

$$u(t) = - L \, x(t)$$

where

$$L = [Q_2 + \hat{\Gamma}^T S(t+1)\hat{\Gamma} + A_3(S(t+1))]^{-1}[\hat{\Gamma}^T S(t+1)\hat{\Phi} + A_2^T(S(t+1))]$$

and

$$S(t) = \hat{\Phi}^T S(t+1)\Phi + Q_1 - L^T[Q_2 + \hat{\Gamma}^T S(t+1)\hat{\Gamma} + A_3(S(t+1))] L + A_1(S(t+1)) =$$

$$= \hat{\Phi}^T S(t+1)[\hat{\Phi} - \hat{\Gamma}L] + Q_1 + A_1(S(t+1)) - A_2(S(t+1)) L =$$

$$= (\hat{\Phi} - \hat{\Gamma}L)^T S(t+1) (\hat{\Phi} - \hat{\Gamma}L) + Q_1 + A_1 + L^T Q_2 L - A_2 L - L^T A_2^T$$

where

$$A_1(S) = \begin{pmatrix} \text{tr } R_{\varphi_1\varphi_1}S & \text{tr } R_{\varphi_1\varphi_2}S & \cdots & \text{tr } R_{\varphi_1\varphi_n}S \\ \text{tr } R_{\varphi_2\varphi_1}S & \text{tr } R_{\varphi_2\varphi_2}S & \cdots & \text{tr } R_{\varphi_2\varphi_n}S \\ \vdots & & & \\ \text{tr } R_{\varphi_n\varphi_1}S & \text{tr } R_{\varphi_n\varphi_2}S & \cdots & \text{tr } R_{\varphi_n\varphi_n}S \end{pmatrix}$$

$$A_2(S) = \begin{pmatrix} \text{tr } R_{\varphi_1\gamma_1}S & \text{tr } R_{\varphi_1\gamma_2}S & \cdots & \text{tr } R_{\varphi_1\gamma_p}S \\ \text{tr } R_{\varphi_2\gamma_1}S & \text{tr } R_{\varphi_2\gamma_2}S & \cdots & \text{tr } R_{\varphi_2\gamma_p}S \\ \vdots & & & \\ \text{tr } R_{\varphi_n\gamma_1}S & \text{tr } R_{\varphi_n\gamma_2}S & \cdots & \text{tr } R_{\varphi_n\gamma_p}S \end{pmatrix}$$

$$A_3(S) = \begin{pmatrix} \text{tr } R_{\gamma_1\gamma_1}S & \text{tr } R_{\gamma_1\gamma_2}S & \cdots & \text{tr } R_{\gamma_1\gamma_p}S \\ \text{tr } R_{\gamma_2\gamma_1}S & \text{tr } R_{\gamma_2\gamma_2}S & \cdots & \text{tr } R_{\gamma_2\gamma_p}S \\ \vdots & & & \\ \text{tr } R_{\gamma_p\gamma_1}S & \text{tr } R_{\gamma_p\gamma_2}S & \cdots & \text{tr } R_{\gamma_p\gamma_p}S \end{pmatrix}$$

where φ_i and γ_i are defined by (4) and $R_{\varphi\gamma}$ by (5)

4. AN EXAMPLE

The equations become especially simple in the case of systems on companion form. Assume

$$
\Phi = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

In this case we get

$$
A_1 = \begin{pmatrix} \text{tr } SR_{\varphi_1 \varphi_1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix}
$$

$$
A_2 = \begin{pmatrix} \text{tr } SR_{\gamma_1 \varphi_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad A_2^T = \begin{pmatrix} \text{tr } SR_{\gamma_1 \varphi_1} & 0 & \dots & 0 \end{pmatrix}
$$

$$
A_3 = \text{tr } SR_{\gamma_1 \gamma_1}
$$

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