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STABILITY OF FEEDBACK SYSTEMS WITH RELAYS OR SATURATIONS

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Stability of Feedback Systems with Relays or Saturations

Per Molander

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1. INTRODUCTION

In an early article ([1]), Anosov gave necessary and sufficient conditions for the local stability of the null solution in a feedback system consisting of a linear, time-invariant link in the forward loop and a relay in the feedback loop. He was followed by Alimov ([2]), who stated sufficient conditions for the global asymptotic stability of the null solution in the same configuration. Since these papers were written, the status of non-linear stability theory has been raised considerably, and it is the object of this paper to apply modern frequency domain stability criteria to the above configuration.

The main work is done in Section 2. In the following section, the results are generalized to systems with an ideal saturation in the feedback loop, a case which may be reduced to the former one by means of a simple loop transformation. Finally, an application to minimum-variance controllers is given.

2. RELAY SYSTEMS

Consider the following single-input, single-output feedback system

$$\begin{cases} \dot{x} = Ax + bu \\ u = -\varphi(y) \\ y = c^T x + du \end{cases} \tag{1}$$

where $\varphi(\cdot)$ is defined by

$$\varphi(\sigma) = \mathrm{sgn}(\sigma) = \begin{cases} 1 & \sigma > 0 \\ -1 & \sigma < 0 \end{cases} \tag{2}$$

There are problems in defining $\varphi(0)$, since the right-hand side of (1a) will be discontinuous no matter how $u(0)$ is chosen, and the standard existence and uniqueness theorems for ordinary differential equations will not be valid. There are ways of overcoming this difficulty by defining generalized solutions to (1). Alternatively, $\varphi(\cdot)$ may be replaced by $\varphi_K(\cdot)$ defined as

$$\varphi_K(\sigma) = \begin{cases} & \sigma > 1/K \\ K\sigma & |\sigma| \le 1/K \\ & \sigma < -1/K \end{cases} \tag{3}.$$

If K is large, this is a good approximation of $\varphi(\cdot)$, which is in itself an idealised mathematical entity with little physical relevance. Besides, the system (1), (2) is not well-defined if $d \neq 0$, which will be the case in Section 3.

A thorough treatment of the well-posedness of the problem is beyond the scope of this paper, and it will be assumed a priori that these questions have been dealt with. The stability problem related to (1), (2) or (1), (3) is to determine conditions on c, A, and b, that ensure stability, locally or in the large (= globally), of the trivial solution. It is assumed that A is strictly Hurwitz (i.e. $\mathrm{Re}\,\lambda(A) < 0$).

2.1 Local Stability of the Null Solution

The conditions for local stability may be derived heuristically as follows. In a neighbourhood of the origin, the system (1),(3) is a linear feedback system with a large gain in the feedback loop. A necessary and sufficient condition for asymptotic stability is that the closed-loop poles lie in the open left half-plane. Thus define

$$G(s) \triangleq c^T(sI-A)^{-1}\, b + d \triangleq \frac{\sum_{i=0}^{n-1} b_i\, s^i}{s + \sum_{j=0}^{n-1} a_i\, s^i} + d \tag{4}$$

i) If $d \neq 0$, the closed-loop poles will tend to the zeros of $G(s)$ as $K \to \infty$.

ii) If $d = 0$, $b_{n-1} \neq 0$, $(n-1)$ of the closed-loop poles will tend to the zeros of $G(s)$, and the remaining one goes to $-\infty$.

iii) If $d = 0$, $b_{n-1} = 0$, $b_{n-2} \neq 0$, $(n-2)$ of the closed-loop poles will tend to the zeros of $G(s)$, and the remaining two poles will go to $-1/2\,(a_{n-1} - b_{n-3}/b_{-2}) \pm i.\,\infty$.

iv) If $d = 0$, $b_{n-1} = 0$, $b_{n-2} = 0$, the closed-loop system is unstable for large enough K.

Necessary conditions for the local stability of the trivial solution of (1), (3) are consequently:

i) $d \neq 0$, and the zeros of $G(s)$ are in the closed LHP, or

ii) $d = 0$, $b_{n-1} \neq 0$, and the zeros of $G(s)$ are in the closed LHP, or

iii) $d = 0$, $b_{n-1} = 0$, $b_{n-2} \neq 0$, $a_{n-1} \geq b_{n-3}/b_{n-2}$, and the zeros of $G(s)$ are in the closed LHP.

Sufficient conditions for stability are i), ii), or iii) with "closed LHP" replaced by "open LHP" and $a_{n-1} > b_{n-3}/b_{n-2}$ in iii).

For a justification of these arguments concerning the system (1), (2), see [1].

2.2 Global Stability of the Null Solution

2.2.1 Alimov's Condition

In this section, d is assumed to be 0. The main sufficient condition for global stability stated by Alimov ([2], eq.(28)) is that there exist a positive definite P satisfying

$$\begin{cases} A^T P + PA = -Q < 0 & (5a) \\ PA^{-1} b = -\mu c, \quad \mu > 0 & (5b) \end{cases}$$

Expressing Q as $qq^T + K$, where (q^T, A) is observable, the Meyer-Kalman-Yakubovich lemma [3] states that such P and q exist if and only if the transfer function $H(s) \triangleq -c^T(sI-A)^{-1}A^{-1}b$ is positive real. Since

$$H(s) = -\frac{c^T A^{-1} b}{s} - \sum_{k=0}^{\infty} \frac{c^T A^k b}{s^{k+2}} = -\frac{c^T A^{-1} b}{s} - \frac{G(s)}{s},$$

it can be inferred that $H(s)$ is P.R. if and only if $\frac{G(0)-G(s)}{s}$ is P.R. This requires, among other things, $\operatorname{Im} G(i\omega) \leq 0, \omega > 0$ and is thus but a special case of the Popov condition; see below.

2.2.2 Frequency Domain Stability Criteria

The reader is assumed to be acquainted with the standard theorems of non-linear stability theory. A complete reference is [4].

Consider again the system given by (1),(3). The forward loop is a linear, time-invariant link, characterized by its transfer function $G(s) = c^T(sI-A)^{-1}b + d$, and the feedback loop is a time-invariant, monotone—non-decreasing non-linearity contained in the sector $[0, K]$, where K is an arbitrarily large positive number. The following stability results are thus easily established:

i) By the circle criterion [5], the feedback system (1), (3) is asymptotically stable in the large if $G(s)$ is positive real ([4], p. 57).

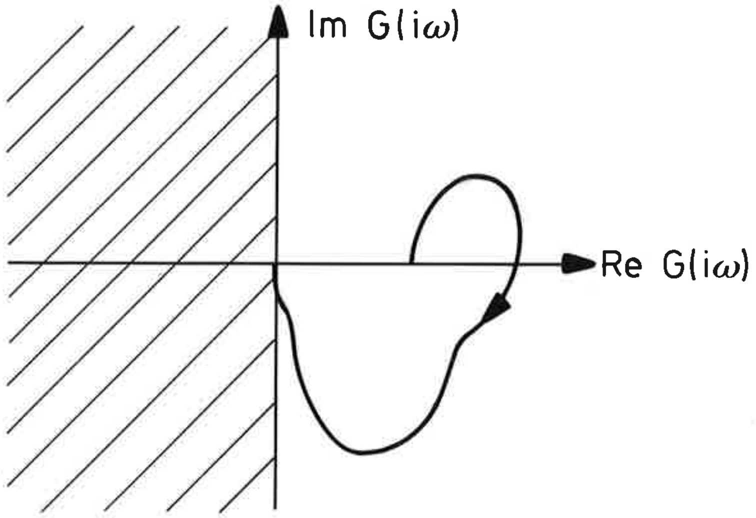


Fig. 1 - Stability by the circle criterion.

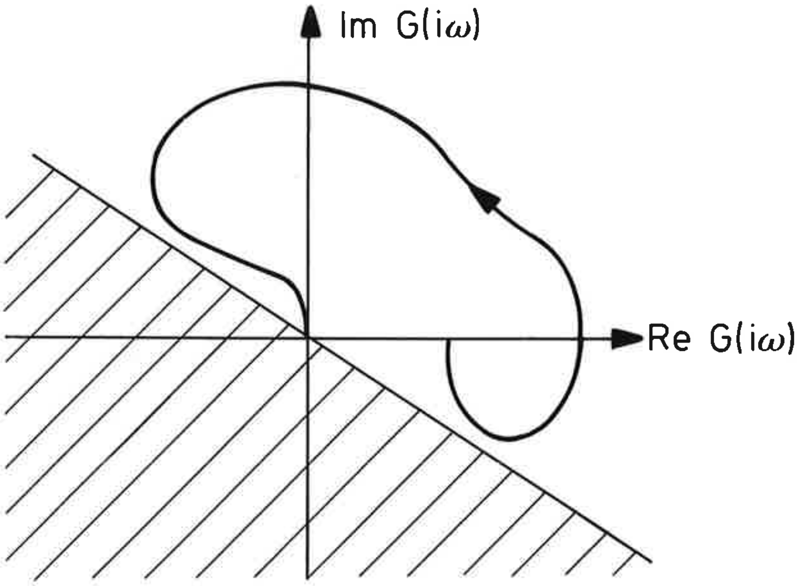


Fig. 2 - Stability by the off-axis circle-criterion.

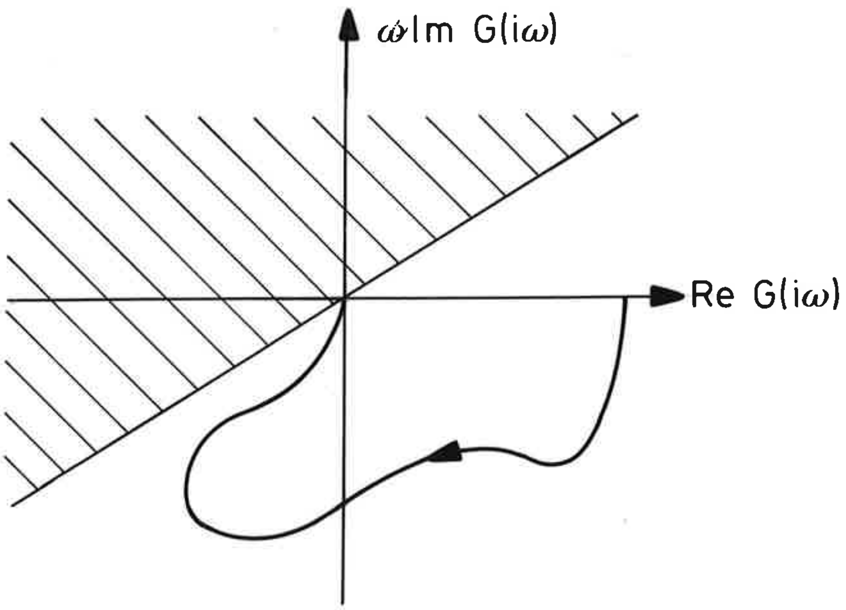


Fig. 3 - Stability by Popov's theorem.

ii) By the off-axis circle criterion [6], global asymptotic stability is ensured if the Nyquist curve of $G(s)$ lies to the right of a straight line through the origin with non-zero slope.

iii) By the Popov theorem [7], the feedback system (1), (3) is globally asymptotically stable if the Popov plot (i.e. $\omega \mathrm{Im}\, G(i\omega)$ versus $\mathrm{Re}\, G(i\omega)$) lies to the right of a straight line through the origin.

Remark 1. The assumption that A be strictly Hurwitz is unnecessarily strict. It can be shown that $G(s)$ may contain one integrator.

Remark 2. iii) implies stability if $\mathrm{Im}\, G(i\omega) < 0$, $\omega > 0$. Thus Alimov's condition (Section 2.2.1) is a special case of the above result. The reason for this can be found in a paper by Yakubovich [8]. Alimov's stability result is based on a Lyapunov function of the Lure-Postnikov form, which is a special case of Popov's Lyapunov function. In the above paper, Yakubovich proves that if stability can be inferred from this Lyapunov function for a special non-linearity, then, under general assumptions, stability is ensured for all non-linearities lying in the same sector, and Popov's frequency domain condition, being necessary and sufficient, is consequently fulfilled.

2.2.3 Multiplier Theory

First, a few definitions will be stated.

Definition 1: Let $\mathcal{H}$ be a Hilbert-space with scalar product $<\cdot|\cdot>$ and $\mathcal{H}_e$ its extension, i.e. $\mathcal{H}_e = \{ h;\ \| P_T f \| < \infty \ \ \forall T \}$; here, P_T denotes the truncation operator. Further, let H be an operator from $\mathcal{H}_e$ to $\mathcal{H}_e$. Then

i) H is passive iff $\exists \beta$ such that $< P_T Hx \mid P_T x > \ \geq \beta \ \ \forall T$,

ii) H is strictly passive iff $\exists \delta > 0$ and $\exists \beta$ such that $< P_T Hx \mid P_T x > \ \geq \delta \ \| P_T x \|^2 + \beta$.

The following theorem holds ([9], p. 182):

Passivity theory: The zero-solution in a feedback system consisting of a finite-gain, passive operator in the forward path and a strictly passive operator in the feedback loop is globally asymptotically stable.

The circly criterion may be viewed as a special case of this theorem. One way of proving Popov's theorem is by inserting the so-called Popov multiplier $(\alpha_1 s + \alpha_2)$, $\alpha_2 \neq 0$ together with its inverse in the foward and feedback loop, respectively, thereby exploiting the time-invariance of the non-linear link. If this link satisfies additional conditions, such as symmetry or incremental conicity, the class of multipliers may be extended. For odd, monotonic, incementally conic non-linearities, a large class of multipliers has been specified by Thathachar et al [10]. This class consists of positive real transfer functions, whose coefficients satisfy certain inequalities. Due to their complexity, these conditions are difficult to check in a practical case. It turns out that the class of admissible multipliers is more easily characterized in the time domain, as will be done below.

Consider the generic configuration of Fig. 4. In Fig. 4b, a multiplier $M(s)$ has been inserted in the forward path and its inverse in the feedback loop. From a stability point of view, the systems are equivalent. The aim of this section is to determine the class of multipliers that, cascaded with a relay, give a passive operator.

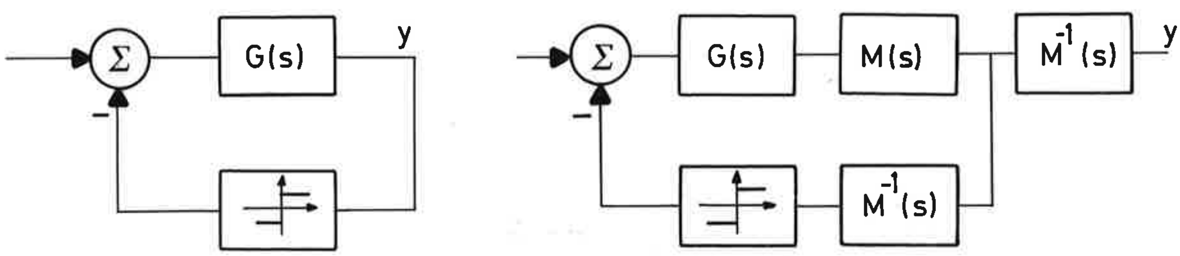


Fig. 4a - The generic confirugation.

Fig. 4b - An equivalent system with multipliers.

Definition: (Continuous time) M_C is the class of linear, time--invariant operators whose impulse response $g_0\delta_0 + g(t)$ $(g(t) = 0,$ $t < 0)$ satisfies $g_0 \geq \int_0^\infty |g(s)|\,ds$.

(Discrete time) M_D is the class of linear, time-invariant operators whose impulse response $g_0\delta_0 + \sum_{i=1}^{\infty} g_i\,\delta_i$ satisfies $g_0 \geq \sum_{i=1}^{\infty} |g_i|$; here $\delta_i(k) = 1$ if $k = i$, else 0.

Theorem. (Continuous time) Let Z be a linear, time-invariant operator and N the relay operator. Then NZ^{-1} is a passive operator if and only if $Z = M + \alpha\frac{d}{dt}$, where $M \in M_C$ and $\alpha \geq 0$.

(Discrete time) Let Z be a linear time-invariant operator and N the relay operator. Then NZ^{-1} is a passive operator if and only if $Z = M + \alpha\nabla$, where $M \in M_D$, $\alpha \geq 0$, and ∇ denotes the backward difference operator.

Definition 2: Denote this class of operators Z by $\overline{M}_C$ and $\overline{M}_D$, respectively.

Proof of the theorem. For simplicity of notation, only the discrete-time case will be proved. The continuous-time version is analogous, but somewhat more technical. The differentiator is omitted, since it is a standard multiplier in this context. Furthermore, the form (2) for the relay will be used; it is easy to see that this does not alter the class of admissible multipliers.

Sufficiency. Let $(u(k))_{k=0}^{\infty}$, $(v(k))_{k=0}^{\infty}$, and $(y(k))_{k=0}^{\infty}$ be sequences in ℓ_e^2 (i.e. locally finite), where $u(\cdot)$ is the input to M^{-1}, $v(\cdot)$ is the input to the relay, and $y(\cdot)$ is the output from the relay. NM^{-1} is a passive operator if and only if $W_T = \sum_{0}^{T} u(k)y(k) \geq 0$ for all $T \geq 0$. Since $y(k) = \mathrm{sgn}\,(v(k))$ and $u(k) = \sum_{i=0}^{k} g_i\, v(k-i)$,

$$W_T = \sum_{k=0}^{T} \operatorname{sgn} v(k) \sum_{i=0}^{k} g_i \; v(k-i) = g_0 \sum_{k=0}^{T} |v(k)| \; +$$

$$+ \sum_{k=0}^{T} \operatorname{sgn} v(k) \sum_{i=1}^{k} g_i \; v(k-i) \geq$$

$$\geq g_0 \sum_{k=0}^{T} |v(k)| - \sum_{k=1}^{T} |g_i| \sum_{k=0}^{T} |v(k)| \geq 0$$

according to the assumption.

Necessity. Assume that $\exists T$ such that $g_0 < \sum_{i=1}^{T} |g_i|$. Choose $v_0 = 1$, $v_i = -\varepsilon \operatorname{sgn}(g_i)$. Then $W_T = g_0 - \sum_{i=1}^{T} |g_i| + O(\varepsilon)$, $\varepsilon \to 0$. Hence the necessity.

Corollary 1. If there is an $M \in \overline{M}_C(\overline{M}_D)$ such that $G(s)M(s)$ $(G(z)\cdot M(z))$ is positive real, then the feedback system (1),(3) is asymptotically stable in the large.

Proof. Since a linear, time-invariant link is passive iff its transfer function is positive real, this is an immediate consequence of the passivity theorem. By means of a well-established inversion procedure (see e.g. [4], p. 109), it can be shown that if M is an admissible multiplier, then so is M^{-1}. Thus, the above corollary may be restated as

Corollary 1'. If there is an $M \in \overline{M}_C(\overline{M}_D)$ such that $G(s)M(s)^{\pm 1}$ $(G(z)M(z)^{\pm 1})$ is positive real, then the feedback system (1), (3) is globally asymptotically stable.

Example 1. Consider a feedback system with the transfer function

$$G(s) = \frac{s(s+1)(s+2)}{(s+10)(s^2+2s+2)(s^2+s+50)}$$

in the forward path and a relay in the feedback loop. As can be seen from the Nyquist diagram, Fig. 5, the Nyquist plot belongs to all four quadrants.

A short computation shows that

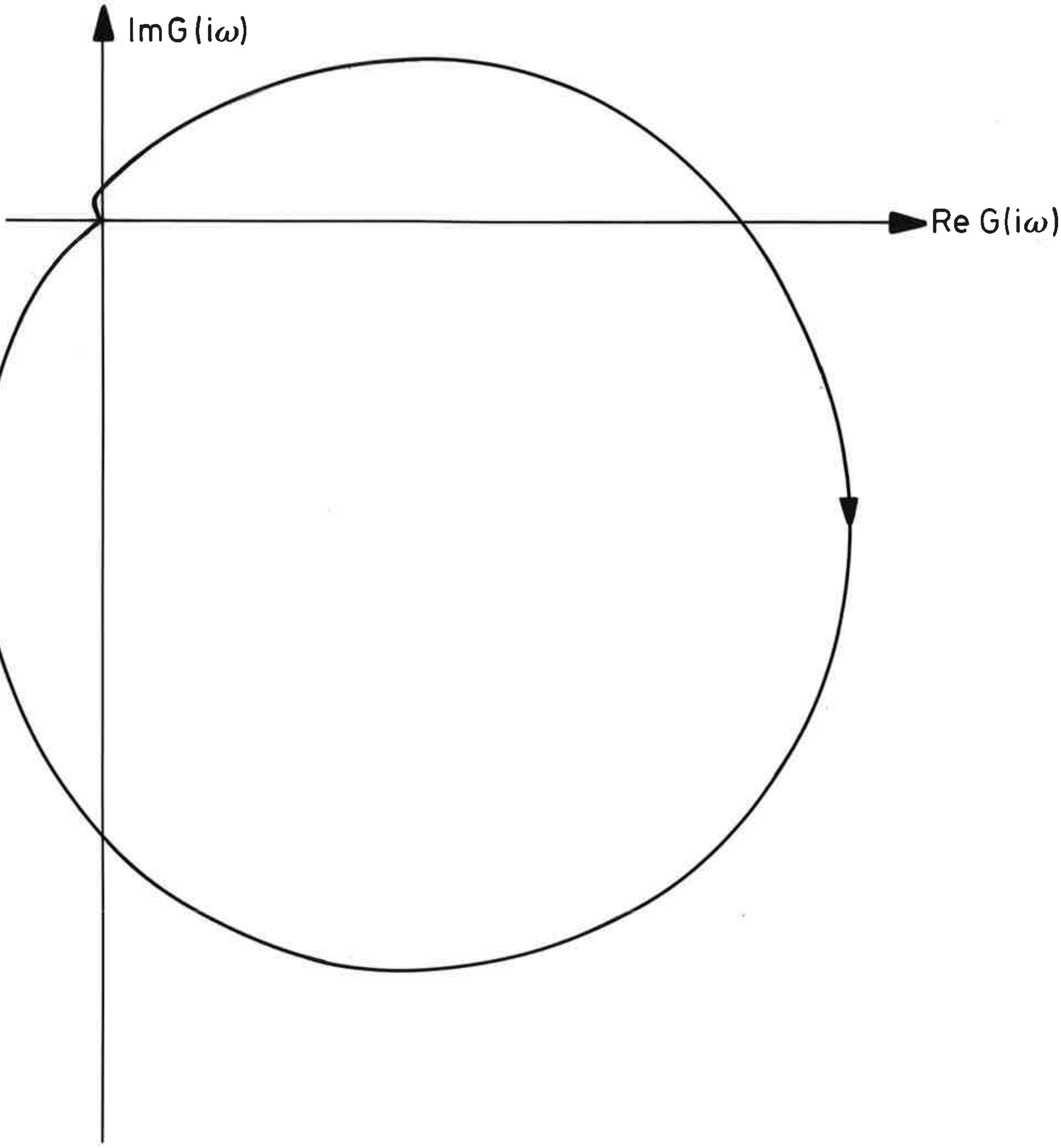


Fig. 5 - The Nyquist plot of $G(s) = \dfrac{s(s+1)(s+2)}{(s+10)(s^2+2s+2)(s^2+s+50)}$.

$$G_M(s) = \frac{(s+10)(s^2+2s+2)}{(s+1)(s+2)}$$

is an admissible multiplier, giving

$$G \cdot G_M(s) = \frac{s}{s^2+s+50} ,$$

which is positive real. The given feedback system is thus globally asymptotically stable.

2.2.4 Tsypkin's Method

For the sake of completeness, a brief account of Tsypkin's exact method for relay systems will be given. For a detailed exposition, see for instance [11].

Let, as usual, $G(s)$ be the transfer function of the linear link and let $G_1(s) = G(s) - G(\infty)$ be its strictly proper part. The Tsypkin locus $T(i\omega)$ is defined by

$$T(i\omega) = \sum_{k=0}^{\infty} \operatorname{Re} G_1((2k+1)i\omega) + i \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \operatorname{Im} G_1((2k+1)i\omega)$$

$$\omega \geq 0$$

For a symmetric limit cycle oscillation with frequency ω_0 to be sustained in the system, the following conditions must be satisfied:

$$\begin{cases} \operatorname{Im} T(i\omega_0) = \frac{\pi}{4} G(\infty) \\ \operatorname{Re} T(i\omega_0) < \frac{\pi}{4\omega_0} \lim\limits_{s\to\infty} (s\, G_1(s)) \end{cases}$$

The limit cycle is stable if $\frac{d}{d\omega} \operatorname{Im}(Ti\omega)\Big|_{\omega=\omega_0}$ is positive.

It should be stressed that the construction of the Tsypkin locus, graphical or analytical, is a rather time-consuming task. However, a great merit of this method is that it is exact and that it gives the possible frequences of self-sustained oscillations in the system.

3. SYSTEMS WITH AN IDEAL SATURATION

Consider a feedback system with a linear, time-invariant link in the forward path and an ideal saturation in the feedback loop. By an ideal saturation is meant

$$\psi(\sigma) = \begin{cases} 1 & \sigma > M \\ \sigma & |\sigma| \leq M \\ -1 & \sigma < -M \end{cases} \tag{6}$$

(For global stability considerations, the value of M is unimportant.) These systems are very important, since, in practice, all control signals are limited.

It is an easy matter to transfer the results from Section 2 to the system (1), (6). Consider Fig. 6.

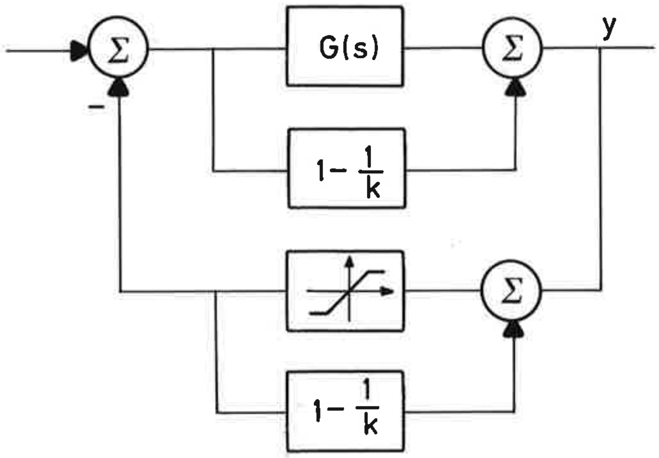


Fig. 6 - A loop transformation.

The transformed feedback system of Fig. 6 is equivalent to the generic configuration. It is left to the reader to verify that the non-linear link in Fig. 6 is the relay (3). The transfer function of the linear part is $G(s) + 1 - \frac{1}{K}$. As an example, the following stability result for the continuous time case is formulated:

Corollary 2. The feedback system (1), (6) is asymptotically stable in the large if there is an $M \in \overline{M}_C$ such that $(G(s) + 1 - \frac{1}{K})M(s)^{\pm 1}$ is positive real for large enough K.

Example 2. Consider a feedback system with the transfer function

$$G(s) = \frac{s^2}{(s^2+0.01)(s^2+100) + \varepsilon(s^3+s^2+s+10)}$$

in the forward path and an ideal saturation in the feedback loop. This is a variant of a famous counterexample to Aizermann's conjecture.

The Nyquist plot of $G(s) + 1$ is given in Fig. 7 for $\varepsilon = 0.03$. What is needed to show overall stability is a multiplier that pulls the bulge in the third quadrant into the right half-plane without effecting the phase at low frequencies. This may be achieved by

$$G_M(s) = \frac{s + 2.27}{s + 45.5},$$

which has its maximum phase-lead around $\omega = 10$ rads/sec, where the bulge is. From the Nyquist plot of $(G(s)+1)G_M(s)$, shown in Fig. 8, it can be inferred that the feedback system is asymptotically stable in the large.

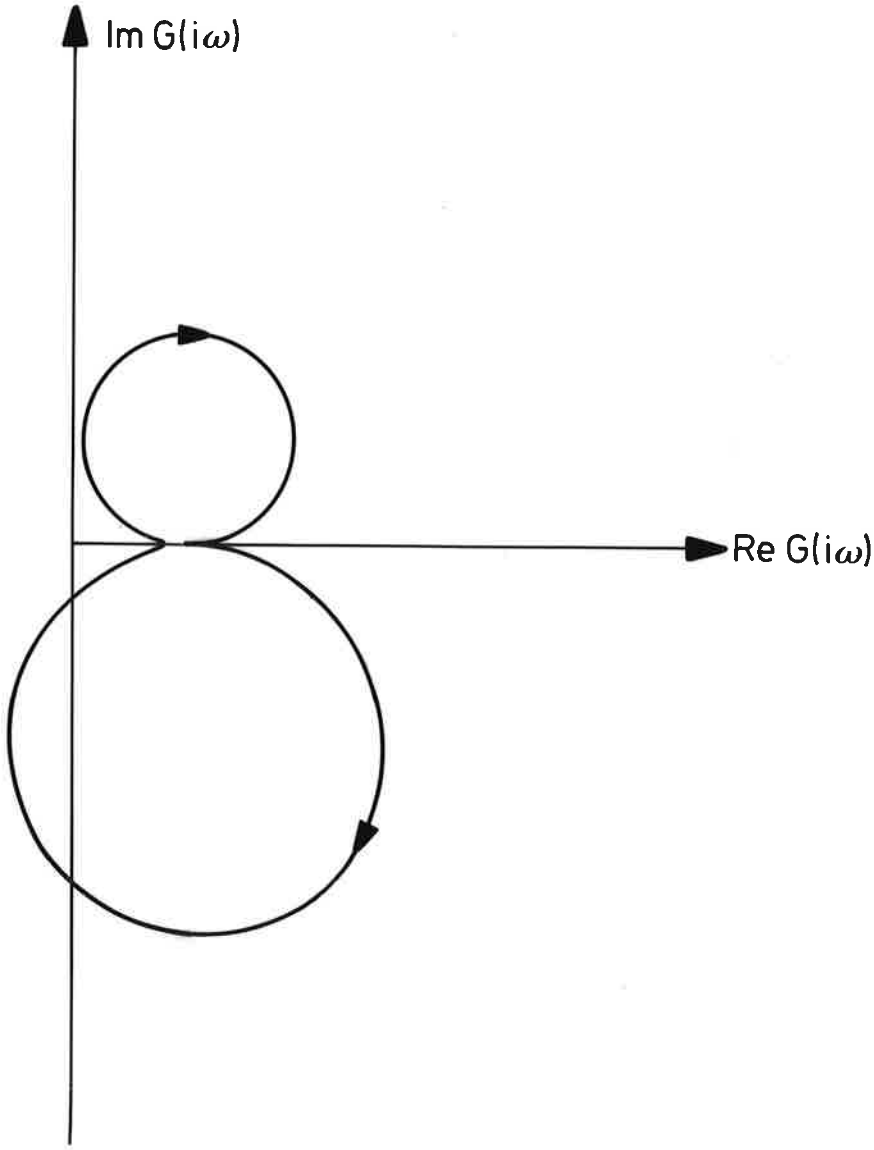


Fig. 7 - The Nyquist plot of

$$G(s)+1 = \frac{s^2}{(s^2+0.01)(s^2+100)+\varepsilon(s^3+s^2+s+10)} + 1.$$

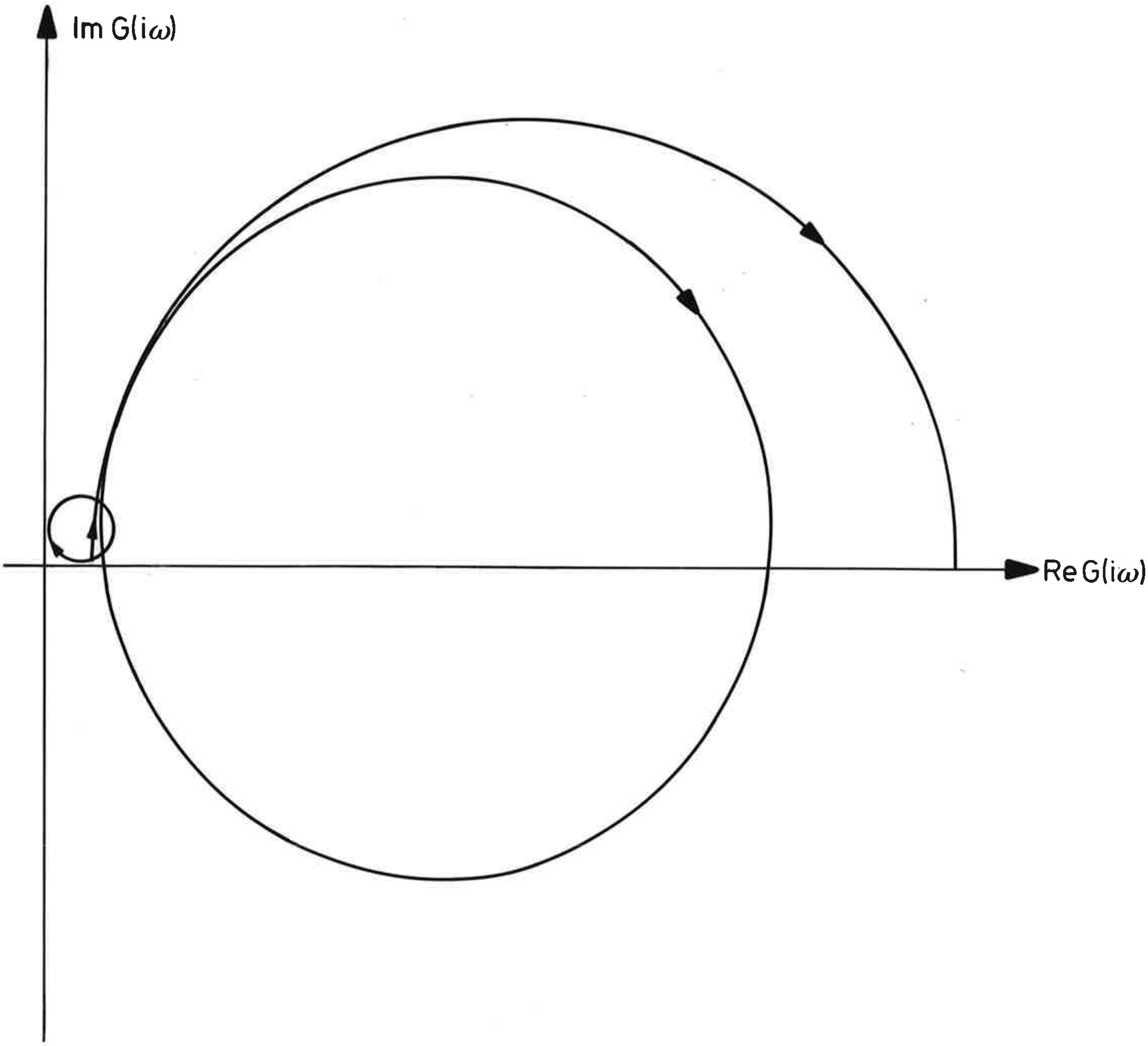


Fig. 8 - The Nyquist plot of $(G(s)+1)\cdot G_M(s)$, $G_M(s) = \dfrac{s + 2.27}{s + 45.4}$.

4. AN EXAMPLE: THE MINIMUM-VARIANCE REGULATOR WITH A BOUNDED CONTROL SIGNAL

Consider the difference equation

$$A^*(q^{-1})y = q^{-k} B^*(q^{-1})u + C^*(q^{-1})e \tag{7}$$

where $e(\cdot)$ is white noise and A^* and B^* are assumed to have all zeros outside the unit circle. The minimum-variance regulator [12] for the system represented by (7) is given by

$$u = - \frac{G^*(q^{-1})}{B^*(q^{-1})F^*(q^{-1})} y \tag{8}$$

where $G^*(q^{-1})$ and $F^*(q^{-1})$ satisfy the identity

$$\begin{cases} C^*(q^{-1}) = A^*(q^{-1}) F^*(q^{-1}) + q^{-k} G^*(q^{-1}) \\ \deg (F^*) = (k-1), \quad \deg (G^*) = (n-1). \end{cases} \tag{9}$$

Now assume that the control signal u is limited, $|u(t)| \leq K$. Two different control strategies are possible: either the minimum-variance regulator is run as usual, or else the bounds on the control signal are taken into account in some fashion.

Consider first the straight-forward regulator with no account for the limit on $|u|$. The block diagram is shown in Fig. 9.

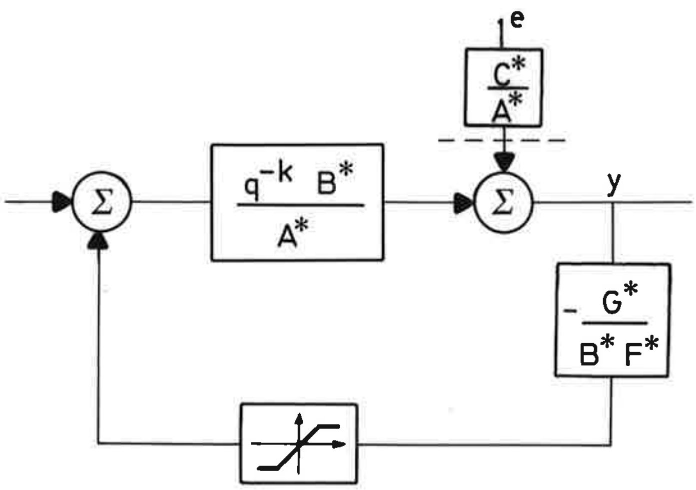


Fig. 9 - The minimum-variance regulator.

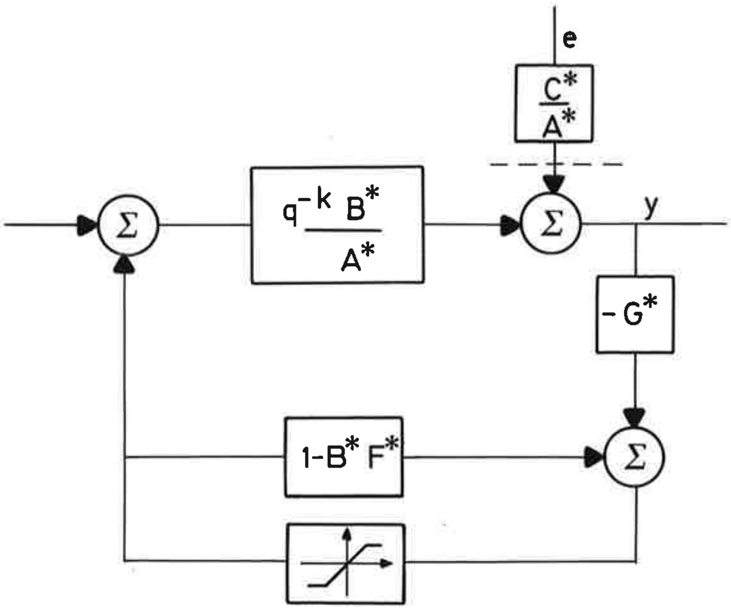
As seen from the non-linearity, the transfer function of the linear part of the system is $q^{-k} \frac{G^*}{A^*F^*}$. Applying the loop transformation of the preceding section shows that the relevant transfer function is

$$q^{-k} \frac{G^*}{A^*F^*} + 1 = q^{-k} \frac{G^* + A^*F^*}{A^*F^*} = \frac{C^*}{A^*F^*} .$$

The following result has thus been established:

<u>Result 1</u>. The system given by (7), (8), with $e \equiv 0$ and $|u| \leq K$ for some $K > 0$, is asymptotically stable in the large if there exists a multiplier M^* in $\overline{M}_D$ such that $\frac{C^*}{A^*F^*} \cdot (M^*)^{\pm 1}$ is positive real. It is interesting to note that the condition is independent of B^*.

There are various ways of taking the limitation on $|u|$ into account. One would be to measure the actual input at each instant and to replace the nominal value of $u(\cdot)$ in (8) by its actual value. This gives the feedback system of Fig. 10.



<u>Fig. 10</u> - A modified minimum-variance regulator.

The modified control law is

$$u = (1 - B^*F^*)\, u_a - G^*y, \tag{10}$$

where u_a denotes the actual value. The relevant transfer function is

$$(B^*F^* - 1) + q^{-k}\frac{B^*}{A^*}\cdot G^* + 1 = B^*\cdot\frac{A^*F^* + q^{-k}G^*}{A^*} = \frac{B^*C^*}{A^*}$$

Result 2. The system (7) with $e \equiv 0$ and $|u| \leq K$, $K > 0$, and the modified regulator (10), is globally asymptotically stable if there is an M^* in $\overline{M}_D$ such that $\frac{B^*C^*}{A^*}\cdot(M^*)^{\pm 1}$ is positive real.

A priori, there is nothing to be said in favour of this modified control law from the stability point of view; the system parameters determine whether one control law is to be preferred to the other.

It the conditions of Result 1 or 2 are not satisfied and the regulator saturates, this may produce self-sustained oscillations in the system. If the regulator saturates only due to initial conditions, this may be avoided by simply letting the transient settle before the regulator is activated. However, if saturation takes place in steady state during normal action due to a large noise variance, other precautions must be taken. A general approach is to replace C^* in (9) by a $\tilde{C}^*$, which satisfies the condition of Result 1 or 2. The transfer function from e to y will be $\frac{C^*}{\tilde{C}^*}\cdot F^*$, giving a larger variance than the optimal regulator, which is the price one has to pay in order to guarantee stability. On the other hand this drawback is to a certain extent illusory, since the control law (8) in presence of saturation is not necessarily optimal.

Finally, a few words chould be said about the case when A^* and B^* do not have all zeros outside the unit circle. Obviously, if A^* is unstable, the system cannot be globally asymptotically stable if u saturates. If B^* has zeros outside the unit circle, the minimum--variance controller will have to be modified in practice in order to eliminate extreme sensitivity to parameter variations. One way of doing this is by changing (8) and (9) into

$$u = - \frac{G^*(q^{-1})}{B_1^*(q^{-1})F^*(q^{-1})} y \tag{8'}$$

and

$$\begin{cases} C^*(q^{-1}) = A^*(q^{-1})F^*(q^{-1}) + q^{-k} B_2^*(q^{-1})G^*(q^{-1}) \\ \deg(F^*) = (n_2 + k - 1), \quad \deg(G^*) = (n-1) \end{cases} \tag{9'}$$

Here, $B_1^*(q^{-1})$ and $B_2^*(q^{-1})$ are the stable and the unstable parts of $B^*(q^{-1})$ respectively, and $n_2 = \deg B_2^*(q^{-1})$. An elementary calculation shows that the condition of Result 1 is unaltered (though with a new F^*) and that $\frac{B^*C^*}{A}$ in the condition of Result w should be replaced by

$$\frac{B_1^* \cdot C^*}{A^*} .$$

5. CONCLUSION

The powerful tools of modern non-linear stability theory has been applied to relay systems. The main drawback of the refined stability results of Section 2.2.3 as compared to the simpler (and more conservative) criteria of Section 2.2.2 is that they provide no algorithm for investigating the existence of the multipliers in question. As a whole, this is a trial-and-error process. However, the given examples should provide an indication of how the problem may be attacked.

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