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# On the Convergence of Iterative Learning Control <sup>★</sup>

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## Abstract

We derive frequency-domain criteria for the convergence of linear iterative learning control (ILC) on finite-time intervals that are less restrictive than existing ones in the literature. In particular, the former can be used to establish the convergence of ILC in certain cases where the latter are violated. The results cover ILC with non-causal filters and provide insights into the transient behaviors of the algorithm before convergence. We also stipulate some practical rules under which ILC can be applied to a wider range of applications.

*Key words:* Learning control; Iterative improvement; Convergence analysis; Transient stability analysis; Stability criteria.

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## 1 Introduction

The main application of iterative learning control (ILC) is to improve the reference tracking performance of a system. In order to reduce the tracking error, the control signal to the system is adjusted in each iteration by using feedback information from previous iterations. In effect, ILC finds an approximate system inverse for a specific reference (Moore et al., 1989). An advantage of ILC is that it does not require an explicit model of the transfer function or even linearity of the system for finding the inverse. Instead, it often uses the actual system as a part of the algorithm. ILC has found successful applications in many different fields (Ahn et al., 2007; Freeman et al., 2012; Sörnmo et al., 2016), where accurate models of the system and disturbances are difficult to obtain.

While the frequency domain is the preferred approach for filter design and analysis of linear ILC (Wang et al., 2014), the widely used convergence criterion applies, only to strictly monotone convergence of the algorithm (the 2-norm of the error between the current control sig-

nal and its final value strictly decreases in each iteration). Moreover, it is not theoretically clear to what extent the frequency criterion is applicable to a practical ILC system where each iteration runs only over a finite-time interval and to ILC systems with non-causal filters. To motivate this study, we demonstrate examples for which the ILC converges but the classical frequency condition cannot provide any indication of the convergence property. Our analysis gives an explanation for this mode of convergence.

We extend the work of Norrlöf and Gunnarsson (2002) by introducing a less conservative criterion, hence reducing the gap between the existing time-domain and frequency-domain criteria. We also provide an analysis of the transient behavior of the algorithm, which proves useful when the convergence is not monotone. The contributions of this article can be summarized as follows:

- Analysis of “convergence on finite-time interval” motivated by practical ILC where the trial length is finite.
- A less conservative frequency domain convergence criterion than the one by Norrlöf and Gunnarsson (2002) is derived (see Theorem 8)

$$\inf_{\rho > 0} \sup_{\omega} |G(\rho e^{i\omega})| < 1.$$

The criterion is applicable to ILC systems with *causal* as well as *non-causal* filters and for strictly monotone convergence coincides with the classical result.

- The connection between time-domain and frequency-domain criteria is established in a rigorous manner

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using Toeplitz operators.

- A frequency domain tool for understanding the transient behavior of ILC—i.e., the wave of convergence/divergence—is introduced.
- A strategy to limit the growth of the transient errors when the convergence is not monotone is proposed.

### 1.1 Previous work

ILC is a two-dimensional process, in the sense that the dynamics are indexed by both time and iteration variables (Kurek and Zaremba, 1993). A standard approach to analysis of linear and a certain class of nonlinear ILC algorithms relies on the lifted-system framework, i.e., considering a time series as a vector (Bristow et al., 2006). Norrlöf (2000) has extensively studied the theory and applications of linear ILC. Time-domain criteria as well as a classical frequency-domain criterion for the convergence of the linear ILC algorithm have been derived by Norrlöf and Gunnarsson (2002).

There have been many attempts to understand and improve the convergence properties of the linear ILC. Longman and Huang (2002) have noted that the algorithm might practically diverge after an initial substantial decay of the tracking error. Elci et al. (2002) have introduced a non-causal filter, namely a zero-phase filter, in the algorithm to improve the transient behavior. The transient properties of the convergence have been studied in more detail by Longman and Huang (2002) and Wang et al. (2014). Longman (2000) and Norrlöf and Gunnarsson (2002) have commented on the potential convergence of the algorithm despite a transient growth of the norm of the error, i.e., when the classical frequency condition is not fulfilled.

### 1.2 Problem description

A general form of the discrete linear first-order ILC algorithm is

$$y_j = T_r r + T_u u_j \quad (1)$$

$$e_j = r - y_j \quad (2)$$

$$u_j = Q(u_{j-1} + L e_{j-1}); \quad (3)$$

see Norrlöf (2000). Here  $j \in \mathbb{Z}_{\geq 0}$  is the iteration index,  $r, y_j$ , and  $e_j \in \ell_2$  are the reference, output, and tracking error signals, respectively,  $u_j \in \ell_2$  is the control signal. The stable systems from reference to output and control signal to output are denoted by  $T_r$  and  $T_u$ , respectively, and  $Q$  and  $L$  are filters to be designed. The choice of  $u_0$  is free. Figure 1 depicts the ILC algorithm. Note that in practice the trial length is finite, i.e., the system is stopped after  $N$  samples and signal values at time  $n \in \{0, \dots, N-1\}$  are stored. The filters  $Q$  and  $L$  do not need to be causal since they operate on the signals of the previous iteration.

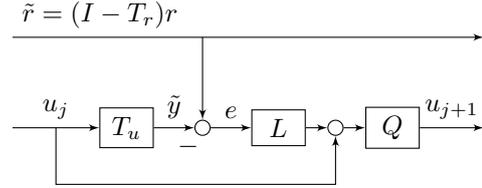


Fig. 1. Block diagram of an iterative learning controller. Here,  $\tilde{y} = y - T_r r$ .

Let us define  $G(e^{i\omega}) := Q(e^{i\omega})(1 - L(e^{i\omega})T_u(e^{i\omega}))$ . The classical sufficient condition for strictly monotone convergence of ILC requires that (see for example Norrlöf and Gunnarsson (2002))

$$|G(e^{i\omega})| < 1, \quad \forall \omega \in [0, 2\pi), \quad (4)$$

where  $L(e^{i\omega})$ ,  $T_u(e^{i\omega})$ , and  $Q(e^{i\omega})$  are the frequency representations of the respective filters.

Given the definition of the ILC algorithm in (1)–(3) and the fact that each iteration runs only over a finite-time interval,  $n \in \{0, \dots, N-1\}$ , our purpose is to find less restrictive conditions for  $G$  that guarantee the convergence of the algorithm, i.e., that the limits  $u_j \rightarrow u_\infty$  and  $e_j \rightarrow e_\infty$  exist for the finite trial length.

The rest of the article is organized as follows: In Sec. 2, we present a motivating example for which the ILC converges but the classical condition cannot provide any indication of the convergence property. The iteration-domain dynamics for ILC are derived in Sec. 3 before we delve into the issue of convergence. Section 4 starts with a formal definition of convergence for iterative procedures and states our convergence results. In Sec. 5, practical aspects concerning the transient behavior of ILC when the convergence is non-monotone is discussed. We propose qualitative measures that characterize the convergence, and discuss the gap between the time- and frequency-domain criteria in Sec. 6. We draw conclusions in Sec. 7. Additionally, a list of useful results and definitions as the background are collected in Appendix A.

## 2 Motivating example

Let us consider the following transfer functions

$$T_u(s) = \frac{1}{(s+1)(s^2+0.8s+16)}, \quad T_r(s) = 0, \quad (5)$$

$$Q(s) = \frac{10}{s+10}, \quad L_d(z) = 10k(1-0.9z^{-1})z^a. \quad (6)$$

We discretize  $T_u(s)$  and filter  $Q(s)$  by the zero-order-hold (ZOH) method (see Åström and Wittenmark, 1997) with sampling time  $h = 0.1$  s. Figure 2 compares the time responses of the systems corresponding to two ILC scenarios where in 1)  $k = 0.8$ ,  $a = 5$  (System I) and in

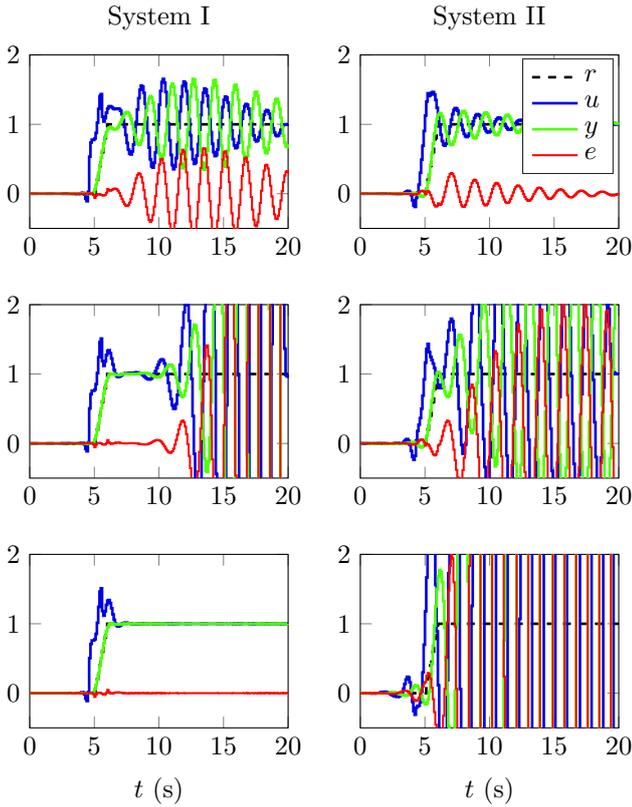


Fig. 2. The 4th, 12th, and 50th iterations for example (5)–(6): The left column shows System I (blue curve in Fig. 3) and the right column shows System II (green curve in Fig. 3). The dashed black, solid green, blue, and red curves correspond to the reference  $r$ , output  $y$ , control signal  $u$ , and error  $e$ , respectively. In the left column, the error signal looks like a growing wave which moves toward plus infinity as the iteration number increases. However, in the right column the wave does not move and the signals grow unbounded in the time region of the trial.

scenario 2)  $k = 0.5$ ,  $a = 8$  (System II). After discretization,  $Q$  is implemented as a zero-phase filter and hence we get

$$G(e^{i\omega}) = Q_d(e^{i\omega})Q_d(e^{-i\omega})(1 - L_d(e^{i\omega})T_{ud}(e^{i\omega})). \quad (7)$$

In Fig. 3, the Bode plots for  $G(e^{i\omega})$  are illustrated. We see that in both scenarios the condition  $|G(e^{i\omega})| < 1$  is violated. Nevertheless, System I appears to converge, at least for the time region of interest, while System II does not. The Bode diagrams corresponding to convergent and non-convergent scenarios may seem counterintuitive at first glance since the one with the highest peak in the gain  $|G(e^{i\omega})|$  corresponds to the convergent case.

Our result in Theorem 8 explains the situation and says that if there exists a  $\rho > 0$  such that  $\sup_{\omega} |G(\rho e^{i\omega})| < 1$ , then we have convergence in the sense that  $u_{\infty}$  and  $e_{\infty}$  exist on the finite interval  $[0, \dots, N)$ . In Fig. 4, where  $\sup_{\omega} |G(\rho e^{i\omega})|$  is plotted against  $\rho$ , it can be seen that

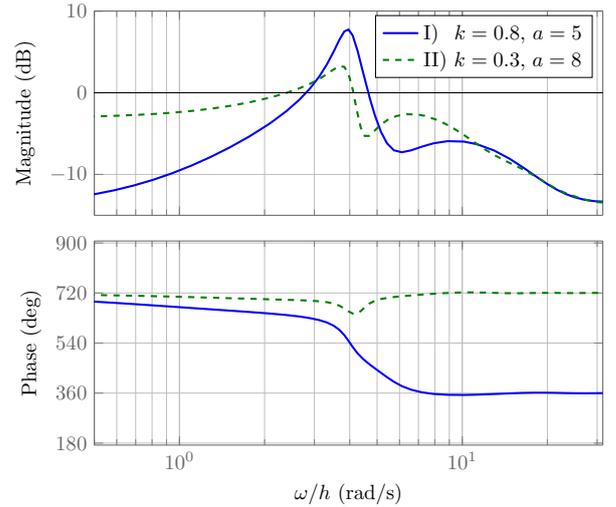


Fig. 3. Comparison of Bode diagrams for  $G(e^{i\omega})$ : The frequency response of System I (left column in Fig. 2) is in blue and System II in green (right column in Fig. 2), respectively. Both systems violate (4).

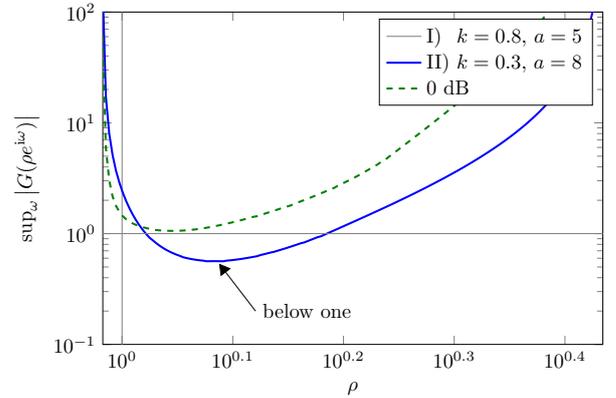


Fig. 4. Comparison of  $\|G(\rho z)\|_{\infty}$  for  $0.96 < \rho < 2.72$ : The blue and green curves correspond to System I and System II, respectively. We prove in Theorem 8 that if the plot of  $\|G(\rho z)\|_{\infty}$  vs.  $\rho > 0$  goes below one, ILC converges on finite time intervals.

the curve for System I goes below 1 for some  $\rho$  and thus the ILC algorithm converges.

### 3 Iteration-domain dynamics

In order to analyze the convergence of the ILC system (1)–(3), we derive the dynamics of the system in the iteration domain. Furthermore, to take into account the assumption of the finite-time intervals, we define the truncated counterparts of the original operators.

Define the truncation operator as

$$(\Pi_k x)[n] = \begin{cases} x[n], & n < k \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

For an operator  $G : \ell_2 \rightarrow \ell_2$ , we define the truncated operator

$$\bar{G} := \Pi_N(I - \Pi_0)G\Pi_N(I - \Pi_0), \quad (9)$$

where  $\Pi_N(I - \Pi_0)$  sets the values of a signal outside of  $\{0, \dots, N-1\}$  to zero and the finite Toeplitz matrix

$$T_{N\{i,j\}}(g) = g[i-j], \quad i, j = 1, \dots, N, \quad (10)$$

where  $g$  is the impulse response of  $G$ .

To summarize the notation, we use the symbols  $G$ ,  $\bar{G}$ ,  $G(z)$ ,  $g$ , and  $T_N(g)$ , which read as a bounded linear time-invariant operator  $\ell_2 \rightarrow \ell_2$ , a bounded linear time-varying operator  $\ell_2 \rightarrow \ell_2$  operating on signals with a finite support  $n \in \{0, \dots, N-1\}$ , the transfer function of a stable LTI system, which gives us the frequency response when evaluated on the unit circle  $z = e^{i\omega}$ , the impulse response of  $G$  obtained by the inverse  $z$ -transform of  $G(z)$ , and the finite Toeplitz matrix corresponding to  $\bar{G}$ , respectively. For more detailed descriptions and some fundamental results, the readers are advised to look at the background in Appendix A.

Using the truncated operators, system equations (1)–(3) can be rewritten as

$$u_{j+1} = \bar{G}u_j + \bar{H}\tilde{r} \quad (11)$$

$$e_j = -\bar{T}_u u_j + \tilde{r}, \quad (12)$$

where  $\bar{G} := \bar{Q}(I - \bar{L}\bar{T}_u)$ ,  $\bar{H} := \bar{Q}\bar{L}$  and  $\tilde{r} := (I - \bar{T}_r)r$ , hence

$$u_j = \sum_{i=0}^{j-1} \bar{G}^i \bar{H}\tilde{r} + \bar{G}^j u_0 \quad (13)$$

$$e_j = (I - \bar{T}_u) \sum_{i=0}^{j-1} \bar{G}^i \bar{H}\tilde{r} - \bar{T}_u \bar{G}^j u_0, \quad (14)$$

where  $\bar{G}^j$  denotes  $j$  times composition of  $\bar{G}$  by itself defined in (A.1).

It is desired that the outcome of the algorithm be independent of the choice of initial input  $u_0$ . Therefore, as  $j$  tends to infinity, the response to the initial conditions  $\bar{T}_u \bar{G}^j u_0$  must vanish. Additionally, the forced response due to the reference signal  $(I - \bar{T}_u) \sum_{n=0}^{j-1} \bar{G}^n \bar{H}\tilde{r}$  should be bounded on the desired interval.

Assuming  $\|T_N(g)\|_2 < 1$ , (13) and (14) converge respectively to

$$u_\infty = (I - \bar{G})^{-1} \bar{H}\tilde{r} \quad (15)$$

$$e_\infty = (I - \bar{T}_u(I - \bar{G})^{-1} \bar{H})\tilde{r}, \quad (16)$$

where  $(I - \bar{G})^{-1}$  is the operator  $\ell_2 \rightarrow \ell_2$  corresponding to  $(I - T_N(g))^{-1}$ . The result can be derived by using Lemmas 12 and 15 in the appendix. Generally, whether the residual  $e_\infty$  is acceptable or not depends on the length of the experiment, the signal  $\tilde{r}$ , and the norm of the operator in (16).

Note that if  $\bar{T}_u$  is right invertible, we can express the error as

$$e_\infty = \bar{T}_u(I - \bar{G})^{-1}(I - \bar{Q})\bar{T}_u^{-1}\tilde{r}$$

Similarly, if  $\bar{L}$  is left invertible and  $\bar{Q} = \bar{q}I$ , we have

$$e_\infty = \bar{L}^{-1}(I - \bar{G})^{-1}(I - \bar{Q})\bar{L}\tilde{r}$$

Therefore,  $\bar{Q} = I$  results in  $e_\infty \equiv 0$ . However, this choice may not fulfill the convergence condition and degrades the robustness of ILC (de Roover, 1996). Hence, there is a trade-off between the convergence property and the residual error.

#### 4 Convergence of iterative procedures

Analysis of linear iterative procedures such as (11) can be well understood using linear system theory (Norri f and Gunnarsson, 2002). First, we formalize the notion of convergence analogously to linear systems. Thereafter, we state time and frequency domain criteria for convergence.

**Definition 1** We say an iterative procedure  $\bar{G}, \bar{H} \neq 0$

$$u_{j+1} = \bar{G}u_j + \bar{H}r_j \quad (17)$$

is convergent iff for all  $u_0 \in \ell_2$  and iteration-independent inputs  $r_j \equiv r \in \ell_2$ , there exists an equilibrium signal  $u_e$  such that for any given  $\epsilon > 0$ , there exists  $J = J(\epsilon, u_0, r) \in \mathbb{Z}_{\geq 0}$  such that

$$\forall j > J \Rightarrow \|u_j - u_e\|_2 < \epsilon.$$

**Definition 2** We say the convergence is strictly monotone if in addition to Definition 1,

$$\|u_{j+1} - u_e\|_2 < \|u_j - u_e\|_2. \quad (18)$$

**Remark 3** Definition 1 translates to uniform asymptotic stability of the lifted system described by matrices  $\mathbf{G} := T_N(g)$  and  $\mathbf{H} := T_N(h)$ ,

$$\mathbf{u}_{j+1} = \mathbf{G}\mathbf{u}_j + \mathbf{H}\mathbf{r}, \quad (19)$$

where  $T_N(g)$  is defined in (10) and  $\mathbf{u}, \mathbf{r} \in \mathbb{R}^N$  are defined as

$$\mathbf{u} := (u[0], \dots, u[N-1])^T,$$

$$\mathbf{r} := (r[0], \dots, r[N-1])^T.$$

Note that the time references of inputs and outputs are assumed to be the same and do not change from one iteration to the next. Therefore, the representation of the lifted system is independent of the relative degree of the system.

**Proposition 4** *The iterative procedure (17) converges according to*

$$\text{Definition 1} \iff \text{rad } \mathbf{G} < 1 \quad (20)$$

$$\text{Definition 2} \iff \bar{\sigma}(\mathbf{G}) < 1 \quad (21)$$

**PROOF.** The first statement is a well-known result for asymptotic stability of linear systems (19); see Rugh (1996). For the proof of sufficiency of the second statements see Norrlöf and Gunnarsson (2002). To show necessity, suppose that (20) holds, i.e.,  $\mathbf{u}_j$  in (19) converges to the limit  $\mathbf{u}_e$  satisfying  $\mathbf{u}_e = \mathbf{G}\mathbf{u}_e + \mathbf{H}\mathbf{r}$ . Suppose to the contraposition that  $\bar{\sigma}(\mathbf{G}) \geq 1$  and let  $\mathbf{G} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$  be a singular decomposition, where  $\mathbf{V} = [\mathbf{V}_1, \dots, \mathbf{V}_n]$ ,  $\mathbf{U} = [\mathbf{U}_1, \dots, \mathbf{U}_n]$  and  $\mathbf{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_n)$  with  $\sigma_1 \geq \dots \geq \sigma_n$ . By choosing  $\mathbf{u}_0 = \mathbf{u}_e + \mathbf{V}_1$ , it follows that  $\mathbf{u}_1 = \mathbf{u}_e + \sigma_1\mathbf{U}_1$ , whereby (18) is violated.

**Lemma 5** *Given  $x \in \ell_2$  and  $\rho \in \mathbb{R}$ , if  $y = \bar{G}x$ , then*

$$\rho^{-[ \cdot ]} y = \bar{G}_\rho(\rho^{-[ \cdot ]} x), \quad (22)$$

where  $\bar{G}_\rho$  is the truncated operator corresponding to the system  $G_\rho(z) := G(\rho z)$  using the def. (9) and  $\rho^{-[ \cdot ]} : \mathbb{Z} \rightarrow \mathbb{R}$  evaluated at  $n$  is equal to  $\rho^{-n}$ .

**PROOF.** Let us denote the impulse response of  $G_\rho$  by  $g_\rho$ . From definition (10), we conclude  $\forall i, j \in [1, N]$

$$T_N\{i, j\}(g_\rho) = g_\rho[i - j] = \rho^{-i+1} g[i - j] \rho^{j-1}.$$

With  $P := \text{diag}(1, \rho, \dots, \rho^{N-1})$ , we get

$$T_N(g_\rho) = P^{-1} T_N(g) P. \quad (23)$$

Using the matrix representation of  $y = \bar{G}x$  according to Lemma 12 in the appendix and multiplying both sides from the left by  $P^{-1}$ , we obtain

$$P^{-1} \mathbf{y} = P^{-1} T_N(g) P P^{-1} \mathbf{x} = T_N(g_\rho) P^{-1} \mathbf{x},$$

which is (22) in its lifted form.

**Lemma 6** *Assume  $G(z)$  is a (not necessarily causal) LTI system and there exists a  $\rho > 0$  such that  $\{|z| = \rho\} \subset \text{ROC}$  of  $G(z)$  and  $\|G(\rho z)\|_\infty < 1$ . Then*

$$\lim_{j \rightarrow \infty} \bar{G}^j u_0 = 0, \quad \forall u_0 \in \ell_2. \quad (24)$$

**PROOF.** We prove the result for the equivalent Toeplitz matrix representation of the system  $\bar{G}$  according to Lemma 12 in the appendix. Note that because of the similarity transformation (23),  $T_N(g)$  has the same spectrum as  $T_N(g_\rho)$ . From Lemmas 14 and 16 in the appendix, it follows

$$\begin{aligned} \text{rad } T_N(g) &= \text{rad } T_N(g_\rho) \\ &\leq \|T_N(g_\rho)\|_2 \leq \|G(\rho z)\|_\infty < 1. \end{aligned} \quad (25)$$

The proof is completed by applying (20).

**Remark 7** *Considering Prop. 4 and Lemma 13 in the appendix, (25) guarantees strictly monotone convergence for the truncated operators if  $\|G(\rho z)\|_\infty < 1$  for  $\rho = 1$ . Therefore, the standard result for the convergence of ILC (Norrlöf and Gunnarsson, 2002) for causal filters is also applicable to non-causal filters.*

The following theorem establishes a frequency-domain criterion for the convergence of the ILC scheme (1)–(3). It is more widely applicable than the standard criterion in the literature (see for example Bristow et al. (2006); Norrlöf and Gunnarsson (2002); Longman (2000)) in the sense that it is not an asymptotic result for the case of  $N \rightarrow \infty$ , covers ILC with non-causal filters, and applies to the convergence according to Definition 1 as well as Definition 2 by setting  $\rho = 1$ .

**Theorem 8** *Given an LTI system  $G(z)$  with the impulse response  $g$ , assume that there exists a  $\rho > 0$  such that  $\{|z| = \rho\} \subset \text{ROC}$  of  $G(z)$  and*

$$\|G(\rho z)\|_\infty < 1. \quad (26)$$

*Then the following limits for ILC system (1)–(3) with interval length of  $N$  hold:*

$$\begin{aligned} \lim_{j \rightarrow \infty} u_j &= u_\infty = (I - \bar{G})^{-1} \bar{H} \tilde{r} \\ \lim_{j \rightarrow \infty} e_j &= e_\infty = (I - \bar{T}_u (I - \bar{G})^{-1} \bar{H}) \tilde{r} \end{aligned}$$

where  $\bar{G} := \bar{Q}(I - \bar{L}\bar{T}_u)$ ,  $\bar{H} := \bar{Q}\bar{L}$  and  $\tilde{r} := (I - \bar{T}_r)r$ .

**PROOF.** The limits of (13) and (14) need to be calculated. For the proof, we use the corresponding matrix representation of the truncated systems, e.g.,  $T_N(g)$  instead of  $\bar{G}$  according to Lemma 12 in the appendix. First note that

$$\begin{aligned} \lim_{j \rightarrow \infty} \sum_{i=0}^{j-1} T_N^i(g) &= P^{-1} \left( \lim_{j \rightarrow \infty} \sum_{i=0}^{j-1} (PT_N(g)P^{-1})^i \right) P \\ &= P^{-1} (I - PT_N(g)P^{-1})^{-1} P = (I - T_N(g))^{-1}. \end{aligned} \quad (27)$$

To derive (27), we have used the fact that  $\|PT_N(g)P^{-1}\| \leq \|G(\rho z)\|_\infty < 1$ , and Lemma 15 in the appendix. Moreover, according to Lemma 6,

$$\lim_{j \rightarrow \infty} T_N^j(g) = 0. \quad (28)$$

Substituting (27) and (28) into the limits of (13) and (14) as  $j \rightarrow \infty$  completes the proof.

## 5 Practical considerations

When  $\|G(z)\|_\infty > 1$ , the ILC lacks the strictly monotone convergence property. Hence, the norm of the signals may grow as iterations proceed. This section provides some insights into this transient behavior and suggests a solution to upper bound the growth of the error under a certain condition.

**Theorem 9** *Given  $G(z)$  with the impulse response  $g$ , let  $u_{j+1} = \bar{G}u_j$  for some  $u_0$  and  $j \in \mathbb{Z}_{\geq 0}$ . Then for every  $\rho \geq 1$  for which  $\{|z| = \rho\} \subset \text{ROC of } \bar{G}(z)$ , there exists a  $C > 0$  such that*

$$\forall n \in \mathbb{Z}, \forall j \in \mathbb{Z}_{\geq 0}, \quad \|\Pi_n u_j\|_2 \leq C \rho^n \|G(\rho z)\|_\infty^j. \quad (29)$$

**PROOF.** We split  $\rho^{-[\cdot]}u_j$  such that

$$\left\| \rho^{-[\cdot]}u_j \right\|_2 = \left\| \Pi_n \rho^{-[\cdot]}u_j \right\|_2 + \left\| (I - \Pi_n) \rho^{-[\cdot]}u_j \right\|_2. \quad (30)$$

Furthermore, note that when  $\rho \geq 1$ ,

$$\rho^{-n} \|\Pi_n u_j\|_2 \leq \left\| \Pi_n \rho^{-[\cdot]}u_j \right\|_2. \quad (31)$$

Combining (30) and (31) results in

$$\rho^{-n} \|\Pi_n u_j\|_2 \leq \left\| \rho^{-[\cdot]}u_j \right\|_2 - \left\| (I - \Pi_n) \rho^{-[\cdot]}u_j \right\|_2. \quad (32)$$

Considering (32) and Lemma 5, we obtain

$$\|\Pi_n u_j\|_2 \leq \rho^n \left\| \rho^{-[\cdot]}u_j \right\|_2 \leq \rho^n \|T_N(g_\rho)\|_2^j \left\| \rho^{-[\cdot]}u_0 \right\|_2. \quad (33)$$

Note that  $\|\rho^{-[\cdot]}u_0\|_2$  is constant. By considering Lemma 16 in the appendix, the final result follows.

**Remark 10** *If for some  $\rho > 1$ ,  $\|G(\rho z)\|_\infty < 1$ , Theorem 9 intuitively means that the iteration operator  $\bar{G}$  can shift the energy distribution of a signal only toward plus infinity. Define  $\tilde{u}_j := u_j - u_e$ , which results in  $\tilde{u}_{j+1} = \bar{G}\tilde{u}_j$ . By equating the right hand side of (29) to*

$\epsilon$ , for a given  $n \in \mathbb{Z}_{\geq 0}$  and  $\forall \epsilon > 0$ , we can find  $J$  such that  $\|\Pi_n \tilde{u}_j\|_2 \leq \epsilon$  for all  $j > J$ . For  $j > J' > J$ , (29) implies that  $n$  can be increased. Hence, the norm of a larger interval is guaranteed to be less than  $\epsilon$ . This resembles a wave traveling to the right. Assuming  $\|G(\rho z)\|_\infty < 1$  for  $0 < \rho < 1$ , a similar relation to (29) can be derived for  $\|(I - \Pi_n)\tilde{u}_j\|_2$ , so the wave moves toward minus infinity.

With the result of Theorem 9 in mind, let us reexamine the example in Sec. 2. In Fig. 2 for the convergent case, the growing wave is pushed toward the right as the number of iterations increases while for the non-convergent case the signals grow unbounded in the time region of the trial.

Assuming the conditions of Theorem 9 are fulfilled for  $\rho > 1$ , we can employ a strategy to only feed “safe inputs” and set the remaining inputs to a bounded signal. This way, a possibly growing tail of the control signal can be truncated, since we know that its energy distribution can only be shifted to plus infinity. Additionally, since we have assumed that (1) is stable, setting the removed part of the signal to a bounded signal is harmless. More specifically, a good choice is

$$u_j[n] = \begin{cases} Q(u_{j-1} + Le_{j-1})[n], & 0 \leq n < j\hat{d} \\ T_u(1)^{-1}(1 - T_r(1))r[n], & j\hat{d} \leq n < N, \end{cases} \quad (34)$$

where the steady-state solution to (1) is used for the unsafe region.

To determine the range of safe inputs, consider the model  $G(z) = cz^{-\hat{d}}$ , i.e.,  $g[n] = c\delta[n - \hat{d}]$  where  $\delta[\cdot]$  is the Dirac delta function. According to this model,

$$g^j[n] = c^j \delta[n - j\hat{d}]. \quad (35)$$

Thus, the norm of the signal is multiplied by  $c$  and shifted  $\hat{d}$  steps along the time axis in each iteration. Obviously, after sufficiently many iterations, the energy of the signal lies outside of the interval of  $n \in \{0, \dots, N - 1\}$ . By setting the right hand side of (29) in Theorem 9 to less than or equal to  $\epsilon$ , we derive

$$n \leq \frac{\ln \epsilon - \ln C}{\ln \rho} - j \frac{\ln \|G(\rho z)\|_\infty}{\ln \rho}.$$

By comparison with model (35), we conclude that a good choice for  $\hat{d}$  in (34) is

$$\hat{d} = - \frac{\ln \|G(\rho z)\|_\infty}{\ln \rho}. \quad (36)$$

Therefore, a rule of thumb for calculating  $\hat{d}$  is obtained by substituting  $\rho$  with  $\rho^* := \arg \min_\rho \|G(\rho z)\|_\infty$  in (36).

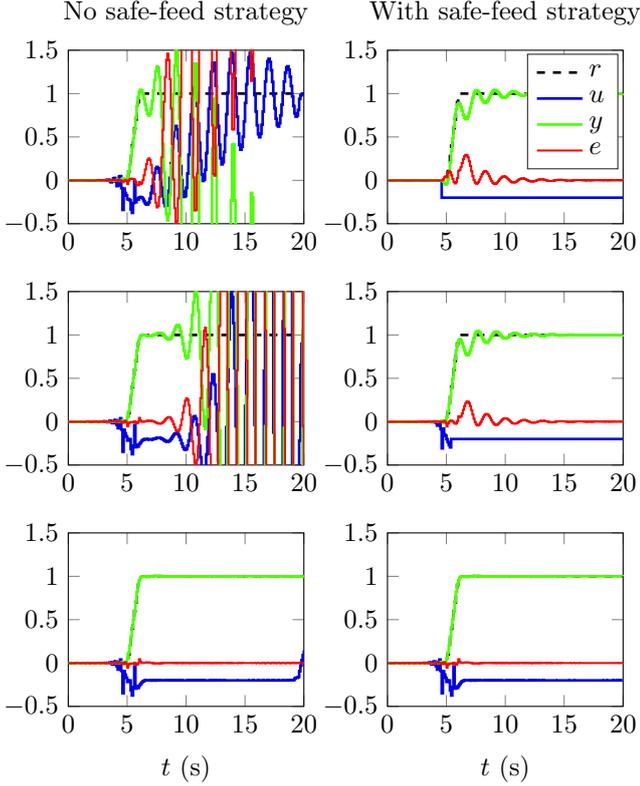


Fig. 5. Safe-feed strategy for non-minimum phase system (37): From top to bottom, iterations 5, 10, and 100 are illustrated. The signals shown are the reference (dashed black), input (blue), output (green), and error (red). For the right pane, the safe-feed strategy (34) is employed with  $\hat{d} = 5$ , while for the left pane it is not.

Figure 5 shows an example of the application of the safe-feed strategy (34). The system is non-minimum phase and  $Q(s)$  is the same as in (6),

$$T_u(s) = \frac{s-5}{(s+1)(s^2+0.8s+16)}, T_r(s) = 0, \quad (37)$$

$$L_d(z) = -6 \frac{1-2.7z^{-1}+2.5z^{-2}-0.8z^{-3}}{1-0.5z^{-1}-0.3z^{-2}} z^5.$$

Without the safe-feed strategy, the amplitude of signals grows rapidly as shown in the left pane of Fig. 5. However, by employing the safe-feed strategy the transient behavior is kept under control.

Early termination schemes are also advisable since, as it has been reported by other authors, divergence might appear after many iterations (Longman and Huang, 2002). In the case of  $\|G(z)\|_\infty \geq 1$ , more precautions must be taken considering the inherent lack of robustness.

## 6 Discussions

A number of qualitative measures for the convergence of (17) can be deduced from the plot of  $\|G(\rho z)\|_\infty$  versus  $\rho$ . First of all, according to (25), the value  $\inf_{\rho>0} \|G(\rho z)\|_\infty$  is an upper bound for the spectral radius of the finite Toeplitz matrix, indicating the exponential decay rate of the slowest transient mode of ILC applied on a finite-time interval. Based on (36), the larger  $\rho^*$  or  $\|G(\rho^* z)\|$  the longer it would take for the growing wave to disappear. We can also define

$$\Omega := \{\rho \mid \|G(\rho z)\|_\infty < 1 \wedge |z| = \rho \subset \text{ROC of } G(z)\},$$

$$\rho_\ell := \inf \Omega, \quad \rho_h := \sup \Omega,$$

i.e.,  $\rho_\ell$  is the first crossing of 0 dB and  $\rho_h$  is the last one. If the control signal is weighted by an exponential function with rate  $\rho < \rho_\ell$ , the weighted signal appears to grow and shift toward plus infinity. On the other hand, weighting by  $\rho > \rho_h$  results in a growing signal that moves toward minus infinity.

The frequency domain criterion provides an intuitive way to judge the convergence of ILC and facilitates the design process. Moreover, its evaluation can be computationally advantageous compared to the time-domain criteria. Iterative algorithms are often employed to solve for the largest eigenvalue or to calculate the  $H_\infty$  norm. Given a system with state dimension  $n$ , the cost of computing the  $H_\infty$  norm scales with  $O(n^3)$  per iteration (Vandenberghe et al., 2005). On the other hand, to compute the largest eigenvalue of a lifted system for the interval length of  $N$ , the cost per iteration for general algorithms is roughly  $O(N^2)$ . Thus, the time-domain criteria scale poorly with the interval length  $N$ , while the frequency domain computation is independent of  $N$ .

Even when  $\rho$  is not set to one, the search over a range of  $\rho$  can be done efficiently. Firstly, note that very large values of  $\rho$  are not interesting since they imply poor transients. Secondly, the valid range of  $\rho$  is constrained by  $\{|z| = \rho\} \subset \text{ROC of } G(z)$ . Let  $|p_1| \leq |p_2| \leq \dots \leq |p_n|$  denote the poles of  $G(z)$ . Since the ROC must be a connected region, if  $G$  is causal  $|p_n| < \rho$  and if  $G$  is non-causal,  $|p_i| < \rho < |p_{i+1}|$  for some  $p_i$ . Thirdly, note that as a consequence of the Hadamard three-circle theorem (Lang, 2013),  $\log \|G(\rho z)\|_\infty$  is strictly convex in  $\log(\rho)$  if  $G(z)$  is not a pure delay or forward shift. These facts allow for a fast evaluation of (26) using, for example, a bisection method.

While the frequency domain representation is widely accepted as an approximation for a practical ILC system where each iteration has a finite duration (Norrlöf and Gunnarsson, 2002; Longman, 2000), its implication for the frequency domain criterion (4) has not been clarified. From Lemma 16 in the appendix, it becomes evident how  $\|G(z)\|_\infty$  approximates  $\bar{\sigma}(\mathbf{G})$ . Namely, the

maximum singular value of a finite Toeplitz matrix is upper bounded by  $\bar{\sigma}(T_\infty(g))$ , which is equal to the infinity norm of the system. Thus, the criterion (4) is sufficient even for the convergence of a practical ILC algorithm. Moreover, when  $g \in \ell_1$  according to Lemma 17, as  $N \rightarrow \infty$  we have equality and hence the criterion (4) in this case is both sufficient and necessary for monotone convergence.

The criterion for non-monotone convergence is closely related to the results by Schmidt and Spitzer (1960), which states that if  $G$  has a finite impulse response (FIR), then

$$\liminf_{N \rightarrow \infty} \text{rad } T_N(g) \leq \limsup_{N \rightarrow \infty} \text{rad } T_N(g) \leq \inf_{\rho > 0} \|G(\rho z)\|_\infty.$$

However, this criterion cannot be used for evaluating the convergence of long sequences, i.e., when  $N$  is large. The reason is that (20) does not guarantee robustness against perturbations. Specifically, the spectral radius cannot reliably describe the convergence behavior of large Toeplitz matrices and instead, the pseudo-spectrum should be considered (Reichel and Trefethen, 1992). In our opinion, this can clarify some of the gaps observed between the theoretical and practical convergence of ILC reported in the literature. The lack of robustness can manifest itself as the divergence of ILC that starts at a distant sample in time after a number of iterations. Hence, either a shorter time interval or an early termination strategy should be considered.

## 7 Conclusion

The time domain criterion states that ILC converges if and only if the spectral radius of  $T_N(g)$  corresponding to the lifted system fulfills  $\text{rad } T_N(g) < 1$ . For strictly monotone convergence, it is necessary and sufficient that the largest singular value is less than one, i.e.,  $\bar{\sigma}(T_N(g)) < 1$ . These conditions have sufficient frequency domain counterparts applicable to ILC systems with causal as well as non-causal filters.

Our main result states that (non-monotone) convergence of ILC on finite-time intervals is implied by

$$\inf_{\rho > 0} \sup_{\omega} |G(\rho e^{i\omega})| < 1,$$

which is less conservative than  $\sup_{\omega} |G(e^{i\omega})| < 1$ . In other words, for non-monotone convergence, the criterion  $\|G(z)\|_\infty < 1$  can be evaluated on any circle in the region of convergence with radius larger than zero. The criterion implies that the spectral radius of the corresponding finite Toeplitz matrix  $T_N(g)$  is less than one. In this case, repeated composition with the operator  $G$  shifts the energy distribution of the control signal toward plus or minus infinity if  $\|G(z)\|_\infty \geq 1$  but  $\|G(\rho z)\|_\infty < 1$  for some  $\rho > 0$ .

Our treatment clarified the source of approximation in using the frequency domain criterion. Practical considerations were also discussed, whereby it is advisable to employ safe-feed and early-termination strategies in an ILC system. Rigorous extensions of the main results to the multi-input-multi-output (MIMO) case as well as analyzing scenarios with non-repetitive disturbances (Mishra et al., 2007; Ruan et al., 2008) are parts of future research. The  $H_\infty$  perspective as it has been taken by Doh and Ryoo (2008) could be helpful for such extensions.

## References

- Ahn, H.-S., Chen, Y.-Q., Moore, K., Nov. 2007. Iterative learning control: Brief survey and categorization. *IEEE Trans. Systems, Man, and Cybernetics, Part C: Applications and Reviews* 37 (6), 1099–1121.
- Åström, K. J., Wittenmark, B., 1997. *Computer-Controlled Systems: Theory and Design*. Prentice Hall, Englewood Cliffs, NJ.
- Böttcher, A., Grudsky, S. M., 2005. *Spectral Properties of Banded Toeplitz Matrices*. SIAM, Philadelphia.
- Bristow, D., Tharayil, M., Alleyne, A., Jun. 2006. A survey of iterative learning control. *IEEE Control Systems* 26 (3), 96–114.
- de Roover, D., Dec. 1996. Synthesis of a robust iterative learning controller using an  $H_\infty$  approach. In: *Proc. 35th IEEE Conf. Decision and Control*. Vol. 3. Kobe, Japan, pp. 3044–3049.
- Doh, T., Ryoo, J. R., 2008. Feedback-based iterative learning control for MIMO LTI systems. *Int. J. Control Automation and Systems* 6 (2), 269.
- Elci, H., Longman, R. W., Phan, M. Q., Juang, J.-N., Ugoletti, R., 2002. Simple learning control made practical by zero-phase filtering: Applications to robotics. *IEEE Trans. Circuits and Systems I: Fundamental Theory and Applications* 49 (6), 753–767.
- Freeman, C. T., Rogers, E., Hughes, A.-M., Burrige, J. H., Meadmore, K. L., 2012. Iterative learning control in health care: electrical stimulation and robotic-assisted upper-limb stroke rehabilitation. *IEEE Control Systems* 32 (1), 18–43.
- Horn, R. A., Johnson, C. R., 2012. *Matrix Analysis*. Cambridge University Press, New York, NY 10013-2473, USA.
- Kreyszig, E., 1989. *Introductory Functional Analysis with Applications*. Vol. 81. Wiley, New York.
- Kurek, J., Zaremba, M., Jan. 1993. Iterative learning control synthesis based on 2-D system theory. *IEEE Trans. Automatic Control* 38 (1), 121–125.
- Lang, S., 2013. *Complex Analysis*. Graduate Texts in Mathematics. Springer, New York.
- Longman, R. W., 2000. Iterative learning control and repetitive control for engineering practice. *Int. J. Control* 73 (10), 930–954.
- Longman, R. W., Huang, Y.-C., 2002. The phenomenon of apparent convergence followed by divergence in

learning and repetitive control. *Intell. Autom. & Soft Comput.* 8 (2), 107–128.

Mishra, S., Coaplen, J., Tomizuka, M., 2007. Precision positioning of wafer scanners segmented iterative learning control for nonrepetitive disturbances [applications of control]. *IEEE Control Systems* 27 (4), 20–25.

Moore, K. L., Dahleh, M., Bhattacharyya, S., 1989. Learning control for robotics. In: *Advances in Computing and Control*. Springer Berlin Heidelberg, pp. 240–251.

Norrlöf, M., Oct. 2000. Iterative learning control: Analysis, design, and experiments. Thesis no. 653, Linköping university, Linköping, Sweden.

Norrlöf, M., Gunnarsson, S., 2002. Time and frequency domain convergence properties in iterative learning control. *Int. J. Control* 75 (14), 1114–1126.

Reichel, L., Trefethen, L. N., 1992. Eigenvalues and pseudo-eigenvalues of Toeplitz matrices. *Linear algebra and its applications* 162, 153–185.

Ruan, X., Bien, Z. Z., Park, K.-H., 2008. Decentralized iterative learning control to large-scale industrial processes for nonrepetitive trajectory tracking. *IEEE Tran. Systems, Man, and Cybernetics-Part A: Systems and Humans* 38 (1), 238–252.

Rugh, W. J., 1996. *Linear System Theory*. Prentice Hall, Upper Saddle River, NJ.

Schmidt, P., Spitzer, F., 1960. The Toeplitz matrices of an arbitrary Laurent polynomial. *Math. Scand.* 8, 15–28.

Sörnmo, O., Bernhardsson, B., Kröling, O., Gunnarsson, P., Tenghamn, R., 2016. Frequency-domain iterative learning control of a marine vibrator. *Control Engineering Practice* 47, 70–80.

Vandenberghe, L., Balakrishnan, V. R., Wallin, R., Hansson, A., Roh, T., 2005. Interior-point algorithms for semidefinite programming problems derived from the KYP lemma. In: Henrion, D., Garulli, A. (Eds.), *Positive Polynomials in Control*. Springer, Berlin, Heidelberg, pp. 195–238.

Wang, D., Ye, Y., Zhang, B., 2014. *Practical Iterative Learning Control with Frequency Domain Design and Sampled Data Implementation*. Springer, Singapore.

## A Background

Some useful results from systems theory are presented in this section. We define

$$\ell_p := \left\{ f : \mathbb{Z} \rightarrow \mathbb{R} \mid \|f\|_{\ell_p}^p := \sum_{n=-\infty}^{\infty} |f(n)|^p < \infty \right\},$$

where  $|\cdot|$  denotes the absolute value.

A linear operator  $G : \ell_2 \rightarrow \ell_2$  is said to be *bounded* if

$$\|G\|_{\ell_2 \rightarrow \ell_2} := \sup_{\|x\|_{\ell_2}=1} \|Gx\|_{\ell_2} < \infty.$$

The product of  $x \in \ell_2$  and  $y \in \ell_2$  is defined as

$$(xy)[n] := x[n]y[n].$$

Given an  $x : \mathbb{Z} \rightarrow \mathbb{R}$ , denote the  $z$ -transform of  $x$  by

$$X(z) = \mathcal{Z}x := \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

and the inverse  $z$ -transform denoted by  $\mathcal{Z}^{-1}$  satisfies  $x = \mathcal{Z}^{-1}X$ . The set of values of  $z$  for which the  $z$ -transform converges absolutely is called the *region of convergence* (ROC) of the  $z$ -transform.

Given a linear *time-invariant* (LTI)  $G : \ell_2 \rightarrow \ell_2$  that is possibly non-causal, denote by  $g : \mathbb{Z} \rightarrow \mathbb{R}$  the impulse response/convolution kernel of  $G$ . In particular, note that  $Gx = g * x$  for any  $x \in \ell_2$ , where  $*$  denotes the convolution operator defined as

$$(x * y)[n] = \sum_{m=-\infty}^{\infty} x[m]y[n - m].$$

We define  $G(z) := \mathcal{Z}g$ . If  $G$  is a bounded LTI operator, then

$$\{|z| = 1\} \subset \text{ROC of } G(z).$$

Define

$$L_\infty := \left\{ G(z) \mid \|G(z)\|_\infty := \sup_{\omega \in [0, 2\pi)} |G(e^{i\omega})| < \infty \right\}.$$

Hereafter, we denote by  $G(z)$  the transfer function corresponding to a bounded LTI operator, i.e.,  $G(z) \in L_\infty$ .<sup>1</sup>

Define  $G^j$  as

$$\begin{aligned} G^0 &= I, \\ G^{j+1} &= G \circ G^j, \end{aligned} \tag{A.1}$$

where  $I$  denotes the identity operator and  $\circ$  the composition of two operators.

**Lemma 11** (*Kreyszig, 1989*) *Given a bounded linear time-invariant operator  $G : \ell_2 \rightarrow \ell_2$ , it holds that*

$$\|G\|_{\ell_2 \rightarrow \ell_2} = \|G(z)\|_\infty.$$

Given the definition of the truncation operator (8), an operator is causal if for all  $k \in \mathbb{Z}$

$$\Pi_k G(I - \Pi_k) = 0. \tag{A.2}$$

<sup>1</sup> In this article, we do not explicitly specify the ROC for stable systems since it can unambiguously be determined.

**Lemma 12** Given  $x \in \ell_2$  and  $\bar{G}$  as defined in (9), if  $y = \bar{G}x$  then

$$\mathbf{y} = T_N(g)\mathbf{x}, \quad (\text{A.3})$$

where  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$  are the input and output signals converted into vectors, e.g.,  $\mathbf{x} = \{x[0], \dots, x[N-1]\}$ .

**PROOF.** Define  $\bar{x} := \Pi_N(I - \Pi_0)x$ . For  $0 \leq i < N$ , we have

$$\begin{aligned} y[i] &= \sum_{j=-\infty}^{\infty} g[i-j]\bar{x}[j] \\ &= \sum_{j=0}^{N-1} g[i-j]x[j] = \sum_{j=0}^{N-1} T_{N\{i,j\}}(g)\mathbf{x}_{\{j\}}, \end{aligned} \quad (\text{A.4})$$

which proves the result.

Note that the operator  $\bar{G}$  can be represented by a matrix by converting the input and output signals to vectors. In this case,  $\bar{G} : \ell_2 \rightarrow \ell_2$  is a linear *time-varying* system with a Toeplitz matrix representation  $T_N(g) \in \mathbb{R}^{N \times N}$ . The infinite-dimensional Toeplitz matrix corresponding to the operator  $G$  is denoted by  $T_\infty(g)$ .

The spectral radius and the largest singular value of  $T \in \mathbb{R}^{N \times N}$  are defined, respectively, as

$$\text{rad}(T) := \max_i |\lambda_i(T)|, \quad (\text{A.5})$$

$$\bar{\sigma}(T) := \sqrt{\text{rad}(TT^*)}. \quad (\text{A.6})$$

where  $\lambda_i(T)$  denotes an eigenvalue and  $T^*$  the transpose of  $T$ .

**Lemma 13** (Horn and Johnson, 2012, Sec. 5.6) If  $T \in \mathbb{R}^{N \times N}$ , then

$$\|T\|_2 := \sup_{\mathbf{u} \neq 0} \frac{\|T\mathbf{u}\|}{\|\mathbf{u}\|} = \bar{\sigma}(T) \quad (\text{A.7})$$

**Lemma 14** (Kreyszig, 1989, Theorem 7.5-2) Given a matrix  $T \in \mathbb{R}^{N \times N}$ ,

$$\text{rad } T \leq \|T^n\|^{\frac{1}{n}}, \quad \forall n \in \mathbb{Z}_{>0}$$

**Lemma 15** (Horn and Johnson, 2012, Sec. 5.6) Given a matrix  $T \in \mathbb{R}^{N \times N}$ , if  $\|T\|_2 < 1$ , then

$$\lim_{j \rightarrow \infty} \sum_{n=0}^j T^n = (I - T)^{-1}.$$

**Lemma 16** (Böttcher and Grudsky, 2005, p. 177) For a transfer function  $G(z)$  and its corresponding Toeplitz matrix  $T_N(g)$  and for all  $n \in \mathbb{Z}_{\geq 0}$ , it holds

$$\|T_N^n(g)\|_2 \leq \|T_N(g)\|_2^n \leq \|T_\infty(g)\|_2^n = \|G(z)\|_\infty^n. \quad (\text{A.8})$$

**Lemma 17** (Böttcher and Grudsky, 2005, Theorem 8.1) If  $g \in \ell_1$ , then

$$\lim_{N \rightarrow \infty} \|T_N^n(g)\|_2 = \|T_\infty(g)\|_2^n = \|G(z)\|_\infty^n. \quad (\text{A.9})$$

For an arbitrary function  $f$ , by  $f[\cdot]$  we denote a mapping  $\mathbb{Z} \rightarrow \mathbb{R}$  which when evaluated at a point  $n \in \mathbb{Z}$  results in  $f(n)$ .