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Optimal H_∞ State Feedback for Systems with Symmetric and Hurwitz State Matrix

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***Abstract*— We address H_∞ state feedback and give a simple form for an optimal control law applicable to linear time invariant systems with symmetric and Hurwitz state matrix. More specifically, the control law as well as the minimal value of the norm can be expressed in the matrices of the system's state space representation, given separate cost on state and control input. Thus, the control law is transparent, easy to synthesize and scalable. If the plant possesses a compatible sparsity pattern, it is also distributed. Examples of such sparsity patterns are included. Furthermore, if the state matrix is diagonal and the control input matrix is a node-link incidence matrix, the open-loop system's property of internal positivity is preserved by the control law. Finally, we give an extension of the optimal control law that incorporate coordination among subsystems. Examples demonstrate the simplicity in synthesis and performance of the optimal control law.**

I. Introduction

Systems with a high density of sensors and actuators often lack centralized information and computing capability. Thus, structural constraints, e.g, on information exchange among subsystems, have to be incorporated into the design procedure of the control system. However, imposing such constraints may greatly complicate controller synthesis.

We address H_∞ structured static state feedback, a problem that is recognized as genuinely hard given arbitrary plant and controller structures. However, we give a simple form for an optimal control law applicable to linear time invariant (LTI) systems with symmetric and Hurwitz state matrix that is distributed if the system possesses a compatible sparsity pattern. Consider the following LTI system

$$\dot{x} = \underbrace{-\mathrm{diag}(1,3,2)}_{A}x + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{B} u + w \qquad (1)$$

where the state x, the control input u and the disturbance w are real valued. The static state feedback controllers

$$L_1 = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0.93 & -0.11 & 0.00 \\ -0.05 & -0.17 & -0.01 \\ 0.04 & 0.16 & -0.26 \end{bmatrix}$$

both minimize the H_∞ norm of the closed-loop system from disturbance w to penalized variables x and u, i.e., when $u = L_1 x$ and $u = L_2 x$, respectively. However, they have different structural properties, e.g., L_1 is sparser than L_2. Furthermore, the feedback law $u = L_1 x$ is distributed as the matrix L_1 has the same structure as the sparse matrix B^T. This is not the case for controller L_2. Controller L_1 can be given on the simple form we propose. More specifically L_1 can be written as $L_1 = B^T A^{-1}$. Controller L_2 is derived by the algebraic Riccati equation (ARE) approach. That is, iteration over an ARE-constraint until the minimal value of the norm is obtained, see [1] for details. Controllers synthesized by the ARE method are often dense, as is the case for controller L_2. Moreover, as the control law we give, i.e., $u = B^T A^{-1} x$, is optimal, it is equal in performance to any centrally derived optimal controller. Additionally, it is transparent in its structure, easy to synthesize and scalable.

In the 1980's, synthesis of controllers that achieve H_∞ norm specifications became a major research area and was formulated in [2]. The state-space based solution approach to the synthesis problem paved the way for optimization tools to be used, e.g., see [3]. The H_∞ norm condition can be turned into a linear matrix inequality (LMI) by the Kalman-Yakubovich-Popov lemma [4], see Lemma 1 in Appendix for the version used in this paper. As the theory on H_∞ control emerged, a decentralized version took form, e.g., see [5]. Imposing general sparsity constraints on the controller might complicate the design procedure. However, design is simplified if the constrained set of controllers is quadratically invariant with respect to the given system [6]. It is also simplified if the closed-loop system is constrained to be internally positive [7]. Our method results in a control law that is equal in performance to the central non-structured controller of the system. This is not the case in the methods previously mentioned. However, they treat more general classes of systems.

The optimal control law $u = B^T A^{-1} x$ only requires some relatively inexpensive matrix calculations for its synthesis, especially for sparse systems. This is in relation to general H_∞ controller synthesis where more expensive computational methods are required. Additionally, its structure is transparent, which is not often the case in H_∞ controller synthesis. The H_∞ framework treats worst-case disturbance as opposed to stochastic disturbance in the H_2 framework. However, the transparent structure and simple synthesis of the derived optimal feedback law might motivate its use even when some characteristics of the disturbance are known, given that the performance criteria are still met. Moreover, we show that it can be extended to incorporate coordination in a system of heterogeneous subsystems, given a linear coordination constraint. The coordinated control law is a superposition

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of a decentralized and a centralized part, where the latter is equal for all agents. This structure might be well suited for distributed control purposes as well. See [8] for a similar problem treated in the H_2 framework. Furthermore, if A is diagonal and $-BB^T$ is Metzler, the closed-loop system with the optimal control law, from disturbance to state, is internally positive. Thus, for such systems the property of internal positivity is preserved in the closed-loop system.

The outline of this paper is as follows. This section is ended with some notation. In Section II, the main results is stated and proved. Section III treats system sparsity patterns that result in a distributed control law. Section IV treats the result on internal positivity while Section V gives an extension of the control law that incorporates coordination. In Section VI, the performance of our optimal control law is compared, by a numerical example, to an optimal controller synthesized by the ARE approach. Concluding remarks are given in Section VII.

The set of real numbers is denoted $\mathbb{R}$ and the space n-by-m real-valued matrices is denoted $\mathbb{R}^{n\times m}$. The identity matrix is written as I when its size is clear from context, otherwise I_n to denote it is of size n-by-n. Similarly, a column vector of all ones is written $\mathbf{1}$ if its length is clear form context, otherwise $\mathbf{1}_n$ to denote it is of length n.

For a matrix M, the inequality $M \geq 0$ means that M is entry-wise non-negative and $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues have negative real part. The matrix M is said to be Metzler if its off-diagonal entries are nonnegative and the spectral norm of M is denoted $\|M\|$. Furthermore, for a square symmetric matrix M, $M \prec 0$ ($M \preceq 0$) means that M is negative (semi)definite while $M \succ 0$ ($M \succeq 0$) means M is positive (semi)definite.

The H_∞ norm of a transfer function $F(s)$ is written as $\|F(s)\|_\infty$. It is well known that this operator norm equals the induced 2-norm, that is

$$\|F\|_\infty = \sup_{v\neq 0} \frac{\|Fv\|_2}{\|v\|_2}.$$

II. An optimal H_∞ state feedback law

Consider a LTI system

$$\dot{x} = Ax + Bu + w \tag{2}$$

where the state matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and Hurwitz and the state $x \in \mathbb{R}^n$ can be measured. Moreover, the control input $u \in \mathbb{R}^m$, disturbance $w \in \mathbb{R}^n$ and matrix $B \in \mathbb{R}^{n\times m}$. Given (2), consider a stabilizing static state feedback law $u := Lx$, where $L \in \mathbb{R}^{m\times n}$. Then, the transfer function of the closed-loop system from disturbance w to penalized variables x and u, is given by

$$G_L(s) = \begin{bmatrix} I \\ L \end{bmatrix} (sI - (A+BL))^{-1}. \tag{3}$$

For (2) with A symmetric and Hurwitz, an optimal H_∞ static state feedback controller L, i.e., a matrix L such that $\|G_L\|_\infty$ is minimized, can be given explicitly in the matrices A and B. This is the main result of this paper and it is stated in the following theorem.

Theorem 1: Consider the system (2) with A symmetric and Hurwitz. Then, the norm $\|G_L\|_\infty$ is minimized by the static state feedback controller $L_* = B^T A^{-1}$. The minimal value of the norm is $\sqrt{\|(A^2+BB^T)^{-1}\|}$.

Proof: Given $\gamma > 0$, the following statements are equivalent.

(i) There exists a stabilizing controller L such that

$$\|G_L\|_\infty = \left\| \begin{bmatrix} I \\ L \end{bmatrix} (i\omega I - A - BL)^{-1} \right\|_\infty < \gamma.$$

(ii) There exist matrices L and $P \succ 0$ such that

$$\begin{bmatrix} (A+BL)^T P + P(A+BL) & P & [I \;\; L^T] \\ P & -\gamma^2 I & 0 \\ [I \;\; L^T]^T & 0 & -I \end{bmatrix} \prec 0.$$

(iii) There exist matrices $X \succ 0$ and Y such that

$$\begin{bmatrix} AX + XA + BY + Y^T B^T & I & [X \;\; Y^T] \\ I & -\gamma^2 I & 0 \\ [X \;\; Y^T]^T & 0 & -I \end{bmatrix} \prec 0.$$

(iv) There exist matrices $X \succ 0$ and Y such that

$$(X+A)^2 + \left(Y + B^T\right)^T \left(Y + B^T\right) - A^2 - BB^T + \gamma^{-2} I \prec 0.$$

(v)

$$-A^2 - BB^T + \gamma^{-2} I \prec 0.$$

(vi)

$$\gamma > \sqrt{\|\left(A^2 + BB^T\right)^{-1}\|}.$$

The equivalence between (i) and (ii) is given by the K-Y-P-lemma, see Lemma 1 given in Appendix. Statement (ii) can be equivalently written as (iii) after right- and left-multiplication with $\mathrm{diag}(P^{-1}, I, I)$ and change of variables $(P^{-1}, LP^{-1}) \to (X, Y)$. The equivalence between (iii) and (iv) is obtained by applying Schur's complement lemma and completion of squares to the inequality in (iii). Choosing $X = -A$ and $Y = -B^T$ shows equivalence between (iv) and (v). It is possible to choose $X = -A$ as A is symmetric and Hurwitz, i.e., $A \prec 0$. Finally, notice that $A^2 + BB^T \succ 0$. Thus, $\left(A^2 + BB^T\right)^{-1} \succ 0$ and

$$\text{(v)} \iff \gamma^2 I \succ \left(A^2 + BB^T\right)^{-1} \iff \text{(vi)}.$$

Given $X = -A$ and $Y = -B^T$, γ is minimized and $L_* = YX^{-1} = B^T A^{-1}$ minimizes the norm in (i). Now, define $\gamma_* := \sqrt{\|\left(A^2 + BB^T\right)^{-1}\|}$ and assume that $\|G_{L_*}\|_\infty \neq \gamma_*$. Then $\|G_{L_*}\|_\infty$ has to be strictly larger than or strictly smaller than γ_*. Consider $\|G_{L_*}\|_\infty > \gamma_*$. This statement contradicts statement (i) and (vi) and is therefore false. Now, consider instead $\|G_{L_*}\|_\infty < \gamma_*$. This statement contradicts that γ is minimized and is therefore also false. Hence, the statement $\|G_{L_*}\|_\infty \neq \gamma_*$ is false and

$$\|G_{L_*}\|_\infty = \sqrt{\|\left(A^2 + BB^T\right)^{-1}\|}.$$

■

Remark 1: The result stated in Theorem 1 can be made more general. However, we only give some comments on this here, the details are left to the reader. One can consider Hw instead of w in (2), where H is a real matrix of appropriate size. Then, the optimal control law is still given by $L_* = B^T A^{-1}$, i.e., its form is not altered by H. However, the value of the norm becomes $\sqrt{\|H^T(A^2+BB^T)^{-1}H\|}$. Notice that if H is a column vector, the expression inside the norm is a scalar. Further, if the considered system is stable and diagonalizable, however not symmetric, a variable transformation can be used in order to be able to apply the result in Theorem 1. If Du, with $D \in \mathbb{R}^{q\times m}$, is penalized instead of u, and $R := D^T D$ is invertible, the control law becomes $L_* = R^{-1}B^T A^{-1}$ and the norm is given by $\sqrt{\|(A^2+BR^{-1}B^T)^{-1}\|}$. If the penalized variables x and u are scaled by scalar nonzero coefficients, the optimal control law will only be scaled by a scalar nonzero coefficient.

Synthesis of optimal state feedback controllers generally requires additional computation beyond what is needed to compute L_* from Theorem 1, i.e., some relatively simple matrix calculations. Moreover, controllers generated by other methods are rarely as transparent as L_*. The transparency simplifies analysis of the controller's structure and enables scalability, which will be exploited in the following section.

In order for Theorem 1 to be applicable, the system of interest has to have a state space representation with symmetric and Hurwitz state matrix A. The symmetry property of A demands that states that affect each other do so with equal rate coefficient. Such representations appear, for instance, in buffer networks and models of temperature dynamics in buildings. We will now give an example of the latter.

Example 1: Consider a building with three rooms as depicted in Fig. 1. The average temperature T_i in each room $i =$ 1, 2 and 3, around some steady state, is given by the following model

$$\begin{aligned}\dot{T}_1 &= -r_1T_1 + r_{12}(T_2-T_1) + u_1 + w_1\\ \dot{T}_2 &= -r_2T_2 + r_{12}(T_1-T_2) + r_{23}(T_3-T_2) + u_2 + w_2\\ \dot{T}_3 &= -r_3T_3 + r_{23}(T_2-T_3) + u_3 + w_3\end{aligned} \tag{4}$$

governed by heat balance. The parameters $r_\bullet$ are constant, real-valued and positive. They are the rate coefficients of the system. For instance, r_{12} is the rate coefficient of the heat transfer through the wall between room 1 and 2. Changes in outdoor temperature and disturbances specific for each room, such as a window is opened, are modeled by disturbances w_i. The average temperatures can be measured as well as controlled, the latter through heating and cooling devices given by control inputs u_i. If (4) is written on form (2), it is easy to see that the corresponding matrix A is symmetric. Thus, Theorem 1 is applicable to (4), assuming that parameters $r_\bullet$ are such that A is also Hurwitz. Given a disturbance, the feedback law with L_* from Theorem 1 tries to keep the average temperature as close to the steady state as possible while minimizing the cost that comes with heating and cooling.

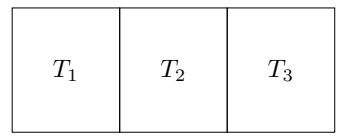

Fig. 1. Schematic of a building with three rooms. The average temperature in each room $i = 1,2$ and 3 is denoted T_i and given by (4).

III. Distributed and scalable

The structure of the optimal controller L_* given in Theorem 1 is clearly dependent on the structure of matrices A and B in (2). For instance, if A is diagonal and B is sparse, L_* has the same sparsity pattern as B^T. Moreover, controller L_* is distributed if (2) possesses a compatible sparsity pattern. This is demonstrated in Example 2 below. It is worthwhile to point out that for some sparsity patterns of (2) the representation $L_*^{-1}u = x$ instead of $u = L_*x$ might be beneficial for computation of u. That is, if B^T is invertible.

Example 2: Consider the following LTI system, containing three subsystems denoted S_1, S_2 and S_3,

$$\begin{aligned}S_1:&\quad \dot{x}_1 = A_1x_1 + B_1u_1 + w_1\\ S_2:&\quad \dot{x}_2 = A_2x_2 + B_2u_1 + B_3u_2 + w_2\\ S_3:&\quad \dot{x}_3 = A_3x_3 + B_4u_2 + w_3\end{aligned} \tag{5}$$

where each subsystem S_i, $i =$1, 2 and 3, has finite state dimension $n_i \geq 1$, each control input u_i, $i =$1, 2 and 3, is a vector of finite length $m_i \geq 1$ and the matrices are of suitable dimension. Furthermore, matrices A_1, A_2 and A_3 are assumed to be symmetric and Hurwitz. Then, Theorem 1 is applicable to (5) and results in the optimal controller

$$L_* = \begin{bmatrix} B_1^TA_1^{-1} & B_2^TA_2^{-1} & 0\\ 0 & B_3^TA_2^{-1} & B_4^TA_3^{-1}\end{bmatrix}. \tag{6}$$

Notice that, if (5) is written on form (2) the optimal controller L_* has the same sparsity pattern as B^T. Thus, each control input vector u_i is only constructed from the states it affects in (5). If we consider each subsystem S_i in (5) to represent an area of the physical system it models, the optimal controller (6) is distributed according to these areas. See Fig. 2 for a graphical representation of the system, drawn in solid lines. Each subsystem S_i is depicted by a circular node while each control input u_i is given by a link connecting the subsystems it affects in (5). Each disturbance w_i is drawn as an arrow that points toward the subsystem it affects in (5).

Now, we will demonstrate the scalability of the optimal control law. Consider that a fourth subsystem denoted S_4, of finite dimension $n_4 \geq 1$, is connected to (5) via a third

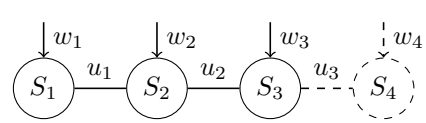

Fig. 2. Graphical representation of (5) in solid lines. Additional subsystem S_4 and control input u_3 in dashed lines.

control input denoted u_3, of finite length $m_3 \geq 1$, as depicted by the dashed lines in Fig. 2. The dynamics of subsystem S_4 and the altered dynamics of subsystem S_3 are then given by

$$\begin{aligned} S_3: &\quad \dot{x}_3 = A_3 x_3 + B_4 u_2 + B_5 u_3 + w_3 \\ S_4: &\quad \dot{x}_4 = A_4 x_4 + B_6 u_3 + w_4 \end{aligned}$$

where matrix A_4 is also assumed to be symmetric and Hurwitz. Then, Theorem 1 is still applicable and the extended optimal controller becomes

$$L_* = \begin{bmatrix} B_1^T A_1^{-1} & B_2^T A_2^{-1} & 0 & 0 \\ 0 & B_3^T A_2^{-1} & B_4^T A_3^{-1} & 0 \\ 0 & 0 & B_5^T A_3^{-1} & B_6^T A_4^{-1} \end{bmatrix}.$$

The expansion of the system does not alter the initial control inputs u_1 and u_2. Thus, for systems with this type of sparsity pattern, the control law $u = L_* x$ is easily scalable. Moreover, the control law is still distributed as the additional control input u_3 is only constructed from states x_3 and x_4.

IV. Preserves internal positivity

We will now consider (2) with diagonal and Hurwitz matrix A and where $-BB^T$ is Metzler. Then, the closed-loop system from disturbance w to state x with the optimal control law $u = L_* x$, from Theorem 1, is internally positive by Lemma 2, given in Appendix. This result is stated in Corollary 1 below and demonstrated in Example 3.

Corollary 1: Consider (2) with A diagonal and Hurwitz. Then, the closed-loop system from w to x with $L_* = B^T A^{-1}$ is internally positive if and only if $-BB^T$ is Metzler.

Proof: Theorem 1 is applicable as A is Hurwitz and clearly symmetric. Now, consider the closed-loop system from w to output $y := x$, with $L_* = B^T A^{-1}$, i.e.,

$$\dot{x} = (A + BL_*)x + w, \quad y = x,$$

where $A + BL_* = A + BB^T A^{-1}$. This system is internally positive by Lemma 2 in Appendix, if and only if $A + BL_*$ is Metzler, as the other matrices are entry-wise nonnegative. As A is diagonal and Hurwitz, i.e, all diagonal elements are negative, it is easy to see that it is necessary and sufficient that $-BB^T$ is Metzler for $A + BL_*$ to be Metzler. ■

Remark 2: If B is a node-link incidence matrix, see [9] for a formal definition of this notion, the matrix product $-BB^T$ is Metzler. The B-matrix given in Example 3 below is an example of a node-link incidence matrix.

Example 3: Consider three buffers of some quantity connected via links with flow u_1 and u_2 as depicted in Fig. 3. The dynamics of the levels in the buffers, around some steady state depicted by the dashed lines in Fig. 3, is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{-\mathrm{diag}\,(1,2,4)}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w. \quad (7)$$

State x_i corresponds to the level in buffer i =1, 2 and 3, respectively. Each buffer has some internal dynamics dependent on its own state, as given by matrix A. However,

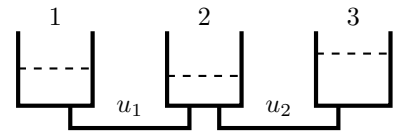

Fig. 3. Three buffers denoted 1, 2 and 3 connected via links with flow u_1 and u_2, respectively. The dashed lines represent some steady state of the system.

with different rate coefficients for the different buffers. We want to construct a control law that minimizes the impact from disturbance w to the penalized variables x and u in the H_∞ norm sense. That is, we want to keep the system at its steady state, i.e, $x_i = 0$ for all i, while also keeping the cost down, i.e, the magnitude of the control input.

Given the matrix B in (7), $-BB^T$ is Metzler. Thus, by Corollary 1, the closed-loop system from w to x with the optimal control law given by Theorem 1, i.e.,

$$L_* = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/4 \end{bmatrix},$$

is internally positive. This implies that, in closed-loop with controller L_*, the states x_i of (7) will always be nonnegative, i.e., the buffer levels will never go below their steady state values, given nonnegative disturbance. To get some further intuition of what controller L_* does, consider control input u_1. It is given by $u_1 = x_1 - x_2/2$. Thus, u_1 is strictly positive if $x_1 > x_2/2$ and the controller L_* redistributes the quantity of buffer 1 and buffer 2 relative to their internal rate coefficients. As in the previous example, L_* has the same sparsity pattern as B^T and thus each control input only considers local information, i.e., from the buffers it connects.

V. Coordination in the H_∞ framework

In this section we will extend the optimal control law given by Theorem 1 in order to include coordination. The problem formulation is as follows. Consider a LTI system of ν subsystems

$$\dot{x}_i = A_i x_i + B_i u_i + w_i, \quad i = 1, \ldots, \nu \quad (8)$$

where A_i, for $i = 1, \ldots, \nu$, is symmetric and Hurwitz. Furthermore, the control inputs u_i have to coordinate in order to fulfill the following constraint

$$u_1 + u_2 + \cdots + u_\nu = 0. \quad (9)$$

Given penalized variables x and u and the coordination constraint in (9), we want to construct an optimal H_∞ static state feedback controller for (8). The resulting control law is given by Corollary 2.

Corollary 2: Consider ν subsystems as in (8) with symmetric and Hurwitz state matrices and coordination constraint (9). Then,

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \quad \text{for } i = 1, \ldots, \nu \quad (10)$$

minimizes the norm of the closed-loop system from w to the penalized variables x and u.

Proof: Rewrite control input u_1 in terms of the other control inputs given (9), i.e.,

$$u_1 = -u_2 - u_3 \ldots - u_\nu, \tag{11}$$

and define $\tilde{u} = [u_2, u_3, \ldots, u_\nu]^T$. Then,

$$u = \underbrace{\begin{bmatrix} -\mathbf{1}_{\nu-1}^T \\ I_{\nu-1} \end{bmatrix}}_{D} \tilde{u}$$

and the overall system of (8) can be written as

$$\dot{x} = \underbrace{\mathrm{diag}(A_1, \ldots, A_\nu)}_{A} x + \underbrace{\mathrm{diag}(B_1, \ldots, B_\nu)}_{B} D\tilde{u} + w$$

with penalized variables x and $u = D\tilde{u}$. Define $R = D^T D = I + \mathbf{1}\mathbf{1}^T$ and notice that $R^{-1} = I - \frac{1}{\nu}\mathbf{1}\mathbf{1}^T$. The optimal control law by Theorem 1, see also Remark 1, is then

$$\begin{aligned} \tilde{u} &= R^{-1} D^T B^T A^{-1} x \\ &= \left(I_{\nu-1} - \frac{1}{\nu}\mathbf{1}_{\nu-1}\mathbf{1}_{\nu-1}^T \right) \begin{bmatrix} -\mathbf{1}_{\nu-1}^T \\ I_{\nu-1} \end{bmatrix}^T B^T A^{-1} x \\ &= \left(\begin{bmatrix} 0 & I_{\nu-1} \end{bmatrix} - \frac{1}{\nu}\mathbf{1}_{\nu-1}\mathbf{1}_{\nu}^T \right) B^T A^{-1} x. \end{aligned}$$

Thus, u_i for $i = 2, \ldots, \nu$, i.e., the elements in $\tilde{u}$, is

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k. \tag{12}$$

Now, consider u_1 again,

$$\begin{aligned} u_1 &\overset{(11)}{=} -\sum_{i=2}^{\nu} u_i \overset{(12)}{=} -\sum_{i=2}^{\nu} \left(B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \right) \\ &= -\left(\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k - B_1^T A_1^{-1} x_1 - \frac{\nu-1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \right) \\ &= B_1^T A_1^{-1} x_1 - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k, \end{aligned}$$

i.e., it has the same structure as (12). Thus, the optimal control law is given by

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k$$

for each subsystem $i = 1, \ldots, \nu$ in (8). ■

Remark 3: The first term of u_i in (10) is a local term, only dependent upon the subsystem i, while the second term is dependent on global information of the overall system. However, as this term is equal for all control inputs u_i, (10) might still be appropriate for distributed control use.

In [8], a similar type of problem is considered, however in the H_2 framework with stochastic disturbances and the necessity of homogeneous subsystems. The optimal control law derived in [8] and the one we suggest in (10) are similar in structure. However, our approach can treat heterogeneous systems in addition to homogeneous ones. On the contrary, it is only applicable to systems with symmetric and Hurwitz state matrix, properties that are not necessary in [8].

VI. Numerical example

Consider a system of the same structure as (1) given in Section I, i.e., a system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= -\underbrace{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &\quad + \underbrace{\begin{bmatrix} -b_1 & 0 & 0 \\ b_2 & b_3 & -b_4 \\ 0 & 0 & b_5 \end{bmatrix}}_{B} \begin{bmatrix} u_{12} \\ u_2 \\ u_{23} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \end{aligned} \tag{13}$$

where $a_i > 0$, for $i = 1$, 2 and 3, and $b_j > 0$, for $j = 1, \ldots, 5$, and penalized variables x and u. We will now compare the optimal controller given by Theorem 1, i.e., L_*, and an optimal controller derived by the ARE-approach, see [1], denoted L_G for global. In the latter approach, we consider the minimal value of the H_∞ norm of (3) given by Theorem 1 and iterate over the ARE-constraint until this minimal value is reached. See [10] for the software used. Controllers L_1 and L_2 given in Section I are examples of controllers L_* and L_G treated here, respectively.

Controllers L_* and L_G are optimal and thus they both obtain the minimal value of the H_∞ norm of (3). Now we want to compare how they affect the closed-loop dynamics more in detail. We randomly generate values of the parameters a_i and b_j in $(0.1, 5]$ and compare the step response of the states of (13) in closed-loop with L_* and L_G. In other words, given constant disturbance of value 1. The average dynamics over 50 such randomly generated systems is shown in Fig. 4. To clarify, we average over the absolute value of the step response in each time instance.

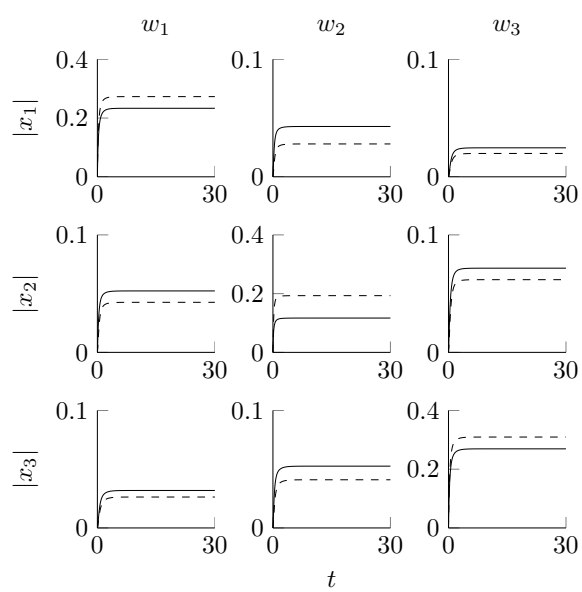

Fig. 4. Average step response for states x_1, x_2 and x_3 for closed-loop systems with controller L_* (solid lines) and L_G (dashed lines).

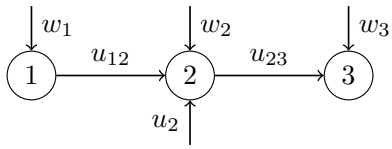

Fig. 5. Associated graph of (13).

The system (13) can be depicted by the graph given in Fig. 5, as described in Section III. If we compare the step responses shown in Figure 4, it seems as if controller L_* is better at attenuating local disturbances than L_G is. With local disturbances we mean the disturbance that points towards the state in Fig. (5). This is at the expense of larger impact on distance. However, overall they are comparable in performance.

We will end this numerical example by commenting on controller L_2 given in Section I, that is an example of controller L_G treated in this numerical example. Some entries of L_2 are small in magnitude compared to the other entries, i.e., entries (2,1), (2,3) and (3,1), where the first number in each parenthesis is the row and the second is the column. However, only entry (3,1) can be replaced with a zero for the controller to still achieve the optimal bound. Furthermore, for systems of much larger dimension than (1), this type of reduction analysis might be difficult.

VII. CONCLUSIONS

We give a simple form for an optimal H_∞ static state feedback law applicable to LTI systems with symmetric and Hurwitz state matrix. More specifically, this simple form is given in the matrices of the system's state space representation which makes the structure of the controller transparent. It also simplifies synthesis and enables scalability of the control law, especially given sparse systems. Furthermore, given compatible system sparsity patterns the control law is distributed. The examples we give consider diagonal or block diagonal state matrices and somewhat more general sparsity patterns of the remaining system matrices. Given some further constraints on the system's matrices the closed-loop system from disturbance to state becomes internally positive. Furthermore, we extend the optimal control law in order to incorporate coordination among subsystems. The resulting coordinated control law is similar for all subsystems. More specifically, for each subsystem, it is a superposition of a local term and an averaged centralized term where the latter is equal for all subsystems involved in the coordination. In conclusion, our control law is well suited for distributed control purposes. Future research directions include to consider saturation constraints on the optimal control law as such are common in the systems intended for its application. Furthermore, to investigate the existence of an analogous optimal control law given output feedback instead of state feedback. For an extension of the result to infinite-dimensional systems, see [11].

APPENDIX

Lemma 1: The Kalman-Yakubovich-Popov lemma
Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega I - A) \neq 0$ and (A,B) controllable, the following two statements are equivalent:
(i)

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \preceq 0$$

$\forall \omega \in \mathbb{R} \cup \{\infty\}$.
(ii) There exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$M + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \preceq 0$$

The corresponding equivalence for strict inequalities holds even if (A,B) is not controllable.

Proof: See [12]. ■

Remark. If the upper left corner of M is positive semidefinite, it follows from (1) and Hurwitz stability of A that $P \succeq 0$ [12].

Lemma 2: The LTI system

$$\dot{x} = Ax + Bv, \quad y = Cx + Dv$$

is internally positive if and only if

i A is Metzler, and
ii $B \geq 0$, $C \geq 0$ and $D \geq 0$.

Proof: See [13]. ■

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