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Published in:
East Asian Journal on Applied Mathematics

DOI:
10.4208/eajam.240611.070811a

2011

East Asian Journal on Applied Mathematics
doi: 10.4208/eajam

Vol. **xx**, No. x, pp. 1-12
xxx 200x

Features of the Nyström Method for the Sherman-Lauricella Equation on Piecewise Smooth Contours

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Abstract. The stability of the Nyström method for the Sherman-Lauricella equation on contours with corner points c_j, $j = 0, 1, \ldots, m$ relies on the invertibility of certain operators A_{c_j} belonging to an algebra of Toeplitz operators. The operators A_{c_j} do not depend on the shape of the contour, but on the opening angle θ_j of the corresponding corner c_j and on parameters of the approximation method mentioned. They have a complicated structure and there is no analytic tool to verify their invertibility. To study this problem, the original Nyström method is applied to the Sherman-Lauricella equation on a special model contour that has only one corner point with varying opening angle θ_j. In the interval $(0.1\pi, 1.9\pi)$, it is found that there are 8 values of θ_j where the invertibility of the operator A_{c_j} may fail, so the corresponding original Nyström method on any contour with corner points of such magnitude cannot be stable and requires modification.

AMS subject classifications: 65R20, 45L05

Key words: Sherman–Lauricella equation, Nyström method, stability.

1. Introduction

Let Γ be a simple closed positively oriented contour in the complex plane $\mathbb{C}$. The Sherman–Lauricella equation

$$\omega(t) + \frac{1}{2\pi i}\int_\Gamma \omega(\tau)\, d\ln\left(\frac{\tau - t}{\overline{\tau} - \overline{t}}\right) - \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\tau)}\, d\left(\frac{\tau - t}{\overline{\tau} - \overline{t}}\right) = f(t), \quad t \in \Gamma. \tag{1.1}$$

where here and subsequently the bar denotes the complex conjugation and ω is an unknown function, plays an important role in various fields of applied mathematics — including elasticity theory, theory of incompressible flows, radar imaging [11–14]. However, at present there is no general analytic solution of Eq. (1.1) available. If the contour Γ is

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smooth, then the integral operators in Eq. (1.1) are compact and the corresponding approximation methods for this equation can be studied without serious difficulties. On the other hand, when the contour Γ has corner points $c_1, c_2, \ldots, c_m$ the stability of the approximation method under consideration usually depends on the invertibility of certain operators $A_{c_0}, A_{c_1}, \ldots, A_{c_{m-1}}$ associated with the method itself and with the parameters of the corner points at hand. As a rule, such operators have a complicated structure, so their invertibility cannot be treated effectively. Nevertheless, apart from the approximation method each operator A_{c_j}, $j = 0, 1, \ldots, m-1$ does not depend on the shape of the contour Γ but on specific parameters of the corner point c_j, so the invertibility of such operators can be studied via connections with the stability of corresponding approximation methods considered on certain special model curves.

In the present paper, we investigate this property for the Nyström method of Ref. [3], and find that in the interval $(0.1\pi, 1.9\pi)$ there are angles for which the operators A_{c_j} are not invertible.

2. The Nyström method and the operators A_{c_j}

Let $\gamma = \gamma(s)$ be a 1-periodic parametrization of Γ. For the sake of simplicity, let us assume that $c_j = \gamma(j/m)$ for all $j = 0, 1, \ldots, m-1$, the function γ is two times continuously differentiable on each interval $(j/m, (j+1)/m)$ and

$$\left|\gamma'\left(\frac{j}{m}+0\right)\right| = \left|\gamma'\left(\frac{j}{m}-0\right)\right|, \quad j = 0, 1, \ldots, m-1.$$

Let us now construct a mesh that will be used in the following discussion. Set $n = qm$ for $q = 1, 2, \ldots$, and for such n note that any corner of Γ is always an end point of a subinterval $(\gamma(r/n), \gamma((r+1)/n))$. Let d be a positive integer and let $0 < \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_{d-1} < 1$ and $0 < \delta_0 < \delta_1 < \ldots < \delta_{d-1} < 1$ be real numbers. Consider two sets of points on Γ — viz.

$$\tau_{lp} = \gamma\left(\frac{l+\varepsilon_p}{n}\right), \quad t_{lp} = \gamma\left(\frac{l+\delta_p}{n}\right), \quad l = 0, 1, \ldots, n-1; p = 0, 1, \ldots, d-1. \tag{2.1}$$

According to [3], the approximate values $\omega(\tau_{lp})$ of an exact solution ω of Eq. (1.1) at the points τ_{lp} are defined by the following system of algebraic equations:

$$\begin{aligned}
&\omega(\tau_{kr}) + \frac{1}{2\pi i}\sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p \omega(\tau_{lp}) \left(\frac{\tau'_{lp}}{\tau_{lp} - t_{kr}} - \frac{\overline{\tau'}_{lp}}{\overline{\tau}_{lp} - \overline{t}_{kr}}\right)\frac{1}{n} \\
&\quad - \frac{1}{2\pi i}\sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p \overline{\omega(\tau_{lp})}\left(\frac{1}{\overline{\tau}_{lp} - \overline{t}_{kr}}\frac{\tau'_{lp}}{n} - \frac{\tau_{lp} - t_{kr}}{(\overline{\tau}_{lp} - \overline{t}_{kr})^2}\frac{\overline{\tau'}_{lp}}{n}\right) \\
&\quad + \frac{1}{(\overline{t}_{kr} - \overline{a})}\frac{1}{2\pi i}\sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p \left(\frac{\omega_{lp}}{(\tau_{lp} - a)^2}\frac{\tau'_{lp}}{n} + \frac{\overline{\omega(\tau_{lp})}}{(\overline{\tau}_{lp} - \overline{a})^2}\frac{\overline{\tau'}_{lp}}{n}\right) = f(t_{kr}) \\
&\quad (k = 0, 1, \ldots, n-1;\ r = 0, 1, \ldots, d-1)
\end{aligned} \tag{2.2}$$

where $\tau'_{lp} := \gamma'((l+\varepsilon_p)/n)$ and w_p for $p = 0,1,\dots,d-1$ are positive numbers such that $w_0+w_1+\dots+w_{d-1} = 1$. Note that the last line of equation (2.2) represents a discretization of the correcting operator $T_{SL} : L_2(\Gamma) \mapsto L_2(\Gamma)$ given by

$$T_{SL}\omega(t) := \frac{1}{(\overline{t}-\overline{a})}\frac{1}{2\pi i}\int_\Gamma \left(\frac{\omega(\tau)}{(\tau-a)^2}d\tau + \frac{\overline{\omega(\tau)}}{(\overline{\tau}-\overline{a})^2}d\overline{\tau} \right), \ t\in\Gamma , \tag{2.3}$$

where a is a fixed point in the domain bounded by the contour Γ. In what follows, this correcting operator is referred to as the Parton and Perlin choice of T_{SL}. In addition, another correcting operator $T_{SL} : L_2(\Gamma) \mapsto L_2(\Gamma)$ used below is given by

$$T_{SL}\omega(t) := \frac{in_t}{2S}\mathrm{Re}\int_\Gamma \left(\omega(\tau) + \frac{1}{\pi i}\int_\Gamma \frac{\omega(z)\,dz}{z-\tau} \right) d\bar{\tau} , \tag{2.4}$$

where S is the arc length of Γ and n_t is the outward unit normal at t on Γ. The operator (2.4) is called the zero average displacement choice of T_{SL} [8], and the role of correcting operators T_{SL} is discussed in [3, 7]. After the approximate values of the solution ω are obtained at the points $\tau_{l,p}$ where $l = 0,1,\dots,n-1$ and $p = 0,1,\dots,d-1$, one can use polynomials or splines to construct an approximate solution ω_n of Eq. (1.1) on the whole curve Γ. In this work, we prefer to use splines of order d from the spline space $S_n^d(\Gamma)$ — cf. [3] or § 5.3 and § 5.5 of [7] for more detail.

Let $P_n : L_2(\Gamma) \to S_n^d(\Gamma)$ be the orthogonal projection on the subspace $S_n^d(\Gamma)$, and let $P_n^\delta : L_\infty(\Gamma) \to S_n^d(\Gamma)$ be the interpolation projector on $S_n^d(\Gamma)$ such that

$$P_n^\delta \varphi\left(\frac{l+\delta_p}{n}\right) = \varphi\left(\frac{l+\delta_p}{n}\right), \quad l = 0,1,\dots,n-1, \quad p = 0,1,\dots,d-1,$$

for all Riemann integrable functions φ. Then the Nyström method represented by Eq. (2.2) is equivalent to the sequence of operator equations

$$A_n^\Gamma \omega_n = P_n^\delta f, \quad n = qm, \quad q = 1,2,\dots , \tag{2.5}$$

where $A_n^\Gamma : S_n^d \to S_n^d$ are the finite dimensional approximation operators described in [3].

Definition 2.1. *The Nyström method in Eq.* (2.2) *or Eq.* (2.5) *is called stable if there is an $N \in \mathbb{N}$ and $m \in \mathbb{R}$ such that for all $n \geq N$ the operators $A_n^\Gamma : S_n^d \to S_n^d$ are invertible and*

$$||(A_n^\Gamma)^{-1}P_n|| \leq m, \quad n \geq N.$$

As is well-known, stability plays a crucial role in numerical analysis. Here it ensures the solvability of the algebraic system involved and convergence of the approximate solution to the solution of the initial equation, for sufficiently large n. Let us now describe the auxiliary operators A_{c_j} responsible for the stability of the above Nyström method.

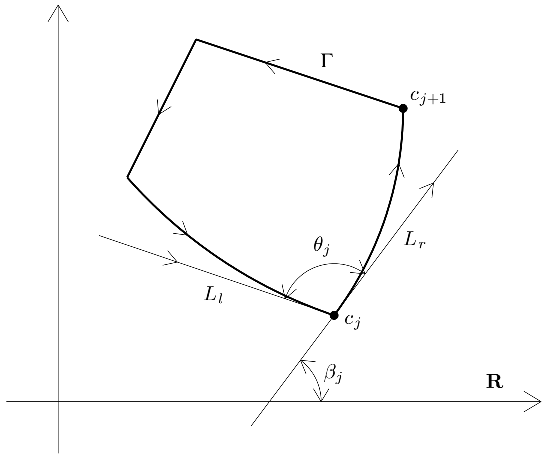

Figure 1: Contours and corners.

Each corner point c_j of Γ can be characterized by two parameters θ_j and β_j, where $\beta_j \in [0, 2\pi)$ is the angle between the right semi-tangent L_r to Γ at the point c_j and the real line $\mathbb{R}$, whereas $\theta_j \in (0, 2\pi), \theta_j \neq \pi$ denotes the angles between the right L_r and the left L_l semi-tangents to Γ at the point c_j (cf. Fig. 1). Let $\mathbf{k} = \mathbf{k}_\theta$ refer to one of the following functions:

$$\mathbf{n}_\theta(z) = \frac{\sinh(\pi - \theta)z - \sinh(\theta - \pi)z}{2 \sinh \pi z}, \tag{2.6}$$

$$\mathbf{m}_\theta(z) = -e^{-i\theta} \frac{z \sin \theta}{\sinh \pi z} e^{-(\theta - \pi)z}, \tag{2.7}$$

considered on the line $L := \{z \in \mathbb{C} : z = x + i/2, \, x \in \mathbb{R}\}$. On the space l_2 of sequences (ξ_k) of complex numbers ξ_k, $k = 0, 1, \ldots$ given by

$$l_2 := \{(\xi_k)_{k=0}^\infty : \sum_{k=0}^\infty |\xi_k|^2 < \infty\},$$

the function $\mathbf{k}$ defines bounded linear operators $A_{r,p}^{\delta_r, \varepsilon_p}$ for $r, p = 0, 1, \ldots, d-1$, with the matrix representation

$$A_{r,p}^{\delta_r, \varepsilon_p}(\mathbf{k}) = \left(\mathbf{k} \left(\frac{k + \delta_r}{l + \varepsilon_p} \right) \frac{1}{l + \varepsilon_p} \right)_{k,l=0}^\infty$$

where ε_r, δ_r are the parameters in the Nyström method Eq. (2.2) or Eq. (2.5). As the next step, one has to construct an operator

$$B^{\delta,\varepsilon}(\mathbf{k}) := \left(w_p A_{rp}^{\delta_r, \varepsilon_p} \right)_{r,p=0}^{d-1},$$

which acts on the Cartesian product of d copies of the space l_2. We also need an additional operator M defined on the space l_2 by

$$M((\xi_k)_{k=0}^{\infty}) := (\overline{\xi}_k)_{k=0}^{\infty},$$

and redefined correspondingly on the Cartesian products of l_2 spaces.

Note that the conditions of the stability of the Nyström method Eq. (2.2) or Eq. (2.5) have been obtained in Ref. [3]. For the convenience of the reader, we reformulate the corresponding result as follows.

Theorem 2.1. *Let $c_0, c_1, \dots, c_{m-1}$ be the corner points of Γ. The Nyström method Eq.* (2.2) *or Eq.* (2.5) *is stable if and only if the operators*

$$A_{c_j} := \begin{pmatrix} I & B^{\delta,\varepsilon}(\boldsymbol{n}_{\theta_j}) \\ -B^{1-\delta,1-\varepsilon}(\boldsymbol{n}_{\theta_j}) & I \end{pmatrix} + \begin{pmatrix} 0 & e^{i2\beta_j}B^{\delta,\varepsilon}(\boldsymbol{m}_{2\pi-\theta_j}) \\ -e^{i2(\beta_j+\theta_j)}B^{1-\delta,1-\varepsilon}(\boldsymbol{m}_{\theta_j}) & 0 \end{pmatrix} M, \quad (2.8)$$

are invertible for all $j = 0, 1, \dots, m-1$.

Thus to have a complete information about the stability of the Nyström method, one has to study the operators A_{c_j}. This is not an easy task, since the operators A_{c_j} have a complicated structure. Nevertheless, certain properties of A_{c_j} can be established as follows.

Lemma 2.1. *The operator A_{c_j} is invertible (Fredholm) if and only if the operator*

$$\widehat{A}_{c_j} = \begin{pmatrix} I & B^{\delta,\varepsilon}(\boldsymbol{n}_{\theta_j}) & 0 & e^{i2\beta_j}B^{\delta,\varepsilon}(\boldsymbol{m}_{2\pi-\theta_j}) \\ -B^{1-\delta,1-\varepsilon}(\boldsymbol{n}_{\theta_j}) & I & -e^{i2(\beta_j+\theta_j)}B^{1-\delta,1-\varepsilon}(\boldsymbol{m}_{\theta_j}) & 0 \\ 0 & -e^{-i2\beta_j}B^{\delta,\varepsilon}(\boldsymbol{m}_{\theta_j}) & I & B^{\delta,\varepsilon}(\boldsymbol{n}_{\theta_j}) \\ e^{-i2(\beta_j+\theta_j)}B^{1-\delta,1-\varepsilon}(\boldsymbol{m}_{2\pi-\theta_j}) & 0 & -B^{1-\delta,1-\varepsilon}(\boldsymbol{n}_{\theta_j}) & I \end{pmatrix} \quad (2.9)$$

is invertible (Fredholm).

Proof. The proof follows from Lemma 1.4.6 of Ref. [7] and Eqs. (23) of Ref. [6].

Next, let $\mathfrak{T}_2$ denote the smallest closed C^*-subalgebra of the algebra of bounded linear operators $\mathfrak{B}(l_2)$ containing all Toeplitz operators $T(a)$ with piecewise constant generating functions a; and recall that on the finitely supported sequences (ξ_k) the operator $T(a)$ is defined by

$$T(a)(\xi_k) = (\eta_j), \quad \eta_j = \sum_{k=0}^{\infty} a_{j-k}\xi_k,$$

where a_k are the Fourier coefficients of the function a.

Lemma 2.2. *Let $\boldsymbol{k}$ refer to the function defined by Eq.* (2.6) *or Eq.* (2.7)*. Then for any corner point c_j the entries of the operator $B^{\delta,\varepsilon}(\boldsymbol{k})$ belong to the algebra $\mathfrak{T}_2$ and the symbol $\mathscr{A}_{A_{r,p}^{\delta_r,\varepsilon_p}(\boldsymbol{k})}$ of the operator $A_{r,p}^{\delta_r,\varepsilon_p}(\boldsymbol{k})$ is*

$$\mathscr{A}_{A_{r,p}^{\delta_r,\varepsilon_p}(\boldsymbol{k})}(z) = \boldsymbol{k}(z), \quad z \in L. \tag{2.10}$$

The proof of this result is lengthy. It can be obtained from considerations found in §5.4 of Ref. [7], but is beyond the main purpose of this paper and so omitted here.

Now let us consider the matrix

$$\mathscr{A}_{B^{\delta,\varepsilon}(\mathbf{k})}(z) := \left(w_p \mathscr{A}_{A_{rp}^{\delta_r,\varepsilon_p}(\mathbf{k})}(z) \right)_{r,p=0}^{d-1} .$$

It follows from Eq. (2.10) that

$$\mathscr{A}_{B^{\delta,\varepsilon}(\mathbf{k})}(z) = (\mathbf{W} \otimes \mathbf{k})(z), \quad z \in L \, ,$$

where $\mathbf{W} := (w_p)_{r,p=0}^{d-1}$ and $\mathbf{W} \otimes \mathbf{k}$ denotes the tensor product of $\mathbf{W}$ and $\mathbf{k}$. From Lemma 2.1, Lemma 2.2 and the representations (2.9) and (2.10), we obtain the following result.

Theorem 2.2. *1. The operator A_{c_j} is Fredholm if and only if the determinant*

$$\det \mathscr{A}_{A_{c_j}}(z) =$$
$$\det \begin{pmatrix} I & \boldsymbol{W} \otimes \boldsymbol{n}_{\theta_j} & 0 & e^{i2\beta_j} \boldsymbol{W} \otimes \boldsymbol{m}_{2\pi-\theta_j} \\ -\boldsymbol{W} \otimes \boldsymbol{n}_{\theta_j} & I & -e^{i2(\beta_j+\theta_j)} \boldsymbol{W} \otimes \boldsymbol{m}_{\theta_j} & 0 \\ 0 & -e^{-i2\beta_j} \boldsymbol{W} \otimes \boldsymbol{m}_{\theta_j} & I & \boldsymbol{W} \otimes \boldsymbol{n}_{\theta_j} \\ e^{-i2(\beta_j+\theta_j)} \boldsymbol{W} \otimes \boldsymbol{m}_{2\pi-\theta_j} & 0 & -\boldsymbol{W} \otimes \boldsymbol{n}_{\theta_j} & I \end{pmatrix} (z)$$
$$\neq 0 \ \textit{for all } z \in L. \tag{2.11}$$

2. The operator A_{c_j} is invertible if and only if

(a) the winding number of the function $\det \mathscr{A}_{A_{c_j}}(z)$*, $z \in L$ is equal to zero, and*

(b) the dimension of the kernel $\dim \ker A_{c_j} = 0$*.*

Corollary 2.1. *For any corner point $c_j \in \Gamma$, the Fredholmness of the operator A_{c_j} is independent of the parameters $\{\varepsilon_j\}$ and $\{\delta_j\}$.*

Proof. The symbol $\mathscr{A}_{A_{c_j}}$ of the operator A_{c_j} depends only on the parameters $\{w_p\}, \theta_j$ and β_j, hence the result.

Despite Corollary 2.1, the parameters $\{w_p\}, \theta_j$ and β_j can still influence the invertibility of the operators A_{c_j}. Note also that similar properties of local operators have been established earlier in other spline approximation methods for Cauchy singular integral equations with conjugation [4].

3. Numerical simulations

Formula (2.11) allows us to study the Fredholm properties of the operator A_{c_j} and to compute the index of the operator A_{c_j}, but on the other hand as yet we do not know of any reliable analytic approach to verify condition (2b) in Theorem 2.2. Surprisingly, the numerical approach is more fruitful. One only needs to use the connections between the invertibility of the operators A_{c_j} and the stability of the Nyström method. In particular, let us consider this approximation method on a model contour $\Gamma_\circ$ parameterized by the parameter s as

$$\gamma(s) = \sin(\pi s)\exp(i\theta(s-0.5))\exp(i\alpha) \quad s \in [0,1], \tag{3.1}$$

where $\theta \in (0, 2\pi)$ and $\alpha \in (\theta/2 - 2\pi, \theta/2)$. This contour has only one corner, located at the origin, with the opening angle θ (cf. Fig. 2).

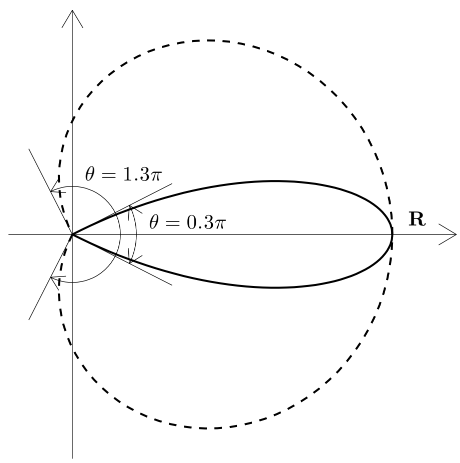

Figure 2: The shape of the curve $\Gamma_\circ$ given by Eq. (3.1) for $\theta = 0.3\pi$ and $\theta = 1.3\pi$ and $\alpha = 0$.

Double application of Theorem 2.1 leads to the following result.

Corollary 3.1. *Let c_j be a corner point of the contour Γ, and let $\theta = \theta_j$ and $\alpha = \beta_j + \theta_j/2 - 2\pi$. The operator A_{c_j} is invertible if and only if the sequence $(A_n^{\Gamma_\circ} P_n)$ is stable.*

Notice that the stability of the corresponding operator sequence $(A_n^{\Gamma_\circ} P_n)$ is directly connected to the condition numbers of the corresponding approximation method, so it can be verified numerically.

In all numerical examples we use adaptively refined meshes on $\Gamma_\circ$. The meshes are constructed from an initial uniform mesh with 20 quadrature panels (subintervals) that are equi-sized in the parameter s. The four panels closest to the corner point (two on each side) are then subdivided up to 60 times with respect to s. Note that the stability of the approximation methods constructed on such sequences of adaptive meshes is connected with the invertibility of additional operators associated with the breakpoints of the mesh [5], but all of these additional operators always seem to be invertible in the present case. The integer d is taken as 16 or 24, and the two sets of points $\{\epsilon_p\}$ and $\{\delta_p\}$ of Eq. (2.1) are both chosen to coincide with the zeros of the Legendre polynomial $P_{16}(x)$ and $P_{24}(x)$ on the canonical interval $x \in [-1, 1]$, scaled and shifted to the interval $x \in [0, 1]$. This corresponds to composite 16- or 24-point Gauss–Legendre quadrature.

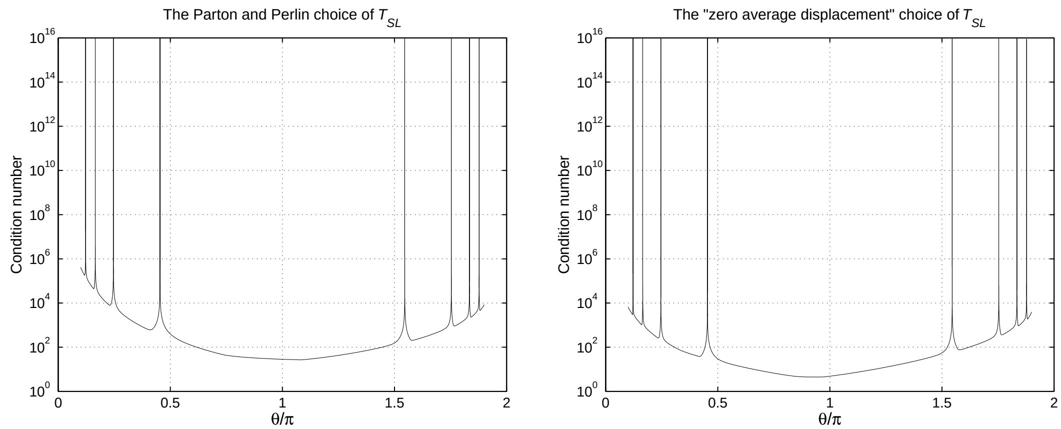

Figure 3: Condition number of the operator $A_n^{\Gamma_\circ}$ for different angles θ when $d = 16$. There are 1280 discretization points on $\Gamma_\circ$. Left: T_{SL} as in Eq. (2.3). Right: T_{SL} as in Eq. (2.4).

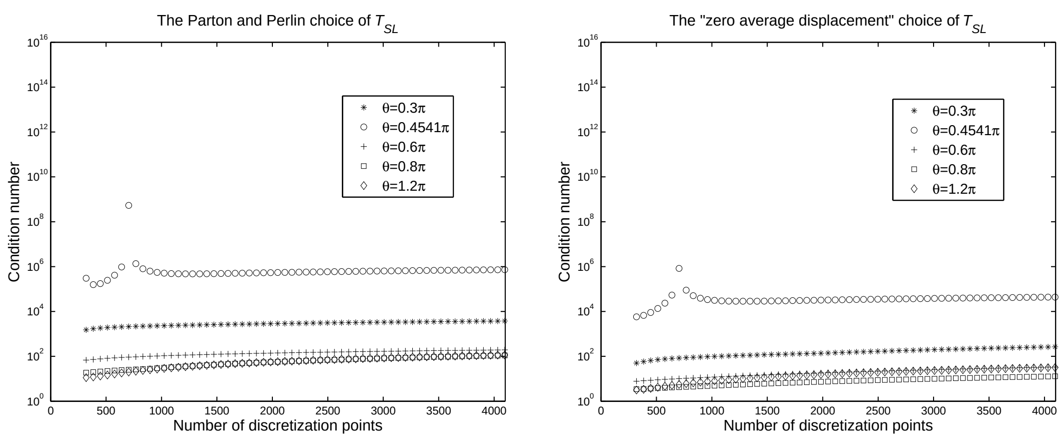

Figure 4: Condition number of $A_n^{\Gamma_\circ}$ for five distinct angles θ under mesh refinement for $d = 16$. Left: results with the Parton and Perlin correcting operator T_{SL} of Eq. (2.3). Right: results with the "zero average displacement choice" T_{SL} of Eq. (2.4).

Fig. 3 shows that for $d = 16$ there are eight points in the interval $[0.1\pi, 1.9\pi]$, symmetrically located with respect to $\theta = \pi$, for which A_{c_j} does not seem to be invertible. Note that the invertibility of the operators A_{c_j} does not depend on the choice of the correcting operator T_{SL} and it is also confirmed by the results of simulations presented in Fig. 3. The right image in Fig. 3 shows that the condition numbers of $A_n^{\Gamma_\circ}$ generally decrease somewhat when the correcting operator T_{SL} of (2.3) is replaced with the operator (2.4). However, the peak points remain the same. To take a closer look at the behaviour of the sequences of condition numbers for fixed angles θ, in Fig. 4 we see that the sequence $(A_n^{\Gamma_\circ})$ is stable under mesh refinement for an angle θ that does not belong to the above set of 8 irreversibility points.

On the other hand, additional numerical simulations show that the inclination angle β_j does not influence the invertibility of A_{c_j}. Thus if one fixes θ_j and rotates the corresponding curve $\Gamma_\circ$ around the origin, the associated condition numbers either remain unchanged or vary very modestly — cf. Table 1. The only notable changes are for the case $\theta = 0.4541\pi$, close to an instability point, and we are not sure of the accuracy of the built-in Matlab function cond for ill-conditioned matrices.

Table 1: Condition number of the operator $A_n^{\Gamma_\circ}$ for four fixed angles θ, with β uniformly distributed in the interval $[0, 2\pi)$ and 14 digits shown. Here $d = 16$, there are 1280 discretization points on $\Gamma_\circ$, and the operator T_{SL} is as in Eq. (2.3).

β	$\theta = 0.3\pi$	$\theta = 0.4541\pi$	$\theta = 0.6\pi$	$\theta = 0.8\pi$
0	2489.1166235746	474643.61824413	116.23440645457	37.512548716057
0.2π	2489.1166235746	474643.61811463	116.23440645457	37.512548716057
0.4π	2489.1166235748	474643.61830215	116.23440645457	37.512548716057
0.6π	2489.1166235748	474643.61827922	116.23440645457	37.512548716057
0.8π	2489.1166235749	474643.61827176	116.23440645457	37.512548716057
π	2489.1166235748	474643.61824181	116.23440645457	37.512548716057
1.2π	2489.1166235745	474643.61810759	116.23440645456	37.512548716057
1.4π	2489.1166235747	474643.61823107	116.23440645457	37.512548716057
1.6π	2489.1166235747	474643.61827487	116.23440645457	37.512548716057
1.8π	2489.1166235747	474643.61826967	116.23440645457	37.512548716057

Note that a different effect is observed in the case $d = 24$ (cf. Fig. 5). Although the number of irreversibility points remains the same, their positions are different. Thus for $d = 16$ the peaks are located at the points (to three significant digits)

$$0.122, 0.166, 0.246, 0.454, 1.546, 1.754, 1.834, 1.878,$$

whereas for $d = 24$ the corresponding peak points are

$$0.125, 0.168, 0.247, 0.454, 1.546, 1.753, 1.832, 1.875.$$

This shows that, in addition to the angle θ_j, the invertibility of the operators A_{c_j} also depends on the choice of the approximation space.

The original Nyström method, based entirely on composite d-point Gauss–Legendre quadrature as outlined in (2.2), was analyzed in Ref. [3]. From a purely practical view-

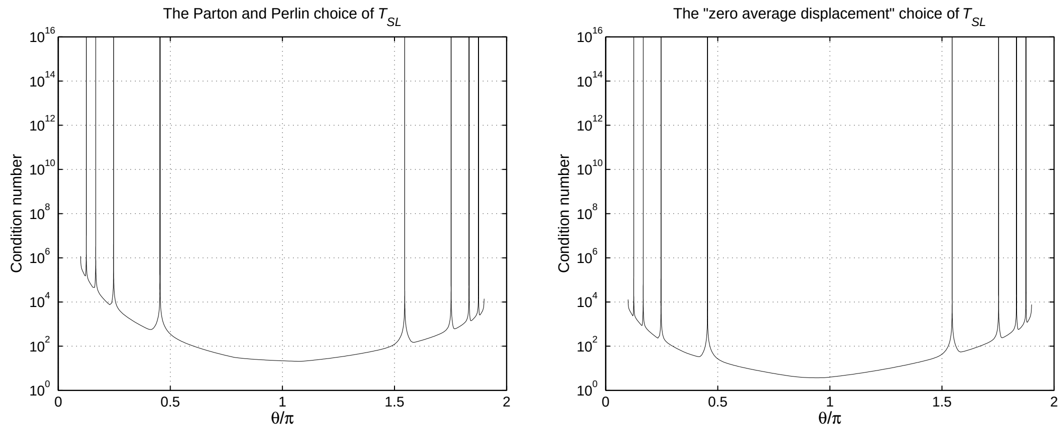

Figure 5: Condition number of the operator $A_n^{\Gamma_\circ}$ for different angles θ when $d = 24$. There are 1440 discretization points on $\Gamma_\circ$. Left: T_{SL} as in Eq. (2.3). Right: T_{SL} as in Eq. (2.4).

point, however, there may be even better approximation strategies to solve the Sherman–Lauricella equation (1.1) on piecewise smooth contours. A problem with composite Gauss–Legendre quadrature and adaptive mesh refinement is that it may require very many discretization points if high accuracy is sought. In addition, inefficiencies occur for discretization points τ_{lp}, $p = 0, \dots, d-1$ that lie close to a corner point c_j when t_{kr} falls close to but on the opposite side of that corner point. The kernel of the integral operator in Eq. (1.1) is not smooth at the point $(t,t) = (c_j, c_j)$, and the Gauss–Legendre quadrature is not optimal for integrating non-smooth functions.

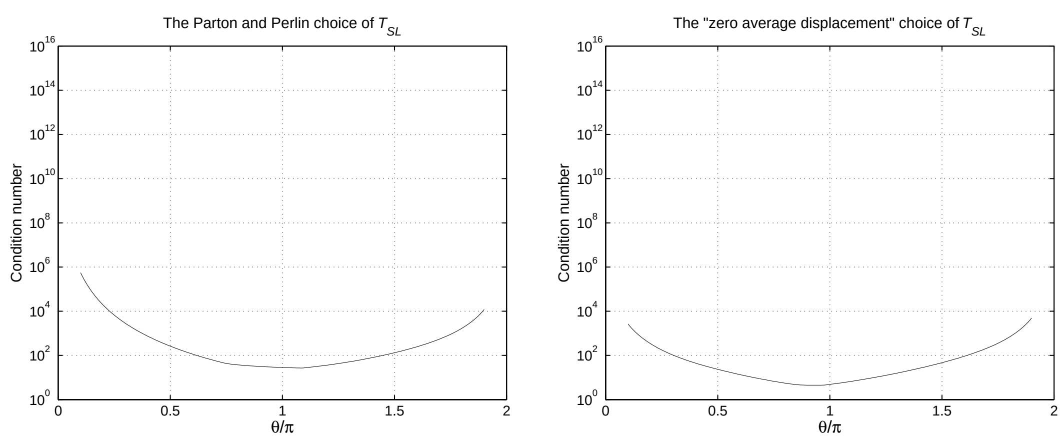

Figure 6: Same as in Fig. 3, but polynomial product integration rather than Gauss–Legendre quadrature is used for interaction on the four panels closest to the corner point.

An interesting option for more accurate discretization within the Nyström method is to use polynomial product integration of degree $d-1$ rather than Gauss–Legendre quadrature, when τ_{lp} and t_{kr} are placed on quadrature panels close to but on opposing sides of the same corner point. Reference may be made to p. 116 of Ref. [1] for general ideas — and to §10.4 of Ref. [9] for an example where polynomial product integration on a few panels within a Nyström scheme, otherwise relying on Gauss–Legendre quadrature, improves the convergence rate of the solution to an integral equation for a biharmonic problem on a non-smooth domain. Fig. 6 shows that polynomial product integration is efficient in the present context, too. The sequence $(A_n^{\Gamma_\circ})$ now seems to be stable for any angle $\theta \in [0.1\pi, 1.9\pi]$.

4. Summary and discussion

The stability of the Nyström method for the Sherman–Lauricella equation on piecewise smooth contours is linked to the invertibility of certain operators A_{c_j}, belonging to an algebra of Toeplitz operators. To study the invertibility of the operators A_{c_j}, we used a numerical approach and a special model contour which has only one corner point with varying opening angle θ_j. For the original Nyström method based on Gauss–Legendre quadrature, we found there are several values of θ_j where the invertibility of the operator A_{c_j} may fail. As a consequence, the original Nyström method on any contour Γ that has corner points with such opening angles is not going to be stable and requires modification. In certain situations, one modification suggested is to replace Gauss–Legendre quadrature with polynomial product integration.

While the focus of the paper is on stability, we end by remarking that improved computational economy of the Nyström method for integral equations on piecewise smooth contours can be obtained with a recently developed scheme [10]. That scheme, in addition to using polynomial product integration, employs a compression technique to restrict integral operators to low-dimensional subspaces – thereby greatly reducing the number of discretization points needed to reach a given accuracy.

Acknowledgments

The work of the first author was partially supported by the University of Brunei Darussalam under Grants `GSR/S&T/19` and `PNC2/2/RG/1(159)`.

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