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Published in:
2016 IEEE 55th Conference on Decision and Control (CDC)

DOI:
10.1109/CDC.2016.7799078

2016

2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

Exchange Economics as an Alternative to Distributed Optimization

Anders Rantzer

Abstract— **Quadratic optimization subject to linear constraints is a fundamental building-block in many other branches of applied mathematics. However, for large-scale systems, where a common global objective function is neither naturally defined nor easily computable, it is natural to view economic equilibrium theory as an alternative approach to design and analysis. Stability and robustness of equilibria can then be studied using the concept of monotonicity. In this paper we prove fundamental monotonicity properties for price dynamics with quadratic utilities. In particular, the main theorem gives quantitative bounds on the size of the monotone cone. Simple examples illustrate the ideas.**

I. INTRODUCTION

A common approach to design of interconnected systems is to define a global cost function and then search for solutions that minimize this cost. See for example [11], [3], [6], [10]. The paradigm has several benefits. The existence of a global objective makes it possible to compare different solution methods and architectures in a rigorous manner. In particular, distributed solutions can be compared to centralized schemes for which theory is well established. A useful approach to synthesis of distributed controllers with this paradigm is dual decomposition [9], which can be viewed as a method to coordinate agents using monetary incentives.

However, optimization of a global cost function also has disadvantages. One difficulty is that optimization over distributed control structures often leads to non-convex problems which are very hard to solve. Another difficulty comes up at an even earlier stage, namely in the choice of optimization criterion. For large-scale control applications such as energy-, traffic- or communication networks, there are many agents involved, each with different needs and interests. Integrating all of them into single optimization objective is hard, if not impossible. This motivates us to consider an alternative paradigm for distributed synthesis, namely "general equilibrium theory" from exchange economics.

Many fundamental ideas in the theory for exchange economics, such as the notion of a Walras equilibrium can be traced back to the 19th century. The study of conditions for existence, uniqueness and stability in the economic literature culminated during the 1950-70s. Results from this period are summarized in a classic book by K.J. Arrow and F.H. Hahn [1] and are also covered in modern textbooks on microeconomics. The concept of monotone operators, which is central to our paper, has a long history in economics. See [5] and references therein.

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Also mathematicians and have expressed interest in the subject. A notable example is S. Smale, who in 1998 stated "Introduction of dynamics into economic theory" as the eight problem in his collection Mathematical problems for the next century [12]. The last ten years has seen a growing interest from the computer science and game theory perspective, where the main challenge is to efficiently compute economic equilibria [8]. Hence the difficulties associated with equilibrium theory should not be underestimated. Our attention will be restricted to the special case that every agent is maximizing a quadratic objective. This is a natural choice for control applications, but not very common in economics. An exception is [4, Figure 2], which nicely illustrates the possibility of multiple equilibria.

The main contribution of this paper is to derive a convex cone of price vectors where the demand function is a monotone operator. Monotonicity guarantees uniqueness and global stability of the equilibrium. After some preliminaries in section II, the main result is presented in section III and proved in section IV.

II. PRELIMINARIES

$\mathbf{P}$ will generally be assumed to be a cone in $\mathbb{R}^n$. Let $\mathbf{S_P} = \{p \in \mathbf{P} : |p| = 1\}$. A *demand function* is a map $\mathbf{z} : \mathbf{P} \to \mathbb{R}^n$ such that $\mathbf{z}(p) = \mathbf{z}(tp)$ for all $t > 0$ and which satisfies *Walras law:* $p^T\mathbf{z}(p) = 0$ for all *price vectors* p.

To model an exchange economy with n products traded by m agents, we may consider demand functions $\mathbf{z}_1, \dots, \mathbf{z}_m$. The price vector $p \in \mathbb{R}^n$ defines relative prices for the products and $\mathbf{z}_i(p)$ specifies the quantities that agent i is willing to buy and sell subject to the budget constraint given by Walras law. The sum $\mathbf{z} = \sum_i \mathbf{z}_i$ is called the *aggregate demand*. When demand equals supply for all products, i.e. the aggregate demand is zero, the market is said to be in equilibrium. Hence a solution p to the equation $\mathbf{z}(p) = 0$ is called an equilibrium price vector.

Dynamic models are used to describe how prices change under non-equilibrium conditions. Normally the relative price for a product tends to fall when supply is bigger than demand. The most common model for price dynamics is the *Tâtonnement process*, which in its simplest form can be stated as

$$\dot{p}(t) = \mathbf{z}(p(t)). \qquad (1)$$

Notice that $\frac{d}{dt}|p(t)|^2 = 2p^T\dot{p} = 2p^T\mathbf{z}(p) = 0$, so the norm $|p(t)|$ is constant and only the relative prices change.

The following is a useful criterion for uniqueness and stability of an equilibrium:

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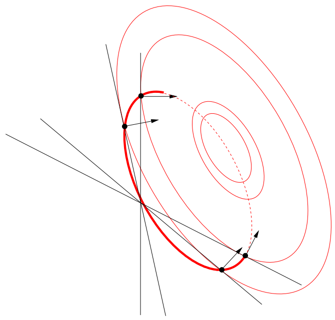

Fig. 1. Level curves of a quadratic cost function $V(z)$ are marked as thin ellipsoids. The straight lines define constraints of the form $p^T z = 0$. In each of the four cases, the value $\mathbf{z}(p)$ has been marked by a point and the direction of p has been marked by a small arrow. The range of $\mathbf{z}$ is also an ellipsoid. The thicker part indicates the values of $\mathbf{z}(p)$ where $p \in \mathbf{P}_R$.

Proposition 1: Given a continuous demand function $\mathbf{z}$, suppose that $\mathbf{P}$ is invariant under the dynamics (1). Assume also that $\mathbf{z}(p_*) = 0$, $p_* \in \mathbf{S_P}$ and $p_*^T \mathbf{z}(p) > 0$ for all other $p \in \mathbf{S_P}$. Then $\lim_{t\to\infty} \frac{p(t)}{|p(0)|} = p_*$ for all solutions to (1) with $p(0) \in \mathbf{P}$. Moreover, if $(p-q)^T[\mathbf{z}(p) - \mathbf{z}(q)] < 0$ for all non-equal $p, q \in \mathbf{S_P}$, then the Euclidean distance between any two solutions decreases monotonically with time.

This is essentially a reformulation of [2, Theorem 2]. For completeness, we include a proof in the appendix.

III. Quadratic Utility Functions

For a positive definite $R \in \mathbb{R}^{n\times n}$ with condition number c, define the scalar product $\langle p, q\rangle := p^T R q$ and the norm $\|p\| := \sqrt{p^T R p}$. Let

$$V(z) = r^T z - \frac{1}{2} z^T R^{-1} z \tag{2}$$

$$\mathbf{z}(p) = \arg \max_{p^T z = 0} V(z) \tag{3}$$

$$\mathbf{P}_R = \left\{ p \,:\, \langle r, p\rangle > [1 + (c-1)^{-2}]^{-1/2} \, \|p\| \, \|r\| \right\}. \tag{4}$$

The following is our main theorem:

Theorem 2: Given $\mathbf{z}, \mathbf{P}_R$ defined by (3)-(4), it holds that

$$\begin{aligned} &(p - r)^T[\mathbf{z}(p) - \mathbf{z}(r)] < 0 && \text{for } p \text{ not parallel to } r \\ &(p - q)^T[\mathbf{z}(p) - \mathbf{z}(q)] < 0 && \text{for all } p, q \in \mathbf{P}_R. \end{aligned}$$

Remark 1. When the condition number c is large, the cone of monotonicity $\mathbf{P}_R$ is small. Conversely, when c is close to one, $\mathbf{P}_R$ is close to a half space.

Remark 2. The range of $\mathbf{z}$ is always an ellipsoid through zero. In fact, it is straightforward to verify that

$$\left| \mathbf{z}(p) - \frac{Rr}{2} \right|_{R^{-1}} = \left| \frac{Rr}{2} \right|_{R^{-1}}$$

for all p. (Here the notation $|x|_M = \sqrt{x^T M x}$ is used.) See Figure 1.

The proof of Theorem 2 is given in the next section after some definitions and preliminary results. However, first we give a some applications.

For positive definite $R_1, \dots, R_m$ with condition numbers $c_1, \dots, c_m$, define the scalar product $\langle p, q\rangle_i := p^T R_i q$ and the norm $\|p\|_i := \sqrt{p^T R_i p}$.

$$V_i(z) = r_i^T z - \frac{1}{2} z^T R_i^{-1} z \tag{5}$$

$$\mathbf{z}_i(p) = \arg \max_{p^T z = 0} V_i(z) \tag{6}$$

$$\mathbf{P}_i = \left\{ p \,:\, \langle r, p\rangle_i > [1 + (c_i - 1)^{-2}]^{-1/2} \, \|p\|_i \, \|r\|_i \right\}. \tag{7}$$

Combination of Theorem 2 with Proposition 1 gives the following corollaries.

Corollary 3: Let $\mathbf{P}$ be a subset of $\cap_{i=1}^m \mathbf{P}_i$ that is invariant under the dynamics $\dot p(t) = \sum_i \mathbf{z}_i(p(t))$. Then the equilibrium equation $\sum_{i=1}^m \mathbf{z}_i(p_*) = 0$ has a unique solution $p_* \in \mathbf{S_P}$ and $\lim_{t\to\infty} \frac{p(t)}{|p_0|} = p_*$ whenever $p(0) \in \mathbf{P}$.

Corollary 4: Suppose that $R_1 = \cdots = R_m$. Then the equilibrium equation $\sum_{i=1}^m \mathbf{z}_i(p_*) = 0$ has the unique solution $p_* = \sum_i r_i$ and $\lim_{t\to\infty} \frac{p(t)}{|p_0|} = p_*$ for all solutions to $\dot p(t) = \sum_i \mathbf{z}_i(p(t))$ with $p(0)$ not parallel to $-\sum_i r_i$.

Example 1. For a simple example, consider two coupled control problems at stationarity. (We save the dynamic case for future work.)

$$\begin{aligned} &\text{minimize}_{x_1,u_1} && |x_1 - w_1|^2 + |u_1|^2 \\ &\text{subject to} && 0 = A_{11}x_1 + A_{12}x_2 + B_1u_1 \end{aligned}$$

$$\begin{aligned} &\text{minimize}_{x_2,u_2} && |x_2 - w_2|^2 + |u_2|^2 \\ &\text{subject to} && 0 = A_{21}x_1 + A_{22}x_2 + B_2u_2 \end{aligned}$$

The two problems are decoupled by introduction of a price vector (p_1, p_2).

$$\begin{aligned} &\text{minimize}_{x_1,x_{12},u_1} && |x_1 - w_1|^2 + |u_1|^2 \\ &\text{subject to} && 0 = A_{11}x_1 + A_{12}x_{12} + B_1u_1 \\ &&& 0 = p_1x_1 - p_2x_{12} \end{aligned}$$

$$\begin{aligned} &\text{minimize}_{x_{21},x_2,u_2} && |x_2 - w_2|^2 + |u_2|^2 \\ &\text{subject to} && 0 = A_{21}x_{21} + A_{22}x_2 + B_2u_2 \\ &&& 0 = p_1x_{21} - p_2x_2 \end{aligned}$$

For a given price vector p, the two agents independently solve their optimization problems. The price vector is said to define an equilibrium if $x_{12} = x_2$ and $x_{21} = x_1$. □

Example 2.

$$R_1 = \begin{bmatrix} 1 & 0.1 & 0 \\ 0.1 & 1.1 & 0.1 \\ 0 & 0.1 & 1 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 1.1 & 0.1 & 0.1 \\ 0.1 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \qquad r_2 = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$$

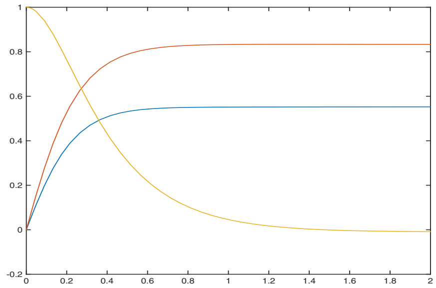

Fig. 2. The price dynamics in Example 1 smoothly finds its equilibrium values. In fact, the theory shows that the distance to the optimal price vector decreases monotonically with time.

For a given price vector p, the two agents independently solve the optimization problem

$$\max_{p^T z=0} r_i^T z - \frac{1}{2} z^T R_i^{-1} z$$

for $i = 1, 2$ respectively. The resulting optimal vectors are

$$\mathbf{z}_i(p) = \left[I - R_i p(p^T R_i p)^{-1} p^T\right] R_i r_i.$$

Simulation of

$$\begin{aligned} \dot{p}(t) &= \mathbf{z}_1(p(t)) + \mathbf{z}_2(p(t)) \\ p(0) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \end{aligned}$$

gives the result shown in Figure 2. □

IV. Monotonicity of Demand Functions

Definition 1. The demand function $\mathbf{z}$ on $\mathbf{P}$ is said to be *monotone*[1] if the inequality $(p-q)^T[\mathbf{z}(p) - \mathbf{z}(q)] \leq 0$ holds for all $p, q \in \mathbf{S_P}$. It is said to be *strictly monotone* if the inequality is strict whenever $p \neq q$.

Definition 2. The demand function $\mathbf{z}$ is said to be *locally monotone* if for every $q \in \mathbf{S_P}$, it is monotone in an open convex cone containing q. *Locally strictly monotone* is defined analogously.

Global monotonicity generally follows from local monotonicity [7]. However, our definition is non-standard, since the inequality is only considered on the non-convex set $\mathbf{S_P}$. Because of this, the following statement has no counterpart for non-proper cones.

Theorem 5: Let $\mathbf{P}$ be a proper convex cone in $\mathbb{R}^n$. A demand function $\mathbf{z}$ on $\mathbf{P}$ is monotone if and only if it is locally monotone. It is strictly monotone if and only it is locally strictly monotone.

Proof. In this proof the scalar product $p^T q$ will be denoted $\langle p, q\rangle$. Let $p, q \in \mathbf{S_P}$ be non-identical. The assumption that $\mathbf{P}$ is proper implies that $\langle p, q\rangle > -1$. Define

$$\begin{aligned} \hat{\theta} &:= \arccos\langle p, q\rangle > 0 \\ r &:= \frac{p - q\cos\hat{\theta}}{\sin\hat{\theta}} \\ s(\theta) &:= r\sin\theta + q\cos\theta \qquad 0 \leq \theta \leq \hat{\theta} \end{aligned}$$

It is then straightforward to verify that $\|r\| = \|q\| = 1$ and $\langle r, q\rangle = 0$. Moreover, $s(0) = q$, $s(\hat{\theta}) = p$ and $s(\theta) \in \mathbf{S_P}$ for all θ. The identity $\langle s(\theta), \mathbf{z}(s(\theta))\rangle = 0$ implies existence of a function $a(\theta)$ such that

$$\langle r, \mathbf{z}(s(\theta))\rangle = a(\theta)\cos\theta \qquad \langle q, \mathbf{z}(s(\theta))\rangle = -a(\theta)\sin\theta$$

It follows for $0 \leq \phi \leq \theta \leq \hat{\theta}$ that

$$\begin{aligned} \langle s(\theta), \mathbf{z}(s(\phi))\rangle &= a(\theta)\left[\sin\theta\cos\phi - \cos\theta\sin\phi\right] \\ &= a(\theta)\sin(\theta - \phi) \end{aligned}$$

$$\langle s(\theta), \mathbf{z}(s(\phi))\rangle + \langle s(\phi), \mathbf{z}(s(\theta))\rangle = \left[a(\theta) - a(\phi)\right]\sin(\theta - \phi)$$

Assuming that $\mathbf{z}$ is locally contractive at q for every $q \in \mathbf{P}$, it follows that the left hand side is positive whenever $\theta - \phi$ is sufficiently small. As a consequence, a must be a strictly increasing function and

$$\langle q, \mathbf{z}(p)\rangle + \langle p, \mathbf{z}(q)\rangle = \left[a(\hat{\theta}) - a(0)\right]\sin(\hat{\theta}) > 0.$$

This completes the proof. □

Theorem 6: A continuously differentiable demand function $\mathbf{z}$ on $\mathbf{P}$ is strictly monotone provided that

$$x^T\left[\frac{\partial\mathbf{z}}{\partial p}(p)\right]x < 0 \tag{8}$$

for all $p \in \mathbf{S_P}$, $x \in \mathbb{R}^n \setminus \{0\}$ with $x^T p = 0$.

Proof. The condition that (8) holds for $x \in \mathbb{R}^n \setminus \{0\}$ with $x^T p = 0$ implies existence of $\lambda, \epsilon > 0$ such that the matrix $\frac{\partial\mathbf{z}}{\partial p}(p) + [\frac{\partial\mathbf{z}}{\partial p}(p)]^T - 2\lambda pp^T + \epsilon I$ is negative definite. Taylor expansion gives

$$\mathbf{z}(q) = \mathbf{z}(p) + \frac{\partial\mathbf{z}}{\partial p}(p)(q-p) + o(|q-p|)$$

and

$$\begin{aligned} &(q-p)^T[\mathbf{z}(q) - \mathbf{z}(p)] \\ &= (q-p)^T\frac{\partial\mathbf{z}}{\partial p}(p)(q-p) + o(|q-p|^2) \\ &\leq \lambda(q-p)^T pp^T(q-p) - \epsilon|q-p|^2 + o(|q-p|^2) \end{aligned}$$

However $|p| = |q| = 1$, so

$$2p^T(q-p) = 2p^T(q-p) + |p|^2 - |q|^2 = -|q-p|^2.$$

In particular, $(q-p)^T[\mathbf{z}(q) - \mathbf{z}(p)]$ is strictly negative for $q \in \mathbf{S_P}$ in a neighbourhood of p. This proves locally strict monotonicity and the claim follows from Theorem 5. □

Lemma 7: Given a positive definite matrix $R \in \mathbb{R}^{n\times n}$ with condition number c, the inequality

$$(x^T Rp)^2 \leq (1 - c^{-1})\, x^T Rx\, p^T Rp \tag{9}$$

holds for all $x, p \in \mathbb{R}^n$ such that $x^T p = 0$.

[1]Our sign convention is consistent with [5], whereas the mathematics literature often has the opposite sign convention, requiring $(p-q)^T[\mathbf{z}(p) - \mathbf{z}(q)] \geq 0$.

Proof. The definition of condition number gives

$$p^T Rp \cdot p^T R^{-1} p \leq c|p|^4 \tag{10}$$

(Actually, this bound is conservative and it is possible to replace condition numbers everywhere in this paper by smaller numbers as long as (10) is not violated.) Given p, x with $p^T x = 0$, define $q = R^{1/2}p$, $s = R^{-1/2}p$ and $y = R^{1/2}x$. Then the inequality (10) can be rewritten as follows:

$$c^{-1}|q|^2|s|^2 \leq (q^T s)^2$$

$$q^T \left(I - \frac{ss^T}{|s|^2}\right)^2 q \leq (1 - c^{-1})|q|^2$$

$$\left(I - \frac{ss^T}{|s|^2}\right) qq^T \left(I - \frac{ss^T}{|s|^2}\right) \leq (1 - c^{-1})|q|^2 I$$

Multiplying the last inequality from left and right by y and using that $s^T y = 0$ gives

$$(q^T y)^2 \leq (1 - c^{-1})|q|^2|y|^2$$

which is just another way of saying (9). □

Lemma 8: Given a number $\alpha \in [0, 1]$, the inequality

$$\left|\frac{a^T b}{|a|^2} a - b\right| \leq |b|\sqrt{1 - \alpha^2}$$

holds for all $a, b \in \mathbb{R}^n$ with $a^T b \geq \alpha|a| \cdot |b|$.

Proof. Let θ be the angle between the two vectors a, b. The inequality $a^T b \geq \alpha|a| \cdot |b|$ then states that $\cos\theta \geq \alpha$. Hence $\sin\theta \leq \sqrt{1 - \alpha^2}$, so the inequality follows from the fact that the left hand side is the distance between b and its orthogonal projection on a. □

Proof of Theorem 2. Straightforward calculations give

$$\begin{aligned}
\mathbf{z}(p) &= \left[I - Rp(p^T Rp)^{-1}p^T\right] Rr \\
r^T\mathbf{z}(p) &= r^T Rr - r^T Rp(p^T Rp)^{-1}p^T Rr \\
&= \min_{t\in\mathbb{R}} \begin{bmatrix} r \\ -tp \end{bmatrix}^T \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} r \\ -tp \end{bmatrix} \\
&= \min_{t\in\mathbb{R}} |r - tp|_R^2
\end{aligned}$$

Moreover, $p^T\mathbf{z}(p) = 0$ and $\mathbf{z}(r) = 0$, so

$$(p - r)^T[\mathbf{z}(p) - \mathbf{z}(r)] = -r^T\mathbf{z}(p) = -\min_{t\in\mathbb{R}} |r - tp|_R^2$$

and the first inequality is proved.

For the second inequality, introduce the notation $\langle p, q\rangle := p^T Rq$ and $\|p\| := \sqrt{\langle p, p\rangle_R}$. To apply Theorem 6, it is sufficient to verify that the inequality (8) holds as long as p stays in the cone $\mathbf{P}$ defined in Theorem 2. Notice that

$$\frac{\partial\mathbf{z}}{\partial p} = \frac{2Rpp^T Rrp^T R}{(p^T Rp)^2} - \frac{Rpr^T R}{p^T Rp} - \frac{p^T Rr}{p^T Rp} R.$$

Hence

$$x^T \left[\frac{\partial\mathbf{z}}{\partial p}(p)\right] x = \frac{2\langle p, r\rangle\langle x, p\rangle^2}{\|p\|^4} - \frac{\langle x, p\rangle\langle r, x\rangle + \langle p, r\rangle\|x\|^2}{\|p\|^2}.$$

Let $\beta = 1 - c^{-1}$ where c is the condition number of R. By Lemma 7, the inequality $\langle x, p\rangle \leq \beta\|x\| \cdot \|p\|$ holds for x such that $x^T p = 0$. The assumption $p \in \mathbf{P}$ means that $\langle p, r\rangle \geq \alpha\|p\| \cdot \|r\|$ with $\alpha = [1 + (c - 1)^{-2}]^{-1/2}$. Hence Lemma 8 gives

$$\left|\frac{\langle p, r\rangle}{\|p\|^2} p - r\right| \leq \sqrt{1 - \alpha^2}\|r\|$$

and

$$\begin{aligned}
&x^T \left[\frac{\partial\mathbf{z}}{\partial p}(p)\right] x \\
&= \frac{\langle p, r\rangle}{\|p\|^2}\left[\frac{\langle x, p\rangle^2}{\|p\|^2} - \|x\|^2\right] + \frac{\langle x, p\rangle}{\|p\|^2}\left\langle \frac{\langle p, r\rangle}{\|p\|^2} p - r, x\right\rangle \\
&\leq \frac{\langle p, r\rangle(\beta^2 - 1)\|x\|^2}{\|p\|^2} + \frac{\langle x, p\rangle}{\|p\|^2}\sqrt{1 - \alpha^2}\|r\| \cdot \|x\| \\
&\leq \frac{\alpha\|p\| \cdot \|r\|(\beta^2 - 1)\|x\|^2}{\|p\|^2} + \beta\sqrt{1 - \alpha^2}\frac{\|x\|^2\|r\|}{\|p\|} \\
&\leq \Big(\underbrace{\alpha(\beta^2 - 1)}_{\alpha c^{-2} - 2\alpha c^{-1}} + \underbrace{\beta\sqrt{1 - \alpha^2}}_{\alpha c^{-1}}\Big)\|x\|^2\|p\|^{-1}\|r\| \\
&< 0 \quad \text{for } x \neq 0
\end{aligned}$$

This completes the proof of Theorem 2. □

V. Conclusions

We have quantified the cone of monotonicity for price dynamics in exchange economics. The cone is large when all agents have similar utility functions and condition numbers are small. If this is not the case, monotonicity is easily lost, which can lead to less predictable dynamics and multiple equilibria. Hence quadratic terms in utility functions for engineering design should be selected with care, and maybe sometimes avoided altogether.

VI. Acknowledgement

The author is grateful to Yongxin Chen for insightful comments.

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APPENDIX

Proof of Proposition 1. Note that

$$\frac{d}{dt}|\mathbf{z}(p(t))|^2 = p^T\dot{p} = p^T\mathbf{z}(p) = 0.$$

Hence the norm $|p(t)|$ remains constant for all t. Assume without loss of generality that $|p(0)| = 1$. Then

$$\frac{d}{dt}|p-p_*|^2 = 2(p-p_*)^T\dot{p} = -2(p-p_*)^T\mathbf{z}(p) = p_*^T\mathbf{z}(p)$$

where the last expression is negative by assumption, so $|p(t)-p_*|$ is decreasing with t. Suppose that $\lim_{t\to\infty}|p(t)-p_*| = \epsilon > 0$. Let δ be the minimal value of $p_*^T\mathbf{z}(p)$ on the compact set $\{p \in \mathbf{S_P} \mid |p-p_*| \geq \epsilon\}$. Then $\frac{d}{dt}|p(t)-p_*|^2 \leq -\delta$ for all t, so the assumption that $\lim_{t\to\infty}|p(t)-p_*| > 0$ must be wrong and $\lim_{t\to\infty}\frac{p(t)}{|p(0)|} = p_*$.

Finally, let $p(t)$ and $q(t)$ be any two solutions of the equation $\dot{p} = -\mathbf{z}(p)$. Then

$$\begin{aligned}\frac{d}{dt}|p-q|^2 &= (p-q)^T(\dot{p}-\dot{q})\\ &= (p-q)^T[\mathbf{z}(q)-\mathbf{z}(p)]\\ &= q^T\mathbf{z}(p) + p^T\mathbf{z}(q) \leq 0,\end{aligned}$$

so $|p(t)-q(t)|$ decreases monotonically. □