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The Use of the $r*$ Heuristic in Covariance Completion Problems

Christian Grussler, Armin Zare, Mihailo R. Jovanović, and Anders Rantzer

***Abstract*— We consider a class of structured covariance completion problems which aim to complete partially known sample statistics in a way that is consistent with the underlying linear dynamics. The statistics of stochastic inputs are unknown and sought to explain the given correlations. Such inverse problems admit many solutions for the forcing correlations, but can be interpreted as an optimal low-rank approximation problem for identifying forcing models of low complexity. On the other hand, the quality of completion can be improved by utilizing information regarding the magnitude of unknown entries. We generalize theoretical results regarding the $r*$ norm approximation and demonstrate the performance of this heuristic in completing partially available statistics using stochastically-driven linear models.**

***Index Terms*— Convex optimization, k-support-norm, low-rank approximation, nuclear norm regularization, state covariances, structured matrix completion problems.**

I. Introduction

Matrix completion problems emerge in many applications (cf. [1]–[3]). In this work, we are interested in a class of structured covariance completion problems which arise as inverse problems in low-complexity modeling of complex dynamical systems. A particular class of models that can be used for this purpose are stochastically-driven linear models. Motivation for this choice arises in the modeling of fluid flows where the stochastically-forced linearized Navier-Stokes equations have proven capable of replicating structural features of wall-bounded shear flows [4]–[9].

The problem of estimating covariances at the output of known linear systems has been previously addressed [10]–[12]. More recently, a modeling and optimization framework was proposed for designing stochastic forcing models of low complexity which account for partially observed statistical signatures [13]. In this setup, the complexity is related to the rank of input correlation structures [13], [14]. This gives rise to a class of structured covariance completion problems that aim to complete partially observed statistical signatures in a way that is consisted with the assumed linear dynamics.

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In addition, the use of the nuclear norm as a convex surrogate for rank minimization [15], [16] has allowed for the development of efficient customized algorithms that handle large-size problems [13], [17]. This approach has particularly proven successful in the low-complexity modeling of turbulent fluid flows [9].

Recently, various benefits and applications of the so-called "$r*$ norm" (also called "k-support-norm"), as a natural extension of the nuclear norm, have been demonstrated [18]–[24]. In particular, its relation with the optimal rank r approximation under convex constraints has been investigated [18], [24]. Herein, we utilize these theoretical results to address the covariance completion problem as a special case of low-rank approximation. We demonstrate the ability of this approach in improving the quality of completion while maintaining (or even lowering) the complexity of the required forcing model compared to the nuclear norm relaxation.

The outline of this paper is as follows. In Section II, we provide a detailed background of the considered covariance completion problem and motivate the use of the $r*$ norm, which is formally introduced in Section III-A. Subsequently, we present two new convex relaxations to our problem in Section III-B. In Section IV, we provide illustrative examples to support our theoretical developments and finally conclude with a summary of contributions in Section V.

II. Problem Formulation

Consider the linear time-invariant (LTI) system with state-space representation

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C x \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{C}^n$ is the state vector, $y(t) \in \mathbb{C}^p$ is the output, $u(t) \in \mathbb{C}^m$ is a zero-mean stationary stochastic process, $A \in \mathbb{C}^{n\times n}$ is Hurwitz, and $B \in \mathbb{C}^{n\times m}$ is the input matrix with $m \le n$. For controllable (A, B), a positive-definite matrix X qualifies as the steady-state covariance matrix of the state vector $x(t)$ if and only if the Lyapunov-like equation

$$A\,X \,+\, X\,A^* \;=\; -\,(B\,H^* \,+\, H\,B^*)\,, \tag{2}$$

is solvable for $H \in \mathbb{C}^{n\times m}$ [25], [26]. Equation (2) provides an algebraic characterization of state covariances of linear dynamical systems driven by white or colored stochastic processes. For white noise u with covariance W, $H = BW/2$ and (2) simplifies to the standard algebraic Lyapunov equation

$$A\,X \,+\, X\,A^* \;=\; -B\,W B^*. \tag{3}$$

The main difference between (2) and (3) is that the right-hand-side, $-BWB^*$ in (3) is negative semi-definite.

We are interested in the setup where the matrix A in (1) is known but due to experimental or numerical limitations, only partial correlations between a limited number of state components are available. Moreover, it is often the case that the origin and directionality of the stochastic excitation u is unknown. Interestingly, the solvability of (2) can be shown to be equivalent to the following rank condition:

$$\operatorname{rank}\begin{bmatrix} AX + XA^* & B \\ B^* & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}. \quad (4)$$

This implies that any positive-definite matrix X is admissible as a covariance of the LTI system (1) if the input matrix B is full row rank [25], which eliminates the role of the dynamics inherent in A. Hence, it is desirable to limit the rank of the input matrix B.

In [13], an optimization framework was developed to account for partially known sampled second-order statistics using stochastically-forced LTI models. In this framework, the complexity of the model is reflected by the rank of the input matrix B, which is bounded by the rank of [13], [14]

$$Z := -(AX + XA^*).$$

Based on this, the structured covariance completion problem is given by

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & \operatorname{rank}(Z) \\ \text{subject to} & AX + XA^* + Z = 0 \\ & (CXC^*) \circ E - G = 0 \\ & X \succ 0, \end{array} \quad (5)$$

in which matrices A, C, E, and G are problem data, and Hermitian matrices $X, Z \in \mathbb{C}^{n\times n}$ are optimization variables. While the steady-state covariance matrix X is required to be positive definite, the matrix Z may have both positive and negative eigenvalues. This is in contrast to the case of white-in-time input u where the matrix Z is positive semi-definite. Entries of G represent partially available second-order statistics and C, the output matrix, establishes the relationship between entries of the matrix X and partially available statistics resulting from experiments or simulations. The symbol $\circ$ denotes elementwise matrix multiplication and E is the structural identity matrix,

$$E_{ij} = \begin{cases} 1, & \text{if } G_{ij} \text{ is available} \\ 0, & \text{if } G_{ij} \text{ is unavailable.} \end{cases}$$

Due to the non-convexity of the rank function, problem (5) is difficult to solve. Typically, the nuclear norm, i.e., the sum of singular values of a matrix, $\|Z\|_* := \sum_i \sigma_i(Z)$, is used as a proxy for rank minimization [15], [16]. This prompts the following convex reformulation:

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & \|Z\|_* \\ \text{subject to} & AX + XA^* + Z = 0 \\ & (CXC^*) \circ E - G = 0 \\ & X \succ 0, \end{array}$$

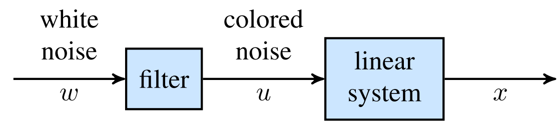


Fig. 1: A cascade connection of an LTI system with a linear filter that is designed to account for the sampled steady-state covariance matrix X.

The solution is used to construct spatio-temporal filters that generate colored-in-time forcing correlations that account for the observed statistics [13], [25], [26]; see Fig. 1.

The nuclear norm is the convex envelope of the rank function over the unit ball $\|Z\|_2 \leq 1$ and in conjunction with incoherence conditions has been utilized to provide theoretical guarantees for standard matrix completion problems [16]. However, for problem (5), the additional structural constraint that arises from the Lyapunov-like equation prevents us from taking advantage of these standard theoretical results. In addition, even though the nuclear norm heuristic achieves a low-rank solution for Z with a clear-cut in its singular values, it may not give good completion of the covariance matrix X. It is thus important to examine more refined convex relaxations that may result in better completion.

In [24], it has been demonstrated that when the magnitudes of unknown entries are significantly smaller than that of the known ones, the nuclear norm often creates regions of large entries which deviate from the ground truth. In the next section, we demonstrate that the $r*$ norm is more suitable if the objective is to keep both the rank and the Frobenius norm of the correlation structures small.

III. Low-rank approximation

We next introduce the $r*$ norm and provide a brief summary of its properties. A more elaborate presentation of these theoretical developments can be found in [24].

A. Preliminaries

In the following, let $\sigma_1(M) \geq \cdots \geq \sigma_{\min\{m,n\}}(M)$ denote the increasingly sorted singular values of $M \in \mathbb{R}^{n\times m}$, counted with multiplicity. Moreover, given a singular value decomposition $M = \sum_{i=1}^{\min\{m,n\}} \sigma_i(M)\, u_i\, v_i^T$ of M, we define $\operatorname{svd}_r(M) := \sum_{i=1}^{r} \sigma_i(M) u_i v_i^T$.

Lemma 1: Let $M \in \mathbb{R}^{n\times m}$ and $1 \leq r \leq q := \min\{m,n\}$. The r norm of the matrix M

$$\|M\|_r := \sqrt{\sum_{i=1}^{r} \sigma_i^2(M)}$$

is unitarily-invariant and its dual-norm is the $r*$ norm

$$\|M\|_{r*} := \max_{\|X\|_r \leq 1} \langle M, X\rangle.$$

It holds that

- $\|M\|_1 \leq \cdots \leq \|M\|_q = \|M\|_F = \|M\|_{q*} \leq \cdots \leq \|M\|_{1*}$
- $\text{rank}(M) \leq r$ if and only if $\|M\|_r = \|M\|_F = \|M\|_{r*}$,

where $\|\cdot\|_F$ denotes the Frobenius norm.

The nuclear norm and the $r*$ norm coincide for $r = 1$. Thus, minimizing $\|\cdot\|_{r*}$ with $r > 1$ can have a more significant influence on decreasing $\|\cdot\|_F$ than $\|\cdot\|_{1*}$. This is also motivated by the following Proposition.

Proposition 1: Let $\mathcal{C} \subset \mathbb{R}^{n\times m}$ be a closed convex set, then

$$\begin{aligned} \inf_{\substack{M\in\mathcal{C}\\ \text{rank}(M)\leq r}} \|M\|_F^2 \;&\geq\; \max_{D\in\mathcal{C}^*} \left[\inf_{M\in\mathcal{C}} \langle D, M\rangle \,-\, \|D\|_r^2\right] \\ &=\; \min_{M\in\mathcal{C}} \|M\|_{r*}^2 \end{aligned} \tag{6}$$

where $\mathcal{C}^* := \{D \in \mathbb{R}^{n\times m} : \inf_{M\in\mathcal{C}} \langle D, M\rangle > -\infty\}$. Let the maximum and the minimum in (6) be achieved by $D^* \in \mathcal{C}^*$ and M^*, respectively.

- If $\sigma_r(D^*) \neq \sigma_{r+1}(D^*)$, then the infimum on the left equals the maximum on the right and $M^* = \text{svd}_r(D^*)$.
- If $\sigma_r(D^*) = \cdots = \sigma_{r+s}(D^*) > \sigma_{r+s+1}(D^*) \neq 0$ for some $s \geq 1$ then $\text{rank}(M^*) \leq r + s$.

Proof: See [24].

Hence, in an ideal situation, i.e. $\sigma_r(D^*) \neq \sigma_{r+1}(D^*)$, r has a strong correlation with the true rank of the matrix that one aims to complete. This will be seen in several examples in Section IV.

We next concentrate on the case of $\sigma_r(D_r^*) = \sigma_{r+1}(D_r^*)$, i.e., $\text{rank}(M_r^*) > r$ where we define

$$\begin{aligned} M_r^* \;&:=\; \operatorname*{argmin}_{M\in\mathcal{C}} \|M\|_{r*}^2 \\ D_r^* \;&:=\; \operatorname*{argmax}_{D\in\mathcal{C}^*} \left[\inf_{M\in\mathcal{C}} \langle D, M\rangle \,-\, \|D\|_r^2\right] \end{aligned}$$

for $1 \leq r \leq \min\{m, n\}$. Assume that $r + 1 \leq \text{rank}(M_r^*) < \text{rank}(M_{r+1}^*)$ and $\|M_r^*\|_F > \|M_{r+1}^*\|_F$. In this very common situation (cf. Section IV), M_r may still be of sufficiently small rank but too high cost. Conversely, M_{r+1} may have sufficiently small cost but too high rank. Therefore, a trade-off between these two solutions is desired, which can be achieved by allowing for a non-integer valued r, i.e.

$$\|\cdot\|_r \;:=\; \sqrt{\sum_{i=1}^{\lfloor r\rfloor} \sigma_i^2(\cdot) \,+\, (r - \lfloor r\rfloor)\,\sigma_{\lceil r\rceil}^2(\cdot)},$$

where $\lfloor r\rfloor := \max\{z \in \mathbb{Z} : z \leq r\}$ and $\lceil r\rceil := \min\{z \in \mathbb{Z} : z \geq r\}$. Letting M_r^* and D_r^* be also defined for $r \in \mathbb{R}$, it remains true that $\text{rank}(M_r^*) \leq \lceil r\rceil + s$ if $\sigma_{\lceil r\rceil}(D_r^*) = \cdots = \sigma_{\lceil r\rceil+s}(D_r^*) > \sigma_{\lceil r\rceil+s+1}(D_r^*)$. Moreover, for $r \in \mathbb{N}$ and $\alpha \in [0, 1]$, $\|\cdot\|_{r+1-\alpha}^2$ is given by the convex combination of $\|\cdot\|_r^2$ and $\|\cdot\|_{r+1}^2$,

$$\|\cdot\|_{r+1-\alpha}^2 \;=\; \alpha\,\|\cdot\|_r^2 \,+\, (1-\alpha)\,\|\cdot\|_{r+1}^2.$$

This indicates the usefulness of the real-valued r norm to achieve the desired trade-off. In conjunction with Lemma 1 it follows that

$$F(D, r) \;:=\; \inf_{M\in\mathcal{C}} \langle D, M\rangle \,-\, \|D\|_r^2$$

is concave. Thus, *Berge's Maximum Theorem* (see [27, p. 116]) implies that the parameter depending set

$$\mathcal{C}^*(r) \;:=\; \operatorname*{argmax}_{D\in\mathcal{C}^*} \left[\inf_{M\in\mathcal{C}} \langle D, M\rangle \,-\, \|D\|_r^2\right]$$

is upper hemicontinuous in r, i.e. for all $r \in [1, \min\{m, n\}]$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \geq 1$

$$|t - r| < \delta \;\Rightarrow\; \mathcal{C}^*(t) \subset \mathcal{B}_\varepsilon\left(C^*(r)\right),$$

where $\mathcal{B}_\varepsilon\left(C^*(r)\right) := \{X : \exists D \in C^*(r) : \|X - D\|_F < \varepsilon\}$. Assume for simplicity that D_r^* is unique, then it follows that a sufficiently small increase of r cannot increase s. Therefore, as for the nuclear regularization [15], one often expects $\text{rank}(M_t^*)$ to look like a staircase function of $t \in [r, r+1]$.

B. Convex reformulation

Based on these theoretical developments, the $r*$ norm can be employed as a convex proxy for the rank function. This prompts the following convex relaxation of the covariance completion problem (5),

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & \|Z\|_{r*}^2 \\ \text{subject to} & AX + XA^* + Z \;=\; 0 \\ & (CXC^*) \circ E - G \;=\; 0 \\ & X \succ 0. \end{array} \tag{7}$$

which can be formulated as the semi-definite program (SDP)

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & \text{trace}(W) \\ \text{subject to} & \begin{bmatrix} I - P & Z \\ Z^* & W \end{bmatrix} \succeq 0 \\ & \text{trace}(P) - n + r \;=\; 0 \\ & AX + XA^* + Z \;=\; 0 \\ & (CXC^*) \circ E - G \;=\; 0 \\ & X \succ 0 \quad P \succeq 0. \end{array} \tag{8}$$

Problem (8) results from taking the Lagrange dual of the SDP characterization of the r norm; see [18] for details.

In the next section we present illustrative examples which demonstrate the benefit of using the $r*$ norm over the nuclear norm. Based on the discussion in Section III-A, and for a fair comparison, we also consider the alternative formulation

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & \|Z\|_F^2 + \gamma\,\|Z\|_* \\ \text{subject to} & AX + XA^* + Z \;=\; 0 \\ & (CXC^*) \circ E - G \;=\; 0 \\ & X \succ 0. \end{array} \tag{9}$$

This formulation has been discussed earlier in [28]. It intends to mimic the behavior of the $r*$ norm and allows us to achieve a trade-off using the tuning parameter γ. Here, $\|Z\|_F^2$ is regularized by the nuclear norm of Z and the parameter γ determines the weight on the nuclear norm.

IV. EXAMPLES

A. Two-dimensional heat equation

We provide an example to compare the performance of the relaxation in problem (7) with the performance of the hybrid objective considered in problem (9). This is based on the two-dimensional heat equation

$$\dot{T} \;=\; \Delta T \;=\; \frac{\partial^2}{\partial x^2} T \,+\, \frac{\partial^2}{\partial y^2} T$$

on the unit square. Inputs are introduced through the Dirichlet boundary conditions of the four edges, i.e., $\xi = [\,\xi_1 \,\cdots\, \xi_4\,]^T$. Finite difference discretization of the Laplacian operator on a uniform grid with step-size $h = \frac{1}{n+1}$ gives

$$\Delta T_{ij} \;\approx\; -\frac{1}{h^2}(4\,T_{ij} - T_{i+1,j} - T_{i,j+1} - T_{i-1,j} - T_{i,j-1}),$$

where T_{ij} is the temperature of the inner grid point on the ith row and jth column of the mesh. Based on this, the dynamic matrix A denotes an $n^2 \times n^2$ Poisson-matrix and the input matrix $B_\xi := [b_{ij}] \in \mathbb{R}^{n^2 \times 4}$ models the boundary conditions as inputs into the state dynamics. Here, $b_{ij} = 0$ except for the following cases:

$$\begin{aligned} b_{i1} &:= 1, \quad \text{for} \quad i = 1, 2, \ldots, n \\ b_{i2} &:= 1, \quad \text{for} \quad i = n, 2n, \ldots, n^2 \\ b_{i3} &:= 1, \quad \text{for} \quad i = n\,(n-1)+1, n\,(n-1)+2, \ldots, n^2 \\ b_{i4} &:= 1, \quad \text{for} \quad i = 1, n+1, \ldots, n\,(n-1)+1 \end{aligned}$$

The dynamics of the discretized heat equation have the following state-space representation

$$\dot{x} \;=\; \frac{1}{h^2}\,A\,x \,+\, \frac{1}{h^2}\,B_\xi\,\xi, \tag{10a}$$

where $x \in \mathbb{R}^{n^2}$ denotes the state. We assume that four input disturbances are generated by the low-pass filter

$$\dot{\xi} \;=\; -\xi \,+\, d. \tag{10b}$$

Here, d denotes a zero-mean unit variance white process.

The steady-state covariance of system (10) can be found as the solution to the Lyapunov equation

$$\tilde{A}\,\Sigma \,+\, \Sigma\,\tilde{A}^* \,+\, \tilde{B}\,\tilde{B}^* \;=\; 0$$

where

$$\tilde{A} \;=\; \begin{bmatrix} A & B_\xi \\ O & -I \end{bmatrix}, \quad \tilde{B} \;=\; \begin{bmatrix} 0 \\ I \end{bmatrix}$$

and

$$\Sigma \;=\; \begin{bmatrix} \Sigma_{xx} & \Sigma_{x\xi} \\ \Sigma_{\xi x} & \Sigma_{\xi\xi} \end{bmatrix}.$$

Here, the sub-covariance Σ_{xx} denotes the state covariance of the discretized heat equation (10a).

We use 16 points to discretize the square region ($n = 4$), and restrict the input to enter through two sides by setting the second and fourth columns of B_ξ to zero. The covariance matrix of the state in (10a) is shown in Fig. 2

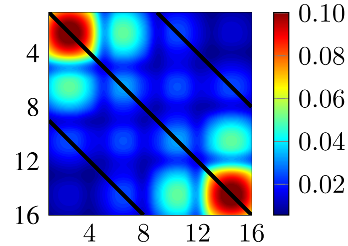


Fig. 2: Interpolated colormap of the true steady-state covariance Σ_{xx} of the discretized heat equation with ━━ indicating the available correlations used in (8) and (9).

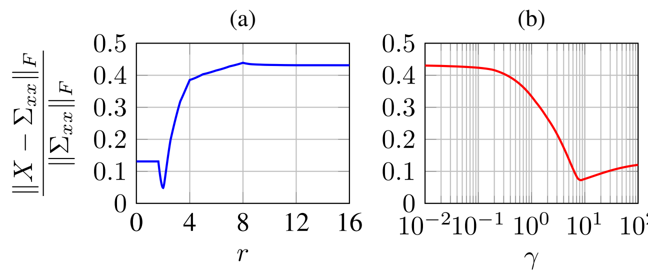


Fig. 3: (a) The r-dependence of the relative Frobenius norm error (percents) between the solution X to (8) and the true covariance Σ_{xx} for the discretized 2D heat equation. (b) The γ-dependence of the relative error between the solution to (9) and the true covariance.

where black lines indicate known correlations that are used as problem data. We conduct a parametric study to determine the influence of r and γ on the solutions of (8) and (9).

Figures 3a and 3b respectively show the r-dependence and γ-dependence of the relative error of solutions to (8) and (9). For problem (8), minimum relative error is achieved with $r = 2$, as expected for a system with two inputs. On the other hand, $\gamma = 8.46$ gives the best completion in problem (9). We note that the optimal solution for (8) results in a relative error which is about a third smaller (4.83% vs. 7.26%) with a corresponding matrix Z of lower rank (2 vs. 3).

Figure 4 shows the recovered covariance matrix of the discretized heat equation resulting from problems (8) and (9) and for various values of r and γ. Figures 4a and 4b correspond to the case of nuclear norm minimization ($r = 1$) and optimal covariance completion ($r = 2$) for problem (8). On the other hand, Figs. 4c and 4d correspond to the solution of the Frobenius norm minimization ($\gamma = 0$) and optimal covariance completion ($\gamma = 8.46$) for problem (9). It is notable that the Frobenius norm minimization does not result in reasonable completion of the covariance matrix. Moreover, the nuclear norm creates off-diagonal spots of relatively large entries where the true covariance matrix is close to zero.

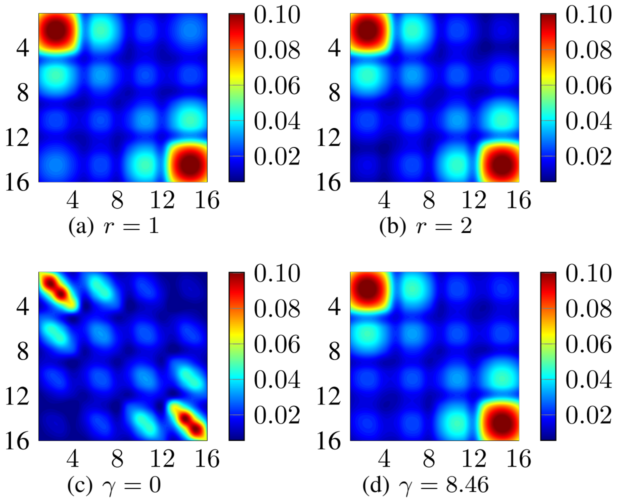


Fig. 4: The recovered state covariance matrix of the heat equation resulting from problem (8) (a, b), and problem (9) (c, d). (a) $r = 1$; (b) $r = 2$; (c) $\gamma = 0$; (d) $\gamma = 8.46$.

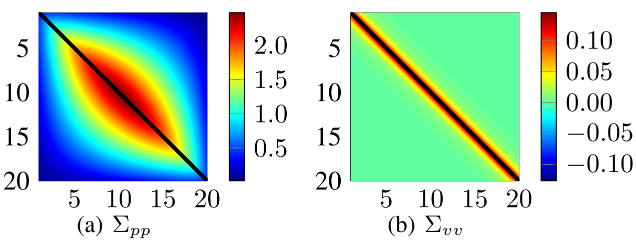


Fig. 5: The steady-state covariance matrices of the (a) position Σ_{pp}, and (b) velocity Σ_{vv}, of masses in MSD system with $n = 20$ masses with ━━ indicating available one-point correlations used in problems (8) and (9).

B. Mass-spring-damper system

We provide an example of a stochastically-forced mass-spring-damper (MSD) system to demonstrate the utility of the $r*$ norm in the completion of diagonally dominant covariance matrices. The state space representation of the MSD system is given by

$$\dot{x} \;=\; A\,x \,+\, B_\xi\,\xi$$
$$A \;=\; \begin{bmatrix} 0 & I \\ -T & -I \end{bmatrix}, \qquad B_\xi \;=\; \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Here, the state vector contains the position and velocity of masses, $x = [\, p^T \;\; v^T \,]^T$, 0 and I are zero and identity matrices of suitable sizes, and T is a symmetric tridiagonal Toeplitz matrix with 2 on the main diagonal and -1 on the first upper and lower sub-diagonals.

Stochastic disturbances are generated by a similar low-pass filter as in the previous example and the steady-state covariance matrix of x is partitioned as

$$\Sigma_{xx} \;=\; \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}.$$

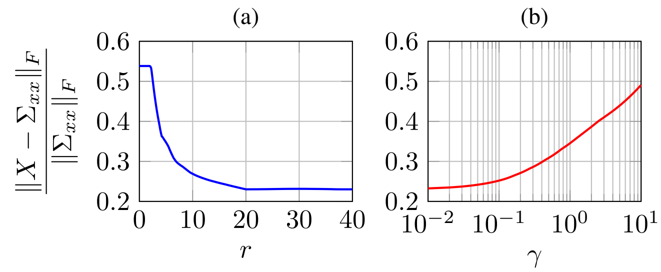


Fig. 6: (a) The r-dependence of the relative error between the solution X to (8) and the true covariance Σ_{xx} for the MSD system with $n = 20$ masses. (b) The γ-dependence of the error between the solution to (9) and the true covariance.

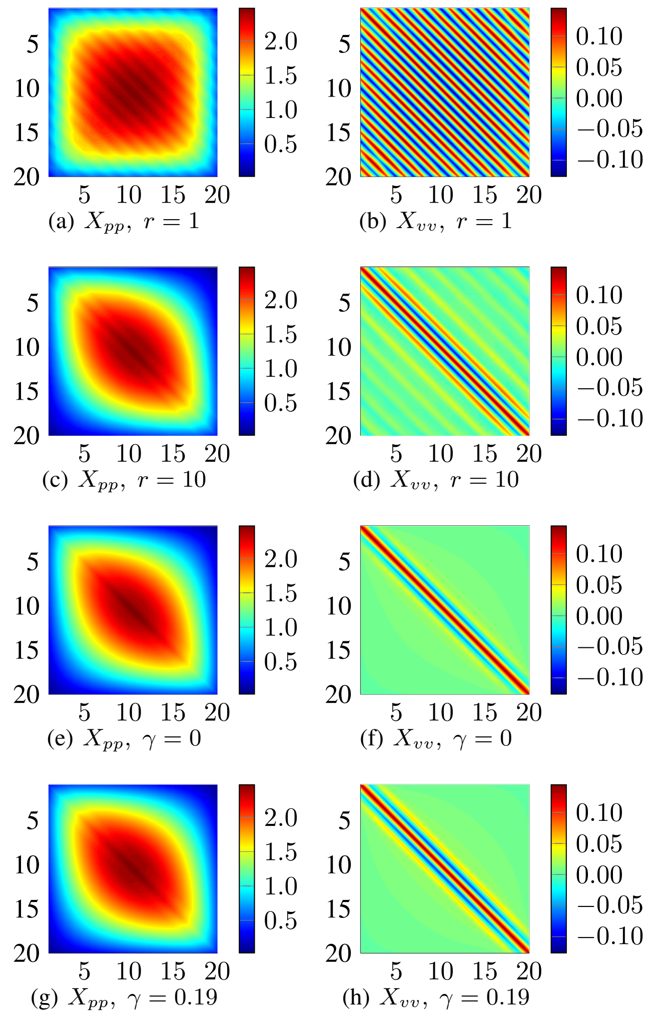
We assume that stochastic forcing affects all masses. For $n = 20$ masses, Fig. 5 shows the covariance matrices of positions Σ_{pp} and velocities Σ_{vv}. We assume knowledge of one-point correlations, i.e., diagonal entries of Σ_{pp} and Σ_{vv}. Note that in this example the covariance matrices are diagonally dominant, especially Σ_{vv}.

Again, we study the respective effect of r and γ on the solutions of (8) and (9). As shown in Fig. 6, these dependencies are monotonic and minimization of the Frobenius norm, which corresponds to solving problem (8) with $r = 2n$ and problem (9) with $\gamma = 0$, results in the best covariance completion (77% recovery). However, in this case the matrix Z is not rank deficient. On the other hand, nuclear norm minimization, which corresponds to solving (8) with $r = 1$ and (9) with $\gamma = \infty$, results in the worst completion (46%).

Figure 7 shows the recovered covariance matrices of position X_{pp} and velocity X_{vv} resulting from optimization problems (8) and (9) with various values of r and γ. While nuclear norm minimization yields poor recovery of the diagonally dominant covariance matrix of velocities Σ_{vv} (cf. Fig. 7b), minimization of the Frobenius norm results in best overall recovery (cf. Figs. 7e and 7f). However, as aforementioned, lack of a surrogate for rank minimization leads to a full-rank matrix Z. An intermediate state with reasonable recovery (73%) can be achieved by solving (8) with $r = 10$ (Figs. 7c and 7d) and (9) with $\gamma = 0.19$ (Figs. 7g and 7h). While the quality of recovery is the same, the matrix Z which results from solving problem (8) is of lower rank (10 vs. 18). Moreover, if one intended to get a solution of rank 18, choosing $r = 18$ would be successful here and by Proposition 1 there is no other solution of smaller Frobenius norm.

V. Conclusions

It has been shown that the success of the $r*$ norm in matrix completion problems is closely related to the true rank and size of unknown entries. To demonstrate this, we focus on a particular application that involves the completion of partially known covariances of complex dynamical systems



(a) $X_{pp},\ r = 1$ (b) $X_{vv},\ r = 1$

(c) $X_{pp},\ r = 10$ (d) $X_{vv},\ r = 10$

(e) $X_{pp},\ \gamma = 0$ (f) $X_{vv},\ \gamma = 0$

(g) $X_{pp},\ \gamma = 0.19$ (h) $X_{vv},\ \gamma = 0.19$

Fig. 7: The recovered covariance matrices of position and velocity in the MSD system with $n = 20$ masses resulting from problem (8) (a, b, c, d), and problem (9) (e, f, g, h). (a, b) $r = 1$; (c, d) $r = 10$; (e, f) $\gamma = 0$; (g, h) $\gamma = 0.19$.

via stochastically-forced linear models. In this, stochastic forcing models of low-complexity are identified that not only complete the partially-observed statistical signatures, but are consistent with the linear dynamics. This amounts to solving a class of structured covariance completion problems which involve minimizing the rank of the correlation structure of excitation sources. Relative to the nuclear norm relaxation, the $r*$ norm exploits an additional degree of freedom which is useful in the completion of diagonally dominant covariances.

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