A Control Application of the Direct Method of Lyapunov

Åström, Karl Johan; Jacob, J. P.

1963

IBM RESEARCH

A CONTROL APPLICATION OF THE DIRECT METHOD OF LYAPUNOV

K. J. Åström

J. P. Jacob

RJ 273

November 5, 1963

SCIENCES

MATERIAL TECHNOLOGIES

MATH AND PROGRAMMING

ENGINEERING STUDIES

MACHINES AND MACHINE ORGANIZATION

PRINTED IN U. S. A.

ACKNOWLEDGEMENT

The authors - both on temporary assignment to Research from the IBM Nordic Laboratory, Sweden - are indebted to Dr. J. E. Bertram who proposed, supervised and commented throughout this study. Dr. Bertram was the head of the Control Systems Research group in San Jose where this work was done.

1. INTRODUCTION:

The purpose of this report is to illustrate via a "case" study how one might use the direct method of Lyapunov to obtain a primitive control law for a complex dynamical system. By a primitive control law, we simply mean a control law which is designed so as to insure that the desired motions of the system are asymptotically stable; this implies that small disturbances to the system result in only small deviations of the motion. Such a control law does not in general produce optimal performance, nor does it explicitly consider system or control constraints. However, there are usually parameters in the primitive control law which can be adjusted so as to operate the system within the constraints and even to approximate optimal behavior. Furthermore, the development of an optimal control law in which constraints are considered directly is both time-consuming and expensive. As a result, in the preliminary phase of a systems study, where the key question is feasibility, and the determination of the "existence" of a stable control law is an essential step, one is often satisfied with a primitive control law.

The particular control problem with which this report is concerned was derived from certain tasks that will be required of a proposed configuration for a manned-orbiting-space-station. In this system, the crew would be housed in a cabin, roughly described as a cylinder twenty feet in length, ten feet in diameter, and weighing approximately twenty-thousand pounds.

In order to establish a reasonably uniform and controlled artificial gravity in the orbiting cabin, it is proposed that two similar bodies be assembled with a connecting cable of sufficient length. Then by spinning the two bodies at constant angular velocity about the center of mass of the system, the gravity in the cabin can be set at a desired level by the proper choice of the cable length and the velocity. The second body of the system will be the frame of the final stage of the rocket used to place the cabin in orbit. The dimensions and weight of this body will be essentially the same as those of the cabin. The connecting cable will be of steel and we shall assume that its length is controllable up to a maximum of 75 meters.

The attitude control of this system must perform a number of functions. Included among these tasks, as Markarian and Clancy [1] have pointed out, are the following:

(a) To maintain precise earth orientation of reconnaissance sensors and antennas;

(b) To maintain desired equilibrium motions against external disturbances (e.g., the gravity gradient and solar pressures) and internal disturbances (e.g., motion of the crew and sloshing of fluids);

(c) To be capable of the extensive attitude maneuvers required for ferry vehicle docking and the many experiments planned.

To efficiently carry out all of these functions, gas jets and/or reaction fly-wheels will be necessary. In addition, it will be necessary to reel the cable in and out as the cabin is maneuvered. Since hardware for this manipulation of the cable length is already necessary equipment, an obvious question is, "Could any of the tasks listed above be performed by just manipulating the cable length?"

Moreover, we limit our studies to plane motions. It is thus assumed that the motions of the bodies are restricted to translations in a plane fixed in inertial space and rotations with respect to an axis orthogonal to this plane. We will thus neglect all problems associated with the orientation of the system and rotations of the bodies around axes in the plane of motion.

The remainder of this report illustrates how the direct method of Lyapunov can be used to aid in answering such a question. Some background material on the direct method is given in Section 2. In Section 3, a mathematical model of the system is developed and an order of magnitude analysis is carried out. With the simplifying assumption that the two bodies can be treated as point masses, a stable control law is developed and the controlled system analyzed in Section 4. In Section 5, the restriction to point masses is relaxed and a control law for this system is developed. An analog computer simulation and Fortran programs for the IBM 7090 which are useful in the analysis of the system are documented in the appendices.

2. SOME BACKGROUND MATERIAL ON THE DIRECT METHOD:

The so-called direct method of Lyapunov permits one to answer questions concerning the stability of the equilibrium solution of a differential equation, utilizing the given form of the equation, but without explicit knowledge of its solution. As usually stated, the direct method applies only to the equilibrium motions of free dynamical systems; that is to say, to deviation about some fixed motion.

Now consider the dynamical system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}(t)) \qquad (2.1)$$

where $\underline{x}$ is the state of the system, $\dot{\underline{x}}$ denotes the time derivative, and the vector $\underline{u}$ is the control input to the system. Let $\underline{x}^*(t)$ be a fixed motion of (2.1) corresponding to a particular control input $\underline{u}^*(t)$. Let

$$\underline{x} = \underline{x}^* + \underline{z}$$

and

$$\underline{u} = \underline{u}^* + \underline{v} \quad .$$

Then, substituting in (2. 1)

$$\dot{\underline{x}}^* + \dot{\underline{z}} = \underline{f}(\underline{x}^* + \underline{z},\ \underline{u}^* + \underline{v}) \quad . \tag{2.2}$$

Now we can rewrite (2. 2) in the following way

$$\dot{\underline{x}}^* + \dot{\underline{z}} = \underline{f}(\underline{x}^*, \underline{u}^*) + \underline{g}(\underline{z}, \underline{v}) \tag{2.3}$$

where the function $\underline{g}$ is defined by

$$\underline{g}(\underline{z}, \underline{v}) = \underline{f}(\underline{x}^* + \underline{z}, \underline{u}^* + \underline{v}) - \underline{f}(\underline{x}^*, \underline{u}^*) \quad .$$

Obviously, for each fixed motion $\underline{x}^*$ resulting from a $\underline{u}^*$, there is a different function $\underline{g}$.

Since $\underline{x}^*$ is a solution of (2. 1), it follows from (2. 3) that

$$\dot{\underline{z}} = \underline{g}(\underline{z}, \underline{v}) \tag{2.4}$$

and

$$\underline{g}(\underline{0}, \underline{0}) = \underline{0} \quad .$$

Thus, with a fixed $\underline{u}(t) = \underline{u}^*(t)$, deviation from the resulting motion $\underline{x}^*(t)$ are described by the dynamical system of (2. 4).

In the context of control we can view the deviation $\underline{z}(t)$ from the desired motion, as an error and the object of control $(\underline{v}(t))$ is to keep the error small in the presence of small persistent perturbations and to return it to a small value following large disturbances of short duration. With this point of view, it is natural to introduce the "direct method" of Lyapunov. Let $V(\underline{z})$ be a "measure" of the error $\underline{z}$. $V(\underline{z})$ is real-valued and to be reasonable, at least locally, it must be positive definite. This means that in some neighborhood Ω of the origin, $\underline{z} = \underline{0}$,

$$V(\underline{z}) > 0 \qquad \text{for } \underline{z} \neq \underline{0}$$

and (2.5)

$$V(\underline{0}) = 0$$

Now define for the system (2.4)

$$\frac{dV(\underline{z})}{dt} = (\text{grad } V) \cdot \underline{g}(\underline{z}) \quad .$$

This function we can compute directly, without a knowledge of the solutions of (2.4). If the system is to keep the error small, then as a minimum we must have

$$\frac{dV}{dt} \leq 0 \qquad \text{for all } \underline{z} \text{ in } \Omega \quad ,$$

and preferably

$$\frac{dV}{dt} < 0 \qquad \text{for all } \underline{z} \text{ in } \Omega, \ \underline{z} \neq \underline{0} \quad .$$

With this somewhat intuitive picture in mind, it is now necessary to be precise in our statements. Let $\underline{\Phi}(t;\underline{z}_o,t_o)$ denote the motion of (2.4) which starts at t_o in state $\underline{z}_o$. Obviously for $t = t_o$, $\underline{\Phi}(t_o;\underline{z}_o,t_o) = \underline{z}_o$.

Definition 1: The solution $\underline{\phi}(t;\underline{0},t_o) = \underline{0}$ is stable if given any $\epsilon > 0$ there is a $\delta > 0$ such that $\|\underline{z}_o\| < \delta$ implies that $\|\underline{\phi}(t;\underline{z}_o,t_o)\| < \epsilon$ for all $t > t_o$.

Definition 2: The solution $\underline{\phi}(t;\underline{0},t_o) = \underline{0}$ is asymptotically stable if in addition to being stable there is an $\eta > 0$ with the property that $\|\underline{z}_o\| < \eta$ implies that $\|\underline{\phi}(t;\underline{z}_o,t_o)\| \to 0$ as $t \to \infty$. (If η includes the whole space, then we say that the solution is asymptotically stable in the large.)

These concepts are physically concerned with the case in which a perturbation suddenly moves the system from its equilibrium solution, but then disappears. If the system is asymptotically stable in the large, the effect of the perturbation tends to disappear regardless of its intensity.

In practice, systems are usually subjected to persistent perturbation, which has led to a concept known as total stability.

Definition 3: The solution $\underline{\phi}(t;\underline{0},t_o) = \underline{0}$ is totally stable if given any ϵ there is $\delta > 0$ and an $\eta > 0$ such that if $\|\underline{z}_o\| < \delta$ and the persistent perturbations $\underline{p}(\underline{z},t)$ are such that $\|\underline{p}(\underline{z},t)\| < \eta$ for all $\underline{z}$ and $t \geq t_o$, then

$$\|\underline{\phi}(t;\underline{z}_o,t_o)\| < \epsilon \qquad \text{for } t \geq t_o \quad .$$

Fortunately if a solution of (2.4) is asymptotically stable this implies that it is totally stable [see 3]. We shall therefore seek to achieve asymptotic stability and in the large where possible. In so doing, we shall make use of the following results:

Theorem 1: The solution of (2.4), $\underline{\phi}(t;\underline{0},t_o) = \underline{0}$ is asymptotically stable in the large if there exists a scalar function $V(\underline{z})$ with continuous first partial derivatives with respect to $\underline{z}$, such that $V(\underline{0}) = 0$ and

(i) $V(\underline{z}) > 0$ for all $\underline{z} \neq \underline{0}$.

(ii) $\frac{dV(\underline{z})}{dt} < 0$ for all $\underline{z} \neq \underline{0}$.

(iii) $V(\underline{z}) \rightarrow \infty$ with $\|\underline{z}\| \rightarrow \infty$.

Experience has shown that in many cases it is only possible to find a positive definite function V whose total derivative is non-positive rather than being strictly negative. For this reason, the following Corollary of Theorem 1 is very useful.

Corollary 1: In Theorem 1, condition (ii) may be replaced by:

(ii) 1 $\frac{dV(\underline{z})}{dt} \leq 0$ for all $\underline{z}$

(ii) 2 $\frac{dV}{dt}(\phi(t;\underline{z}_o, t_o)$ does not vanish identically in $t \geq t_o$ for any t_o and any $\underline{z}_o \neq 0$.

These results are derived and discussed in a paper by Kalman and Bertram. [2]

Results on the size of the region of asymptotic stability have been given by LaSalle [3] and are of particular use in problems of the sort to be considered in this report. In particular:

Theorem 2: If the region R defined by $V(\underline{z}) \leq c$ is bounded and if condition (i) and (ii) of Theorem 1 hold for all $\underline{z}$ in R , $\underline{z} \neq 0$, then R is contained in the region of asymptotic stability.

The preceding is only a sketchy outline of the direct method, but it summarizes the essential points for its application. The obvious and well-known difficulty encountered in the application of the direct method is the choice of an appropriate Lyapunov function $V(\underline{z})$. While there are no known methods for the generation of such a function for a general dynamical system, experience has shown that for the conservative systems involved in attitude control [4] the total energy of the system is often adequate.

3. THE MATHEMATICAL MODEL OF THE SYSTEM:

We will assume that the system can be described as two rigid bodies M_1 and M_2 connected by a cable with negligible mass whose length u can be controlled. Furthermore, to illustrate the techniques, we will only consider plane motions, i.e., the masses move in a plane and rotate only with respect to an axis orthogonal to the plane of motion. The parameters m_1 and m_2 represent the masses of the two bodies, and J_1 and J_2 their moments of inertia with respect to axes through their centers of mass and orthogonal to the plane of motion. The cable is attached to the bodies at points p_1 and p_2 at distances a_1 and a_2 from the centers of mass. The cable is assumed to obey Hooke's law, i.e., the force is proportional to the relative elongation

$$F = k \frac{\Delta \ell}{\ell} \quad .$$

The spring constant k is given by

$$k = aE$$

where a is the cross section of the cable and E is Young's modulus. In this preliminary study we will neglect the effect of the cable rubbing against the bodies.

To describe the motion of the bodies M_1 and M_2, we introduce a coordinate system OXY with the origin O at the center of mass of the system, and with the direction of the axes fixed with respect to inertial space. If we neglect the gravity gradient and the variation of the centrifugal forces over the system, then OXY is an inertial system. To describe the orientation of the rigid bodies relative to the coordinate system, we introduce the notation of Figure 1.

The system has four degrees of freedom: two for the position of one of the rigid bodies (say M_1) and one each for the rotations of the bodies M_1 and M_2. (Given the position of one of the masses, the position of the other is defined by the choice of the coordinate system.)

To obtain the equations of motion, we will use the Lagrangian formalism. The kinetic energy T of the system is

$$2T = m_1\dot{r}_1^2 + m_2\dot{r}_2^2 + m_1(r_1\dot{\phi})^2 + m_2(r_2\dot{\phi})^2 + J_1(\dot{\phi}+\dot{\phi}_1)^2 + J_2(\dot{\phi}+\dot{\phi}_2)^2 \tag{3.1}$$

The potential energy P, which is non-zero only when the cable is stretched, is given by

$$2P = \frac{kL^2 h(L)}{u} \tag{3.2}$$

where u is the unstretched length of the cable, k is the spring constant for a cable of unit length, L is the elongation of the cable, and h, the Heavyside function, is defined such that

$$h(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

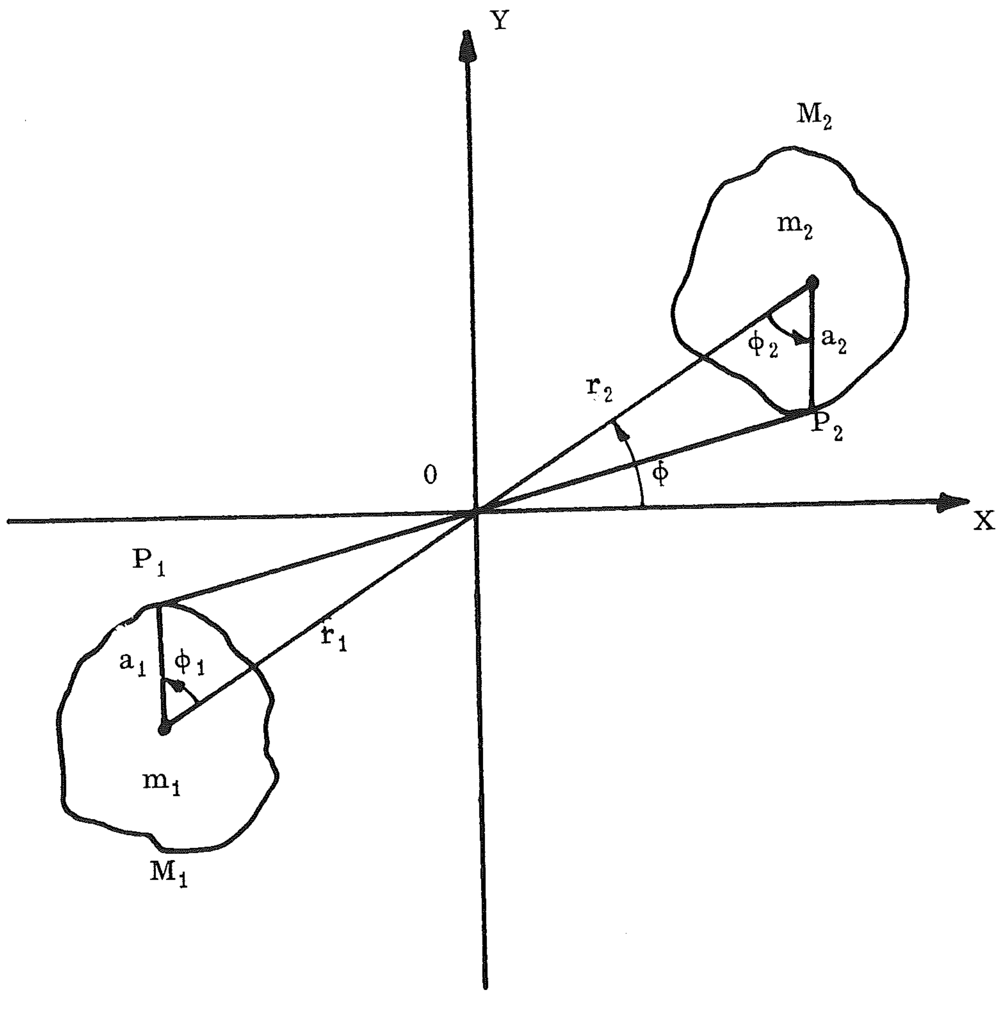


Figure 1.

The elongation, L , of the cable is just

$$L = C - u \tag{3.3a}$$

where C is the distance between the points P_1 and P_2 in Figure 1,

$$C = (A^2 + B^2)^{\frac{1}{2}} \tag{3.3b}$$

where

$$A = r_1 + r_2 - a_1 \cos\phi_1 - a_2 \cos\phi_2 \tag{3.3c}$$

and

$$B = a_1 \sin\phi_1 + a_2 \sin\phi_2 \quad . \tag{3.3d}$$

It is convenient to introduce the angle Θ defined by

$$A = C\cos\Theta \quad , \quad B = C\sin\Theta \ .$$

Due to the choice of coordinates, we have the following relation between r_1 and r_2 ; $m_1 r_1 = m_2 r_2$.

Now introducing the Lagrangian $\Gamma = T - P$, we find the equations of motion of the system,

$$m_1\ddot{r}_1 - m_1\dot{\phi}^2 r_1 + \frac{k}{u} L\,h(L)\cos\Theta = 0 \tag{3.4a}$$

$$J_1(\ddot{\phi}_1 + \ddot{\phi}) + \frac{k}{u} L\,h(L)\sin(\phi_1 + \Theta) = 0 \tag{3.4b}$$

$$J_2(\ddot{\phi}_2 + \ddot{\phi}) + \frac{k}{u} L\,h(L)\sin(\phi_2 + \Theta) = 0 \tag{3.4c}$$

$$m_1 \frac{d}{dt}(r_1^2\dot{\phi}) + m_2\frac{d}{dt}(r_2^2\dot{\phi}) + J_1(\ddot{\phi}_1 + \ddot{\phi}) + J_2(\ddot{\phi}_2 + \ddot{\phi}) = 0 \tag{3.4d}$$

Carrying out the differentiation indicated in (3.4d), and solving system 3.4 for $\ddot{\phi}$, we get

$$\ddot{\phi} = \frac{(\frac{k}{u})Lh(L)[a_1\sin(\phi_1 + \Theta) + a_2\sin(\phi_2 + \Theta)] - 2m_1 r_1 \dot{r}_1 \dot{\phi} - 2m_2 r_2 \dot{r}_2 \dot{\phi}}{m_1 r_2^2 + m_2 r_2^2}$$

but, since $m_1 r_1 = m_2 r_2$, then

$$\ddot{\phi} = \frac{\frac{k}{u} Lh(L)[a_1\sin(\phi_1 + \Theta) + a_2\sin(\phi_2 + \Theta)] - 2m_1 r_1 \dot{\phi}(\dot{r}_1 + \dot{r}_2)}{m_1 r_1 (r_1 + r_2)}$$

For ease of notation, let $\ddot{\phi} = F$.

If we now let

$$\begin{aligned} x_1 &= r_1 \\ x_2 &= \dot{r}_1 \\ x_3 &= \phi_1 \\ x_4 &= \dot{\phi}_1 \\ x_5 &= \phi_2 \\ x_6 &= \dot{\phi}_2 \\ x_7 &= \phi \\ x_8 &= \dot{\phi} \end{aligned} \qquad (3.5)$$

the equations of motion (3.4a - 3.4d) can be written as eight first order differential equations.

$$\dot{x}_1 = x_2 \tag{3.6a}$$

$$\dot{x}_2 = x_1 x_8^2 - \frac{k}{m_1 u} L h(L) \cos \Theta \tag{3.6b}$$

$$\dot{x}_3 = x_4 \tag{3.6c}$$

$$\dot{x}_4 = - \frac{k}{J_1 u} a_1 L h(L) \sin(x_3 + \Theta) - F \tag{3.6d}$$

$$\dot{x}_5 = x_6 \tag{3.6e}$$

$$\dot{x}_6 = - \frac{k}{J_2 u} a_2 L h(L) \sin(x_5 + \Theta) - F \tag{3.6f}$$

$$\dot{x}_7 = x_8 \tag{3.6g}$$

$$\dot{x}_8 = F \tag{3.6h}$$

where

$$L = [(x_1 (1+ \frac{m_1}{m_2}) - a_1 \cos x_3 - a_2 \cos x_5)^2 + (a_1 \sin x_3 + a_2 \sin x_5)^2]^{\frac{1}{2}} - u \tag{3.7}$$

$$F = \frac{\frac{k}{u} L h(L)[a_1 \sin(x_3 + \Theta) + a_2 \sin(x_5 + \Theta)] - 2 m_1 x_1 x_2 x_8 (1+ \frac{m_1}{m_2})}{m_1 x_1^2 (1+ \frac{m_1}{m_2})} \tag{3.8}$$

and

$$\Theta = \tan^{-1} [\frac{a_1 \sin x_3 + a_2 \sin x_5}{x_1 (1+\frac{m_1}{m_2}) - a_1 \cos x_3 - a_2 \cos x_5}] \tag{3.9}$$

These equations describe the motion of the system.

4. A SIMPLIFIED CASE:

To illustrate the techniques that we plan to employ and to provide some insight into the behavior of the general system, let us first consider a very simple version of the problem. In this example we assume that the two bodies M_1 and M_2 may be represented by point masses m_1 and m_2, respectively, and that $m_1 = m_2 = m$.

The objective of this example is to show via the direct method of Lyapunov that the desired motion of the system can be made asymptotically stable by just controlling the length of the cable.

In this simplified case, the system has only one degree of freedom (the position of one of the point masses) and the equations of motion reduce to

$$\ddot{r} - r\dot{\phi}^2 + \frac{k}{mu}(2r - u)\,h(2r - u) = 0 \qquad (4.1)$$

$$r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = 0 \qquad (4.2)$$

If we let

$$\begin{aligned} x_1 &= r \\ x_2 &= \dot{r} \\ x_3 &= \phi \\ x_4 &= \dot{\phi} \end{aligned} \qquad (4.3)$$

then (4.1) and (4.2) can be written as four first order differential equations,

$$\dot{x}_1 = x_2 \qquad (4.4a)$$

$$\dot{x}_2 = x_4^2 x_1 - \frac{k}{mu}(2x_1 - u)\,h(2x_1 - u) \qquad (4.4b)$$

$$\dot{x}_3 = x_4 \qquad (4.4c)$$

$$\dot{x}_4 = -2\,\frac{x_2 x_4}{x_1} \qquad (4.4d)$$

(a) The desired motion: The motion which is desired and whose asymptotic stability we wish to insure in the following:

$$
\begin{aligned}
x_1^*(t) &= a \\
x_2^*(t) &= 0 \\
x_3^*(t) &= \omega t \\
x_4^*(t) &= \omega
\end{aligned}
\qquad (4.5)
$$

In other words, we want M_1 and M_2 to move at radius a with constant angular velocity ω about the center of mass of the system.

Substituting (4.5) into the equations of motion (4.4), yields (see (4.4b)) that the unstretched length of the cable u^* which produces the desired motion is

$$u^* = 2a \cdot \frac{k}{k+ma\omega^2} \qquad (4.6)$$

(b) The dynamical equations for the motion relative to the desired motion: In order to study the asymptotic stability of the desired motion (4.6), it is necessary to obtain the dynamical equations of motion relative to the desired motion. To do this, introduce the perturbations from the desired motion

$$
\begin{aligned}
x_1 &= z_1 + a \\
x_2 &= z_2 \\
x_3 &= z_3 + \omega t \\
x_4 &= z_4 + \omega \\
u &= u^* + v
\end{aligned}
\qquad (4.7)
$$

Now, substituting (4.7) into (4.4), we find

$$\dot{z}_1 = z_2 \tag{4.8a}$$

$$\dot{z}_2 = (z_1+a)(z_4+\omega)^2 - \frac{k}{m(u^*+v)}\,[2(z_1+a)-u^*-v]\;h(2z_1+2a-u^*-v) \tag{4.8b}$$

$$\dot{z}_3 = z_4 \tag{4.8c}$$

$$\dot{z}_4 = -\frac{2z_2(z_4+\omega)}{(z_1+a)} \tag{4.8d}$$

In terms of these variables, we see that $z_1 = z_2 = z_3 = z_4 = 0$ defines the desired solution. Furthermore, we see that this is an equilibrium solution of (4.8). Actually we are only interested in the asymptotic stability of the state variables z_1, z_2, z_4, the deviations in r, $\dot{r}$, and $\dot{\phi}$ from the desired solution. Consequently, since z_3, the deviation in ϕ, does not enter into the right hand members of equations (4.8a), (4.8b), and (4.8d), we can drop (4.8c) from further consideration.

(c) Development of a class of primitive control laws via the direct method of Lyapunov: As a Lyapunov function we choose a quantity which is equal to the total energy in the system minus an appropriate constant so that it vanishes at $\underline{z} = \underline{0}$; thus,

$$\begin{aligned} V(z_1, z_2, z_4) &= (T+P) - \text{constant} \\ &= m z_2^2 + m(z_1+a)^2(z_4+\omega)^2 + \frac{k}{2u^*}(2z_1+2a-u^*)h(2z_1+2a-u^*) \\ &\quad - [m a^2\omega^2 + \frac{k(2a-u^*)^2}{2u^*}] \;. \end{aligned} \tag{4.9}$$

Unfortunately, V is not a positive definite function of z_1, z_2, z_4. Due to the particular form of the equations of motion, the variable z_4 can, however, be eliminated. Integration of (4.8d) yields

$$(z_1 + a)^2 (z_4 + \omega) = a^2 \omega \quad , \tag{4.10}$$

which can also be obtained directly from the conservation of angular momentum. Using this relation to eliminate z_4, the equations of motion become

$$\dot{z}_1 = z_2 \tag{4.11a}$$

$$\dot{z}_2 = \frac{a^4 \omega^2}{(z_1 + a)^3} - \frac{k}{m(u^* + v)} [2(z_1 + a) - u^* - v] \; h(2(z_1 + a) - u^* - v) \quad . \tag{4.11b}$$

If we can show that the trivial solution, $z_1 = z_2 = 0$, of these equations is asymptotically stable, we have also proven that the trivial solution of (4.8), $z_1 = z_2 = z_4 = 0$, is asymptotically stable because

$$z_1 \rightarrow 0 \quad \text{and} \quad z_2 \rightarrow 0$$

implies by (4.10) that

$$z_4 \rightarrow 0 \quad .$$

To show the asymptotic stability of the null solution of (4.11) we use the function V of (4.9) with z_4 eliminated by (4.10) as a Lyapunov function, i.e.,

*)

$$V(z_1, z_2) = m z_2^2 + m \frac{a^4 \omega^2}{(z_1 + a)^2} + \frac{k}{2u^*} (2z_1 + 2a - u^*)^2 \, h(2z_1 + 2a - u^*)$$

$$- [m a^2 \omega^2 + k \frac{(2a-u^*)^2}{2u^*}] \qquad (4.12)$$

We can easily verify that

(i) $V(0,0) = 0$

(ii) $V(z_1, z_2) > 0$ for $z_1, z_2 \neq 0$ and $z_1 > -a$

(iii) $V(z_1, z_2) \to \infty$ $z_1 \to -a$ or $z_1 \to +\infty$

and also as $|z| \to \infty$.

Further, we notice that the total derivative of $V(z_1, z_2)$,

$$\frac{dV}{dt} = \frac{\partial V}{\partial z_1} \dot{z}_1 + \frac{\partial V}{\partial z_2} \dot{z}_2$$

becomes

$$\frac{dV}{dt} = z_2 \, k[\frac{1}{u^*} (2z_1 + 2a - u^*) h(2z_1 + 2a - u^*)$$

$$- \frac{1}{u^* + v} (2z_1 + 2a - u^* - v) \, h(2z_1 + 2a - u^* - v)] \quad . \qquad (4.13)$$

To analyze the sign of $\frac{dV}{dt}$ in the z_1, z_2 plane, we consider the four cases

I. $2z_1 + 2a - u^* \leq 0$, $2z_1 + 2a - u^* - v \leq 0$

then

$$\frac{dV}{dt} = 0 \; .$$

II. $2z_1 + 2a - u^* \le 0 \quad , \quad 2z_1 + 2a - u^* - v > 0$

then

$$v < 0$$

and

$$\frac{dV}{dt} = -k \cdot \frac{z_2}{u^* + v} (2z_1 + 2a - u^* - v) \quad .$$

III. $2z_1 + 2a - u^* > 0 \quad , \quad 2z_1 + 2a - u^* - v \le 0$

then

$$v > 0$$

and

$$\frac{dV}{dt} = k \cdot \frac{z_2}{u^*} (2z_1 + 2a - u^*) \quad .$$

IV. $2z_1 + 2a - u^* > 0 \quad , \quad 2z_1 + 2a - u^* - v > 0$

$$\frac{dV}{dt} = 2k \cdot \frac{z_2 v}{u^*(u^* + 0)} (z_1 + a)$$

From this summary, it is clear that $\frac{dV}{dt}$ will be negative definite in cases II, III, and IV if v is chosen to have the form

$$v = -f(z_2) \quad , \qquad (4.14)$$

where $f(x)$ is a function with the following properties:

(i) $f(0) = 0$

(ii) $x f(x) > 0 \quad , \quad x \ne 0$

(iii) $|f(x)| < u^* \quad .$

Choosing the control law (4. 14), we can then summarize the investigation of the sign of $\frac{dV}{dt}$ in Figure 2.

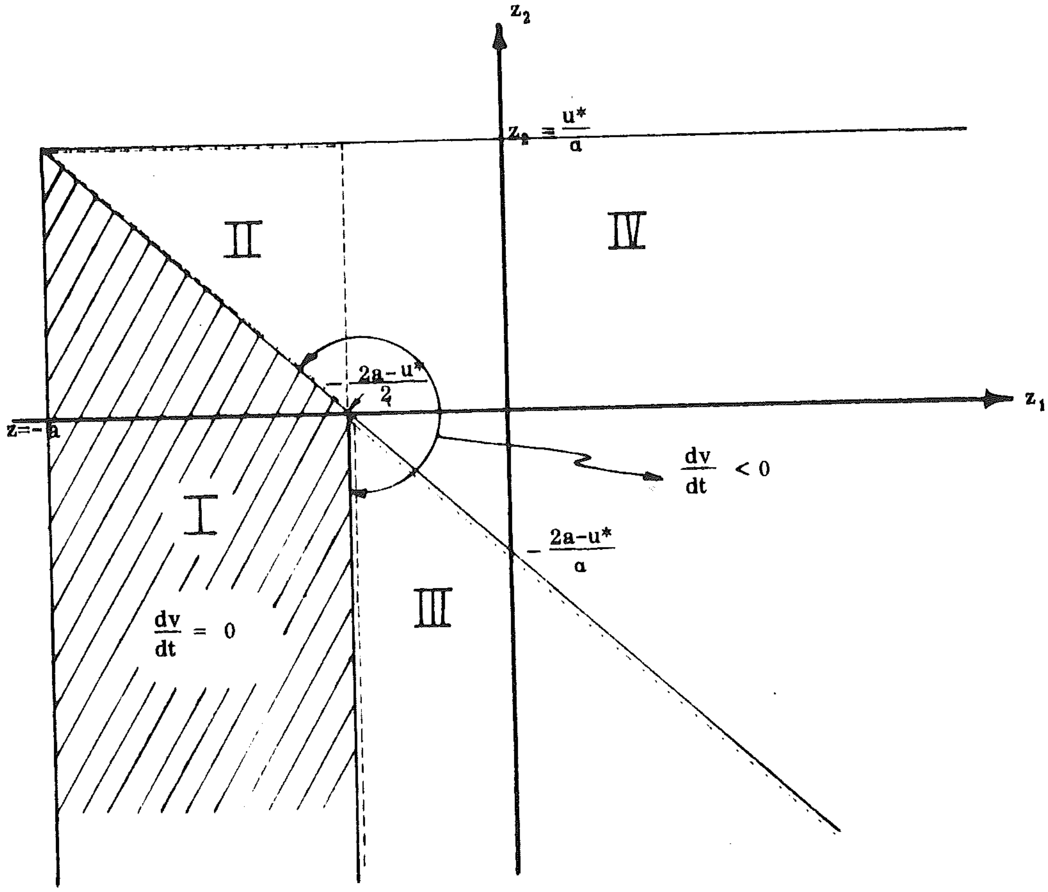

z_2
$z_2 = \frac{u^*}{a}$
II
IV
$-\frac{2a-u^*}{2}$
z_1
z=-a
$\frac{dv}{dt} < 0$
I
$-\frac{2a-u^*}{a}$
III
$\frac{dv}{dt} = 0$

Figure 2.

To show the asymptotic stability of the solution $z_1 = z_2 = 0$, it remains to be shown that:

1. Starting in the region $z_1 > -a$ all motions remain in this region.

2. There are no closed trajectories of (13) in the region where $\frac{dV}{dt} = 0$.

To show (1), we note that $V(z_1, z_2) \to \infty$ as $z_1 \to -a$, which means that infinite energy is required to get $z_1 < -a$.

To show (2), we notice that in the shaded region that the equations of motion (4.11) can be integrated in closed form, yielding trajectories

$$z_2^2 + \frac{a^4\omega^2}{(z_1+a)^2} = c^2 \tag{4.15}$$

It is obvious that these trajectories cannot be closed in the shaded region and we have thus shown that the control (4.14) gives an asymptotically stable system.

Remark: It should be physically obvious that r and $\dot{r}$ cannot be stabilized asymptotically unless there is some angular momentum. This follows since for $a\omega^2 = 0$ in the previous equation, the arguments would all break down.

(d) The choice of a particular control law: So far we have only been concerned with the problem of finding a class of control laws that yields an asymptotically stable system. We will now show that by choosing a particular member of this class we can obtain suitable dynamic behavior of the system. We will consider the control law :

$$
v = \begin{cases} -\alpha\beta & z_2 \geq \beta \\ -\alpha z_2 & |z_2| < \beta \\ \alpha\beta & z_2 \leq \beta \end{cases}
$$

In particular, we shall consider the asymptotic stability of the motion

$$
\begin{aligned}
r &= a = 30 \text{ meters} \\
\dot{r} &= 0 \\
\dot{\phi} &= \omega = 0.5 \text{ radians/sec}
\end{aligned}
$$

and with the physical parameters

$$
\begin{aligned}
m &= 10^4 \text{ Kilograms} \\
k &= 1.284 \times 10^8 \text{ Newtons.}
\end{aligned}
$$

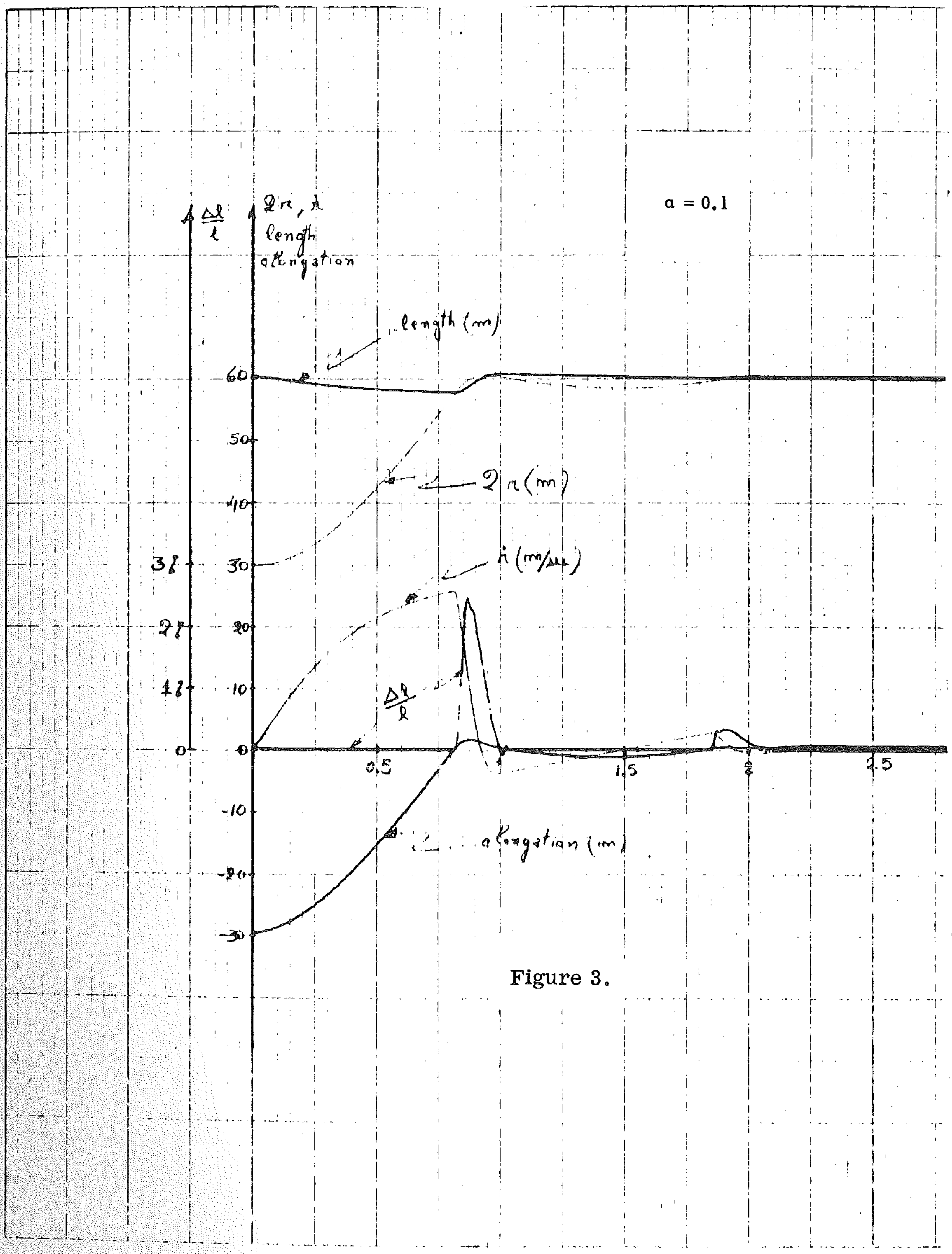
Note that this motion gives an artificial gravity in the cabin of 0.76g.

Figures 3 - 7 show the time response of the system when it is released from the initial conditions

$$
\begin{aligned}
r(0) &= 15 \text{ meters} \\
\dot{r}(0) &= 0 \\
\dot{\phi}(0) &= 2 \text{ radians/sec}
\end{aligned}
$$

At these initial conditions, the cabin has an artificial gravity of 6.1g.

A constraint of considerable physical importance is the fact that the cable would break if the percent elongation were ever to exceed 1%. From Figures 3 and 7 we see that for $\alpha = 0.1$ that this limit is exceeded but for $\alpha = 10$ the percent elongation does not exceed 0.6%. From this data, it would seem that $\alpha = 10$ represents a reasonable control law.



a = 0.1
Δl/l
2r, ṙ
length
elongation
length (m)
2r (m)
ṙ (m/sec)
Δl/l
elongation (m)
60
50
40
30
20
10
0
-10
-20
-30
0.5
1
1.5
2
2.5

Figure 3.

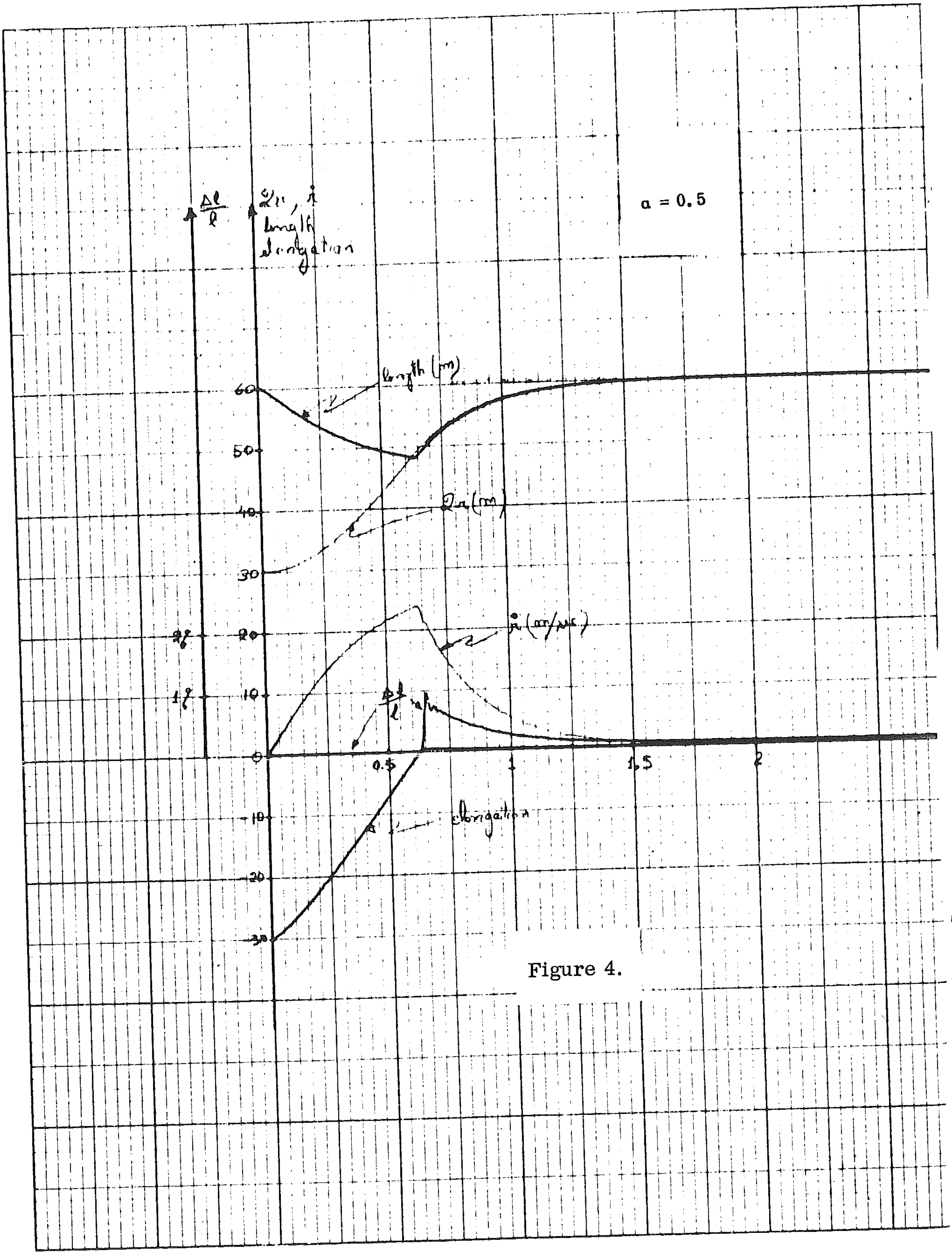


Figure 4.

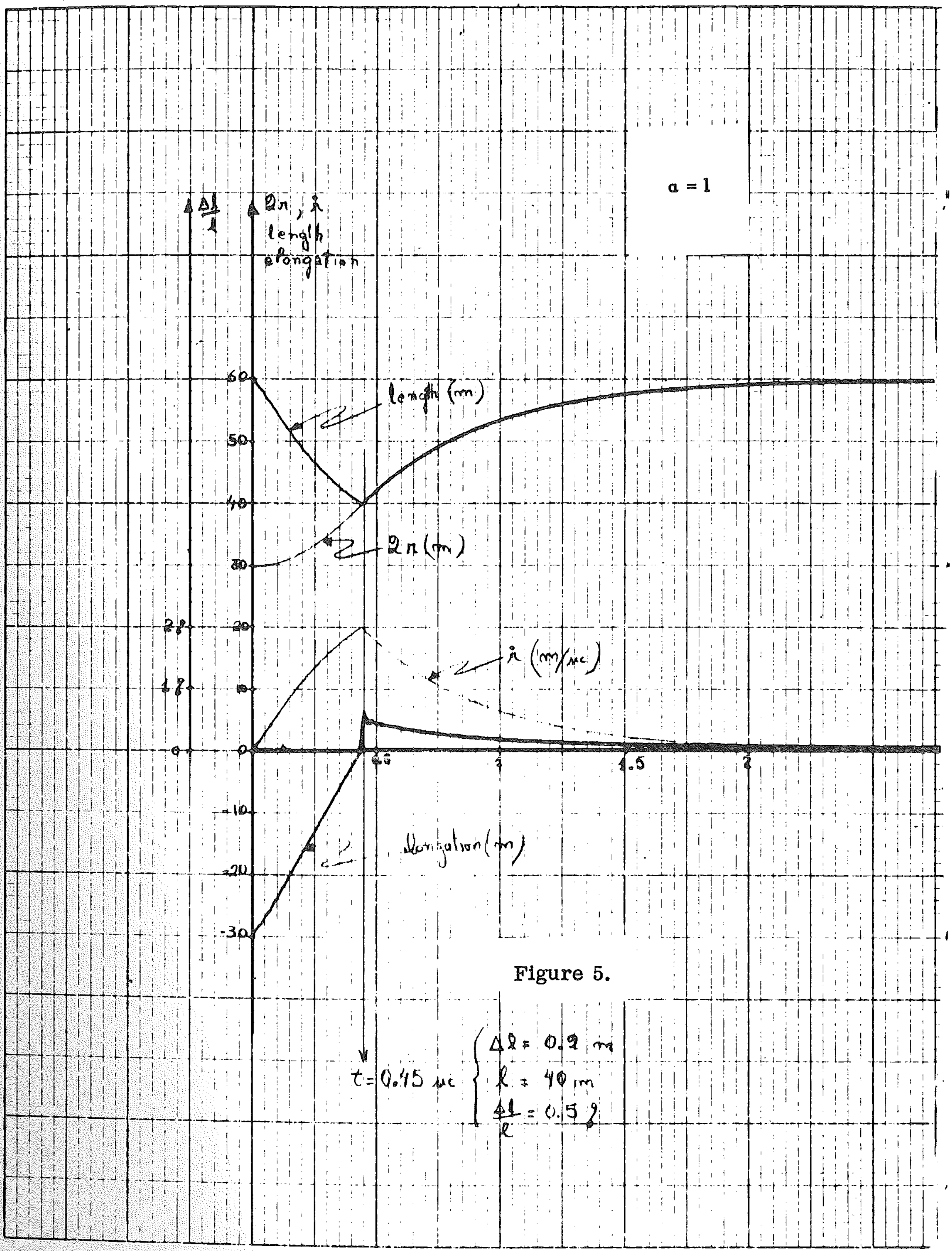

a = 1
Δl/l
2r, ṙ
length
elongation
60
50
40
30
20
10
0
-10
-20
-30
28
14
length (m)
2r (m)
ṙ (m/µs)
elongation (m)
0.5
1
1.5
2
t = 0.45 µs
Δl = 0.2 m
l = 40 m
Δl/l = 0.5 %

Figure 5.

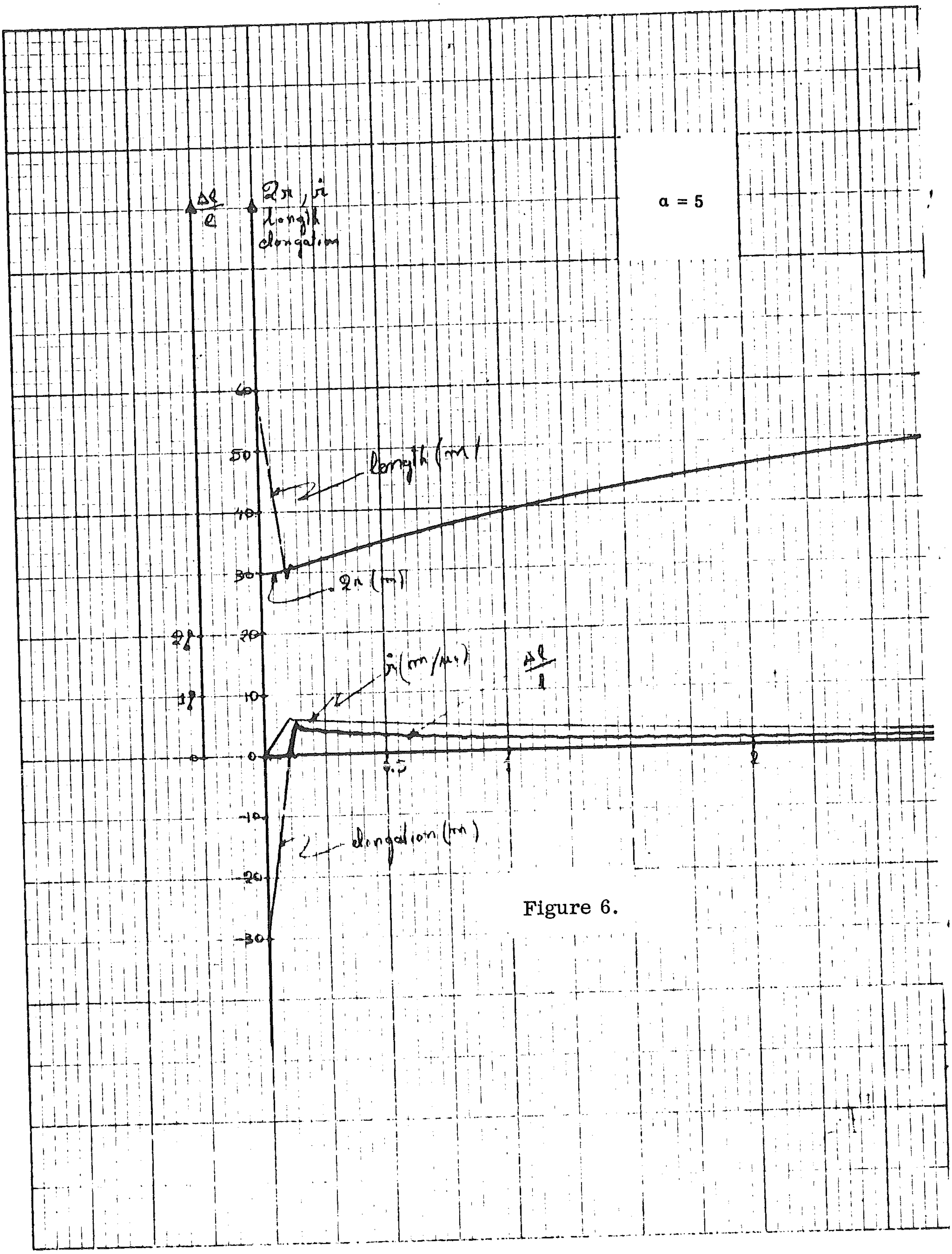

a = 5
Δl/l
2r, ṙ
length
elongation
60
50
40
30
20
10
0
-10
-20
-30
length (m)
2r (m)
ṙ (m/μs)
elongation (m)
0.5
1
2

Figure 6.

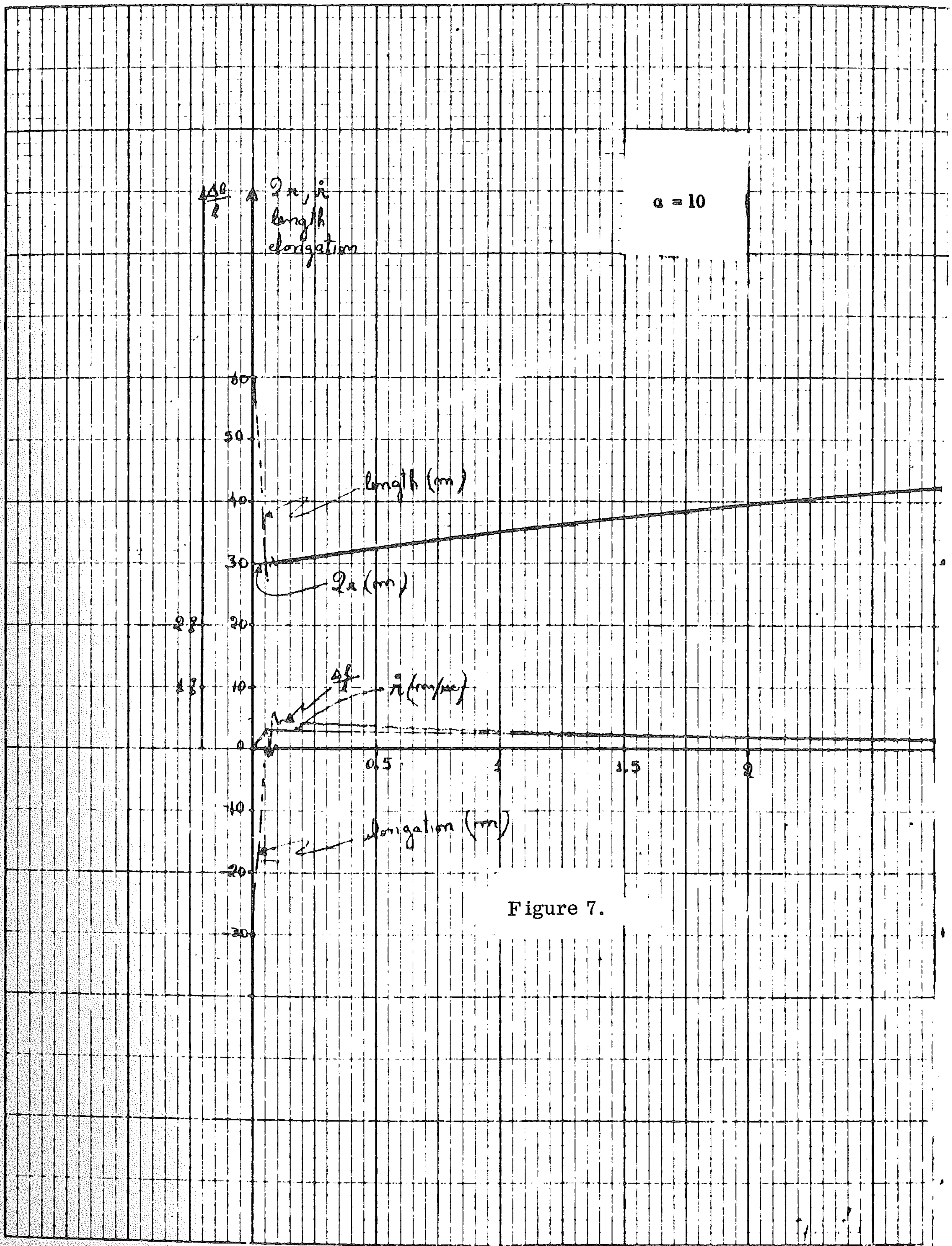


Figure 7.

However, to qualitatively find the influence of the parameter α we linearize the equations of motion around the equilibrium solution. Neglecting terms of second and higher order in z_2, we find

$$\ddot{z}_1 + 2 \frac{a}{u^*} \frac{k}{m u^*} \alpha \dot{z}_1 + (3\omega^2 + \frac{2k}{m u^*}) z_1 = 0 .$$

Introducing the numerical values used, we find that for $\alpha > 0.1$ the characteristic roots are

$$\lambda_1 \approx 214 \alpha$$

$$\lambda_2 \approx 2/\alpha$$

The smallest root will thus decrease with increasing α. This is clear from the Figures 3-7. With the orders of magnitude of α considered, it is clear that the mode corresponding to the highest eigenvalues will tend to zero very quickly after an initial disturbance and that the following motion is then governed by the mode associated with the small eigenvalue. This can also be verified quantitatively from the Figures 3-7.

5. ANALYSIS OF THE COMPLETE PROBLEM:

We will now return to the complete problem whose mathematical model was given in Section 3. We will follow the same path as in the simplified example of the previous section. The equations of motion of the system are given by (3.6).

(a) The desired motion: The motion which is desired and whose asymptotic stability we wish to insure is the following:

$$x_1^* = a$$
$$x_2^* = x_3^* = x_4^* = x_5^* = x_6^* = 0 \qquad (5.1)$$
$$x_7^* = \omega t$$
$$x_8^* = \omega$$

where a and ω are given. In other words, we want the arms lined up with the center of masses m_1 and m_2, and the bodies M_1 and M_2 moving at constant angular velocity ω about the center of mass of the system.

Substituting the solution (5.1) into the equations of motion (3.6), yields the condition (see 3.6b) that the length of cable $u = u^*$ which produces the desired motion (5.1) is

$$u^* = \frac{a(1+\frac{m_1}{m_2}) - a_1 - a_2}{1 + \frac{m_1 a \omega^2}{k}} \qquad (5.2)$$

(b) <u>The dynamical equations for motions relative to the desired motion</u>: In order to study the asymptotic stability of the desired motion (4), it is necessary to obtain the dynamical equations for motions relative to the desired motion. To do this, we let

$$x_1 = z_1 + a \qquad (5.3a)$$
$$x_2 = z_2 \qquad (5.3b)$$
$$x_3 = z_3 \qquad (5.3c)$$
$$x_4 = z_4 \qquad (5.3d)$$
$$x_5 = z_5 \qquad (5.3e)$$

$$x_6 = z_6 \tag{5.3f}$$

$$x_7 = z_7 + \omega t \tag{5.3g}$$

$$x_8 = z_8 + \omega \tag{5.3h}$$

$$u = u^* + v \tag{5.3i}$$

Now substituting (5.3) into (3.6), we find

$$\dot{z}_1 = z_2 \tag{5.4a}$$

$$\dot{z}_2 = (z_1 + r)(z_8 + \omega)^2 - \frac{k}{m_1(u^*+v)} L\, h(L) \cos\Theta \tag{5.4b}$$

$$\dot{z}_3 = z_4 \tag{5.4c}$$

$$\dot{z}_4 = - \frac{k}{J_1(u^*+v)} a_1 L\, h(L) \sin(z_3 + \Theta) - F \tag{5.4d}$$

$$\dot{z}_5 = z_6 \tag{5.4e}$$

$$\dot{z}_6 = - \frac{k}{J_2(u^*+v)} a_2 L\, h(L) \sin(z_5 + \Theta) - F \tag{5.4f}$$

$$\dot{z}_7 = z_8 \tag{5.4g}$$

$$\dot{z}_8 = F \tag{5.4h}$$

where

$$L = [((z_1 + a)(1+\frac{m_1}{m_2}) - a_1 \cos z_3 - a_2 \cos z_5)^2 + (a_1 \sin z_3 + a_2 \sin z_5)^2]^{\frac{1}{2}} - v - u^* \tag{5.5}$$

$$F = \frac{\frac{k}{u^*+v} L\, h(L)[a_1 \sin(z_3 + \Theta) + a_2 \sin(z_5 + \Theta)] - 2m_1(z_1 + a)(z_8 + \omega) z_2 (1+\frac{m_1}{m_2})}{m_1 (z_1 + a)^2 (1+ \frac{m_1}{m_2}} \tag{5.6}$$

and

$$\Theta = \tan^{-1}\left[\frac{a_1 \sin z_3 + a_2 \sin z_5}{(z_1 + a)(1 + \frac{m_1}{m_2}) - a_1 \cos z_3 - a_2 \cos z_5}\right] . \tag{5.7}$$

We find immediately that the trivial equilibrium solution $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = z_8 = 0$ of (5.7) is the desired equilibrium solution whose asymptotic stability we wish to insure. However, we also find that the following are equilibrium solutions of (5.7):

$$\begin{aligned} z_1 &= z_2 = 0 \\ z_3 &= 2n\pi , \quad n = \ldots, -2, -1, 1, 2, \ldots \\ z_4 &= 0 \\ z_5 &= 2n\pi , \quad n = \ldots, -2, -1, 1, 2, \ldots \\ z_6 &= z_7 = z_8 = 0 \end{aligned} \tag{5.8}$$

These equilibrium solutions are the result of cable "wrap around" and are to be avoided.

(c) <u>Development of control via the second method of Lyapunov</u>: Again we will choose as the Lyapunov function a quantity which is proportional to the total energy of the system for $u = u^*$. The total energy is

$$V(r_1, \dot{r}_1, r_2, \dot{r}_2, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi, \dot{\phi}) = T + P_o - \text{constant} \tag{5.9}$$

where

$$2T = m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_1 (r_1 \dot{\phi})^2 + m_2 (r_2 \dot{\phi})^2 + J_1 (\dot{\phi} + \dot{\phi}_1)^2 + J_2 (\dot{\phi} + \dot{\phi}_2)^2 \tag{5.10}$$

and

$$2P_o = \frac{k}{u_o^*} L_o^2 h(L_o) \tag{5.11}$$

and where the parameter L_o is

$$L_o = [(r_1+r_2 - a_1 \cos \phi_1 - a_2 \cos \phi_2)^2 + (a_1 \sin \phi_1 + a_2 \sin \phi_2)^2]^{\frac{1}{2}} - u^* . \tag{5.12}$$

The constant is chosen in such a way that the energy is zero for the desired equilibrium solution (5. 1).

The existence of several equilibria of the perturbed equations is due to the non-physical assumption made in Section 3 that the cable does not wrap up around the bodies. We also notice that the right member of the equations is independent of $z_7 = \phi$. Therefore, since we are not concerned about the values of ϕ, we can thus ignore the state z_7 and only consider the states z_1, z_2, z_3, z_4, z_5, z_6, and z_8.

Again we find that the total energy is not positive definite. To obtain a positive definite function, we proceed in the same way as in Section 4 and eliminate the variables ϕ and $\dot{\phi}$. Integrating (3. 6h), we get

$$m_1 r_1^2 \dot{\phi} + m_2 r_2^2 \dot{\phi} + J_1(\dot{\phi}+\dot{\phi}_1) + J_2(\dot{\phi}+\dot{\phi}_2) = [m_1 a^2 (1+\frac{m_1}{m_2}) + J_1 + J_2]\,\omega, \tag{5.13}$$

which also can be obtained directly from the conservation of angular momentum. Using (5. 10), we can now eliminate ϕ from the equations of motion. We further notice that $\dot{\phi}$ does not appear in the right member of the equations of motion and that the choice of coordinate systems gives a linear relation between r_1 and r_2. We can thus express the equations of motion in terms of the six variables $r_1, \dot{r}_1, \phi_1, \dot{\phi}_1, \phi_2$ and $\dot{\phi}_2$. (Since we never use these equations explicitly, we will not write them down.) We will now show that the function (5.9) is positive definite in these variables and that we can find a control law for the system such that the motion is asymptotically stable. Notice that in this case we cannot show global stability as the total energy vanishes for all the solutions (5.8).

d. <u>The non-negative definiteness of V:</u> To show that V is positive definite in $r, \dot{r}, \phi_1, \dot{\phi}_1, \phi_2$ and $\dot{\phi}_2$, we will simply eliminate $\dot{\phi}$ from (5.9) and (5.13) and make a series expansion of the function obtained.

The kinetic energy (5.10) can be written as

$$2T = M\dot{r}_1^2 + M(r_1\dot{\phi})^2 + J_1(\dot{\phi}+\dot{\phi}_1)^2 + J_2(\dot{\phi}+\dot{\phi}_2)^2 \qquad (5.10a)$$

where

$$M = m_1(1+\frac{m_1}{m_2}) \qquad (5.14)$$

Rewriting equation (5.13), we get

$$Mr_1^2 + J_1(\dot{\phi}+\dot{\phi}_1) + J_2(\dot{\phi}+\dot{\phi}_2) = J\omega \qquad (5.13a)$$

where

$$J = Ma^2 + J_1 + J_2 \quad . \qquad (5.15)$$

Elimination of ϕ between (5.10a) and (5.13a) gives

$$2T = M\dot{r}_1^2 + \frac{J^2\omega^2 - (J_1\dot{\phi}_1 + J_2\dot{\phi}_2)^2}{J + 2Maz_1 + Mz_1^2} + J_1\dot{\phi}_1^2 + J_2\dot{\phi}_2^2.$$

A series expansion around the equilibrium solution gives

$$2T = J\{\omega^2 - 2\omega^2\gamma_3\frac{z_1}{a} + \omega^2\gamma_3(4\gamma_3 - 1)\left(\frac{z_1}{a}\right)^2 + \gamma_3\left(\frac{z_2}{a}\right)^2 +$$

$$+ \gamma_1(1-\gamma_1)\dot{z}_4^2 - 2\gamma_1\gamma_2\dot{z}_4\dot{z}_6 + \gamma_2(1-\gamma_2)\dot{z}_6^2 + 0(z_i^3)\}$$

where

$$\gamma_1 = \frac{J_1}{J}, \quad \gamma_2 = \frac{J_2}{J}, \quad \gamma_3 = \frac{Ma^2}{J}$$

Similarly a series expansion of the potential energy around the equilibrium solution (5.1) gives

$$2P_o = \frac{k}{u^*}\{(\ell' - u^*)^2 + 2\beta(\ell' - u^*)z_1 + \beta^2 z_1^2 +$$

$$+ (\ell' - u^*)[a_1(1 + \frac{a_1}{\ell_1})z_3^2 + 2\frac{a_1 a_2}{\ell'}z_3 z_5 + a_2(1 + \frac{a_2}{\ell_2})z_5^2] +$$

$$+ 0(z_i^3)\},$$

where

$$\ell' = a\beta - a_1 - a_2.$$

Now collecting the terms we get

$$2T + 2P = J\omega^2 + \frac{k}{u^*}(\ell' - u^*)^2 + z_1^2\left(\beta^2 \frac{k}{u^*} + J\omega^2 \gamma_3 (4\gamma_3 - 1)\frac{1}{a^2}\right)$$

$$+ J\gamma_3 \left(\frac{z_2}{a}\right)^2 + \frac{k}{u^*}(\ell' - u^*)\left[a_1\left(1+\frac{a_1}{\ell'}\right) z_3^2 + 2\frac{a_1 a_2}{\ell'} z_3 z_5\right.$$

$$\left. + a_2\left(1+\frac{a_2}{\ell'}\right) z_5^2 \right] + J\left[\gamma_1 (1-\gamma_1)\, z_4^2 - 2\gamma_1\gamma_2\, z_4^2 z_6 + \gamma_2 (1-\gamma_2)\, z_6 \right]$$

$$+ 0(z_i^3),$$

where the terms which are linear in z_1 cancel due to (5.2). It is now obvious that T + P and thus also $V(z_1, z_2, z_3, z_4, z_5, z_6)$ is positive definite. We further notice that $V(z_1, z_2, z_3, z_4, z_5, z_6)$ is periodic in z_3 and z_5. We can thus conclude that the function V vanishes for all the equilibrium solutions (5.8) and that it is positive definite around these equilibria.

(d) <u>The sign of the total derivative of V.</u> Now, to obtain the total derivative of V

$$\frac{dV}{dt} = \sum_{i=1}^{10} \frac{\partial V}{\partial x_i} \dot{x}_i$$

we need the following terms

$$\frac{\partial V}{\partial r_1} = \frac{\partial Po}{\partial r_1} + m_1 r_1 \dot{\phi}^2 , \qquad \dot{r}_1 = \dot{r}_1$$

$$\frac{\partial V}{\partial \dot{r}_1} = m_1 \dot{r}_1 , \qquad \ddot{r}_1 = r_1 \dot{\phi}^2 - \frac{1}{m_1}\frac{\partial P}{\partial r_1}$$

$$\frac{\partial V}{\partial r_2} = \frac{\partial Po}{\partial r_2} + m_2 r_2 \dot{\phi}^2 , \qquad \dot{r}_2 = \dot{r}_2$$

$$\frac{\partial V}{\partial r_2} = m_2 \dot{r}_2 \quad , \quad \ddot{r}_2 = r_2\dot{\phi}^2 - \frac{1}{m_2}\frac{\partial P}{\partial r_2}$$

$$\frac{\partial V}{\partial \phi_1} = \frac{\partial Po}{\partial \phi_1} \quad , \quad \dot{\phi}_1 = \dot{\phi}_1$$

$$\frac{\partial V}{\partial \phi_1} = J_1(\dot{\phi}+\dot{\phi}_1) \quad , \quad \ddot{\phi}_1 = -\ddot{\phi} - \frac{1}{J_1}\frac{\partial P}{\partial \phi_1}$$

$$\frac{\partial V}{\partial \phi_2} = \frac{\partial Po}{\partial \phi_2} \quad , \quad \dot{\phi}_2 = \dot{\phi}_2$$

$$\frac{\partial V}{\partial \phi_2} = J_2(\dot{\phi}+\dot{\phi}_2) \quad , \quad \ddot{\phi}_2 = -\ddot{\phi} - \frac{1}{J_2}\frac{\partial P}{\partial \phi_2}$$

$$\frac{\partial V}{\partial \phi} = 0 \quad , \quad \dot{\phi} = \dot{\phi}$$

$$\frac{\partial V}{\partial \phi} = m_1 r_1^2 \dot{\phi} + m_2 r_2^2 \dot{\phi} + J_1(\dot{\phi}+\dot{\phi}_1) + J_2(\dot{\phi}+\dot{\phi}_2),$$

$$\ddot{\phi} = \frac{\frac{\partial P}{\partial \phi_1} + \frac{\partial P}{\partial \phi_2} - 2m_1 r_1 \dot{r}_1 \dot{\phi} - 2m_2 r_2 \dot{r}_2 \dot{\phi}}{m_1 r_1^2 + m_2 r_2^2}$$

Now combining terms we find

$$\frac{dV}{dt} = \dot{r}_1\left(\frac{\partial Po}{\partial r_1} - \frac{\partial P}{\partial r_1}\right) + \dot{r}_2\left(\frac{\partial Po}{\partial r_2} - \frac{\partial P}{\partial r_2}\right) + \dot{\phi}_1\left(\frac{\partial Po}{\partial \phi_1} - \frac{\partial P}{\partial \phi_1}\right)$$

$$+ \quad \dot{\phi}_2\left(\frac{\partial Po}{\partial \phi_2} - \frac{\partial P}{\partial \phi_2}\right)$$

but r_1 and r_2 are linearly related so

$$\frac{dV}{dt} = \dot{r}_1\left(1 + \frac{m_1}{m_2}\right)\left(\frac{\partial Po}{\partial r_1} - \frac{\partial P}{\partial r_1}\right) + \dot{\phi}_1\left(\frac{\partial Po}{\partial \phi_1} - \frac{\partial P}{\partial \phi_1}\right) +$$

$$+ \phi_2\left(\frac{\partial Po}{\partial \phi_2} - \frac{\partial P}{\partial \phi_2}\right).$$

Since

$$\frac{\partial L}{\partial r_1} = \frac{\partial Lo}{\partial r_1}, \quad \frac{\partial L}{\partial \phi_1} = \frac{\partial Lo}{\partial \phi_1}, \quad \frac{\partial L}{\partial \phi_2} = \frac{\partial Lo}{\partial \phi_2}$$

we get

$$\frac{dV}{dt} = k\left\{\left[\left(1+\frac{m_1}{m_2}\right)\frac{\partial Lo}{\partial z_1} z_2 + \frac{\partial Lo}{\partial z_3} z_4 + \frac{\partial Lo}{\partial z_5} z_6\right]. \right.$$

$$\left. \cdot\left[\frac{Lo}{u^*} h(Lo) - \frac{L}{u^* + V} h(L)\right]\right\}$$

$$= kZ\left[\frac{Lo}{u^*} h(Lo) - \frac{L}{u^* + v} h(L)\right]$$

where

$$Z = \left(1+\frac{m_1}{m_2}\right) z_2 \cos\Theta + a_1 z_4 \sin(z_3 + \Theta) +$$

$$+ a_2 z_6 \sin(z_5 + \Theta)$$

To analyze the sign of $\dot{V}$ we will now separately consider the following cases.

I. $L_o \leq 0$, $L = L_o + V \leq 0$, then $\frac{d\dot{V}}{dt} = 0$

II. $L_o \leq 0$, $L > 0$

then $V > 0$

and $$\frac{dV}{dt} = -kZ_1 \frac{L}{u^* + V}$$

III. $L_o > 0$, $L \leq 0$

then $V < 0$

and $$\frac{dV}{dt} = kZ \frac{Lo}{u^*}$$

IV. $L_o > 0$, $L > 0$

then $$\frac{dV}{dt} = kZ \frac{V (L_o - u^*)}{u^*(u^* + V)}$$

It is now easily seen that $\frac{dV}{dt}$ will be negative definite in the cases II, III and IV if the control signal V is chosen as

$$v = -f(Z)$$

where the function f(z) has the properties (i), (ii) and (iii) as stated on page 19.

It now remains to analyze the behaviour of the system in the region I. In this region the equations of motion can be integrated. As they represent the motion of two free bodies it is obvious that there can be closed trajectories in this region, e.g.

$$r_1 = \text{const.}$$

$$\phi = 0$$

$$\phi_1 = \omega t$$
$$\phi_2 = \omega t$$

where

$$r_1\left[(1+\frac{m_1}{m_2}) - a_1 - a_2\right] \geq f\,(a_1\omega_1 + a_2\omega_2)$$

In this case, the desired motion thus is not globally asymptotically stable. However, in a small neighborhood of the desired solution we have case IV and the total derivation of V is thus negative definite. Again we point out that this phenomena is partly due to the unphysical assumption that the rope does not wind upon the bodies. If this is taken into account, it will rule out at least the particular class of solutions given above.

(e) The properties of a particular control law: So far we have only obtained a class of control laws which leads to an asymptotically stable system. We will now briefly consider a particular law, namely the linear saturated control law.

In table 1 we have shown a solution obtained for a = 10. We find that the statevariables ϕ_1, $\dot{\phi}_1$, ϕ_2 and $\dot{\phi}_2$ associated with the angular motions of the rigid bodies are very slightly coupled and very slightly damped. The motion of these states is very close to that of two coupled slightly damped oscillators. The period of the oscillation is approximatively 3 seconds and the damping is such that after 10 complete oscillations the amptitude of the motion 0.0326 compared to the initial amptitude 0.0350. The control law proposed is thus not very effective in damping out small angular motions of the system. This is easily seen from the linearized version of the equations of motions.

TABLE I.

t sec	r_1 m	$\dot{r}_1$ m-sec	ϕ_1	$\dot{\phi}_1$	ϕ_2	$\dot{\phi}_2$	ϕ	$\dot{\phi}$
0.0	30.0000	0.0000	0.0350	0.0000	0.0350	0.0000	0.0000	0.5000
2.0	30.0015	0.0096	-0.0156	0.0651	-0.0111	0.0732	1.0004	0.4992
4.0	29.9990	-0.0027	-0.0193	-0.0551	-0.0288	-0.0488	2.0005	0.5005
6.0	29.9970	0.0101	0.0292	-0.0132	0.0318	-0.0430	3.0000	0.5004
8.0	29.9988	0.0035	-0.0060	0.0560	0.0076	0.0837	4.0002	0.4992
10.0	29.9952	0.0011	-0.0189	-0.0268	-0.0399	-0.0179	5.0006	0.5003
12.0	29.9965	0.0116	0.0154	-0.0286	0.0235	-0.0738	6.0001	0.5006
14.0	29.9963	-0.0014	0.0080	0.0360	0.0224	0.0767	7.0002	0.4994
16.0	29.9938	0.0071	-0.0172	0.0107	-0.0414	0.0133	8.0007	0.5000
18.0	29.9964	0.0078	-0.0011	-0.0443	0.0166	-0.0852	9.0003	0.5007
20.0	29.9942	-0.0011	0.0222	0.0141	0.0310	0.0519	10.0000	0.4997
22.0	29.9941	0.0111	-0.0154	0.0445	-0.0326	0.0412	11.0005	0.4996
24.0	29.9957	0.0017	-0.0144	-0.0546	-0.0054	-0.0768	12.0003	0.5007
26.0	29.9928	0.0037	0.0311	-0.0034	0.0334	0.0139	12.9997	0.5000
28.0	29.9951	0.0104	-0.0126	0.0631	-0.0164	0.0626	14.0001	0.4993
30.0	29.9943	-0.0014	-0.0209	-0.0528	-0.0216	-0.0532	15.0001	0.5005
32.0	29.9929	0.0090	0.0309	-0.0156	0.0302	-0.0279	15.9994	0.5002
34.0	29.9954	0.0055	-0.0070	0.0620	0.0024	0.0739	16.9994	0.4992
36.0	29.9930	0.0007	-0.0211	-0.0355	-0.0339	-0.0223	17.9993	0.5003
38.0	29.9940	0.0110	0.0212	-0.0253	0.0226	-0.0623	18.9985	0.5004
40.0	29.9948	0.0004	0.0026	0.0447	0.0188	0.0717	19.9982	0.4992

Linearization around the desired equilibrium gives

$$\ddot{\phi} + \frac{1}{1+\frac{v}{u^*}} A \phi = 0 \quad , \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 6.85 & 0.807 \\ 0.606 & 8.058 \end{bmatrix}$$

The matrix A has the eigenvalues

$$\gamma_1 = 6.53 \text{ and } \gamma_2 = 8.38$$

The corresponding periods are

$$T_1 = 2.17 \text{ sec} \qquad T_2 = 2.46 \text{ sec.}$$

From the above equation we immediately find that the control law introduces terms of the third order in the statevariables, which explains the low damping of small motions.

REFERENCES:

1. Markarian, H., Clancy, T.F.: "Preliminary Considerations for Two Body MODS Attitude Control", IBM Federal Systems Division Report 952-5497-0.

2. Kalman, R.E., Bertram, J.E.: "Control System Analysis and Design via the 'Second Method' of Lyapunov", Journal of Basic Engineering, June 1960, p. 371-399.

3. LaSalle, J.P.: "Stability and Control", SIAM Journal on Control, Ser. A, Vol. 1, Number 1, p. 3-15.

4. Lee, E.B.: "Discussion of Satellite Attitude Control", ARS Journal, June 1962, p. 981-982.

5. Henrici, P.: Discrete Variable Methods in Ordinary Differential Equations, John Wiley & Sons, New York-London, 1961.

APPENDIX I) Digital Computer Simulations

A 7090 Fortran program (see following 3 pages) has been written to integrate the equations of motion. To the 8 state variables considered in our problem, time has been added as a ninth state and the system of differential equations given by (5.4) is solved through a 4th. order classical Runge-Kutta method (e.g., [5] p. 120). The computation of a thousand consecutive states takes approximately one hundredth of a second.

The printout of this program is one line giving ordinately the nine state variables and, in the next line, giving the corresponding Lyapunov function, its time derivative and the control signal v.

```
*R001-2  J.P. JACOB            DEPT 978              JUNE 13
*      XEQ
*      LABEL
CSPLB
C THE FOLLOWING PROGRAM IS A 7090 FORTRAN PROGRAM TO SOLVE, BY A FOURTH ORDER
C RUNGE-KUTTA ROUTINE, A SYSTEM OF 9 FIRST ORDER DIFFERENTIAL EQUATIONS, AS
C SPECIFIED IN THE ATTACHED REPORT.
C
C WE NOW PROCEED WITH THE DEFINITION OF FORTRAN NAMES RELATED TO THE REPORT
C
C I) PROBLEM CONSTANTS
C FORTRAN NAME  REPORT NAME           DEFINITION OR UNIT
C
C A1          LOWER CASE A, INDEX 1     RADIUS 1 IN METERS
C A2          LOWER CASE A, INDEX 2     RADIUS 2 IN METERS
C AM1         LOWER CASE M, INDEX 1     MASS 1 IN KG.
C AM2         LOWER CASE M,INDEX 2      MASS 2 IN KG.
C XA          LOWER CASE ALPHA          CONTROL GAIN
C XK          LOWER CASE K              NEWTONS
C U            U ASTERISK              INITIAL LENGTH OF CABLE, IN METERS
C YSF1        X ASTERISK, INDEX 1      EQUILIBRIUM RADIUS R1
C YSF8        X ASTERISK, INDEX 8      EQUILIBRIUM ANGULAR SPEED
C
C II) AUXILIARY CONSTANTS
C A(I)        RUNGE-KUTTA AUXILIARY CONSTANTS (SEE HENRICI,S BOOK)
C P(I)                   IDEM
C H           INTEGRATION STEP SIZE
C B           TIME COMPUTATION ENDS     SECONDS
C K           COUNTING FOR PRINTING PURPOSES
C NU          COUNTING TO COMPLETE THE FOUR STEPS OF RUNGE-KUTTA ROUTINE
C
C III) PROBLEM VARIABLES
C NOTE= THE DIMENSION OF THE STATE IS 9. TIME IS THE NINTH STATE VARIABLE
C
C ETA         INITIAL STATE (ETA(1) IN METERS, ETA(2) IN M/SEC, ETA(3) IN RADIANS,
C             ETA(4) IN RD/SEC, ETA(5) IN RADIANS, ETA(6) IN RD/SEC, ETA(7) IN RD,
C             ETA(8) IN RD/SEC, ETA(9) IN SEC )
C YS          SYSTEM STATE (SAME D'MENSIONS AS ETA)
C CAPV        LYAPUNOV FUNCTION (CAPITAL V)
C TETA        ANGLE TETA
C V           CONTROL LENGTH (LOWER CASE V)
C VDOT        TIME DERIVATIVE OF THE LYAPUNOV FUNCTION
C XF          ANGULAR ACCELERATION (CAPITAL F)
C XLHL        ELONGATION (CAPITAL L TIMES THE HEAVYSIDE FUNCTION OF L)
C XU          CABLE,S LENGTH (LOWER CASE MU)
C
C ALL OTHER VARIABLES ARE AUXILIARY VARIABLES FOR THE RUNGE-KUTTA ROUTINE
C IN THIS PROGRAM, TWO SUBROUTINES ARE CONSTRUCTED, ONE (DICK) TO GET THE
C FUNCTION F APPEARING IN THE RIGHT HAND SIDE OF THE BASIC SYSTEM
C                     YSDOT=F(YS) ,
C
C AND ANOTHER (GETLHL) TO GET THE PRODUCT OF THE ELONGATION L BY THE HEAVYSIDE
C FUNCTION OF L.
C
C THE COMPUTATION TIME REQUIRED FOR THIS PROGRAM IS APPROXIMATELY 0.01 OF AN
C HOUR FOR EACH 1000 CALCULATIONS OF THE STATE. IN OTHER WORDS, THE COMPUTATION
C TIME T IN  HUNDREDS OF AN HOUR IS GIVEN BY
C                           T=(B - ETA(9))/(1000*H)
C
      DIMENSION ETA(9),YS(9),FI(9),A(4),YARG(9),RK(9),P(4)
      READ INPUT TAPE 5,101,(ETA(I),I=1,9)
```

```
      READ INPUT TAPE 5,102,(A(I),I=1,4),(P(I),I=1,4)
      READ INPUT TAPE 5,103,H,B,A1,A2,AM1,AM2,XK,YSF1
      READ INPUT TAPE 5,101,XA
  101 FORMAT (5F10.7)
  102 FORMAT (8F5.1)
  103 FORMAT (F9.6,3F5.2,4E12.3)
      YSF8=(AM1*(1.0+AM1/AM2)*ETA(1)**2*ETA(8)+0.4*AM1*A1**2*(ETA(8)+ETA
     1(3))+0.4*AM2*A2**2*(ETA(8)+ETA(5)))/(AM1*(1.0+AM1/AM2)*YSF1**2+0.4
     1*AM1*A1**2+0.4*AM2*A2**2)
      U=(YSF1*(1.0+AM1/AM2)-A1-A2)/(1.0+AM1*YSF1*YSF8**2/XK)
      CONST=0.5*U/XK*(AM1*YSF1*YSF8**2)**2+AM1*YSF8**2*(YSF1**2(1.0+AM1/
     1AM2)+0.4*(A1**2+AM2/AM1*A2**2))*0.5
      K=0
      DO 10 I=1,9
   10 YS(I)=ETA(I)
   11 DO 20 I=1,9
      YARG(I)=YS(I)
   20 FI(I)=0.0
      NU=1
      GO TO 30
   21 NU=NU+1
      DO 22 I=1,9
   22 YARG(I)=YARG(I)+0.5*P(NU)*H*RK(I)
   30 TETA=ATANF((A1*SINF(YARG(3))+A2*SINF(YARG(5)))/(YARG(1)*(1.0+AM1/A
     1M2)-A1*COSF(YARG(3))-A2*COSF(YARG(5))))
      XU=U - XA*((1.0+AM1/AM2)*YARG(2)*COSF(TETA)+A1*YARG(4)*SINF(YARG(3
     1)+TETA)+A2*YARG(6)*SINF(YARG(5)+TETA))
      XU=MIN1F(MAX1F(XU,0.00001),130.0)
      CALL GETLHL (YARG,A1,A2,AM1,AM2,XLHL,XU)
      XF=(XK/XU*XLHL*(A1*SINF(YARG(3)+TETA)+A2*SINF(YARG(5)+TETA))-2.0*A
     1M1*YARG(1)*YARG(2)*YARG(8)*(1.0+AM1/AM2))/(AM1*YARG(1)**2*(1.0+AM1
     1/AM2))
      IF (K)91,91,95
   91 GO TO (93,95,95,95),NU
   93 V=XU - U
      CALL GETLHL (YARG,A1,A2,AM1,AM2,XLOHLO,U)
      VDOT=XK/XA*V*(XLHL/XU - XLOHLO/U)
      CAPV=0.5*(AM1*(1.0+AM1/AM2)*(YS(2)**2+(YS(1)*YS(8))**2)+0.4*AM1*A1
     1**2*(YS(8)+YS(4))**2+0.4*AM2*A2**2*(YS(8)+YS(6))**2)+0.5*(XK/U*XLO
     1HLO**2) -CONST
      WRITE OUTPUT TAPE 6,105,CAPV,VDOT,V
  105 FORMAT (1X3E18.7)
      IF (XLHL) 200,200,201
  200 WRITE OUTPUT TAPE 6,106
  106 FORMAT (23H NOW THE CABLE IS LOOSE    )
  201 CONTINUE
   95 CALL DICK(YARG,RK,TETA,XF,A1,A2,XK,XU,AM1,AM2,XLHL)
      DO 40 I=1,9
   40 FI(I)=FI(I)+1.0/6.0*A(NU)*RK(I)
      IF(NU-4)21,50,50
   50 DO 60 I=1,9
   60 YS(I)=YS(I)+H*FI(I)
      K=K+1
      IF(K-10) 80,90,90
   90 WRITE OUTPUT TAPE 6,104,(YS(I),I=1,9)
  104 FORMAT (1X9F12.7)
      K=0
   80 IF(YS(9)-B)11,11,70
   70 CALL EXIT
      END
*     LABEL
```

```
CDICK
      SUBROUTINE DICK(YARG,RK,TETA,XF,A1,A2,XK,XU,AM1,AM2,XLHL)
      DIMENSION YARG(9),RK(9)
      RK(1)=YARG(2)
      RK(2)=YARG(1)*YARG(8)**2-XK/(AM1*XU)*XLHL*COSF(TETA)
      RK(3)=YARG(4)
      RK(4)=-XK/XU*1.0/(0.4*AM1*A1)*XLHL*SINF(YARG(3)+TETA)-XF
      RK(5)=YARG(6)
      RK(6)=-XK/XU*1.0/(0.4*AM2*A2)*XLHL*SINF (YARG(5)+TETA)-XF
      RK(7)=YARG(8)
      RK(8)=XF
      RK(9)=1.0
      RETURN
      END
*     LABEL
CGETLHL
      SUBROUTINE GETLHL  (YARG,A1,A2,AM1,AM2,XLHL,XU)
       DIMENSION YARG(9)
      DIST=SQRTF((YARG( 1)*(1.0+AM1/AM2)-A1*COSF(YARG(3))-A2*COSF(YARG(5
     1)))**2+(A1*SINF(YARG(3))+A2*SINF(YARG(5)))**2)
      IF(DIST-XU) 1,1,2
    1 XLHL=0.0
      GO TO 10
    2 XLHL=DIST-XU
   10 RETURN
      END
```

APPENDIX II) Analog Computer Simulations.

A couple of different analog computer simulations were run to back up some of the ideas related to this problem. This appendix will just describe, programwise, these simulations.

The computer used was an Electronic Associated 131-R PACE and the notations or conventions used below are the standard ones for E.A.'s machines.

II. A) Simulation of the simplified case.

Referring to equations 4.11.a and 4.11.b and to the numerical values indicated in section 4(d) the following system of scaled equations - where the brackets indicate analog computer voltages, as usual - has been programmed:

$$\frac{d(2x_1)}{dt} = 2(x_2) \quad \text{------------------------} \quad \text{II. 1.1}$$

$$\frac{d(x_2)}{dt} = \frac{8 \times 2.025 \times 10^5}{10^6} \left[\frac{1}{\frac{(2x_1)(2x_1)(2x_1)}{10^4} / 10^2} \right] +$$

$$- \frac{1.284 \times 10^4}{10^2} \left[\frac{(\text{elongation})}{\text{length}/10^2} \; H\,(\text{elongation}) \right] \quad \text{II. 1.2}$$

$$(\text{length}) = u^* - \alpha(x_2) \quad \text{--------------------} \quad \text{II. 1.3}$$

$$(\text{elongation}) = (2x_1) - (\text{length}) \quad \text{-------------} \quad \text{II. 1.4}$$

A time scaling of 2, which makes the analog computer solution twice slower than the real time, transforms the equations to:

$$\frac{d(2x_1)}{d\tau} = (x_2) \quad \text{------------------------} \quad \text{II. 2.1}$$

$$\frac{d(x_2)}{d\tau} = 0.8100 \left[\frac{1}{\frac{(2x_1)(2x_1)(2x_1)}{10}} / 10^2 \right] +$$

$$- 0.6420 \times 10^2 \left[\frac{(\text{elongation})}{\text{length}/10^2} \quad H\,(\text{elongation}) \right] \text{--- II. 2. 2}$$

$$(\text{length}) = u^* - \alpha(x_2) \text{ ----------------- II. 2.3}$$

$$(\text{elongation}) = (2x_1) - (\text{length}) \text{ ------------- II. 2.4}$$

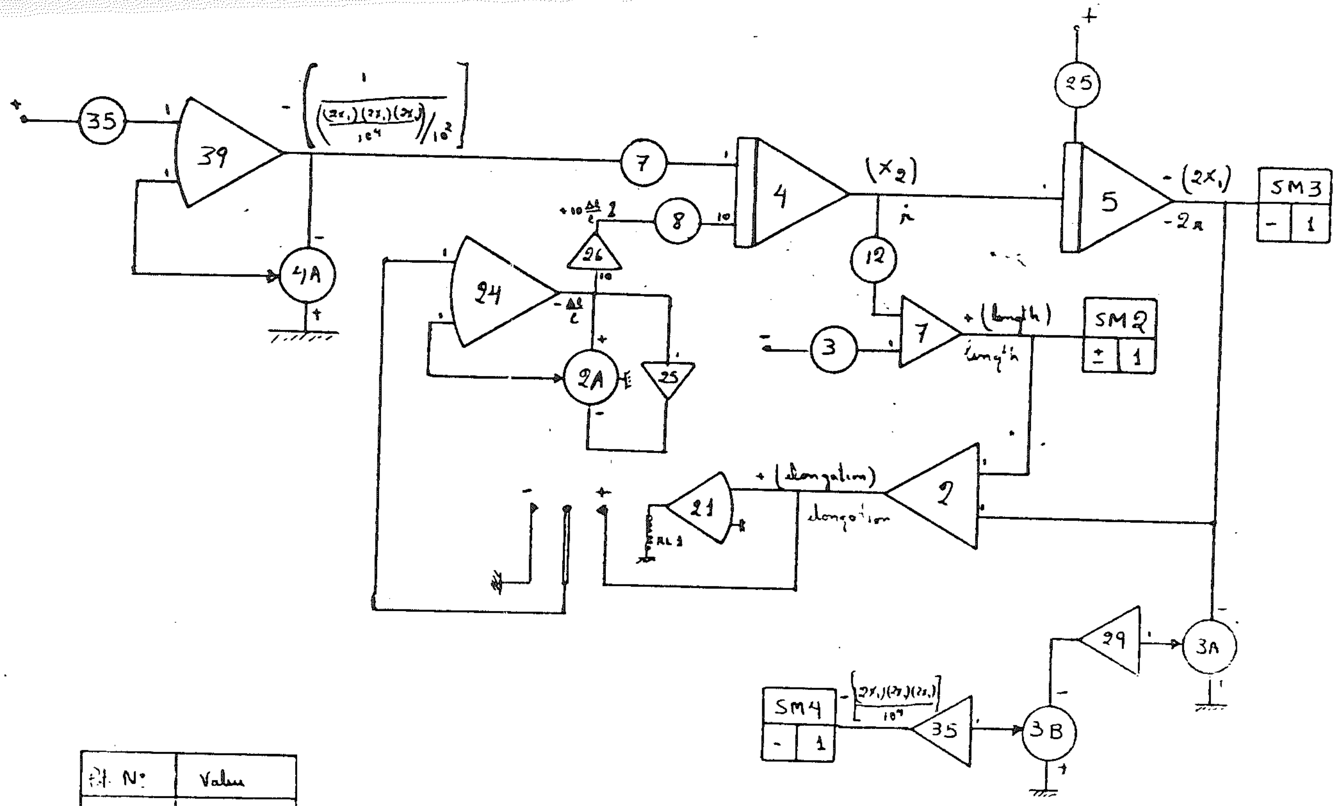
The diagram for these equations is shown on next page. Notice that amplifier no. 24 performs exactly the division specified in the rightmost term of equation 11.2.2.. The output of amplifier 24 is, in volts,

$$\left[\frac{(\text{elongation})}{(\text{length})\ /10^2} \right] = \left[\frac{100\ (\text{elongation})}{(\text{length})} \right]$$

Therefore 1 volt in the output of 24 corresponds to $\frac{\Delta \ell}{\ell} = 1\%$

II. B) Simulation of the complete problem

Only the final set of time scaled-amplitude scaled equations for the complete case will be presented here. The nume rical values used for the parameters are those described next page (see APACHE). Before introducing these scaled equations one must mention that the analog computer diagram was checked against the equations with the help of an APACHE code program (Analod Programming And CHEcking, see IBM Research Note N J 44). The very non-linear nature of this problem could make it serve almost as a counter-example of the utility of APACHE. If however, the function



Pt. N°	Value
3	0.5996
7	0.8100
8	0.6420
12	α
25	2x(0)
35	0.0100

generators are all checked independently of the patching, APACHE can and has been of some help for us.

A copy of our APACHE program can be found in next pages. The following few comments will halp its understanding. Firstly, notice that the initial conditions given to the problem are not only different from the example we are giving here but also they are not consistent. For instance, the initial elongation is not equal to the initial distance between attaching points minus the initial length of the cable. These two facts are not program mistakes: a) the first one is justified by the fact that the initial conditions outlined for Apache are "static check" conditions, i.e., just intended for a check of the patching and potentiometer settings. If these are correct, then any initial values can be given to the state variables for the actual runs; b) the second one, i.e., the unconsistency, is justified by the fact that Apache will compute the initial values of all variables standing at the left hand side of the equations' statements, given those of the right hand side and this computed initial values are used by Apache. Further, a diagnostic is given which tells the programmer this substitution. The procedure is very practical to save the programmer the work of computing by hand the initial condition of a variable which is an algebraic function of other ones with known initial values.

The Apache equations rather than those described by system (3.6) are used in the analog simulation. The difference is in a conversion from radians to degrees since the resolvers in the analog computer work for angles in degrees (and scaled 0.5 volts/degree).

```
 1   COMMENT                                                          JPJ
 2   SPACE LABORATORY PROBLEM                                         JPJ
                                                                      JPJ
 3   PARAMETERS                                                       JPJ
 4    A1=5                                                            JPJ
 5   A2=7                                                             JPJ
 6   J1=0.4*M1*A1*A1                                                  JPJ
 7   J2=0.4*M2*A2*A2                                                  JPJ
 8   K=1.284E+8                                                       JPJ
 9   M1=1.5E+4                                                        JPJ
10   M2=1.0E+4                                                        JPJ
                                                                      JPJ
11   VARIABLES                                                        JPJ
12   X1=40,100                                                        JPJ
13   X2=0,50                                                          JPJ
14   X3=30,100                                                        JPJ
15   X4=0,200                                                         JPJ
16   X5=45,100                                                        JPJ
17    X6=0,200                                                        JPJ
18   X7=0,100                                                         JPJ
19   X8=50,100                                                        JPJ
20   ELONG=0,0.1                                                      JPJ
21   MU=90,100                                                        JPJ
22   LHL=1,1                                                          JPJ
23   COS(TETA)=0.9999,1                                               JPJ
24   SIN(X3TET)=0.5067,1                                              JPJ
25   F=0,100/3.75,EXACT                                               JPJ
26   SIN(X5TET)=0.7120,1                                              JPJ
27   C=91,100                                                         JPJ
28   JACK=0,100/6,EXACT                                               JPJ
29   HANK=0,10/6,EXACT                                                JPJ
30   MIH=0,50E+2                                                      JPJ
31   ERIC=0,50E+4                                                     JPJ
32   JEAN=0,1.0E+4                                                    JPJ
33   BILL=0,1.0E+10                                                   JPJ
34   BOB=1.0E+7,1.0E+9/2.67,EXACT                                     JPJ
                                                                      JPJ
35   COMMENT                                                          JPJ
36           THE AUXILIARY VARIABLES WHICH APPEAR HERE WILL BE        JPJ
                                                                      JPJ
```

```
37   OBTAINED FROM RESOLVERS AND FUNCTION GENERATORS. AS CAN BE SEEN         JPJ
38   IN THE COMPUTER DIAGRAM, THEY ARE COS(TETA), SIN(X3TET),                JPJ
39   SIN(X5TET) AND C. TO GET C ONE NEEDS THE SINUS AND COSINUS OF BOTH      JPJ
40   X3 AND X5.                                                              JPJ
                                                                             JPJ
41                  VARIABLE LHL, WHERE HL STANDS FOR THE HEAVYSIDE          JPJ
42   FUNCTION H(L), IS OBTAINED WITH THE HELP OF A RELAY SWITCH.             JPJ
                                                                             JPJ
                                                                             JPJ
43   BETA=2                                                                  JPJ
44   EQUATIONS                                                               JPJ
45   DER1(X1)=X2                                                             JPJ
46   DER1(X2)=6.28*6.28/360/360*X1*X8*X8 - K/M1*ELONG*COS(TETA)              JPJ
47   DER1(X3)=X4                                                             JPJ
48   DER1(X4)= - 360/6.28*K/J1*A1*ELONG*SIN(X3TET) - F                       JPJ
49   DER1(X5)=X6                                                             JPJ
50   DER1(X6)= -360/6.28*K/J2*A2*ELONG*SIN(X5TET) - F                        JPJ
51   DER1(X7)=X8                                                             JPJ
                                                                             JPJ
52   DER1(X8)=F                                                              JPJ
53   ELONG=LHL/MU                                                            JPJ
54   LHL= C - MU                                                             JPJ
55   JACK=A1*SIN(X3TET) + A2*SIN(X5TET)                                      JPJ
56   HANK=ELONG*JACK                                                         JPJ
57   MIH=X1*X2                                                               JPJ
58   ERIC=MIH*X8                                                             JPJ
59   JEAN=X1*X1                                                              JPJ
60   BILL=K*HANK - 2*M1*(1 + M1/M2)*ERIC                                     JPJ
61   BOB=M1*(U+M1/M2)*JEAN                                                   JPJ
62   F=BILL/BOB                                                              JPJ
63   END                                                                     JPJ
```

Here follows the complete set of equations, as prepared for analog computation, with a time scale factor β=10.

$$\frac{d(x_1)}{d\tau} = 0.05\,(2x_2) \qquad \text{II. B. 1}$$

$$\frac{d(2x_2)}{d\tau} = 0.608\left[\frac{(x_1)}{100} \cdot \frac{(x_8)(x_8)}{100}\right] - 1.71\left[\frac{(10^3\,\text{elong})(100\ \cos\,\Theta)}{100}\right]$$

II. B. 2

$$\frac{d(0.5x_3)}{d\tau} = 0.1(0.5x_4) \qquad \text{II. B. 3}$$

$$\frac{d(0.5x_4)}{d\tau} = -12.2\left[\frac{(10^3\ \text{elong})(100\ \sin\ X\ \ \text{TET})}{100}\right] - 0.0134\,(3.75F)$$

II. B. 4

$$\frac{d(0.5x_5)}{d\tau} = 0.1(0.5x_6) \qquad \text{II. B. 5}$$

$$\frac{d(0.5x_6)}{d\tau} = -13.2\left[\frac{(10^3\,\text{elong})(100\ \sin\ X5\ \text{TET})}{100}\right] - 0.0134\,(3.75F)$$

II. B. 6

$$\frac{d(x_7)}{d\tau} = 0.1(x_8) \qquad \text{II. B. 7}$$

$$\frac{d(x_8)}{d\tau} = 0.0267(3.75F) \qquad \text{II. B. 8}$$

$$(6\ \text{Jack}) = 0.3(100 \sin X3\ \text{TET}) + 0.42\ (100 \sin X5\ \text{TET})$$ II. B. 9

$$(0.6\text{x}10^2\ \text{Hank}) = \left[\frac{(10^3\ \text{elong})(6\ \text{Jack})}{100}\right]$$ ---------- II. B. 10

$$(2\text{x}10^2\ \text{Mih}) = \left[\frac{(x_1)(2x_2)}{100}\right]$$ -------------- II. B. 11

$$(2\text{x}10^4\ \text{Eric}) = \left[\frac{(2 \text{ x } 10^{-2}\ \text{Mih}) \text{ x } (x_8)}{100}\right]$$ ------ II. B. 12

$$(10^2\ \text{Jean}) = \left[\frac{(x_1)(x_1)}{100}\right]$$ ------------- II. B. 13

$$(10^8\ \text{Bill}) = 0.0213(0.6\text{x}10^2\ \text{Hank}) - 3.75\ (2\text{x}10^{-4}\ \text{Eric})$$ II. B. 14

$$(2.67\text{x}10^{-7}\ \text{Bob}) = (10^{-2}\ \text{Jean})$$ ------------- II. B. 15

$$(3.75\text{F}) = \left[\frac{(10^{-8}\ \text{Bill})}{(2.67\text{x}10^{-7}\ \text{Bob})/100}\right]$$ ------------ II. B. 16

$$(10^2\ \text{elong}) = \left[\frac{\text{LHL}}{\text{mu}/100}\right]$$ ----------------- II. B. 17

$$(\text{A}) = 2.5(x_1) - 0.05(100 \cos x_3) - 0.07\ (100 \cos x_5)$$ - II. B. 18

$$(\text{B}) = 0.05(100 \sin x_3) + 0.07(100 \sin x_5)$$ ------- II. B. 19

$$(\text{mu}) = u^* + (v)$$ --------------------- II. B. 20

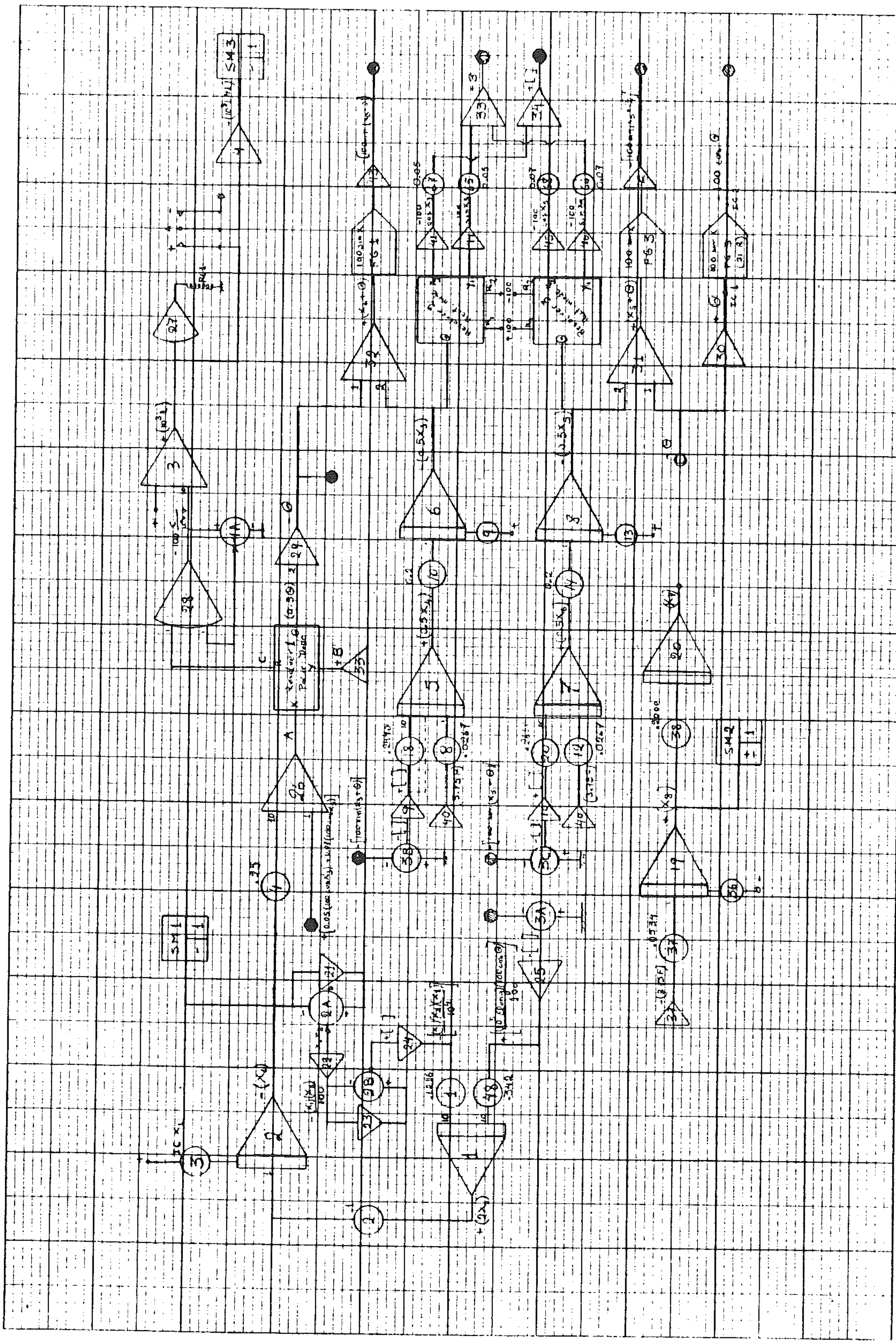
$$(v) = -\alpha\left[1.25\left(\frac{(2x_2)(100 \cos \Theta)}{100}\right) + 10\left(\frac{(0.5x_4)(100 \sin X3TET)}{100}\right) + 14\left(\frac{0.6x_8)(100 \sin X5\ TET)}{100}\right)\right] \qquad \text{II. B. 21}$$

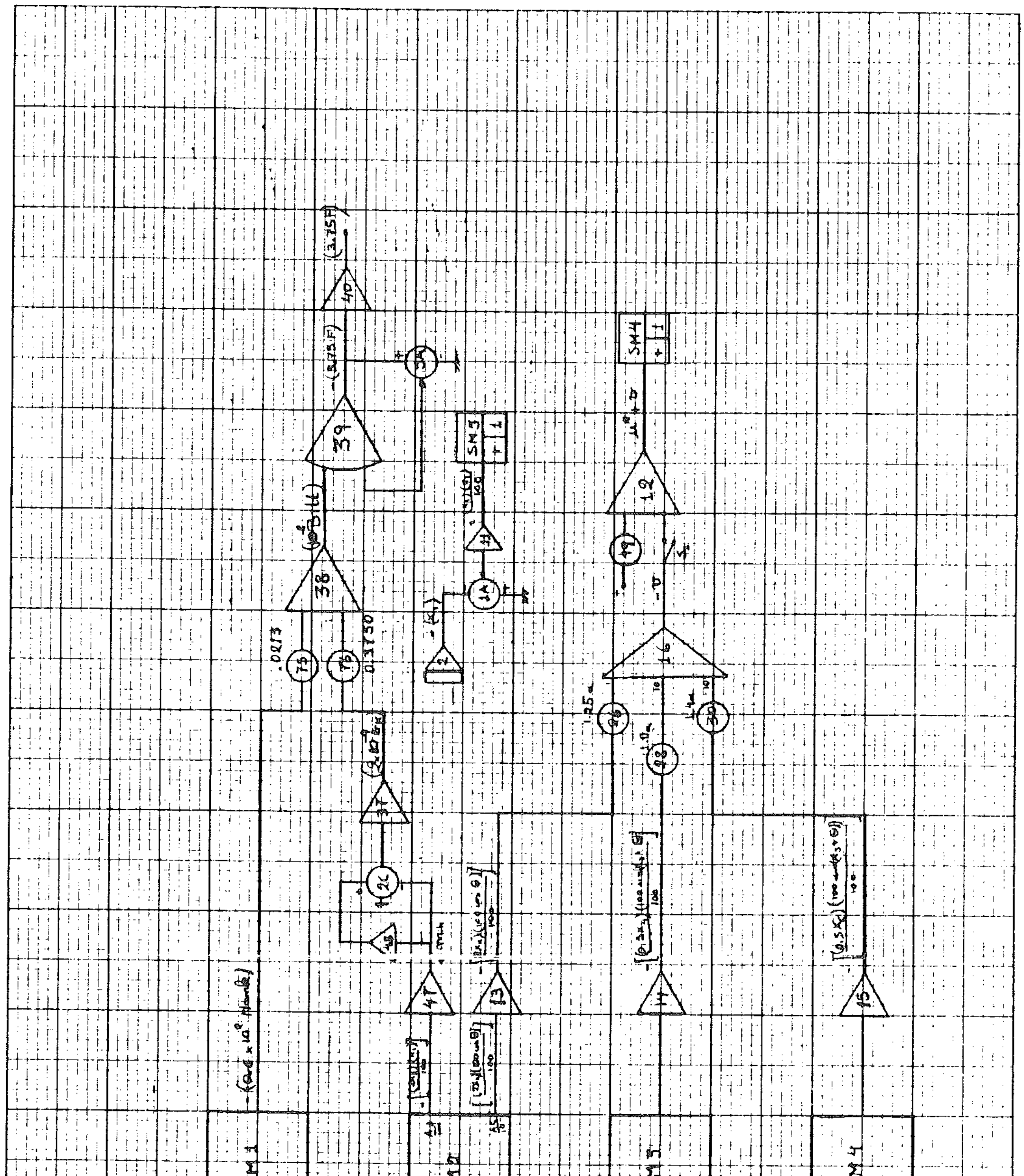
The diagram corresponding to these equations follows next pages. Some comments about the equations and the diagram have to be made here to show the nature of this problem and its non fitness for analog computation.

By observing the first eight differential equations, one sees that the coefficients range from 10^{-2} to 26. This immediately places the problem out of the normal range of analog computers, where the extreme coefficients should have a maximum ratio of 100 to 1. All the big coefficients, however, are multiplying terms containing the relative elongation in equations II. B. 4 and II. B. 6. This relative elongation is underscaled (by a factor of 5 - assuming its maximum to be 2 %). But the nature of the elongation is such that it is produced from the difference of two voltages: (C) (the distance between the attaching points of the cable) and the length of the cable. This difference is of the order of a few hundredths or tenths of milli-volts and the analog computer obviously superposes an equal noise to this voltage. It would be impossible to amplify this by a thousand times. What is done, however, is to write the equation . .

$$\text{elongation} = \frac{c-\ell}{u^{*}+v} = \frac{c}{u^{*}+v} - 1$$

and write this equation analog computer-wise, therefore decreasing considerably the noise - but not eliminating it entirely. This noise shakes continuously servo-multiplier 3 and introduces imprecisions in the solution. Notice that in the equilibrium solution this fact is of no concern, since we mainly (and hopefully "only") use the differential equations II. B. 1 , II B. 2.





which are well fitted for analog computation.

Another source of imprecision is that the control law v may not be zero when it should, because of offset of function generators giving sinus and cosinus functions and also of the electronic multipliers. This fact is more accentuated the greater is α, of course. This will not only introduce errors in the correction of the length of the cable, but it can also occasion short intervals of time during which the Lyapunov function and its time derivative have the same sign.

The malignous effects of both the above mentioned effects did not hesitate in showing up when we tried a few runs in this simulation. . .