A Column-Pivoting Based Strategy for Monomial Ordering in Numerical Gröbner Basis Calculations

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Overview

This work is about solving systems of polynomial equations arising in many geometric vision problems.

Problem: State-of-the-art polynomial methods use Gröbner basis techniques, but are still numerically unstable in many cases or slow.

We previously proposed to make a change of basis in $\mathbb{C}[\mathbf{x}]/I$ to improve the conditioning of the Gröbner basis computation [BJÅ07]. This was done using a large and expensive SVD decomposition.

A new efficient algorithm: Based on a clarification of the chain of matrix operations needed to compute the Gröbner basis, we are able to (i) factorize a typically much smaller sub matrix and (ii) substitute the expensive SVD factorization with the cheaper QR-factorization with column pivoting. This yields a simultaneous factorization and numerically sound choice of monomial basis for $\mathbb{C}[\mathbf{x}]/I$.

Contribution:

- A simplified derivation of how to change basis.
- A new strategy for how to change basis using fast and numerically stable QR factorization with column pivoting.

Problem Statement

Find the complete set of solutions to a system of equations on the following form

$$\begin{aligned} c_{11}\mathbf{x}^{\alpha_1} + c_{12}\mathbf{x}^{\alpha_2} + \ldots + c_{1n}\mathbf{x}^{\alpha_n} &= 0, \\ &\vdots \\ c_{m1}\mathbf{x}^{\alpha_1} + c_{m2}\mathbf{x}^{\alpha_2} + \ldots + c_{mn}\mathbf{x}^{\alpha_n} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_n}$ are a given set of monomials with $\mathbf{x}^{\alpha_k} = x_1^{\alpha_{k1}} \cdot \cdots \cdot x_p^{\alpha_{kn}}$. Ensure high numerical accuracy in the process.

Motivation

- Polynomial equations arise in *e.g.* minimal cases of structure from motion and in global optimisation.
- Numerical stability of existing solvers in many cases poor.
- The Gröbner basis technique for equation solving not yet fully understood.

1 Equation 1 Variable

$x^3 + 4x^2 + x - 6 = 0$ (1 Equation, 1 variable) $\xrightarrow{A}$ $\begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -1 \\ 0 & 1 & -4 \end{bmatrix}$ (Companion matrix) $\xrightarrow{B}$ $x_1 = 1$, $x_2 = -2$, $x_3 = -3$ (Solutions)

A. Copy polynomial coefficients into the companion matrix. B. The eigenvalues of the companion matrix are the solutions.

m Equations n Variables

$25xy - 15x - 20y + 12$, $x^2 + y^2 - 1 = 0$ (2 Equations, 2 variables) $\xrightarrow{A}$ $125y^3 - 75y^2 - 45y + 27 = 0$, $25xy - 15x - 20y + 12 = 0$, $x^2 + y^2 - 1 = 0$ (Gröbner basis) $\xrightarrow{B}$ $\frac{1}{125}\begin{bmatrix} 0 & -60 & 65 & -63 \\ 125 & 100 & 100 & 45 \\ 125 & 75 & 75 & 45 \\ 0 & 125 & -125 & 175 \end{bmatrix}$ (Action matrix) $\xrightarrow{C}$ $x_1 = -4/5, y_1 = 3/5$, $x_2 = 4/5, y_2 = -3/5$, $x_3 = 4/5, y_3 = 3/5$ (Solutions)

A. Generalized gaussian elimination. B. Copy elements from the Gröbner basis to form a generalized companion matrix. C. The eigenvalues and eigenvectors of the action matrix yield the solutions.

Gröbner Bases Using Matrix Factorization

Buchberger's algorithm computes a Gröbner basis, but to make it numerically stable we make use of the following

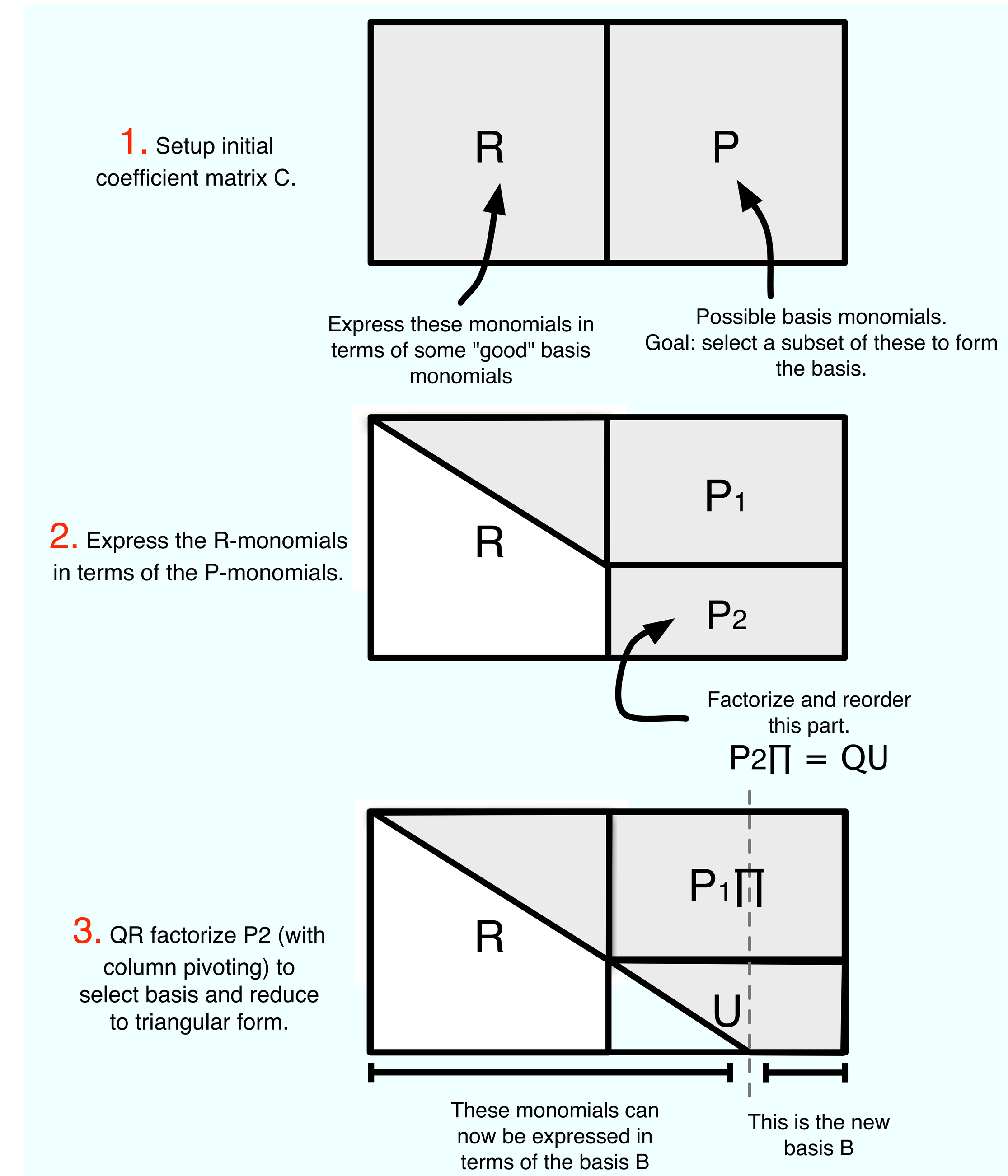
INSIGHT: Computing a Gröbner basis ≈ solving an underdetermined linear system.

and reformulate it using matrices. Equation 1 can be written using matrix notation as

$$\mathbf{C}\begin{pmatrix} \mathbf{x}^{\alpha_1} \\ \vdots \\ \mathbf{x}^{\alpha_n} \end{pmatrix} = 0. \tag{2}$$

With this, we can use a variation of the Buchberger algorithm working in two phases: (i) add a large number of new equations by multiplying the original equations with a hand-selected set of monomials, (ii) use numerical linear algebra to express higher order monomials in terms of a set of lower order monomials (the basis for $\mathbb{C}[\mathbf{x}]/I$).

Our contribution concerns the second phase and can be outlined like so:



Pose with Hybrid Features

- Pose estimation with unknown focal length and mixed 2D and 3D feature data. [JBKÅ07]
- Minimal case: 3 correspondences to known 3D points, 1 correspondence to a known camera.
- 4 unknowns, 36 solutions.
- Maximum total degree 6.
- After expansion: 980 equations in 873 monomials, total degree 10.
- For this problem we had to use truncation of the Gröbner basis to get a working solver. See paper for details.

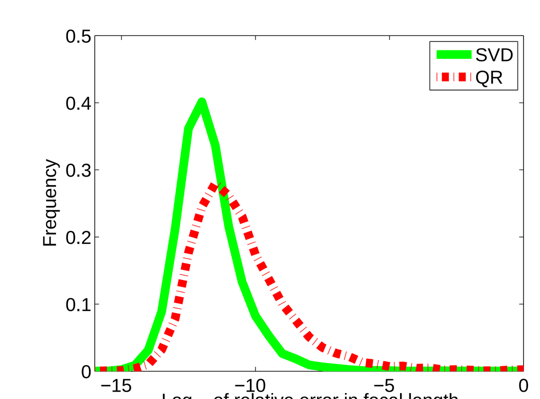


FIGURE 1: Relative error in focal length for pose estimation with unknown focal length (mixed 2D and 3D feature data)

Three View Triangulation

- Optimal L_2-triangulation by calculation of all stationary points [SSN05].
- 3 unknowns, 50 solutions.
- 3 equations with total degree 6.
- After expansion: 225 equations in 209 monomials with total degree 9.

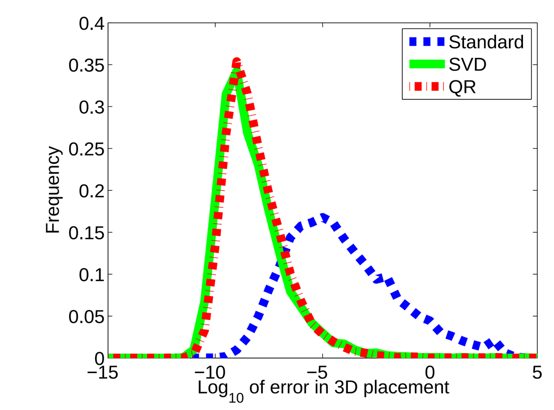


FIGURE 2: Histogram over the error in 3D placement of the unknown point obtained using optimal three view triangulation.

Code available at: www.maths.lth.se/vision/downloads

Speed Comparison

Method	Time per call / ms	Relative time
SVD	41.685	1
QR	10.937	0.262
Standard	8.025	0.193

TABLE 1: Time consumed in the solver part for the problem of three view triangulation. The comparison is made for the three different methods. The time is an average over 1000 calls.

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