On some topics in operator theory

An unfinished story about mathematical control

Rydhe, Eskil

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An unfinished story about mathematical control

by Eskil Rydhe

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DOCTORAL THESIS

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Abstract:
This thesis considers differentiation of non-negative, fractional order, composed with Hardy space-type Hankel operators. H^2-boundedness is characterized in terms of a reproducing kernel thesis. The setting of operator-valued symbols is considered, in which H^2-boundedness is characterized in terms of Carleson embeddings, provided that the order of differentiation is strictly positive. Some new results are deduced for the zeroth order. The complexity of the Carleson embedding conditions is demonstrated by means of examples. Natural corresponding factorization theorems are proved. Some results are phrased in terms of control theory. An attempt is made at describing Hilbert space contraction semigroups which can be modeled by a weighted backward shift.

Keywords: Carleson embeddings, complex analysis, control theory, functional models, Hankel operators, harmonic analysis, operator theory, Triebel–Lizorkin spaces, vector-valued analytic functions

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Cover illustration: For $\alpha > 0$, and an analytic polynomial f, let $\tilde{f}(z) = f(\bar{z})$, and define $\Phi(\alpha, f) = D^{-\alpha}\left((D^{\alpha} f) \otimes \tilde{f}\right)$. The cover illustrates the functions $\|\Phi(\alpha, f)|_{\{|z|=1\}}\|_{\mathcal{S}^1}$, where f is fixed, and α ranges from 0 to 2, as the colors range from purple to red. Numerical evidence suggests that the quadratic form $\Phi(\alpha, \cdot)$ maps $H^2(\mathcal{H})$ contractively into $H^1(\mathcal{S}^1)$ whenever $\alpha > 0$. Boundedness of the corresponding bilinear form would have implied that $H^1(\mathcal{S}^1)$ has a square function characterization.

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> "You seem to be doing a lot of calculations. [...] I used to do calculations, but now it's just going straight to TeX. [...] What are you calculating? [...] You should find a simpler problem."
>
> – M. T. Lacey
>
> In monologue with the author

While the above quote may seem disheartening, I rather received it as a provocation to stubbornly pursue that which seized my attention. My time as a graduate student has taught me two things. The first is that a thesis is not written by itself.

The second is that, typically, it is also not written by any oneself. For the last few years I have depended upon a great many people for inputs on mathematics, meta-mathematics, academic life, non-academic life, meta-life, nonsense, and most of all, the matter of completing a PhD while trying to maintain a shred of sanity. Attempting to mention everyone would of course be foolish.

Sandra Pott. If you are reading this, then I guess that we are approaching the finish line. I have often felt how every step of the way has seemed to lead me down a different path than the one that you had intended. While allowing me this great freedom, you have somehow always succeeded in providing me with relevant advice. My experience with supervisors is limited of course, but this ability of yours is something that I will always admire. Thank you for your support, that helped me become a PhD student, write a thesis with which I'm mighty pleased, and obtain a postdoc in Leeds.

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All my friends, colleagues, and teachers at the Centre for Mathematical Sciences, who have supported me in different ways. In almost alphabetical order: Yacin Ameur, Wafaa Assaad, Christer Bennewitz, Marcus Carlsson, Jacob Stordal Christiansen, Thomas Edlund, Kjell Elfström, Magnus Fontes, Gudrun Gudmundsdottir, Sigmundur Gudmundsson, Annika Hansdotter, Rasmus Henningsson, Per-Anders Ivert, Kerstin Johnsson, Hanna Källén, Sara Maad Sasane, Bartosz Malman, Arne Meurman, Fatemeh Mohammadi, Dag Nilsson, Jonas Nordström, Anders Olofsson, Jan-Fredrik Olsen, Karl-Mikael Perfekt, Mikael Persson Sundqvist, Anna-Maria Persson, Tomas Persson, Pelle Pettersson, Maria Carmen Reguera, Kerstin Rogdahl, Ebba Ruhe, Amol Sasane, Andrei Stoica, Douglas Svensson Seth, Francisco Villarroya Alvarez, Joe Viola, Erik Wahlén, Carl-Gustav Werner, Frank Wikström, Jens Wittsten, Mikael Abrahamsson. Whatever it may be, learning together, searching together, teaching together, lunching together, making pancakes together, asking tricky questions, giving elegant answers, making me feel like an amazing mathematician, making me feel like an awful mathematician, proofreading my papers, solving my problems, listening to my complaints, everything counts.

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Barbara. Thank you for joining me, and believing in me throughout this mess of an adventure. There are some parts of my madness that I wish you wouldn't have had to put up with, but I could not be more grateful to have you there to keep me on track. You have seen more than anyone. Borne more than anyone.

Jorun and Nore. If you read this, then I most certainly I hope that you have become old enough to not be struck by hubris. You two are my greatest teachers.

Populärwissenschaftliche Zusammenfassung

Wir stellen uns einen dünnen Metallstab vor, der auf eine inhomogene Temperaturverteilung erhitzt wurde. Wir stellen uns nun weiter vor, dass dieser Stab bis auf die Endpunkte ganz isoliert ist. Wenn wir den Stab nun in ein großes Becken mit Wasser von null Grad Celsius senken, so wird die Wärmeenergie an den Enden aus dem Stab abfliessen. Dadurch wird die Stab abkühlen, und am Ende wird er die gleiche Temperatur wie das umgebende Wasser im Becken haben.

Wenn wir die Position eines Punktes auf dem Stab mit der Koordinate $0 \leq x \leq 1$ spezifizieren, und mit t die Zeit bezeichnen, die seit dem Eintauchen des Stabes im Wasser vergangen ist, so ist für jede Zeit $t \geq 0$ die Temperatur T am Punkt x einer *Funktion* $f_t(x)$:

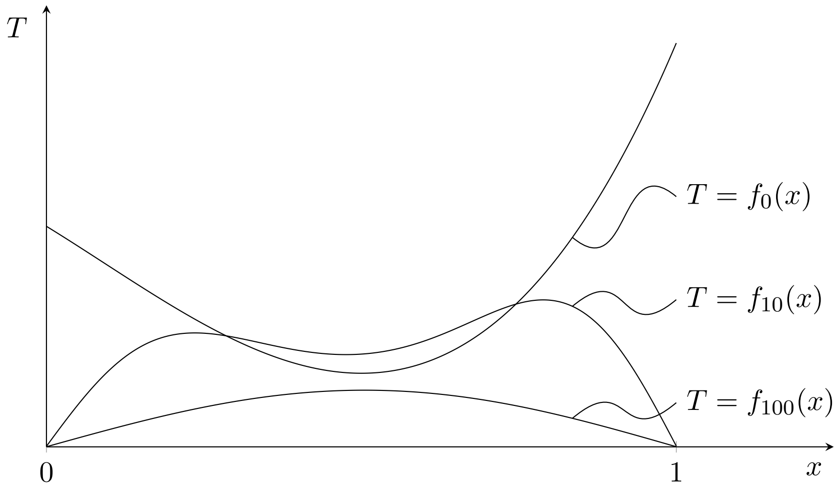


Abbildung 1: Die Temperatur der Stange bei $t = 0$, $t = 10$ und $t = 100$.

Dieser Stab kann als *(lineares) System* modelliert werden. Es ist eine grundlegende Annahme in den Naturwissenschaften , dass, wenn der Zustand eines Systems zu einem gegebenen Zeitpunkt bekannt ist, es (zumindest theoretisch) möglich ist, den Zustand des Systems zu jedem Zeitpunkt danach vorherzusagen. Der

Zustand des Stabes zum Zeitpunkt $t = 0$ wird durch die Temperatur-Funktion f_0 beschrieben. Nun sollten wir somit in der Lage sein, f_t für jedes $t > 0$ mit Hilfe von f_0 zu bestimmen. Eine mathematische Formulierung dieses Umstandes besagt, dass es für jedes $t > 0$ eine Abbildung $H(t)$ geben sollte, die die Temperaturverteilung f_0 zum Zeitpunkt 0 auf die Temperaturverteilung f_t zum Zeitpunkt t abbildet, d.h. $H(t)f_0 = f_t$.

Die Familie $(H(t))_{t\geq 0}$ ist ein Beispiel einer sogenannten *Halbgruppe*. Eine Halbgruppe kann man als eine mathematische Beschreibung sein der Entwicklung eines Systems im Laufe der Zeit verstehen.

Ein Beispiel für ein solches abstraktes System, das sich im Laufe der Zeit weiterentwickelt, ist wie folgt: Für eine Funktion f, die für $x \geq 0$ definiert ist, können wir

$$(S(t)f)(x) = \begin{cases} f(x-t) & \text{för } t \leq x, \\ 0 & \text{för } 0 \leq x < t. \end{cases}$$

definieren. Die Wirkung von $S(t)$ ist, dass der Graph $y = f(x)$ um t Längeneinheiten nach rechts versetzt wird:

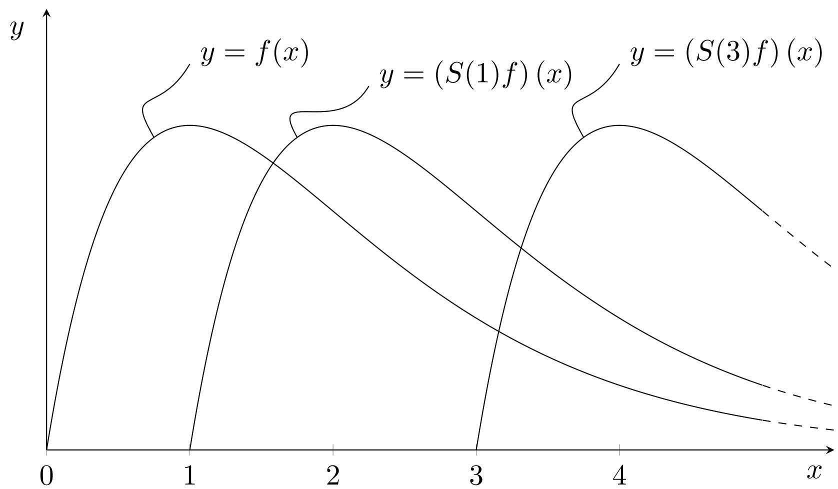


Abbildung 2: Der Operator $S(t)$ verschiebt den Graph $y = f(x)$ um t Längeneinheiten nach rechts.

Die Halbgruppe $(S(t))_{t\geq 0}$ wird die *Halbgruppe der Rechtsverschiebungen* genannt. Obwohl diese zunächst als ein sehr einfaches oder künstliches Beispiel erscheinen mag, hat sie doch sich als äusserst wichtig erwiesen, weil eine große Klasse von Halbgruppen mit Hilfe dieser Halbgruppe der Rechtsverschiebungen beschrieben werden kann. In mathematischer Sprache heißt dies, dass die Halbgruppe der Rechtsverschiebungen ein *universelles Modell* für die Klasse der kontraktiven, vollständig nichtunitären Halbgruppen ist.

Eine Methode, ein System zu studieren, besteht darin, zu versuchen, den Zustand des Systems zu jeder Zeit t zu bestimmen. Dies kann schwierig sein und bringt oft auch viel mehr Information, als von Interesse ist. Um zum Beispiel des gekühlten Stabes zurückzukehren, so ist es vielleicht von Interesse, die Temperatur an einem bestimmten Punkt und Wärmefluss aus dem Stab zu kennen. Wenn wir statt des ganzen Systems nur eine solche einfache Funktion des Systems studieren, so sagen wir, dass wir das System *beobachten*. Für jede mögliche Art von Beobachtung gibt es ein Abbildung C, die den durch die Funktion f_t beschriebenen Zustand des Systems zur Zeit t auf eine Zahl Cf_t abbildet, die der beobachteten Größe entspricht.

Dieser sogenannte *Beobachtungsoperator* C ist ein mathematisches Objekt und kann als solches rein mathematisch definiert werden, d.h. ohne Verbindung zur physischen Realität. Ein Problem in der mathematischen Systemtheorie ist, dass sich einige mathematisch definierte Beobachtungsoperatoren sehr schlecht verhalten. Solche Beobachtungsoperatoren heißen *nicht zulässig*.

Es ist von Interesse, herauszufinden, ob ein gegebener Beobachtungsoperator zulässig ist oder nicht. Besonders interessant ist es, die Klasse von Beobachtungsoperatoren zu verstehen, die für Systeme, deren Zeitentwicklung durch die Halbgruppe der Rechtsverschiebungen beschrieben wird, zulässig sind, weil dies auch Information für die Zulässigkeit von Beobachtungsoperatoren auf anderen Systemen geben kann.

Eine Möglichkeit, herauszufinden, welche Beobachtungsoperatoren für die Halbgruppe der Rechtsverschiebungen zulässig sind, ist, die sogenannten *Hankeloperatoren* Γ_c zu studieren:

Betrachten wir eine Ebene im Raum, und sei f ein Punkt in dieser Ebene. Dieser Positionsvektor f wird nun auf eine Weise verdreht, die vom Beobachtungsoperator C abhängt. Der resultierende Positionsvektor $\tilde{f}$ wird dann zurück auf die Ebene projiziert, auf der wir angefangen haben, siehe Figur 3. Einen Hankeloperator Γ_c kann man nun als ein unendlichdimensionales Analog des eben beschriebenen Verfahren verstehen. Neben einem kleinen Ausflug in Richtung universaler Modelle für bestimmte Klassen von Halbgruppen handelt diese Arbeit im Wesentlichen von verallgemeinerten Hankeloperatoren.

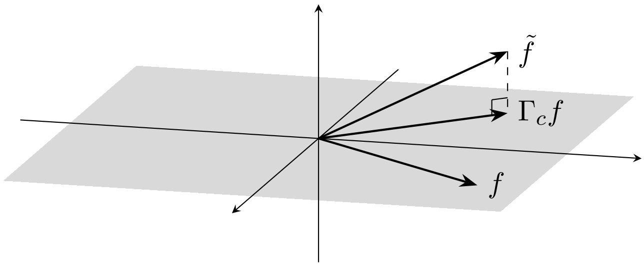
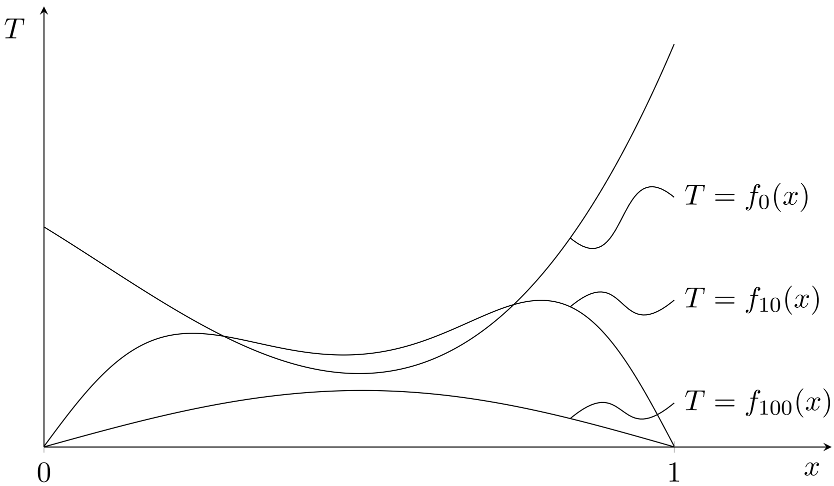


Abbildung 3: Endlich-dimensionale Interpretation eines Hankeloperators.

Populärvetenskaplig sammanfattning

Vi föreställer oss en metallstav som hettats upp till en, låt oss säga inhomogen, temperatur. Vi föreställer oss också att staven är isolerad runtom, men inte i ändpunkterna. Om vi sänker ned staven i en stor bassäng med nollgradigt vatten så kommer termisk energi att lämna staven genom dess ändar. Därmed svalnar staven, och till slut kommer den att ha samma temperatur som den omgivande bassängen.

Om vi anger positioner på staven med en koordinat $0 \leq x \leq 1$, och låter t beteckna den tid som passerat sedan staven nedsänktes i vattnet, så beskrivs temperaturen T vid varje $t \geq 0$ av en *funktion* $f_t(x)$:



Figur 1: Stångens temperatur vid $t = 0$, $t = 10$ och $t = 100$.

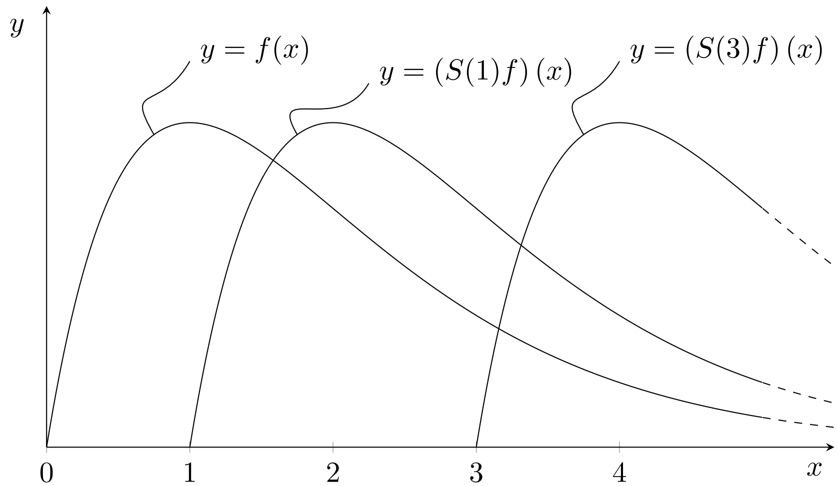
Den svalnande stången modelleras med fördel som ett *(lineärt) system*. Ett centralt antagande inom naturvetenskaperna är att om tillståndet hos ett system vid ett givet tillfälle är känt, så är det (åtminstone teoretiskt) möjligt att förutsäga systemets tillstånd vid alla senare tillfällen. Den svalnande stångens tillstånd vid tiden $t = 0$ beskrivs av temperaturfunktionen f_0. Enbart utifrån f_0 bör vi alltså kunna bestämma f_t för varje $t > 0$. En matematisk formulering av detta är att det för varje $t > 0$ ska finnas en *avbildning* $H(t)$ som avbildar temperaturen vid tiden 0 på temperaturen vid tiden t, dvs. $H(t)f_0 = f_t$.

Familjen $(H(t))_{t\geq 0}$ är ett exempel på en så kallad *semigrupp*. En semigrupp kan sägas vara den matematiska beskrivningen av hur ett system utvecklar sig över tid.

Ett exempel på ett abstrakt system som utvecklar sig över tid är följande: Givet en funktion f definierad för $x \geq 0$ så kan vi definiera

$$(S(t)f)(x) = \begin{cases} f(x-t) & \text{för } t \leq x, \\ 0 & \text{för } 0 \leq x < t. \end{cases}$$

Verkan av $S(t)$ är att grafen $y = f(x)$ förskjuts t längdenheter åt höger:



Figur 2: Operatorn $S(t)$ förskjuter grafen $y = f(x)$, t längdenheter åt höger.

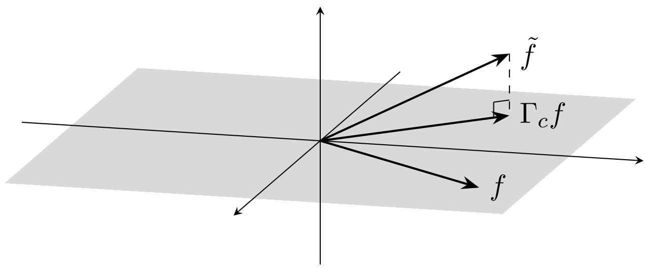
Semigruppen $(S(t))_{t\geq 0}$ kallas för *högerskiftsemigruppen*. Denna kan framstå både som mycket enkel och som artificiell, men har visat sig vara synnerligen viktig, eftersom en stor klass av semigrupper låter sig beskrivas enbart i termer av högerskiftsemigruppen. På matematiska heter det att högerskiftsemigruppen är en *universalmodell* för klassen av fullständigt ickeunitära semigrupper.

Ett sätt att studera system är att försöka bestämma hela tillståndet hos systemet. Detta kan vara svårt, och ger samtidigt ofta betydligt mer information än vad som är intressant. För att återvända till exemplet med den svalnande stången så är det kanske av intresse att känna till temperaturen i en given punkt, eller värmeflödet ut ur stången. Om vi väljer att studera en sådan enklare egenskap hos systemet så kallas detta för att vi *observerar* systemet. För varje typ av observation finns det en avbildning C som avbildar ett tillstånd, beskrivet av en funktion f_t, på ett tal Cf_t motsvarande den storhet som observeras.

Den så kallade *observationsoperatorn* C är ett matematiskt objekt, och kan som sådant definieras rent matematiskt, dvs. utan koppling till verkligheten. Ett problem inom matematisk systemteori är att vissa matematiskt definierade observationsoperatorer beter sig mycket illa. Motsvarande observationer kallas för *icke tillåtna*.

Det är av intresse att på förhand kunna avgöra om en avbildning motsvarar en tillåten observation. Det är särskilt intressant att förstå vilka observationer som är tillåtna för systemet vars utveckling beskrivs av högerskiftsemigruppen, eftersom detta också kan ge information om vilka observationer som är tillåtna för andra system.

Ett sätt att avgöra vilka observationer som är tillåtna för högerskiftsemigruppen är att studera så kallade *Hankeloperatorer* Γ_c: Betrakta ett plan i rummet, och låt f vara en punkt i detta plan. Ortsvektorn för f vrids enligt en regel som beror på C. Den nya vektorn $\tilde{f}$ projiceras därefter tillbaka på planet i vilket vi började, se Figur 3. En Hankeloperator Γ_c kan sägas vara en oändligtdimensionell analog till den beskrivna proceduren. Förutom en liten avstickare mot universalmodeller för klasser of semigrupper så handlar denna avhandling väsentligen om operatorer av Hankelliknande typ.



Figur 3: Ändligtdimensionell tolkning av en Hankeloperator.

List of publications

[**Paper I**] B. Jacob, E. Rydhe, and A. Wynn, *The weighted Weiss conjecture and reproducing kernel theses for generalized Hankel operators*, J. Evol. Equ. **14** (2014), no. 1, 85–120, doi: 10.1007/s00028-013-0209-z. Reproduced with permission of Springer Science+Business Media.

[**Paper II**] E. Rydhe, *An Agler-type model theorem for C_0-semigroups of Hilbert space contractions*, J. London Math. Soc. (2) **93** (2016), no. 2, 420–438, doi: 10.1112/jlms/jdv067. Reproduced with permission of the London Mathematical Society.

[**Paper III**] ______, *Vectorial Hankel operators, Carleson embeddings, and notions of BMOA*, Geom. Funct. Anal. **27** (2017), no. 2, 427–451, doi: 10.1007/s00039-017-0400-4. `arXiv:1604.05505`.

[**Paper IV**] ______, *Two more counterexamples to the infinite-dimensional Carleson embedding theorem*, Accepted for publication in Int. Math. Res. Not. IMRN (2017), doi: 10.1093/imrn/rnx120. `arXiv:1608.06728`. Reproduced with permission of the Oxford University Press.

[**Paper V**] ______, *On the characterization of Triebel–Lizorkin type space of analytic functions*, preprint (2016). `arXiv:1609.09229`.

It is my intention to reproduce all papers in their most recently published form, as of the 8th of May, 2017, with reservations regarding corrected typos, created typos, and editorial tweaks.

My contribution to [**Paper I**] is the sections 4 and 5.

Table of contents

Preface

Preface

The publications reproduced in this thesis tell stories of Hankel operators, model theory, Carleson embeddings, and spaces of vector-valued analytic functions. But there is also a story which is perhaps not always visible in the publications. A story about mathematical control theory. In this introductory part, I will try to connect the different topics of the thesis in a somewhat informal way, by telling this hidden story. Rigor is saved for the actual papers.

The reader is warned: I will try to keep my notation consistent with my publications, but I make no promises.

Let $\mathcal{T} = (\mathcal{T}(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded linear transformations on a separable Hilbert space X. We call $\mathcal{T}$ a C_0-*semigroup*. Its generator A is a closed operator whose domain $\mathcal{D}(A)$ is dense in X. We equip $\mathcal{D}(A)$ with the natural graph norm, i.e. $\|x\|_{\mathcal{D}(A)} = \|x\|_X + \|Ax\|_X$. Within the scope of this thesis, the significance of C_0-semigroups is that for any $x_0 \in \mathcal{D}(A)$, the function $x : [0,\infty) \ni t \mapsto T(t)x_0 \in \mathcal{D}(A)$ is the unique solution in $C^1([0,\infty), X) \cap C^0([0,\infty), \mathcal{D}(A))$ to the abstract initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0.$$

Let Y be another separable Hilbert space and $C : \mathcal{D}(A) \to Y$ be a bounded linear operator. Consider the *control system*[1]

$$\dot{x} = Ax, \quad y = Cx, \quad x(0) = x_0 \in \mathcal{D}(A). \tag{1}$$

We call C an *observation operator*, y an *output*, and the map $\Psi_0 : x_0 \mapsto y$ an *output map*. Given a semigroup $\mathcal{T}$ (or a generator A), we say that an observation operator C is *admissible*[2] if there exists $M > 0$ such that

$$\forall\, x_0 \in \mathcal{D}(A) \quad \int_0^\infty \|CT(t)x_0\|_Y^2 \, dt \leq M^2 \|x_0\|_X^2. \tag{2}$$

[1]This control system is of a particularly simple kind. See for example [19] for a more extensive discussion of the subject.

[2]In the literature, such operators might be called *infinite time admissible* [21], or L^2-*admissible and stable* [19]. For our discussion, "admissible" is sufficiently precise.

Admissibility can be interpreted as a type of stability property, where the norm of the output is controlled by the norm of the initial state of the system. But this property implies of course the much more striking feature that the output map has a unique bounded extension to an operator $\Psi : X \to L^2(\mathbb{R}_+, Y)$, i.e. *any initial state of the system gives rise to a well-defined output of certain regularity*. For this reason, admissibility is an important notion when discussing well-posedness of control systems.

We assume for a moment that $\mathcal{T}$ is a semigroup of contractions. If λ lies in the right half-plane $\mathbb{C}_+$, then $C(\lambda - A)^{-1}x_0 = \hat{y}(\lambda)$, where $\hat{y}$ denotes the Laplace transform of y. By the Paley–Wiener theorem, e.g. [18], the Laplace transform maps $L^2(\mathbb{R}_+, Y)$ isometrically (up to a multiplicative constant) onto the Hardy space $H^2(\mathbb{C}_+, Y)$. It thus follows by basic boundary properties of Hardy space functions, that admissibility of C implies the resolvent growth condition

$$\sup_{\mathrm{Re}\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2} \, \|C(\lambda - A)^{-1}\|_{X \to Y} < \infty.$$

If $\mathcal{T}$ is a C_0-semigroup which is not contractive, then there still exists some $\omega > 0$ and some $K > 0$ such that

$$\|C(\lambda - A)^{-1}\|_{X \to Y} \leq \frac{K}{(\mathrm{Re}\lambda)^{1/2}} \quad \text{whenever} \quad \mathrm{Re}\lambda > \omega. \tag{3}$$

The *Weiss conjecture* states that this condition is also sufficient for C to be admissible [22]. While this conjecture has been disproved through several counter examples, [10, 12], there are still important special cases where it holds:

Proposition 1 ([9]). *If $Y = \mathbb{C}$, then the Weiss conjecture is true whenever $\mathcal{T}$ is contractive.*

We mention two particularly important special cases of Proposition 1: The statement was proved for normal semigroups in [22], and for the right-shift on $L^2(\mathbb{R}_+)$, henceforth denoted by $\mathcal{S}$, in [17]. These two cases together imply the general statement of Proposition 1, by use of the model theory of Sz.-Nagy and Foiaş, in particular [20, Chapter VI]. It has later been observed that if $\mathcal{T}$ is contractive and C satisfies (3), then the same is true if we replace $\mathcal{T}$ with the completely non-unitary regularization given by $T_\epsilon(t) = e^{-\epsilon t}T(t)$ (where $\epsilon > 0$). Being completely non-unitary, this semigroup admits a simpler model, and it follows from [17] alone that C is admissible with respect to $\mathcal{T}_\epsilon$, with M in (2) independent of ϵ. Proposition 1 now follows by letting $\epsilon \to 0$.

Proposition 1 is equivalent to Fefferman's theorem on $H^1 - BMOA$ duality (in one variable). In the special case of normal semigroups, it is equivalent to the Carleson embedding theorem. Thus, Proposition 1 is a formidable statement. Even so, the theory of admissible operators appears to be far from nearing completion, as is realized when one considers general output spaces:

Proposition 2 ([8]). *Let Y be a separable Hilbert space. If $\mathcal{T}$ is normal and analytic, then* (3) *implies* (2).

Any normal and analytic semigroup is also contractive, by the spectral mapping theorem together with normality. Whether or not Proposition 2 remains true under the weaker hypothesis "normal and contractive" appears to be an open problem at this time. On the other hand, the hypothesis of normality can be relaxed:

Proposition 3 ([14]). *Let Y be a separable Hilbert space. If $\mathcal{T}$ is analytic and contractive, then* (3) *implies* (2).

Proposition 3 fails if we omit the hypothesis of contractivity, [12]. However, there are examples of non-contractive analytic semigroups for which the Weiss conjecture holds, [14, Theorem 5.2].

For the semigroup $\mathcal{S} \otimes id_Y$ acting on $L^2(\mathbb{R}_+, Y)$, the Weiss conjecture fails:

Proposition 4 ([10]). *Let Y be a separable Hilbert space. If $\mathcal{T} = \mathcal{S} \otimes id_Y$ is the right shift semigroup acting on $L^2(\mathbb{R}_+, Y)$, then there exists an observation operator C for which* (3) *is satisfied, while* (2) *is not.*

The present thesis originates from a generalization of admissibility, first considered in [6]: Given $\alpha \geq 0$, and a contractive semigroup $\mathcal{T}$, we say that an observation operator C is *α-admissible* if there exists $M > 0$ such that

$$\forall\, x_0 \in \mathcal{D}(A) \quad \int_0^\infty \|CT(t)x_0\|_Y^2 t^\alpha \, dt \leq M^2 \|x_0\|_X^2. \tag{4}$$

This condition implies that (1) is well-posed in a slightly different sense: Every initial state $x_0 \in X$ gives rise to a well-defined output y, but instead of $L^2(\mathbb{R}_+, Y)$, we now have $L^2(\mathbb{R}_+, Y, t^\alpha \, dt)$-regularity.

Up to a constant, $C(\lambda - A)^{-1-\alpha} x_0 = (t^\alpha y)\hat{}(\lambda)$. This is a consequence of the functional calculus for sectorial operators, e.g. [7]. If C is α-admissible, then the corresponding output satisfies $t^\alpha y \in L^2(\mathbb{R}_+, t^{-\alpha} \, dt)$. It follows from the

Paley–Wiener theorem for Bergman spaces, e.g. [5], that α-admissibility implies the condition

$$\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{(1+\alpha)/2} \|C(\lambda - A)^{-1-\alpha}\|_{X \to Y} < \infty. \tag{5}$$

The *weighted Weiss conjecture* is the statement that (5) implies (4).

The weighted Weiss conjecture was disproved by Wynn [23]. It fails for $\mathcal{S}$, even in the scalar case. My own contribution to [**Paper I**] is a positive result on characterization of admissibility with respect to $\mathcal{S}$ in terms of a resolvent condition. Instead of modifying the power of the resolvent and the rate of growth in (3), one should modify the space upon which the resolvent operator acts:

Theorem 5 ([**Paper I**, Corollary 5.9]). *Let A denote the infinitesimal generator of $\mathcal{S}$, and let $\alpha \geq 0$. Then $C \in \mathcal{D}(A)^*$ is α-admissible with respect to $\mathcal{S}$ if and only if*

$$\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2} \left\|C(\bar{\lambda} I - A)^{-1}\right\|_{L^2(\mathbb{R}_+, t^\alpha\, dt)^*} < \infty.$$

Remark 0.1. The above result is of course not original for the case $\alpha = 0$. Nevertheless, we include it in the statement, in order to simplify our discussion.

This seems like a natural point to introduce the function theoretic interpretation of admissibility with respect to $\mathcal{S}$. The space $\mathcal{D}(A)^*$ can be identified with the class of analytic functions $c : \mathbb{C}_+ \to \mathbb{C}$ such that $\frac{c(z)}{1+z} \in H^2(\mathbb{C}_+)$, via the relation

$$Cx_0 = \int_{-\infty}^{\infty} \hat{x}_0(iy)\overline{c(iy)}\, dy.$$

This is essentially the Riesz representation theorem for Hilbert spaces. For c as above, we define the Hankel operator Γ_c on a dense subset of $H^2(\mathbb{C}_+)$ by $\Gamma_c f = P_+(c\tilde{f})$, where P_+ is the orthogonal projection from $L^2(i\mathbb{R})$ onto $H^2(\mathbb{C}_+)$, and $\tilde{f}(z) = f(\bar{z})$. It is fairly simple to show that $C \in \mathcal{D}(A)^*$ is admissible with respect to $\mathcal{S}$ if and only if Γ_c extends to a bounded linear operator on $H^2(\mathbb{C}_+)$.

Given $\lambda \in \mathbb{C}_+$, the $H^2(\mathbb{C}_+)$ reproducing kernel at λ is the function

$$k_\lambda(z) = \frac{1}{\bar{\lambda} + z}.$$

A well known result by Bonsall [2], is that Γ_c extends to a bounded linear operator on $H^2(\mathbb{C}_+)$ if and only if it is bounded on reproducing kernels, i.e.

$$\sup_{\lambda \in \mathbb{C}_+} \frac{\|\Gamma_c k_\lambda\|_{H^2(\mathbb{C}_+)}}{\|k_\lambda\|_{H^2(\mathbb{C}_+)}} < \infty. \tag{6}$$

We say that Hankel operators on $H^2(\mathbb{C}_+)$ satisfy a *reproducing kernel thesis*, RKT for short. The beautiful thing is now that (6) is equivalent to (3). Moreover, this is also quite easy to prove. This proves Theorem 5 for $\alpha = 0$, or equivalently, Proposition 1 for $\mathcal{T} = \mathcal{S}$.

For $\alpha > 0$, we need to introduce one more operator. Let $f : \mathbb{C}_+ \to \mathbb{C}_+$ be an analytic function, and define the corresponding fractional derivative by $D^\alpha f = (t^\alpha \check{f})\hat{}$, where $\check{f}$ denotes the inverse Laplace transform of f. Now $C \in \mathcal{D}(A)^*$ is 2α-admissible with respect to $\mathcal{S}$ if and only if $D^\alpha \Gamma_c$ extends to a bounded linear operator on $H^2(\mathbb{C}_+)$. The resolvent condition (5) is equivalent to the operator $\Gamma_c D^\alpha$ being bounded on reproducing kernels. It is well-known that for any fixed c, the operators $D^\alpha \Gamma_c$ and $\Gamma_c D^\alpha$ are simultaneously bounded, and that this happens if and only if $D^\alpha c \in BMOA(\mathbb{C}_+)$, a result obtained by Janson and Peetre [13]. However, the operator $\Gamma_c D^\alpha$ is bounded on reproducing kernels if and only if $D^\alpha c$ belongs to a Besov-type space, which is much bigger than $BMOA(\mathbb{C}_+)$. This is the essence of the Wynn counter example [23]. The technical result behind Theorem 5 is that the operator $D^\alpha \Gamma_c$ is bounded on reproducing kernels if and only if $D^\alpha c \in BMOA(\mathbb{C}_+)$. In conclusion, operators of the type $D^\alpha \Gamma_c$ satisfy an RKT, while operators of the type $\Gamma_c D^\alpha$ do not.

It is of course natural to ask oneself if there is perhaps some other class of semigroups for which the weighted Weiss conjecture holds. The answer is indeed yes. Another theorem by Wynn states that if $\mathcal{T}$ is normal and $C \in \mathcal{D}(A)^*$ satisfies (5), then C is α-admissible[3] [24]. Recently, this result was significantly improved to the case where Y is a general Hilbert space and the cogenerator $T = (I + A)(I - A)^{-1}$ is β-hypercontractive for some $\beta > 1$, [11]. This was achieved using results in [**Paper I**], combined with a functional model for hypercontractive operators [1]. The objective of [**Paper II**] was to better understand semigroups satisfying this hypothesis. The outcome was a characterization in terms of the elements of the semigroups. The flaw of [**Paper II**] is that I still believe there is a more simple characterization, but this is conditional to some properties of certain special functions. I conjecture that this simpler result is true, and I hope to see it resolved by myself or by somebody else.

Another natural question to ask is if Theorem 5 can be used to understand general semigroups of contractions by use of model theory. This seems to be a question which requires the development of a lot of new theory. With the case

[3] Technically speaking, this was done under the assumption that $\alpha \in (0, 1)$. We also point out that [24] uses a different, yet equivalent, resolvent condition.

$\alpha = 0$ in mind [9], a natural first step could be to understand boundedness properties of operators $D^\alpha \Gamma_{c\#} : f \mapsto D^\alpha P_+(\langle \tilde{f}, \tilde{c} \rangle_{\mathcal{H}})$, or equivalently, their adjoints $\Gamma_c D^\alpha : f \mapsto P_+(c\widetilde{D^\alpha f})$. Here $\mathcal{H}$ is an auxiliary separable Hilbert space and c is an $\mathcal{H}$-valued analytic function. The operators are defined on dense subsets of $H^2(\mathbb{C}_+, \mathcal{H})$ and $H^2(\mathbb{C}_+)$ respectively. In the case where $\alpha = 0$, the $\mathcal{H}$-valuedness makes little difference, quite opposite to the case where $\alpha > 0$.

The endeavor to understand boundedness of $D^\alpha \Gamma_{c\#}$ resulted in [**Paper III**, **Paper IV**, **Paper V**]. These were originally intended to become a single paper, but eventually it seemed unreasonably long. Since most of the literature I needed for these papers consider analytic functions on a disc rather than a half plane, so did I.

In [**Paper III**], boundedness of operators $D^\alpha \Gamma_{c\#}$ is characterized in terms of a certain Carleson embedding condition. In the process, it was realized that the techniques used are also applicable to operators $D^\alpha \Gamma_\phi$, where ϕ is a general operator-valued analytic function. The main result also implies results on the operators Γ_ϕ. These are notoriously difficult to understand. Perhaps the highlight of this thesis is the affirmative resolution of the conjecture by Nazarov, Treil, and Volberg [16], that there exists an operator-valued measure for which the corresponding analytic Carleson embedding is bounded while the anti-analytic ditto is not.

In [**Paper III**], the space $BMOA_{\mathcal{C}}(\mathcal{L})$ is defined in order to have the property that $D^\alpha \Gamma_\phi$ is $H^2(\mathcal{H})$-bounded if and only if $D^\alpha \phi \in BMOA_{\mathcal{C}}(\mathcal{L})$. The space $BMOA_{\mathcal{C}\#}(\mathcal{L})$ is defined by $H^2(\mathcal{H})$-boundedness of $D^\alpha \Gamma_{\phi\#}$. It then follows from a result by Davidson and Paulsen [4], that $BMOA_{\mathcal{C}}(\mathcal{L}) \subsetneq BMOA_{\mathcal{C}\#}(\mathcal{L})$.

Now, return to the less general case of $\mathcal{H}$-valued symbols. It is not quite clear what a reproducing kernel thesis for $D^\alpha \Gamma_{c\#}$ would even mean. More specifically, what would be the "kernels" on which we test our operators? A simple guess would be to use functions of the type $k_\lambda \otimes x$, $\lambda \in \mathbb{D}$, $x \in \mathcal{H}$. The corresponding condition would read

$$\sup_{\lambda \in \mathbb{D},\, x \in \mathcal{H} \setminus \{0\}} \frac{\| D^\alpha \Gamma_{c\#}(k_\lambda \otimes x) \|_{H^2(\mathbb{D})}}{\|x\|_{\mathcal{H}} \|k_\lambda\|_{H^2(\mathbb{D})}} < \infty.$$

It follows from the case $\mathcal{H} = \mathbb{C}$, that the above condition is satisfied if and only if $D^\alpha c \in BMOA_{\mathcal{W}}(\mathcal{H})$, the class of weak $BMOA$.

It is easy to see that $BMOA_{\mathcal{C}\#}(\mathcal{H}) \subset BMOA_{\mathcal{W}}(\mathcal{H})$, but is the inclusion strict? As it turns out, the answer is positive if and only if a particular type of counter example exists to the infinite dimensional Carleson embedding theorem. The existence of a counter example has been established through abstract means, by Nazarov, Pisier, Treil, and Volberg [15], but it is not clear that this example has the required properties. In [**Paper IV**], an older, dyadic, counter example due to Nazarov, Treil, and Volberg [16] is adapted to the complex analytic setting. While the example is explicitly constructed, I was not able to completely avoid abstract methods in order to obtain the particular form which is needed. Nevertheless, $BMOA_{\mathcal{C}\#}(\mathcal{H}) \subsetneq BMOA_{\mathcal{W}}(\mathcal{H})$.

A different approach to understanding the operators $D^\alpha\Gamma_c$ and $D^\alpha\Gamma_{c\#}$ is taken in [**Paper V**]. By simple duality considerations, these are bounded if and only if $c \in (H^2(\mathbb{D})\hat{\otimes}D^\alpha H^2(\mathbb{D},\mathcal{H}))^*$ and $D^\alpha c \in (H^2(\mathbb{D},\mathcal{H})\hat{\otimes}D^\alpha H^2(\mathbb{D}))^*$ respectively. A remarkable result, claimed by Cohn and Verbitsky [3], is that $D^\alpha H^1(\mathbb{D}) = H^2(\mathbb{D}) \cdot D^\alpha H^2(\mathbb{D})$. This implies that when $\mathcal{H} = \mathbb{C}$, $D^\alpha\Gamma_c$ are $D^\alpha\Gamma_{c\#}$ simultaneously bounded, and that this happens precisely when $D^\alpha c \in BMOA = (D^{-\alpha}(H^2(\mathbb{D}) \cdot D^\alpha H^2(\mathbb{D})))^*$, thus reproducing the Janson–Peetre result [13]. For $\mathcal{H}$-valued functions we obtain that

$$D^\alpha H^1(\mathbb{D},\mathcal{H}) = H^2(\mathbb{D}) \cdot D^\alpha H^2(\mathbb{D},\mathcal{H}),$$

and

$$H^2(\mathbb{D},\mathcal{H}) \cdot D^\alpha H^2(\mathbb{D}) \subsetneq D^\alpha H^1(\mathbb{D},\mathcal{H}),$$

where the second result depends on the results in [**Paper III**]. If one desires only to understand boundedness properties of the operator $D^\alpha\Gamma_{c\#}$, then this result seems rather weak in comparison to the results in [**Paper III**]. However, [**Paper V**] has several noteworthy features. The factorization seems interesting in itself, and even in the case where $\mathcal{H} = \mathbb{C}$, I find it quite surprising. But if one defines the Hardy spaces in the correct way, then the result actually holds not only when the functions take values in a Hilbert space, but rather in an arbitrary Banach space. Another point of the paper is that in the original article by Cohn and Verbitsky, one important step in the proof was left to the reader, perhaps a little too easily. The major part of [**Paper V**] is devoted to clarifying this step.

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