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Stress intensity factors for a crack in front of an inclusion

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Abstract

A stable numerical algorithm is presented for an elastostatic problem involving a crack close to and in front of an inclusion interface. The algorithm is adaptive and based on an integral equation of Fredholm's second kind. This enables accurate analysis also of rather difficult situations. Comparison with stress intensity factors of cracks close to straight infinite bimaterial interfaces are made.

Key words: Bimaterial, crack, interface, stress intensity factor, integral equation of Fredholm type

1 Introduction

The analysis of stress states close to straight cracks in front of straight infinite bimaterial interfaces has received considerable attention in the fracture mechanics community during the 1990s. The numerical approaches vary: Ahmad [1], Lim and Kim [2], Lim and Lee [3], Meguid, Tan and Zhu [4] prefer the finite element method. The reported accuracy in their solutions seems to be on the order of 0.5 per cent. Chen [5], who relies on a singular integral equation, gets a somewhat poorer accuracy. Wang and Ståhle [6], on the other hand, stabilize a singular integral equation through a Chebychev transform and achieve a substantial improvement in accuracy for geometries where the crack is reasonably well separated from the interface. Unfortunately, their algorithm is not adaptive. The number of expansion terms needed for a given relative accuracy increases rapidly as the crack approaches the interface. Further, both the algorithms of Chen [5] and of Wang and Ståhle [6] are based on Greens functions especially tailored for a bimaterial with a straight interface. It is not obvious how their algorithms can be modified to treat more general geometries.

Surprisingly, we have not been able to find any work in the literature analyzing cracks close to, or terminating at, the interface of an inclusion. Is this a less interesting case? or is it perhaps considered too difficult? As we shall see in this paper, stress intensity factors for a crack close to a straight infinite interface and for a crack close to the interface of an inclusion can be very different even though the material parameters are the same.

Singular integral equations naturally occur when crack problems are modeled using potential theory. Algorithms based on such equations are often unstable. This is so since singular integral operators, in these contexts, only are invertible under certain conditions. While an approximate numerical solution can be found on a coarse mesh, the quality of the solution is likely to decrease as the mesh is refined. Stabilization is needed. Preferably also adaptivity. In this paper we shall achieve both these goals.

One way to stabilize a singular integral equation is to transform it into an integral equation of Fredholm's second kind via an analytic right or left inverse. In previous work on cracks and inclusions [7], and on interface cracks [8], we constructed analytic inverses for parts of singular integral operators that describe self-interaction. This is efficient for cracks and inclusions that are well separated. As cracks and inclusions approach each other, however, otherwise smooth parts of leading operators that describe crack-inclusion interaction become ill-behaved and the scheme encounters difficulties. This paper differs from reference [7] in that an analytic right inverse is found for the entire leading integral operator, not just for its singular part. While the construction of this larger analytical inverse is more expensive and involved, it pays off in the sense that much more extreme cases can be studied.

The paper is organized as follows: A standard singular integral equation (3-4) for an unknown density Ω is derived from potential theory in Section 2. The leading singular operator is denoted K. The analytic right inverse to K, denoted K^*, is constructed in Section 3 along with a transformed density $\Omega_{\rm II}$. Substitution of $\Omega_{\rm II}$ into the original singular integral equation gives the Fredholm equation (20). Section 3 ends with some technicalities concerning the numerical evaluation of integral operators. Formulas for extracting stress intensity factors from the density Ω are given in Section 4. Section 5 discusses the implementation of the algorithm. The paper ends with numerical examples in Section 6.

2 A standard singular integral equation

Consider now a material consisting of an infinite elastic medium with two dimensional bulk and shear moduli κ_1 and μ_1 placed in a cartesian coordinate system. An inclusion with elastic moduli κ_2 and μ_2 is centered at the origin. The interface of the inclusion is bonded and denoted $\Gamma_{\rm bo}$. Close to the inclusion there is a crack denoted $\Gamma_{\rm cr}$. The starting point of $\Gamma_{\rm cr}$ is $\gamma_{\rm s}$ and the endpoint of $\Gamma_{\rm cr}$ is $\gamma_{\rm e}$. The union of $\Gamma_{\rm bo}$ and $\Gamma_{\rm cr}$ is Γ. The stress at infinity is $\sigma^\infty = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$. We would like to compute stress fields and stress intensity factors in the material subject to three different imposed stresses at infinity. These are $\sigma^\infty_{xx} = (1, 0, 0)$, $\sigma^\infty_{yy} = (0, 1, 0)$, and $\sigma^\infty_{xy} = (0, 0, 1)$.

We will start out with a representation of the Airy stress function based on the uppercase potentials Φ and Ψ [9] in the form

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)} + \frac{\alpha}{2}, \tag{1}$$

and

$$\Psi(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\Omega(\tau)\rho(\tau)}\mathrm{d}\bar\tau}{(\tau - z)} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar\tau\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)^2} + \beta\,, \tag{2}$$

where $\Omega(z)$ is an unknown density on Γ and $\rho(z)$ is a weight function given by equation (10) below. Once Φ is assumed to take the form (1), the expression (2) for Ψ enforces continuity of traction across Γ. The constants α and β in (1) and (2) represent the forcing terms in our formulation. For imposed stresses σ_{xx}^{∞}, σ_{yy}^{∞}, and σ_{xy}^{∞} the constants take the values $\alpha = 1/2$ and $\beta = -1/2$, $\alpha = 1/2$ and $\beta = 1/2$, and $\alpha = 0$ and $\beta = i$.

The requirements of continuity of displacement along $\Gamma_{\rm bo}$ and zero traction along $\Gamma_{\rm cr}$ lead to a singular integral equation

$$(K + CM_3)\,\Omega(z) = -\left(B\alpha + C\frac{\bar{n}}{n}\bar{\beta}\right), \qquad z \in \Gamma\,, \tag{3}$$

accompanied with the closure condition

$$Q\Omega = 0\,. \tag{4}$$

Here K is a singular integral operator given by

$$K\Omega(z) = A\rho\Omega(z) + \frac{B}{\pi i}\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)}, \qquad z \in \Gamma\,, \tag{5}$$

M_3 is a compact integral operator,

$$M_3\Omega(z) = \frac{1}{2\pi i}\left[\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)}\right.$$

$$\left. + \frac{\bar{n}}{n}\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\bar{\tau} - \bar{z})} + \int_\Gamma \frac{\overline{\Omega(\tau)\rho(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})} + \frac{\bar{n}}{n}\int_\Gamma \frac{(\tau - z)\overline{\Omega(\tau)\rho(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})^2}\right], \qquad z \in \Gamma\,, \tag{6}$$

where n is the normal unit vector, and Q is an operator from Γ into the complex numbers given by

$$Q\Omega = \frac{1}{\pi i}\int_\Gamma \Omega(\tau)\rho(\tau)\mathrm{d}\tau\,. \tag{7}$$

The functions $A(z)$, $B(z)$, and $C(z)$ are piece-wise constant and given by

$$\begin{aligned} A(z) &= 1, \quad z \in \Gamma_{\rm bo}, \quad \text{and} \quad A(z) = 0, \quad z \in \Gamma_{\rm cr},\\ B(z) &= d_1, \quad z \in \Gamma_{\rm bo}, \quad \text{and} \quad B(z) = 1, \quad z \in \Gamma_{\rm cr},\\ C(z) &= d_2, \quad z \in \Gamma_{\rm bo}, \quad \text{and} \quad C(z) = -1, \quad z \in \Gamma_{\rm cr}. \end{aligned} \tag{8}$$

The constants d_1 and d_2 are given by

$$d_1 = \left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right) \Big/ \left(\frac{1}{\mu_2} + \frac{1}{\kappa_2} + \frac{1}{\mu_1} + \frac{1}{\kappa_1}\right),$$

and

$$d_2 = \left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \Big/ \left(\frac{1}{\mu_2} + \frac{1}{\kappa_2} + \frac{1}{\mu_1} + \frac{1}{\kappa_1}\right). \tag{9}$$

The weight function ρ is given by

$$\rho(z) = (A - B)(z - \gamma_{\rm s})^{-0.5}(z - \gamma_{\rm e})^{-0.5}\,. \tag{10}$$

It is worth pointing out that our constants d_1 and d_2 are simply related to bimaterial parameters introduced by other investigators. For example, in terms of the parameters a, b, and c of equation (3.12) in Sherman [10] we have $d_1 = b/a$ and $d_2 = -c/a$. In terms of the parameters α and β in Dundurs [11] we have $d_1 = \beta$ and $d_2 = \alpha - \beta$.

3 A Fredholm integral equation

We now intend to rewrite the system (3) and (4) as one Fredholm integral equation of the second kind. For this we need a few new functions and operators. Let

$$A^*(z) = \frac{A(z)}{A^2(z) - B^2(z)}, \qquad \text{and} \qquad B^*(z) = \frac{B(z)}{A^2(z) - B^2(z)}. \tag{11}$$

Let K^* be an operator whose action on a function $f(z)$ is defined by

$$K^* f(z) = \frac{A^*(z) f(z)}{\rho(z)} - \frac{B^*(z)}{\pi i} \int_\Gamma \frac{f(\tau) \mathrm{d}\tau}{\rho(\tau)(\tau - z)}. \tag{12}$$

Let $P_{\rm cr}$ and $P_{\rm bo}$ be two projection operators which project onto $\Gamma_{\rm cr}$ and onto $\Gamma_{\rm bo}$, respectively. Let f be in L^2. The following relations are proved using techniques in Paragraphs 107 and 117 of Muskhelishvili [9] in Appendix I of Toya [12] and in Section 4 of Helsing and Peters [7].

$$\begin{aligned}
K^* P_{\rm cr} B &= B^*(z - 0.5\gamma_{\rm s} - 0.5\gamma_{\rm e}) - A^* d_1 (1 - d_1) \rho^{-1}\,, && (13)\\
K^* P_{\rm bo} B &= A^* d_1 (1 - d_1) \rho^{-1}\,, && (14)\\
Q B^* &= 1, && (15)\\
Q \circ K^* f &= 0, && (16)\\
K \circ K^* &= I, && (17)\\
K^* \circ K &= I - B^* \circ Q. && (18)
\end{aligned}$$

We now introduce the representation

$$\Omega = K^*(C \Omega_{\rm II} - B\alpha)\,. \tag{19}$$

The system given by (3) and (4) is equivalent with the following single integral equation of Fredholm's second kind

$$(I + M_3 \circ K^* C)\, \Omega_{\rm II}(z) = M_3 K^* B \alpha - \frac{\bar{n}}{n} \bar{\beta}, \qquad z \in \Gamma\,. \tag{20}$$

We need the following result for the numerical evaluation of the operators of this equation: Let f be a smooth function. Then

$$K^* B f(z) = f B^*(z - 0.5\gamma_{\rm s} - 0.5\gamma_{\rm e}) - \frac{B^*(z)}{\pi i} \int_\Gamma \frac{B(\tau)(f(\tau) - f(z)) \mathrm{d}\tau}{\rho(\tau)(\tau - z)}\,, \tag{21}$$

and

$$K^* C f(z) = K_{\rm I}^* C f(z) + f A^*(d_1 + d_2)(1 - d_1)\rho^{-1}\,, \tag{22}$$

where

$$K_{\rm I}^* C f(z) = -f B^*(z - 0.5\gamma_{\rm s} - 0.5\gamma_{\rm e}) - \frac{B^*(z)}{\pi i} \int_\Gamma \frac{C(\tau)(f(\tau) - f(z)) \mathrm{d}\tau}{\rho(\tau)(\tau - z)}\,. \tag{23}$$

4 Stress intensity factors

The density $\Omega(z)\rho(z)$ has a one over square root singularity at the tips of an open crack. The complex valued stress intensity factor $F_{\rm I} + iF_{\rm II}$ at the crack-tip $\gamma_{\rm s}$ can be defined as

$$F_{\rm I} + iF_{\rm II} = \frac{i\sqrt{2\pi}}{g} \lim_{z \to \gamma_{\rm s}} \overline{\Omega(z)\rho(z)} \sqrt{\delta s(z)}\,, \tag{24}$$

and at the crack-tip $\gamma_{\rm e}$

$$F_{\rm I} + iF_{\rm II} = -\frac{i\sqrt{2\pi}}{g} \lim_{z \to \gamma_{\rm e}} \overline{\Omega(z)\rho(z)} \sqrt{\delta s(z)}\,. \tag{25}$$

Here $\delta s(z)$ is arclength measured from the closest crack-tip and g is a normalization factor which varies with different authors.

5 A stable algorithm

We have implemented the main equation (20) as a 1500 line FORTRAN program using a Nyström scheme with adaptive composite quadrature as described in Helsing [13]. With a Nyström scheme we mean that the density $\Omega_{\rm II}$ is represented with its values at quadrature points and that the integral equation should be satisfied at those same quadrature points. On quadrature panels that do not contain crack-tips we use 16-point Gauss-Legendre quadrature. On quadrature panels that do contain crack-tips we use a 16-point quadrature rule based on interpolation with weighted polynomials. The linear system of equations is solved with the GMRES iterative solver [14] and the iterations are terminated when the residual is less than 10^{-14}. In the numerical examples below we start with 192 uniformly distributed discretization points, solve, refine adaptively, and solve again until the convergence of the stress intensity factors stops. This typically happens at the order of 1000 discretization points. We also performed series of massively overresolved calculations where we started with around 1000 uniformly distributed discretization points and repeated the adaptive process until convergence stopped, which then happened at the order of 2000 points. The converged values for stress intensity factors presented in the next section have been confirmed by regular as well as by massively overresolved calculations. The GMRES solver typically converges to the desired accuracy in 20 iterations.

6 Numerical examples

In a simple numerical example we let the inclusion at the origin be a disk with radius R. We let the crack be straight, have length $2a$ and be placed on the x-axis at a distance c away from the disk interface. In this way $\gamma_{\rm s}$ is at $x = R + c$ and $\gamma_{\rm e}$ is at $x = R + c + 2a$. A uniform stress σ_{yy}^{∞} applied at infinity. See Figure 1. The normalization factor g of equations (24-25) is chosen as $g = \sigma_{yy}^{\infty}\sqrt{\pi a}$. The elastic moduli are chosen so that $d_1 = -0.20775$ and $d_2 = -0.70309$. This corresponds to the moduli of the aluminum-epoxy bimaterial treated by Wang and Ståhle [6].

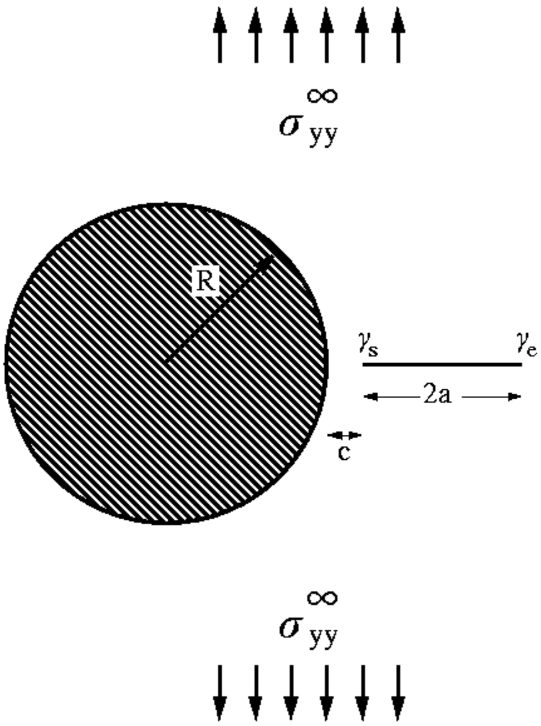

Figure 1: A straight crack of length $2a$ placed a distance c away from an elastic disk with radius R in an infinite elastic material. The crack starts at the tip $\gamma_{\rm s}$ and ends at the tip $\gamma_{\rm e}$. A uniform stress σ_{yy}^{∞} is applied at infinity.

Table 1: Stress intensity factor $F_{\rm I}$ and q_2 for a straight crack of length $2a$ separated a distance c from a circular inclusion of radius R centered at the origin. The crack lies along the x-axis. A uniform stress σ_{yy}^{∞} is applied at infinity. The elastic moduli, $d_1 = -0.20775$ and $d_2 = -0.70309$, correspond to an aluminum inclusion in an epoxy matrix.

a/R	c/a	$F_{\rm I}(\gamma_{\rm s})$	q_2	$F_{\rm I}(\gamma_{\rm e})$
1	0.01	0.26095394929	0.54998029733	0.8958760871255
1	0.001	0.1733317621	0.5303383567	0.8935785663583
1	0.0001	0.119456159	0.530609550	0.8933373890490
1	0.00001	0.08230243	0.53072653	0.8933127617873
1	0.000001	0.05669172	0.5307248	0.8933102749727
1	0.0000001	0.0390505	0.530723	0.893310025149
0.1	0.01	0.05153341931	0.10861060103	0.3621801535193
0.1	0.001	0.0351308779	0.1074889670	0.3602450541708
0.1	0.0001	0.024227319	0.107614770	0.3600498094337

An interesting parameter in this context is λ given by

$$\cos(\pi\lambda) - \frac{2d_2(1-\lambda)^2}{1-d_1} + \frac{d_1+d_2-d_1^2}{1-d_1^2} = 0\,. \tag{26}$$

The stress singularity ahead of the tip of a crack perpendicular to and terminating at an interface is of the order $r^{-\lambda}$, where r is the distance to the crack-tip [15]. In the present example we have $\lambda = 0.33810837120263$.

The stress intensity factor $F_{\rm I}$ at the crack tip $\gamma_{\rm s}$ will "approach zero due to the strong block from the stiff material aluminum" [6] as the separation distance c decreases. Wang and Ståhle [6] suggest a "fitting equation" for the crack close to a straight infinite bimaterial interface

$$F_{\rm I}(\gamma_{\rm s}) \approx q_1 \left(\frac{c}{a}\right)^{(0.5-\lambda)+0.5(0.5-\lambda)^2}\,, \tag{27}$$

where q_1 is a constant. According to Figure 2 in their paper this fitting equation seems reasonable in the interval $0.001 \le c/a \le 0.01$.

For our setup we propose a parameter q_2 defined by

$$q_2 = F_{\rm I}(\gamma_{\rm s}) \left(\frac{c}{a}\right)^{(\lambda-0.5)}\,. \tag{28}$$

The parameter q_2 approaches a constant as $c/a \to 0$. The difference between the magnitude of the exponents in equation (27) and equation (28) is about eight per cent for the aluminum-epoxy composite. Values of $F_{\rm I}(\gamma_{\rm s})$ and q_2 for various combinations of the ratios a/R and c/a are displayed in Table 1. As we can see, q_2 of equation (28) converges quite rapidly for $c/a < 0.01$. We also observe that the ratio a/R does influence the value of $F_{\rm I}(\gamma_{\rm s})$ and $F_{\rm I}(\gamma_{\rm e})$ considerably. When the crack is in front of and perpendicular to a straight infinite bimaterial interface there is just one geometric parameter, namely c/a.

In a second, more strongly inhomogeneous, example we choose the elastic moduli so that $d_1 = 0.22678$ and $d_2 = 0.75778$. This gives $\lambda = 0.92574923570657$ and corresponds to

Table 2: Stress intensity factor F_{I} and q_2 for a straight crack of length $2a$ separated a distance c from a circular inclusion of radius R centered at the origin. The crack lies along the x-axis. A uniform stress σ_{yy}^{∞} is applied at infinity. The elastic moduli, $d_1 = 0.22678$ and $d_2 = 0.75778$ correspond to an epoxy inclusion in a boron matrix.

a/R	c/a	$F_{\mathrm{I}}(\gamma_{\mathrm{s}})$	q_2	$F_{\mathrm{I}}(\gamma_{\mathrm{e}})$
1	0.01	4.13573205889	0.582175497540	1.318521763103
1	0.001	8.6120023268	0.45483767552	1.359174492211
1	0.0001	19.56613723	0.3877110263	1.37870481228
1	0.00001	47.0744264	0.349976480	1.3894226129
1	0.000001	117.30692	0.32721125	1.395853648
1	0.0000001	298.8340	0.3127414	1.39993675

the moduli of the epoxy-boron bimaterial also treated by Wang and Ståhle [6]. According to Figure 3 in their paper the fitting equation (27) seems reasonable in the interval $0.003 \leq c/a \leq 0.03$. Values of $F_{\mathrm{I}}(\gamma_{\mathrm{s}})$ and q_2 for various combinations of the ratios a/R and c/a are displayed in Table 2. The numbers illustrate, again, that equation (28) is asymptotically correct.

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