On the Interior Stress Problem for Elastic Bodies

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On the interior stress problem for elastic bodies

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Abstract

The classic Sherman-Lauricella integral equation and an integral equation due to Muskhelishvili for the interior stress problem are modified. The modified formulations differ from the classic ones in several respects: both modifications are based on uniqueness conditions with clear physical interpretations and, more importantly, they do not require the arbitrary placement of a point inside the computational domain. Furthermore, in the modified Muskhelishvili equation the unknown quantity, which is solved for, is simply related to the stress. In Muskhelishvili's original formulation the unknown quantity is related to the displacement. Numerical examples demonstrate the greater stability of the modified schemes.

1 Introduction

The task of computing the elastic field inside an unconstrained body subjected to external stress is a basic one in applied mechanics. A variety of numerical methods exist, leading to the solution of systems of linear equations. Finite element methods and integral equation methods are two examples. A problem, for any method, is a certain undeterminacy in the solution – when stress is applied, displacement is not unique.

There are standard ways to get a well-posed problem. In a finite element program one can prescribe also the displacement at some points (to prevent rigid body movements). In the context of integral equations, the integral operator can be completed with an extra operator, containing another arbitrary point, which makes the solution unique. The choice of particular representations of the unknown fields and placements of arbitrary points will, of course, affect the stability of a numerical code. With a direct solver and for simple problems this may not be an issue. The computational work only depends on the size of the system matrix. With the faster, iterative, solvers used by many engineers today and in difficult situations, the stability of the code and the condition number of the system matrix is suddenly important. The lower the condition number, the faster the solver will converge. A stable algorithm can give a solution with better quality.

This paper focuses on the Sherman-Lauricella integral equation and an integral equation due to Muskhelishvili for the interior stress problem in two-dimensional elastostatics. The classic way to get a unique solution for these equations is to complete them with an operator B containing an arbitrary point z^* (Sherman 1940). We show that the convergence

properties of iterative algorithms based on these equations can be sensitive to the placement of z^*. To remedy this situation, we introduce a uniqueness condition with a clear physical interpretation. This leads us to a new operator B which is free from the arbitrary point z^*. We then derive a modification of the Muskhelishvili equation which is better suited to compute stress fields. Numerical experiments indicate that our modified equations give more efficient algorithms.

2 Potential representation

A finite, linearly elastic, body occupies a domain D. Its two-dimensional elastic bulk and shear moduli are κ and μ. The boundary of the body is denoted Γ and is given positive (counter-clockwise) orientation. Traction is prescribed at Γ. We would like to compute the deformation and the stress field inside D.

Let U denote the Airy stress function. Since U satisfies the biharmonic equation inside D it can be represented as

$$U = \Re\mathrm{e}\left\{\bar{z}\phi + \chi\right\}, \tag{1}$$

where the potentials ϕ and χ are single valued analytic functions of the complex variable $z = x + iy$. For a thorough discussion of the complex variable approach to elasticity problems, see Muskhelishvili (1953), Sokolnikoff (1956), Mikhlin (1957), and Parton and Perlin (1982). For our purposes it is sufficient to observe a few relations that link the complex potentials to quantities of physical interest: The displacement (u_x, u_y) in the material satisfies

$$u_x + iu_y = \left(\frac{1}{2\mu} + \frac{1}{\kappa}\right)\phi - \frac{1}{2\mu}\left(z\overline{\phi'} + \overline{\psi}\right), \tag{2}$$

where $\psi = \chi'$. The integral of traction (t_x, t_y) along a curve $\gamma(s)$ can be obtained from the relation

$$\int_{s_0}^{s} (t_x + it_y)\mathrm{d}s = -\Big|_{s_0}^{s} i\left(\phi + z\overline{\phi'} + \overline{\psi}\right), \tag{3}$$

where s denotes arclength along $\gamma(s)$. Complex differentiation of the expression (2) along the tangent to $\Gamma(s)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}z}(u_x + iu_y) = \left(\frac{1}{2\mu} + \frac{1}{\kappa}\right)\Phi - \frac{1}{2\mu}\left(\overline{\Phi} - \frac{\bar{n}}{n}z\overline{\Phi'} - \frac{\bar{n}}{n}\overline{\Psi}\right), \tag{4}$$

and differentiation with respect to arclength in (3) gives

$$t_x + it_y = \Phi n + \overline{\Phi} n - z\overline{\Phi'}\bar{n} - \overline{\Psi}\bar{n}, \tag{5}$$

where $\Phi = \phi'$, $\Psi = \chi''$, and $n = n_x + in_y$ is the outward unit normal vector on Γ. The components of the stress tensor can be computed via

$$\sigma_{xx} + \sigma_{yy} = 4\Re\mathrm{e}\{\Phi\}, \tag{6}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2(\bar{z}\Phi' + \Psi), \tag{7}$$

A natural starting point for elastostatic problems is to represent the potentials ϕ and ψ, or Φ and Ψ, in the form of Cauchy-type integrals

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{(\tau - z)}\,, \qquad z \in D\,, \tag{8}$$

and

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\rho(\tau)\mathrm{d}\tau}{(\tau - z)}\,, \qquad z \in D\,, \tag{9}$$

or

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Omega(\tau)\mathrm{d}\tau}{(\tau - z)}\,, \qquad z \in D\,, \tag{10}$$

and

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Xi(\tau)\mathrm{d}\tau}{(\tau - z)}\,, \qquad z \in D\,, \tag{11}$$

where ω and ρ, or Ω and Ξ, are unknown layer densities on Γ. Values of the potentials ϕ, ψ, Φ, and Ψ on Γ are defined as limits of ϕ, ψ, Φ and Ψ in D as Γ is approached. Since the equations of elasticity now are satisfied everywhere, it remains only to solve the problem which consists of enforcing the boundary condition of prescribed traction $(t_x^{\mathrm{pr}}, t_y^{\mathrm{pr}})$ along Γ. This can be done in various ways, leading to various integral equations.

3 The classic Sherman-Lauricella integral equation

An classic choice for the interior stress problem is to choose the unknown layer density ρ of (9) in the following way

$$\rho(z) = \overline{\omega(z)} - \bar{z}\omega'(z)\,. \tag{12}$$

The choice (12) makes ψ of (9) assume the form

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\tau)\mathrm{d}\bar{\tau}}{(\tau - z)} + \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\tau)}\mathrm{d}\tau}{(\tau - z)} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\tau}\omega(\tau)\mathrm{d}\tau}{(\tau - z)^2}\,. \tag{13}$$

The requirement of precribed traction on Γ leads, via (3), to the Lauricella integral equation for ω

$$(I + M_{\mathrm{SL}})\,\omega(z) = g(z)\,, \qquad z \in \Gamma\,, \tag{14}$$

accompanied with the solvability conditions that $g(z)$ must be single valued and

$$Q_1 g = 0\,. \tag{15}$$

In (14-15) the notation $g(z)$ has been introduced for the integral of traction along Γ from a point $z(s_0)$ as

$$g(z) = i\int_{s_0}^{s(z)} t\mathrm{d}s\,, \qquad z \in \Gamma\,,$$

where $t = t_x^{\mathrm{pr}} + it_y^{\mathrm{pr}}$, and the operator Q_1 is a mapping from Γ to $\mathbb{R}$, defined by

$$Q_1 g = \frac{1}{S}\Re\mathrm{e}\left\{\int_\Gamma g(z)\mathrm{d}\bar{z}\right\}, \tag{16}$$

where S is the perimeter of the body, and $M_{\rm SL}$ is a compact integral operator given by

$$M_{\rm SL}\omega(\tau) = \frac{1}{2\pi i}\left[\int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{(\tau - z)} - \int_\Gamma \frac{\omega(\tau)\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})} - \int_\Gamma \frac{\overline{\omega(\tau)}\mathrm{d}\tau}{(\bar{\tau} - \bar{z})} + \int_\Gamma \frac{(\tau - z)\overline{\omega(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})^2}\right]. \qquad (17)$$

Consider now the integral operator B suggested by Sherman (1940) and defined by

$$B\omega(z) = \left(\frac{1}{(z - z^*)} - \frac{1}{(\bar{z} - \bar{z}^*)} + \frac{(z - z^*)}{(\bar{z} - \bar{z}^*)^2}\right)\frac{1}{\pi i}\Re\mathrm{e}\left\{\int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{(\tau - z^*)^2}\right\}, \qquad z \in \Gamma, \qquad (18)$$

where z^* is an arbitrary point in D. Parton and Perlin (1982) suggest a simpler operator B

$$B\omega(z) = \frac{1}{(\bar{z} - \bar{z}^*)}\frac{1}{\pi i}\Re\mathrm{e}\left\{\int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{(\tau - z^*)^2}\right\}, \qquad z \in \Gamma. \qquad (19)$$

Addition of the operator B to the left hand side of (14) gives the Sherman-Lauricella integral equation

$$(I + M_{\rm SL} + B)\,\omega(z) = g(z), \qquad z \in \Gamma. \qquad (20)$$

Uniqueness of the solution to (20), with the choice (18) for B, is proven in paragraph 56 of Mikhlin (1957). Uniqueness, with the choice (19), is proven in paragraph 19 of Parton and Perlin (1982).

Once equation (20) is solved for ω, various quantities of physical interest can be computed. The displacement on Γ, for example, can be obtained from

$$u_x + iu_y = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\kappa}\right)(I + M_1)\omega(z) - \frac{g(z)}{2\mu}, \qquad z \in \Gamma, \qquad (21)$$

where

$$M_1\omega(z) = \frac{1}{\pi i}\int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{(\tau - z)}, \qquad z \in \Gamma. \qquad (22)$$

From the viewpoint of numerical efficiency, the choice between (18) and (19) for B is perhaps not so important. Greenbaum, Greengard and Mayo (1992) use (20) with (18). Greengard, Kropinski, and Mayo (1996) and Strandberg (1999) use (20) with (19). No author comments on the relative merits of the two choices. The next section presents yet another choice B. As we shall see in the last section, this new "twist" can make a substantial difference.

4 A new operator B

Equation (14) with the solvability condition (15) does not have a unique solution. The operator on the left-hand side of (14) is rank-one deficient. On the unit disk, for example, a null-vector is $\omega_{\rm n} = iz$. We suggest the uniqueness condition

$$Q_1(I + M_1)\omega = 0. \qquad (23)$$

We see, from (16) and (21), that the condition (23) has a physical interpretation in terms of average tangential displacement on Γ. Two useful relations are

$$Q_1(I + M_{\rm SL})\omega = 0\,, \tag{24}$$

$$Q_1 in = 1\,. \tag{25}$$

We are now in the position to propose a new equivalent formulation for (14) and (23), assuming that (15) holds. The new formulation is based on the choice

$$B\omega(z) = \frac{in}{2} Q_1(I + M_1)\omega(z)\,, \qquad z \in \Gamma\,. \tag{26}$$

This choice for B differs from the choices (18) and (19) in two respects: it has a clear physical interpretation and, more importantly, it does not involve the arbitrary point z^*. The Sherman-Lauricella equation now reads

$$\left(I + M_{\rm SL} + \frac{in}{2} Q_1(I + M_1)\right)\omega(z) = g(z)\,, \qquad z \in \Gamma\,. \tag{27}$$

Equation (27) trivially follows from (14) and (23). To prove the converse, we apply Q_1 from the left in (27) and use the relations (24-25) and (15). This gives (23). Subtraction of (23) from (27) gives back (14).

Uniqueness of the solution to (27) can be proven using the same technique as in paragraph 56 of Mikhlin (1957) and observing the relations

$$\phi(z) = \frac{1}{2}(I + M_1)\omega(z)\,, \qquad z \in \Gamma\,, \tag{28}$$

and

$$Q_1 iz = 2A/S\,, \tag{29}$$

where A is the area of the body.

5 A modified Muskhelishvili equation

An interesting extension for the interior stress problem is to let it involve a problem exterior to D. The exterior problem is one where the prescribed traction on Γ is zero and the stress at infinity is zero. We shall seek Φ and Ψ such that the two problems are solved simultaneously. Clearly, Φ and Ψ are zero outside D. This follows from the uniqueness of the solution to the second fundamental exterior problem in the plane (Mikhlin 1957), and implies that Ω of (10) and Ξ of (11) are boundary values of analytic functions in D. Now we choose Ω to be the value of Φ on Γ, and choose Ξ in such a way that the traction is zero outside Γ and jumps a quantity t as Γ is crossed. The jump condition makes Ψ of (11) take the form

$$\Psi(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\Phi(\tau)}\mathrm{d}\bar\tau}{(\tau - z)} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar\tau\Phi(\tau)\mathrm{d}\tau}{(\tau - z)^2} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar n\bar t\mathrm{d}\tau}{(\tau - z)}\,, \qquad z \in D\,. \tag{30}$$

The requirement that the traction outside Γ is zero leads to the following integral equation for Φ on Γ

$$(I - M_3)\,\Phi(z) = \frac{\bar n t(z)}{2} + \frac{\bar n}{n}\frac{1}{2\pi i}\int_\Gamma \frac{nt\mathrm{d}\bar\tau}{(\bar\tau - \bar z)}\,, \qquad z \in \Gamma\,, \tag{31}$$

accompanied with the solvability condition

$$Q_2 \bar{n} t = 0\,, \tag{32}$$

where

$$Q_2 f = -\frac{1}{2A} \Re\mathrm{e} \left\{ \int_\Gamma f(z) \bar{z} \mathrm{d}z \right\}. \tag{33}$$

In (31) M_3 is an integral operator given by

$$M_3 \Phi(z) = \frac{1}{2\pi i} \left[\int_\Gamma \frac{\Phi(\tau)\mathrm{d}\tau}{(\tau - z)} + \frac{\bar{n}}{n} \int_\Gamma \frac{\Phi(\tau)\mathrm{d}\tau}{(\bar{\tau} - \bar{z})} + \int_\Gamma \frac{\overline{\Phi(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})} + \frac{\bar{n}}{n} \int_\Gamma \frac{(\tau - z)\overline{\Phi(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})^2} \right]. \tag{34}$$

Equation (31) can be viewed as the derivative of the conjugate of equation (7) in paragraph 54 of Mikhlin (1957). That equation was originally derived by Muskhelishvili.

A quantity of physical interest, which can be computed once (31) is solved for Φ, is the complex tangential derivative of the displacement on Γ

$$\frac{\mathrm{d}}{\mathrm{d}z}(u_x + i u_y) = \left(\frac{1}{\kappa} + \frac{1}{\mu} \right) \Phi(z) - \frac{\bar{n} t}{2\mu}\,, \qquad z \in \Gamma\,. \tag{35}$$

The following Lemma will be useful for proving equation (39) below.

Lemma 5.1

$$Q_2 i = 1\,, \tag{36}$$

$$Q_2 (I - M_3)\Phi = 0\,. \tag{37}$$

Proof: Equation (36) is proven by applying Gauss' theorem. Equation (37) is proven by expressing $(I - M_3)\Phi$ explicitly in terms of analytic potentials and then applying Cauchy's theorem. □

Equation (31) with the solvability condition (32) does not have a unique solution. The operator on the left hand side of (31) is rank-one deficient. Imaginary constants are null-solutions. For any solution Φ we can form a new solution as $\Phi + i\alpha$, where α is a real constant. See paragraph 54 of Mikhlin (1957) for a similar result for the null-space of an operator derived for the potential ϕ. Here we propose the uniqueness condition

$$Q_2 \Phi = 0\,. \tag{38}$$

We see, from (33) and (35), that the condition (38) has the same physical interpretation as the condition (23).

We are now in the position to propose a new formulation for (31) and (38), assuming that (32) holds.

Theorem 1 *Given the solvability condition (32), equation (31) and the uniqueness condition (38) are equivalent to the the following Fredholm equation of the second kind*

$$\left(I - M_3 + i Q_2\right) \Phi(z) = \frac{\bar{n} t(z)}{2} + \frac{\bar{n}}{n} \frac{1}{2\pi i} \int_\Gamma \frac{n t \mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})}\,, \qquad z \in \Gamma\,. \tag{39}$$

Proof: Equation (39) trivially follows from (31) and (38). To prove the converse, we apply Q_2 from the left in (39) and use the relations (36-37) and (32). This gives (38). Subtraction of (38) from (39) gives back (31). □

Uniqueness of the solution to (39) can be proven using the method of paragraph 54 in Mikhlin (1957). First one proves that an assumed homogeneous solution, Φ_0, to (39) has to be an imaginary constant. Then (38), which is implied by (39), gives that this constant is zero.

6 Numerical comparison between formulations

In this section we undertake a comparison between algorithms for the classic Sherman-Lauricella equation (20) with the choice (19) for B, for the modified formulation (27), and for the modified Muskhelishvili equation (39). The algorithms are of Nyström type based on composite 16-point Gaussian quadrature and the GMRES iterative solver (Saad and Schultz 1986). The iterations are terminated when the residual is as small as it can get, which typically means $2 \cdot 10^{-15}$. Compensated summation (Kahan 1965; Higham 1996) is used for the computation of matrix-vector multiplications and inner products in the GMRES iterative solver. For details on how to regularize the Cauchy-type singular operator M_1 of (22), see Helsing and Jonsson (1999).

For setups with smooth boundaries and analytical solutions, such as loaded circular or elliptic disks, it is hard to say which equation leads to the best algorithm. Algorithms based on the three equations all require only a few GMRES iterations for full convergence. Non-trivial examples are needed in order to detect differences in performance.

When comparing the performance of the algorithms below, we need a reference quantity q_{ref} to measure accuracy against. We have decided to use the L^2 norm of the hydrostatic stress on Γ, that is

$$q_{\text{ref}} = \left(\int_\Gamma \left(\sigma_{xx}(z) + \sigma_{yy}(z) \right)^2 \mathrm{d}s \right)^{\frac{1}{2}} , \tag{40}$$

as such a reference quantity.

Example 1: A symmetric starfish. We first consider a body in the shape of a nine-armed starfish parameterized by

$$z(t) = (1 + 0.36 \cos 9t) e^{it} , \qquad 0 \leq t < 2\pi . \tag{41}$$

The load is chosen as

$$g(z) = z^2 . \tag{42}$$

We start out with testing the algorithm for the classic equation (20) with B as in (19). The starfish of (41) is symmetric with respect to the origin. A natural choice for the arbitrary point is therefore $z^* = 0$. With this choice for z^* the algorithm for the classic equation (20) requires 2000 discretization points to reach a relative error in q_{ref} of 10^{-12}. Upon increased resolution the quality of the solution slowly gets worse. See Figure 1. The number of GMRES iterations required is 25. The sensitivity to the placement of the arbitrary point z^* turns out to be quite large in this example. When the position of $z^* = 0$ is changed a tiny distance to $z^* = 0.01i$, which still is far away from the boundary of the starfish contour, the number of GMRES iterations needed for convergence more than doubles, see Figure 2.

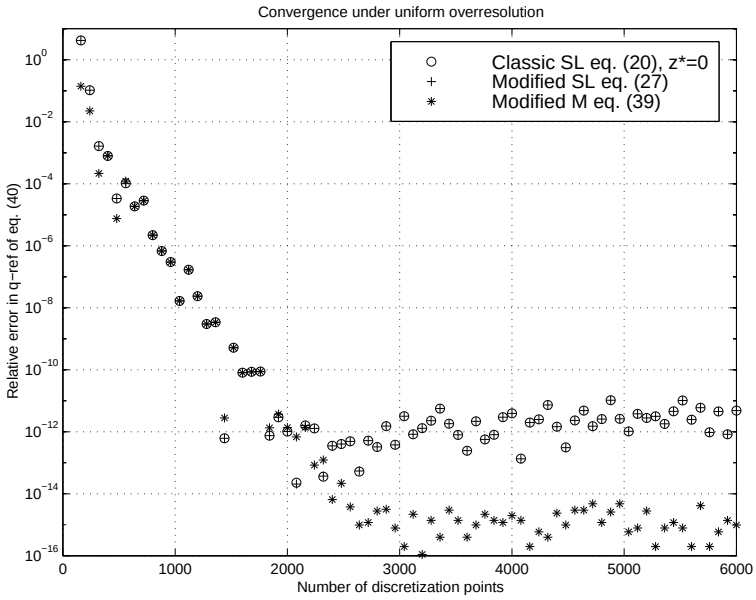

Figure 1: Example 1. Convergence of the reference quantity q_{ref} of (40), defined as the Euclidean norm of the hydrostatic stress on the boundary, for algorithms based on the classic equation (20) with $z^* = 0$, the modified formulation (27), and the modified equation (39). The correct value, $q_{\mathrm{ref}} = 71.79088302407723$ was computed using quadruple precision arithmetic.

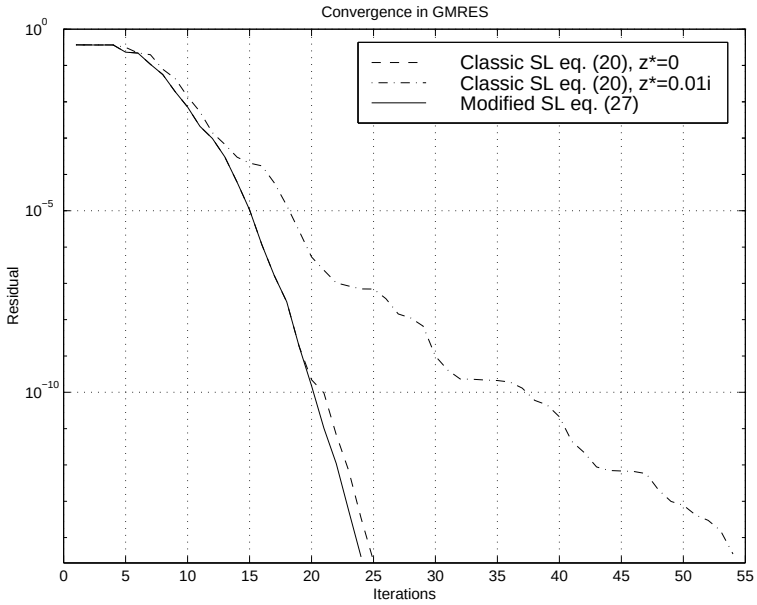

Figure 2: Example 1. The iteration history from GMRES for the the classic equation (20) and the modified formulation (27). The number of discretization points is 2080. The iterations are terminated when the residual is less than $5 \cdot 10^{-15}$.

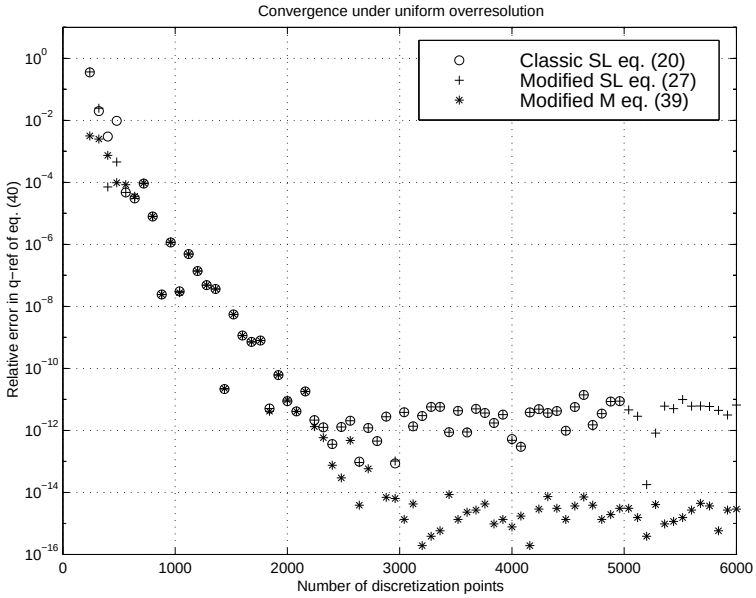

Figure 3: Example 2. Convergence of the reference quantity $q_{\rm ref}$ of (40) for algorithms based on the classic equation (20) with z^* placed at the center of gravity, the modified formulation (27), and the modified equation (39). The correct value, $q_{\rm ref} = 73.451300874308??$ was computed using quadruple precision arithmetic.

The algorithm for the modified formulation (27) gives results which are almost identical to those of (20) with the optimal choice $z^* = 0$. See Figure 1. The algorithm for the modified formulation requires also about 2000 discretization points for a relative error in result $q_{\rm ref}$ of about 10^{-12}. Solving the system of linear equations takes 24 GMRES iterations, which is one iteration less than for (20) with the optimal choice $z^* = 0$. See Figure 2.

The algorithm for the modified equation (39) has, by far, the best stability properties in this example. Like the previous formulations it gives a relative error in $q_{\rm ref}$ of about 10^{-12} for 2000 discretization points, but as the resolution is increased the relative error in $q_{\rm ref}$ decreases further and stabilizes on about $2 \cdot 10^{-15}$. Equation (39) also gives the best results for underresolved calculations. See Figure 1.

Example 2: An irregular starfish. The starfish of (41) has an obvious symmetry point $z = 0$ which, as we have seen, is the optimal choice for the arbitrary point z^*. To investigate the properties of the three algorithms under more general conditions we perturb the geometry in the previous example so that all arms of the starfish have different shapes

$$z(t) = (1 + 0.1 \sin t + 0.36 \cos 9t)e^{it}\,, \qquad 0 \le t < 2\pi\,. \tag{43}$$

The load is changed to

$$g(z) = z^2 - \frac{izSQ_1z^2}{2A}\,, \tag{44}$$

so that the solvability conditions (15) and (32) still are satisfied.

The convergence properties for the three algorithms in this example turn out to be similar to those in Example 1. See Figure 3. The main difference is that GMRES now

requires 105 iterations for full convergence instead of the 25 iterations in Example 1. An increase in the number of iterations could be expected since the irregular starfish of (43) constitutes a more difficult geometry than the symmetric starfish of (41).

The modified equation (39) still gives the most stable convergence and still performs best for underresolved calculations. The algorithm for the the classic equation (20), with reasonable placements of the arbitrary point z^*, still gives results which are similar to, or only slightly worse than, those of the modified formulation (27). See Figure 3. One difference is worth pointing out: The number of GMRES iterations needed for convergence with the classic equation (20), and for fully resolved calculations, is less sensitive to the placement of z^* in Example 2 than in Example 1. There seems to be no interior point that is obviously optimal.

7 Conclusions and outlook

We conclude that an algorithm based on the modified formulation (27) has shown to be equally or more efficient for interior stress problems than an algorithm based on the classic Sherman-Lauricella equation (20). The chief advantage with (27) over (20) is that (27) omits the need for the arbitrary point z^*. A non-optimal choice for z^* can in certain situations greatly deteriorate the performance of iterative algorithms based on the classic equation (20). In addition, removal of the arbitrary point z^* is an advantage from a coding viewpoint.

We, further, conclude that an algorithm based on the modified Muskhelishvili equation (39) shows superior stability properties compared to both the classic (20) and the modified (27) Sherman-Lauricella equations and that (39) is best for underresolved calculations. This is so, since in (39) we solve for the stress potential Φ on Γ directly, while in (27) we solve for the density ω. The density ω is related to Φ on Γ via a transform containing differentiation. Differentiation is in itself an ill-conditioned operation.

In a forthcoming paper we intend to apply (39) to the problem of computing so called notch intensity factors of loaded rectangular specimens.

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