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Forest fires on $\mathbb{Z}_+$ with ignition only at 0

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Abstract. We consider a version of the forest fire model on graph G, where each vertex of a graph becomes occupied with rate one. A fixed vertex v_0 is hit by lightning with the same rate, and when this occurs, the whole cluster of occupied vertices containing v_0 is burnt out. We show that when $G = \mathbb{Z}_+$, the times between consecutive burnouts at vertex n, divided by $\log n$, converge weakly as $n \to \infty$ to a random variable which distribution is $1 - \rho(x)$ where $\rho(x)$ is the Dickman function.

We also show that on transitive graphs with a non-trivial site percolation threshold and one infinite cluster at most, the distributions of the time till the first burnout of *any* vertex have exponential tails.

Finally, we give an elementary proof of an interesting limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \binom{n}{k} (-1)^k \log k - \log\log n = \gamma.$$

1. Introduction and results

Consider the following forest fire model on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Let $\eta_x(t) \in \{0, 1\}$ be the state of site $x \in \mathbb{Z}_+$ at time $t \geq 0$, and we say that site x is *vacant* if $\eta_x = 0$ and *occupied*, if $\eta_x = 1$. The vacant sites become occupied with rate 1; once they are occupied, they can only be "burnt" by a fire spread from a neighbour, which reverses them to the original vacant state. Imagine that there is a constant source of fire attached to site 0. Hence, whenever site 0 becomes occupied, the whole connected cluster of occupied sites containing 0 is instantaneously burnt out. Denote the process we obtain as $\{\eta_x(t)\}$. We are interested in the dynamics of process $\{\eta_x(t)\}$, as time passes by, under the assumption that all sites are initially vacant, i.e. $\eta_x(0) = 0$ for all x.

Note that our model differs from more classical versions presented in van den Berg and Járai (2005) and van den Berg and Brouwer (2006), where *each* occupied site can be ignited at rate λ, and then the cluster containing this site disappears.

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On the other hand, our model on $\mathbb{Z}_+$ turns out to be a special case of the one studied in van den Berg and Tóth (2001), where some of the results, independently obtained in the present paper, are also given. This covers, for example, the recursion (1.2), and also most of Lemma 1.2, but none of the limiting statements derived in Theorems 1.9 and 2.1. Forest fire models have also been recently considered on Erdős-Rényi random graphs, see Ráth and Tóth (2009).

Let $T_x(i)$, $i = 1, 2, \dots$, be the consecutive times when site x is burnt for the i-th time, and let $T_x(0) = 0$. Let $\tau_n(i) = T_n(i) - T_n(i-1)$ for $i \geq 1$. We can easily show that for a fixed n, $\tau_n(i)$'s are i.i.d. random variables; this can be done by induction on n. Indeed, the times of burnouts at $(n+1)$ depend only on T_n's and the Poisson arrival process at site $(n+1)$ itself. Since for each j necessarily $T_{n+1}(j) = T_n(i)$ for some i, $T_{n+1}(j)$'s are renewal times, and hence $\tau_{n+1}(j)$'s are i.i.d. as well.

Now we would like to find the distribution of $\tau_{n+1}(i)$'s. For site 0 this is trivial as the burn-out times constitute a Poisson process, so that

$$\mathbb{P}(\tau_0 > u) = e^{-u},\ u \geq 0.$$

Reasonably easy one can also obtain

$$\mathbb{P}(\tau_1 > u) = (u+1)e^{-u},$$

so that τ_1 has $\Gamma(2,1)$ distribution with density ue^{-u}; similarly

$$\mathbb{P}(\tau_2 > u) = \frac{(2u^2 + 10u + 7)e^{-u} + e^{-3u}}{8}.$$

From the above calculations we conclude that

$$\begin{array}{rclrclrcl} \mathbb{E}\,(\tau_0) & = & 1, & \mathbb{E}\,(\tau_1) & = & 2, & \mathbb{E}\,(\tau_2) & = & 8/3; \\ \mathbb{V}\mathsf{ar}\,(\tau_0) & = & 1, & \mathbb{V}\mathsf{ar}\,(\tau_1) & = & 2, & \mathbb{V}\mathsf{ar}\,(\tau_2) & = & 8/3. \end{array}$$

Incidentally, this suggests that $\mathbb{V}\mathsf{ar}\,(\tau_n) = \mathbb{E}\,\tau_n$, which, however, turns out to be incorrect, as follows from Remark 1.7.

For a general n, let $\varphi_n(t) = \mathbb{E}\,e^{t\tau_n}$ be the moment generating function of random variable τ_n. Suppose that sites n and $n+1$ have just been burnt, and without loss of generality reset the time to $t = 0$. Let $\eta \sim \exp(1)$ be the time till the next Poisson arrival at site $(n+1)$. The next burnout at site $(n+1)$ will be either at time $t = \tau_n$ if $\eta \leq \tau_n$, or at a later time otherwise; in the latter case due to the memoryless property of the Poisson process the time between τ_n and the next burnout at $(n+1)$, denoted by $\tilde{\tau}_{n+1}$, will have the same distribution as τ_{n+1} itself. Therefore, given τ_n,

$$\tau_{n+1} = \tau_n + \left\{ \begin{array}{ll} 0, & \text{if } \eta \leq \tau_n; \\ \tilde{\tau}_{n+1}, & \text{if } \eta > \tau_n. \end{array} \right.$$

Consequently,

$$\begin{array}{rcl} \mathbb{E}\,\left(e^{t\tau_{n+1}} \,|\, \tau_n\right) & = & e^{t\tau_n}\left[(1-e^{-\tau_n}) \cdot 1 + e^{-\tau_n} \cdot \mathbb{E}\,\left(e^{t\tilde{\tau}_{n+1}} \,|\, \tau_n\right)\right] \\ & = & e^{t\tau_n} - e^{(t-1)\tau_n} + e^{(t-1)\tau_n} \cdot \mathbb{E}\,\left(e^{t\tilde{\tau}_{n+1}}\right) \end{array}$$

using the fact that $\tilde{\tau}_{n+1}$ is independent of τ_n. Taking the expectation on both sides, we obtain

$$\varphi_{n+1}(t) = \varphi_n(t) - \varphi_n(t-1) + \varphi_n(t-1)\varphi_{n+1}(t)$$

whence

$$\varphi_{n+1}(t) = \frac{\varphi_n(t) - \varphi_n(t-1)}{1 - \varphi_n(t-1)}. \tag{1.1}$$

Let $u_n(t) = \varphi_n(t) - 1$. Then

$$u_{n+1}(t) = -\frac{u_n(t)}{u_n(t-1)}. \tag{1.2}$$

Since $\varphi_0(t) = \int_0^\infty e^{tx} e^{-x}\, dx = 1/(1-t)$ yielding $u_0(t) = t/(1-t)$, we can easily iteratively compute $u_n(t)$. For example,

$$u_1(t) = \frac{t(2-t)}{(1-t)^2},\ u_2(t) = \frac{t(2-t)^3}{(1-t)^3(3-t)}$$

which is consistent with our previous calculations of the distributions of τ_1 and τ_2.

Lemma 1.1. *For $n = 1, 2, \ldots$*

$$u_{n-1}(t) = \frac{t \cdot (2-t)^{\binom{n}{2}} \cdot (4-t)^{\binom{n}{4}} \cdot (6-t)^{\binom{n}{6}} \ldots}{(1-t)^{\binom{n}{1}} \cdot (3-t)^{\binom{n}{3}} \cdot (5-t)^{\binom{n}{5}} \ldots}$$

with the convention that $\binom{n}{k} = 0$ whenever $k > n$. Thus τ_n is a mixture of Gamma random variables with the moment generating function

$$\varphi_{n-1}(t) \quad = \quad 1 + t \prod_{k=1}^{n} (k-t)^{(-1)^k \binom{n}{k}}$$

defined for all $t < 1$.

Proof. By induction, using (1.2) and the fact that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. ■

Lemma 1.2. *Let $\mu_n = \mathbb{E}\,\tau_n$. Then for $n = 1, 2, \ldots$*

$$\log \mu_{n-1} \quad = \quad \sum_{i=1}^{n} \binom{n}{i} (-1)^i \log i;$$

$$\mathbb{E}\left(\tau_{n-1}^2\right) \quad = \quad 2\mu_{n-1} \sum_{i=1}^{n} \binom{n}{i} \frac{(-1)^i}{i}.$$

Moreover,

$$\lim_{n\to\infty} \log\left(\frac{\mu_n}{\log n}\right) = \gamma$$

where $\gamma = 0.577\ldots$ is the Euler constant.

The following two lemmas will be proved in Section 3.

Lemma 1.3. *Let*

$$A_{n-1} \quad = \quad \sum_{i=1}^{n} \binom{n}{i} (-1)^i \log i$$

then $\lim_{n\to\infty}(A_n - \log\log n) = \gamma$*, where $\gamma = 0.577\ldots$ is the Euler constant.*

Remark 1.4. After this paper has been written, we learned (Bálint Tóth, personal communications) that the above limit is in fact derived in Flajolet and Sedgewick (1995), Theorem 4, though no explicit proof was given there. Thus we shall give a reasonably short and elementary self-contained proof of this convergence.

Lemma 1.5. *Let*

$$a(n,m)=\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k^m}.$$

Then

$$(a)\quad a(n,m)=\sum_{1\le i_1\le i_2\le\cdots\le i_m\le n}\frac{1}{i_1 i_2\dots i_m};$$

$$(b)\quad a(n,m)\le(\log n+1)^m \text{ for all } n\ge 1;$$

$$(c)\quad a(n,m)=\frac{\log^m n}{m!}+O(\log^{m-1} n) \text{ for a fixed } m \text{ as } n\to\infty.$$

Remark 1.6. The quantities $a(n,m)$ are closely related to the Stirling numbers of the second kind:

$$\left\{\begin{array}{c} n\\ m\end{array}\right\}=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m,$$

and up to a coefficient of proportionality coincide with “negative-positive” Stirling numbers in Branson (2006), see equations (68) and (78) there.

Proof of Lemma 1.2. The first part follows immediately from Lemma 1.1 and the properties of moment-generating functions; the second part follows from Lemma 1.3. ■

Remark 1.7. Lemma 1.2 together with part (c) of Lemma 1.5 yield that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left(\frac{\tau_n}{\log n}\right) &= e^{\gamma}\approx 1.78\dots\\ \lim_{n\to\infty}\mathbb{E}\left(\frac{\tau_n}{\log n}\right)^2 &= 2e^{\gamma}\approx 3.56\dots\end{aligned}$$

whence for large n, $\mathbb{V}\mathsf{ar}\,(\tau_n)\ne\mathbb{E}\,\tau_n$.

Theorem 1.8. *Let $\xi_n=\tau_n/\log n$. Then as $n\to\infty$*

$$\xi_n\xrightarrow{\mathcal{D}}\xi$$

(meaning convergence in distribution) where ξ is a random variable with mean $\mathbb{E}\,\xi=\gamma'$, and the moment generating function

$$\varphi_\xi(s)\quad\equiv\quad\mathbb{E}\,e^{s\xi}=1+\exp\{\mathrm{Ei}(s)\}\,. \tag{1.3}$$

Here $\gamma'=e^{\gamma}=1.781...$ and

$$\mathrm{Ei}(s)=\int_{-\infty}^{s}\frac{e^x}{x}\,\mathrm{d}x=\gamma+\log s+\sum_{m=1}^{\infty}\frac{s^m}{m\cdot m!}$$

is the exponential integral (understood in terms of the Cauchy principal value; see Abramowitz and Stegun (1965), Section 5.1 and formula 5.1.10).

Proof. Observe that

$$\begin{aligned}\log\left[\frac{u_{n-1}(t)}{t\mu_{n-1}}\right] &= \sum_{k=1}^{n}\binom{n}{k}(-1)^k\log\left(1-\frac{t}{k}\right)=\sum_{m=1}^{\infty}\frac{t^m}{m}\left[\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^m}\right]\\ &= \sum_{m=1}^{\infty}\frac{a(n,m)\,t^m}{m}\end{aligned}$$

where

$$a(n,m)=\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k^m}.$$

For the moment generating function of ξ_{n-1} we have

$$\log(\mathbb{E}\,e^{s\xi_{n-1}}-1)=\log u_{n-1}(s/\log n) = \log s+\log\frac{\mu_{n-1}}{\log n}+\sum_{m=1}^{\infty}\frac{a(n,m)}{\log^m n}\frac{s^m}{m},$$

consequently for any $N\geq 1$, using part (b) of Lemma 1.5,

$$\begin{aligned}\Delta_{n-1}(s) &:= \left|\log(\mathbb{E}\,e^{s\xi_{n-1}}-1)-\mathrm{Ei}(s)\right|\\ &= \left|\log(\mathbb{E}\,e^{s\xi_{n-1}}-1)-\gamma-\log s-\sum_{m=1}^{\infty}\frac{s^m}{m\cdot m!}\right|\\ &\leq \left|\log\frac{\mu_{n-1}}{\log n}-\gamma\right|+\sum_{m=1}^{N}\left|\frac{a(n,m)}{\log^m n}-\frac{1}{m!}\right|\frac{s^m}{m}\\ &\quad+\sum_{m=N+1}^{\infty}\frac{s^m}{m\cdot m!}+\sum_{m=N+1}^{\infty}\left(1+\frac{1}{\log n}\right)^m\frac{s^m}{m}.\end{aligned}$$

Fix an $\varepsilon>0$. Assuming $|s|\leq 1/2$, we can choose N so large that the last two summands are smaller than $\varepsilon/2$ each. Now for a *fixed* N by Lemma 1.2 and part (c) of Lemma 1.5 the first two terms of the RHS of $\Delta_{n-1}(s)$ go to 0 as $n\to\infty$. Consequently, $\limsup_{n\to\infty}\Delta_n(s)\leq\varepsilon$. Since ε is arbitrary, we conclude that for $|s|\leq 1/2$

$$\lim_{n\to\infty}\mathbb{E}\,e^{s\xi_{n-1}}=1+\exp\{\mathrm{Ei}(s)\}\,.$$

By Theorem 3 in Curtiss (1942), if the sequence of moment-generating functions corresponding to random variables ξ_n converges point-wise to a limit function $\varphi_\xi(s)$ on *some* interval around 0, then there is a random variable ξ such that $\xi_n\to\xi$ in distribution and $\varphi_\xi(s)$ is its moment generating function. This finishes the proof. ∎

Theorem 1.9. *Random variable ξ defined in Theorem 1.8 has the density function $f(x)$ and the survival function $\rho(x)=\mathbb{P}(\xi>x)$ satisfying*

$$\begin{aligned} f(x) &= 0, & x\leq 1;\\ \tfrac{\mathrm{d}}{\mathrm{d}x}(xf(x)) &= -f(x-1), & x>1,\end{aligned}\tag{1.4}$$

and

$$\begin{aligned} \rho(x) &= 1, & x\leq 1;\\ x\rho'(x) &= -\rho(x-1), & x>1,\end{aligned}\tag{1.5}$$

so that $\rho(x)$ is the Dickman function.

Proof. Let us denote by $\psi(t) = \mathbb{E}\, e^{it\xi} = \varphi_\xi(it)$, then we have

$$t\psi'(t) = \psi(t)e^{it} - e^{it}.$$

Using *formally* the inversion formula and the fact that for a random variable $Y \equiv 1$, $\mathbb{E}\, e^{itY} = e^{it}$, we have

$$\begin{aligned} \frac{1}{2\pi}\int (t\psi(t))' e^{-itx}\,\mathrm{d}t &= \frac{1}{2\pi}\int [\psi(t)e^{it} + \psi(t)]e^{-itx}\,\mathrm{d}t - \delta_{x-1} \\ &= \frac{1}{2\pi}\int \psi(t)e^{-it(x-1)}\,\mathrm{d}t + \frac{1}{2\pi}\int \psi(t)e^{-itx}\,\mathrm{d}t - \delta_{x-1} \end{aligned}$$

(where δ_x denotes the Dirac delta-function.) Using integration by parts on the left (again, formally) we have

$$\begin{aligned} ix\frac{1}{2\pi}\int t\psi(t)e^{-itx}\,\mathrm{d}t &= -x\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2\pi}\int \psi(t)e^{-itx}\,\mathrm{d}t\right] = -x\frac{\mathrm{d}}{\mathrm{d}x}f(x) \\ &= f(x-1) + f(x) - \delta_{x-1} \end{aligned}$$

yielding $(xf(x))' = \delta_{x-1} - f(x-1)$. Integrating this equality from $-\infty$ to x, and denoting $F(x) = \mathbb{P}(\xi \le x)$, we obtain

$$xF'(x) = 1_{x\ge 1} - F(x-1)$$

implying $\rho(x) = 1 - F(x)$ satisfies $x\rho'(x) = -\rho(x-1)$ for $x \ge 1$ as required.

To prove the above results rigorously, first observe that the Dickman function $\rho(u)$ has the following properties: (1) it is positive and decreasing on the $[1,\infty)$; (2) it is infinitely differentiable on $[0,\infty]$ except at integer points; (3) $\rho(u) \le 1/\Gamma(u+1)$ for $u \ge 1$ (see e.g. Tenenbaum, 1995 for its properties). Consequently,

$$F(u) := \begin{cases} 0, & u < 1; \\ 1 - \rho(u), & u \ge 1 \end{cases}$$

is the cumulative distribution function of some continuous random variable ζ which density is supported on $[1,\infty)$. Multiplying the second equation in (1.5) by te^{tx} and integrating, we obtain

$$\int_1^\infty txF'(x)e^{tx}\,\mathrm{d}x = \int_1^\infty (1 - F(x-1))te^{tx}\,\mathrm{d}x \tag{1.6}$$

Integrating by parts the RHS of (1.6), we have

$$\begin{aligned} &t\lim_{x\to\infty}\rho(x-1)e^{tx} - [1 - F(0)]e^t + \int_1^\infty e^{tx}F'(x-1)\,\mathrm{d}x \\ &= 0 - e^t + e^t\int_0^\infty e^{tx}F'(y)\,\mathrm{d}y = e^t(\varphi_\zeta(t) - 1) \end{aligned}$$

where $\varphi_\zeta(t) = \mathbb{E}\, e^{t\zeta}$ is the moment generating function of ζ. On the other hand, the LHS of (1.6) equals

$$t\frac{\mathrm{d}}{\mathrm{d}t}\int_1^\infty F'(x)e^{tx}\,\mathrm{d}x = t\varphi_\zeta'(t).$$

This yields $t\varphi_\zeta'(t) = e^t(\varphi_\zeta(t) - 1)$ and $\varphi_\zeta(0) = 1$, a general solution to which has a form

$$\varphi_\zeta(t) = 1 + C_1 s \exp\left(\sum_{m=1}^{\infty} \frac{t^m}{m \cdot m!}\right)$$

for some constant C_1. To identify C_1, we will use the fact that $\varphi_\zeta(-z) = \mathbb{E}\, e^{-z\zeta} \to 0$ as $z \to \infty$ (as $\zeta \geq 0$). Using Taylor expansion for e^t we obtain

$$\varphi_\zeta(-z) = 1 - C_1 z \exp\left(-\int_0^z \frac{1 - e^{-t}}{t}\, \mathrm{d}t\right).$$

Now, formulas 5.1.1 and 5.1.39 in Abramowitz and Stegun (1965) for the function $E_1(z)$ give

$$\begin{aligned} E_1(z) &= \int_z^\infty \frac{e^{-t}}{t}\, \mathrm{d}t, \\ \int_0^z \frac{1 - e^{-t}}{t}\, \mathrm{d}t &= E_1(z) + \log z + \gamma \end{aligned} \tag{1.7}$$

yielding

$$\varphi_\zeta(-z) = 1 - C_1 \exp\left(-\gamma - \int_z^\infty \frac{e^{-t}}{t}\, \mathrm{d}t\right).$$

Since the integral goes to 0 as $z \to \infty$, we conclude that $C_1 = e^\gamma$. Thus φ_ζ coincides with the expression given by (1.3) and by the uniqueness theorem, ξ must have the same distribution as ζ, from which the Theorem follows. ■

Here are a few observations about the distribution of ξ. Trivially we have $F(x) = 0$ for $x \leq 0$; thus using (1.5) for $0 \leq x \leq 1$ we have $F'(x) = 0$ whence

$$F(x) = 0, 0 \leq x \leq 1$$

as well. Consequently, for $1 \leq x \leq 2$, we have $xF'(x) = 1$ so that

$$F(x) = \log x, 1 \leq x \leq 2.$$

Therefore, by induction we can obtain piece-wise smooth density function of ξ:

$$f(x) = \begin{cases} 0, & x \leq 1; \\ 1/x, & 1 < x \leq 2; \\ \frac{1 - \log(x-1)}{x}, & 2 < x \leq 3; \\ \ldots \end{cases}$$

Unfortunately, there is no explicit formula in elementary functions for $f(x)$ on an interval $[n, n+1]$ for $n \geq 2$.

Our next statement deals with residual waiting times for the renewal process generated by consecutive burnouts at site n.

Proposition 1.10. *Let $\eta_{t,n}$ be the time till the next burnout at site n after time $t > 0$. Then $\eta_{t,n}/\log n \xrightarrow{\mathcal{D}} \bar\eta_n$ as $t \to \infty$ and $\bar\eta_n \xrightarrow{\mathcal{D}} \bar\eta$ as $n \to \infty$, where $\bar\eta$ has a generalized Dickman distribution GD(1), see Penrose and Wade (2004), i.e. the same distribution as $U_1 + U_1U_2 + U_1U_2U_3 + \ldots$ with U_i being i.i.d. uniform $[0,1]$ random variables.*

Proof. As we already know, the times between consecutive burnouts $\tau_n^{(i)}$, $i = 1, 2, \dots$ are i.i.d. and have a common distribution of τ_n. Let $\xi_n^{(i)} = \tau_n^{(i)} / \log n$, and let $F_n(\cdot)$ be the common cumulative distribution function of $\xi_n^{(i)}$, which is the same as for the random variable ξ_n defined in Theorem 1.8. As it is well-known, see e.g. Durrett (1996), Chapter 3.4, the residual waiting times for the renewal process generated by $\xi_n^{(i)}$ converge in distribution to a non-negative random variable $\bar\eta_n$ such that

$$\mathbb{P}(\bar\eta_n \le x) = \frac{1}{\mathbb{E}\,\xi_n} \int_0^x (1 - F_n(u))\,\mathrm{d}u, \quad \text{for all } x \ge 0.$$

(We need to verify that the distribution F_n is non-arithmetic, however this easily follows from the fact that τ_n is a continuous random variable, which is a mixture of Gamma distributions, as implied by Lemma 1.1.)

Let F be the cumulative distribution function of ξ, as defined in the proof of Theorem 1.9, and $\bar\eta$ be a non-negative random variable such that

$$\mathbb{P}(\bar\eta \le x) = \frac{1}{\mathbb{E}\,\xi} \int_0^x (1 - F(u))\,\mathrm{d}u = e^{-\gamma} \int_0^x \rho(u)\,\mathrm{d}u, \quad \text{for all } x \ge 0. \tag{1.8}$$

Then, since $\int_0^x (1 - F_n(u))\,\mathrm{d}u \le \int_0^\infty (1 - F_n(u))\,\mathrm{d}u = \mathbb{E}\,\xi_n$,

$$\begin{aligned} |\mathbb{P}(\bar\eta_n \le x) - \mathbb{P}(\bar\eta \le x)| &\le \left| \frac{1}{\mathbb{E}\,\xi} \int_0^x (F_n(u) - F(u))\,\mathrm{d}u \right| \\ &\quad + \left| \frac{1}{\mathbb{E}\,\xi} - \frac{1}{\mathbb{E}\,\xi_n} \right| \int_0^x (1 - F_n(u))\,\mathrm{d}u \\ &\le e^{-\gamma} \int_0^x |F_n(u) - F(u)|\,\mathrm{d}u + \left| \frac{\mathbb{E}\,\xi_n}{\mathbb{E}\,\xi} - 1 \right| \to 0 \end{aligned}$$

where the first summand tends to 0 by the dominated convergence theorem since $F_n(x) \to F(x)$ pointwise by Theorems 1.8 and 1.9, and the second one vanishes because of Lemma 1.2. Therefore $\bar\eta_n \xrightarrow{\mathcal{D}} \bar\eta$.

We finish the proof by noting that the distribution in (1.8) coincides with the distribution of $\sum_{i=1}^\infty \prod_{j=1}^i U_j$, see Chamayou (1973). ■

We conclude by noting that similar distributions (called *Dickman-type* distributions) show up in some other probabilistic models, including e.g. minimal directed spanning trees as well as number-theory related problems, see Penrose and Wade (2004) and references therein. Another interesting application is in economics, related to plot-size distributions: see Exner and Šeba (2008), formula (4), which is identical to that for $\bar\eta$.

2. Generalizations

One can consider a similar forest fire model on an arbitrary connected locally-finite graph G with the vertex set $V(G)$ and one special vertex $v_0 \in V(G)$ which is called *the origin*. Let $\eta_x(t) \in \{0, 1\}$ be the state of site $x \in V(G)$ at time $t \ge 0$; again the site x is vacant (occupied resp.) if $\eta_x = 0$ ($\eta_x = 1$ resp.). Vacant sites become occupied at rate 1; they remain occupied until they are burnt out, which makes them vacant again. For definiteness, at time 0 all sites are vacant. As before, only site v_0 is constantly hit by lightning, hence whenever it becomes occupied all the sites in the cluster of occupied sites containing v_0 are instantaneously burnt out.

Unfortunately, this model turns out to be not so interesting, provided that the critical percolation threshold p_c for site percolation on G is strictly smaller than 1, which is true on many graphs. Recall that if $\theta_{v_0}(p) = \theta(p)$ denotes the probability that site v_0 belongs to an infinite cluster of occupied sites given that *each* site is independently occupied with probability p, then the critical percolation threshold is defined by

$$p_c = \sup\{p : \ \theta(p) = 0\}$$

(see for example Grimmett, 1999).

We claim that if $p_c < 1$, then in our forest fire model infinitely many sites can be burnt in a finite time. Indeed, fix a $p \in (p_c, 1)$, and let

$$S = S(p) = -\log(1 - p). \tag{2.1}$$

Then with probability at least $\frac{1-p}{p}\,\theta(p) > 0$ site v_0 becomes occupied in time exceeding S (at which point it is immediately burnt), and by that time there will be already an infinite cluster attached to v_0, so that it will burn some arbitrarily far away vertices.

As it is well-known, on many graphs ($\mathbb{Z}^d$, $d \geq 2$, regular trees, some others) the number of infinite occupied clusters can be either 0, 1, or ∞ (see Newman and Schulman, 1981); also it is known that on $\mathbb{Z}^d$, $d \geq 2$, and some infinite Cayley graphs (but not a regular tree) the infinite cluster, whenever present, must be unique; see Burton and Keane (1989), Häggström and Peres (1999), and also Chapter 8.9 in Grimmett (1999) and Theorem 4 in Chapter 5.1 in Bollobás and Riordan (2006). Additionally, suppose that the graph is *transitive*, that is to say that graph G viewed from any vertex $v \in V(G)$ is isomorphic to graph G viewed from v_0; this in turn would imply using the FKG inequality for the connectivity function (Grimmett, 1999, Chapter 8.5) that the probability that an arbitrary chosen vertex v is burnt out in time S exceeds $(1-\gamma)(1-p)$ where

$$\gamma := 1 - \frac{\theta(p)^2}{p} \in (0,1). \tag{2.2}$$

We can generalize this argument as follows.

Theorem 2.1. *Suppose that graph G is connected, transitive, the critical point for the site percolation $p_c = p_c(G) < 1$ and that there can be at most one infinite cluster on G. Fix an arbitrary $v \in V(G)$ and let η be the time till its first burnout in our forest fire model. Then for any $p \in (p_c, 1)$*

$$\mathbb{P}\left(\eta > x\right) \leq \gamma^{-1}\left[x(1-p)+1\right]e^{-\lambda x} \quad \text{for all } x > 0 \tag{2.3}$$

where γ is given by (2.2), and $\lambda = \lambda(\gamma) > 0$ is the smallest positive solution of

$$\varphi(\lambda) = \gamma^{-1} \tag{2.4}$$

with $\varphi(t) = \left[1 - \frac{t}{(1-p)^{1-t}}\right]^{-1} = [1 - te^{S(1-t)}]^{-1}$ and S being defined by (2.1).

Remark 2.2. The function $\varphi(t)$ satisfies the following properties:

- $\varphi(0) = 1$;

- $\varphi(t)$ is positive and finite on $[0, t_{\max})$ where $t_{\max} = t_{\max}(S)$ is the smallest positive solution of $1 = te^{S(1-t)}$, that is
$$t_{\max}(S) = \begin{cases} 1, & \text{for } S \le 1, \\ -\mathsf{LambertW}(-Se^{-S})/S & \text{for } S > 1 \end{cases}$$
where $\mathsf{LambertW}$ is the Lambert W function;
- $t_{\max} \le S^{-1}$ and hence $\varphi'(t) \propto (1 - tS) > 0$ for $t < t_{\max}$ (easy to check);
- $\varphi(t) \uparrow +\infty$ as $t \uparrow t_{\max}$.

Therefore, the solution to (2.4) indeed exists for any $0 < \gamma < 1$.

Proof of Theorem 2.1. As we have already established, the probability that an arbitrary vertex v is burnt out in time S is at least (2.2); this would be obviously also true even if some of the vertices $v \in V(G) \setminus \{v_0\}$ were already occupied at time 0. Denote by $T(1), T(2), \dots$ the times of ignitions of vertex v_0, set $T(0) = 0$ and let $\tau(n) = T(n) - T(n-1)$ be the (exponentially(1) distributed) times between consecutive burnouts. Let $N = N(x)$ be the number of intervals $\tau(i)$ of length at least S entirely lying inside $[0, x]$, that is

$$N(x) = \mathsf{card}\{i: \ \tau_i \ge S, \ T_i \le x\}$$

To get a handle on $N(x)$, we will use the renewal theory approach. Let

$$\begin{aligned} i_0 &= 0, \\ i_k &= \min\{i > i_{k-1}: \ \tau_i \ge S, \ \tau_j < S \ \forall j \in (i_{k-1}, i)\}, \ k = 1, 2, 3, \dots \end{aligned}$$

Then $T(i_k)$ form a renewal process, and

$$N(x) = \max\{k: \ T(i_k) \le x\} = \max\{k: \ \nu_1 + \nu_2 + \dots + \nu_k \le x\}$$

where $\nu_k := T(i_{k+1}) - T(i_k)$ are i.i.d. random variables, and if $\varphi_\nu(t) = \varphi(t)$ denotes its moment generating function which we will need later, then, by conditioning on τ_1 and using the memoryless property, we obtain

$$\begin{aligned} \varphi(t) &= \mathbb{E}\, e^{tT(i_1)} = \mathbb{E}\left[e^{tT(i_1)} 1_{\tau_1 \le S}\right] + \mathbb{E}\left[e^{tT(i_1)} 1_{\tau_1 > S}\right] \\ &= \mathbb{E} \int_0^S e^{tu + T(i_1)} e^{-u}\, \mathrm{d}u + \int_S^\infty e^{tu} e^{-u}\, \mathrm{d}u \\ &= \frac{1}{1-t}\left[\left(1 - e^{-(1-t)S}\right)\varphi(t) + e^{-(1-t)S})\right] \end{aligned}$$

yielding

$$\varphi(t) = \frac{1}{1 - te^{S(1-t)}}$$

which is defined for all $t < t_{\max}$. In particular, $\mathbb{E}\,\nu = \varphi'(0) = e^S$, and thus we expect $N(x)$ to be typically around $xe^{-S} = x(1-p)$.

On the other hand, by the arguments preceding the statement of the Theorem, conditioned on $N(x)$, the probability that v has *not* been burnt out in time x is smaller than $\gamma^{N(x)}$, hence

$$\mathbb{P}(\eta > x) \le \mathbb{E}\,\gamma^{N(x)} = \sum_{n=0}^{\infty} \gamma^n \mathbb{P}(N(x) = n).$$

We split the sum above into two parts and estimate it as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty}\gamma^n\mathbb{P}(N(x)=n) &\le \sum_{n=0}^{\lfloor x(1-p)\rfloor-1}\gamma^n\mathbb{P}(N(x)=n) \\
&\quad + \sum_{n=\lfloor x(1-p)\rfloor}^{\infty}\gamma^n\mathbb{P}(N(x)=n) \\
&\le \sum_{n=0}^{\lfloor x(1-p)\rfloor-1}\gamma^n\mathbb{P}(N(x)\le n) \\
&\quad +\gamma^{\lfloor x(1-p)\rfloor}\mathbb{P}\left(N(x)\ge\lfloor x(1-p)\rfloor\right) \\
&\le \sum_{n=0}^{\lfloor x(1-p)\rfloor-1}\gamma^n\mathbb{P}(\nu_1+\cdots+\nu_{n+1}\ge x)+\gamma^{x(1-p)-1} \qquad (2.5)\\
&\le \gamma^{-1}\left[x(1-p)\max_{m\in\{1,\dots,\lfloor x(1-p)\rfloor\}}\gamma^m\mathbb{P}(\nu_1+\cdots+\nu_m\ge x)\right. \\
&\qquad\qquad \left. +\,\gamma^{x(1-p)}\right].
\end{aligned}$$

From Markov inequality, we have for any $t>0$

$$\gamma^m\mathbb{P}(\nu_1+\cdots+\nu_m\ge x)\le e^{\Lambda(t,m)} \text{ where } \Lambda(t,m)=m\log\gamma+m\log\varphi(t)-tx.$$

We will bound $\log\left[\gamma^m\mathbb{P}(\nu_1+\cdots+\nu_m\ge x)\right]$ by $\max_{0\le m\le x(1-p)}\min_{t>0}\Lambda(t,m)$. From well-known properties of the MGF we know that $\log\varphi(t)$ and hence $\Lambda(t,m)$ is convex in t, therefore the latter achieves a unique minimum at point $t^*=t^*(x/m)$ where $t^*(\alpha)$ solves the equation

$$\frac{\varphi'(t^*(\alpha))}{\varphi(t^*(\alpha))}=\alpha.$$

Also, for $m\le x(1-p)$ we have $t^*\ge 0$ as $\partial\Lambda(t,m)/\partial t\,|_{t=0}=m\varphi'(0)/\varphi(0)-x=m(1-p)^{-1}-x\le 0$, yielding $\min_{t\ge 0}\Lambda(t,m)=\Lambda(t^*(x/m),m)$. Additionally, $t^*\left(\frac{1}{1-p}\right)=0$, $t^*(\alpha)$ is increasing in α as $\mathrm{d}\log\varphi(t)/\,\mathrm{d}t$ is increasing, and it is easy to check in our case $\varphi(t^*(\alpha))\to\infty$ as $\alpha\to\infty$.

On the other hand,

$$\begin{aligned}
\frac{\mathrm{d}\Lambda(t^*(x/m),m)}{\mathrm{d}m} &= \log\gamma+\log\varphi(t^*)+\left[m\frac{\varphi'(t^*)}{\varphi(t^*)}-x\right]\frac{\mathrm{d}t^*(x/m)}{\mathrm{d}m}\\
&= \log\left[\gamma\varphi\left(t^*\left(\frac{x}{m}\right)\right)\right].
\end{aligned}$$

The RHS of this expression decays in m; moreover as $m\downarrow 0$, $\frac{x}{m}\to+\infty$ resulting in $\varphi(t^*(x/m))\to+\infty$ and $\left.\frac{\mathrm{d}\Lambda(t^*(x/m),m)}{\mathrm{d}m}\right|_{m=0}=+\infty$. At the same time, for $m=x(1-p)$ we have $t^*(x/m)=0$ hence $\left.\frac{\mathrm{d}\Lambda(t^*(x/m),m)}{\mathrm{d}m}\right|_{m=x(1-p)}=\log\gamma<0$. Therefore, the maximum of $\Lambda(t^*(x/m),m)$ is achieved at some intermediate m and this maximum equals $-\lambda x$ where $\lambda=t^*(x/m)$ solves $\frac{\mathrm{d}\Lambda(t^*(x/m),m)}{\mathrm{d}m}=0$, i.e. equation (2.4). Finally, observe that

$$\begin{aligned}\gamma^{x(1-p)} &= \exp\{\Lambda(0,x(1-p))\} = \exp\{\Lambda(t^*((1-p)^{-1}),x(1-p))\}\\ &\le \exp\left\{\max_{0\le m\le x(1-p)} \Lambda(t^*(x/m),m)\right\} = e^{-\lambda x}.\end{aligned}$$

Now (2.5) yields (2.3). ■

3. Proofs of the combinatorial results

Proof of Lemma 1.3. Observe that

$$\begin{aligned}A_{n-1} &= \sum_{i=2}^{n}\binom{n}{i}(-1)^i \log i = \sum_{i=2}^{n}\binom{n}{i}(-1)^i\left[\log\frac{2}{1}+\log\frac{3}{2}+\dots+\log\frac{i}{i-1}\right]\\ &= \sum_{i=2}^{n}(-1)^i\left[\binom{n}{i}-\binom{n}{i+1}+\binom{n}{i+2}-\dots\pm\binom{n}{n}\right]\log\frac{i}{i-1}\\ &= \sum_{i=2}^{n}(-1)^i\binom{n-1}{i-1}\log\frac{i}{i-1} = \sum_{k=1}^{n-1}(-1)^{k-1}\binom{n-1}{k}\log\frac{k+1}{k},\end{aligned}$$

hence

$$A_n = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\log\frac{k+1}{k}.$$

To estimate the above quantity, we use the partial fractions method the way it is employed in Sondow (2003), equation (8), and in Guillera and Sondow (2008), Example 5.8,

$$\frac{n!}{x(x+1)\dots(x+n)} = \frac{1}{x} - \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{1}{x+k}.$$

Consequently,

$$A_n = \int_0^1\left[\frac{1}{x}-\frac{n!}{x(x+1)\dots(x+n)}\right]\mathrm{d}x. \tag{3.1}$$

(In fact, there is yet another formula for A_n in Prudnikov et al., 1986, 5.5.1, saying that

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\log\frac{k+a}{k+b} = -\log\frac{a}{b}+\int_0^1(t^{a-1}-t^{b-1})(1-t)^n\frac{\mathrm{d}t}{\log t}$$

hence $\log A_n = \lim_{b\downarrow 0}\left[\log b^{-1} - \int_0^1(1-t^{b-1})(1-t)^n(\log t)^{-1}\,\mathrm{d}t\right]$. Unfortunately, we could not estimate this limit and hence decided to work directly with (3.1).)

Let us rearrange (3.1) as follows:

$$A_n = \int_0^1\left[\frac{1}{x}-\frac{1}{x(1+x/1)(1+x/2)\dots(1+x/n)}\right]\mathrm{d}x$$

Using standard Taylor series expansion for $|x|<1$ we have

$$\log\left((1+x)\left(1+\frac{x}{2}\right)\dots\left(1+\frac{x}{n}\right)\right) = \sum_{m=1}^{\infty}\frac{x^m}{m}(-1)^{m-1}H_{n,m}$$

where $H_{n,m} = \sum_{k=1}^n k^{-m}$ are the generalized harmonic numbers. Moreover

$$\begin{aligned} H_n \equiv H_{n,1} &= \gamma + \log n + \frac{1}{2n} + O(n^{-2}), \\ H_{n,m} &= \zeta(m) - \frac{1}{(m-1)n^{m-1}} + O(n^{-m}),\ m = 2, 3, \ldots \end{aligned} \tag{3.2}$$

with $\zeta(s) = \sum_{k=1}^\infty k^{-s}$ being the Riemann zeta function and $\gamma = 0.577...$ the Euler constant. The first equality in (3.2) follows from the asymptotic for the Digamma function $\psi(x) = \frac{\mathrm{d}\log\Gamma(x)}{\mathrm{d}x}$ (see 6.3.2 and 6.3.18 in Abramowitz and Stegun, 1965) while the second one is an elementary consequence of the fact that

$$\zeta(m) - H_{n,m} = \sum_{k=n+1}^\infty \frac{1}{k^m}, \text{ while } \int_{n+1}^\infty \frac{\mathrm{d}x}{x^m} < \sum_{k=n+1}^\infty \frac{1}{k^m} < \int_n^\infty \frac{\mathrm{d}x}{x^m}.$$

By changing the variables $x = y/\log n$ in the integral, we obtain

$$A_n = \int_0^{\log n} \frac{\mathrm{d}y}{y}\left[1 - \frac{1}{\exp(B_n(y))}\right]$$

where

$$B_n(y) = y + \frac{y}{\log n}\left(\gamma + \frac{1}{2n} + O\left(n^{-2}\right)\right) - \frac{y^2}{2(\log n)^2}\left(\zeta(2) - \frac{1}{n} + O(n^{-2})\right) + \ldots$$

Integrating separately on $[0,1]$ and $[1, \log n]$ we obtain

$$\begin{aligned} A_n &= \int_1^{\log n} \frac{\mathrm{d}y}{y} - \int_1^{\log n} \frac{e^{-B_n(y)}\,\mathrm{d}y}{y} + \int_0^1 \frac{\mathrm{d}y}{y}\left[1 - e^{-B_n(y)}\right] \\ &= \log\log n - \int_1^\infty \frac{e^{-y}\,\mathrm{d}y}{y} + \int_0^1 \frac{1-e^{-y}}{y}\,\mathrm{d}y + o(1) \\ &= \log\log n + \gamma + o(1), \end{aligned}$$

by plugging $z = 1$ into (1.7), taking into account that for $y \in [0,1]$

$$1 - e^{-B_n(y)} = (1 - e^{-y}) + \frac{\gamma y e^{-y}}{\log n} + y \times O((\log n)^{-2}),$$

and at the same time

$$\begin{aligned} \left|\int_1^{\log n} \frac{e^{-B_n(y)}\,\mathrm{d}y}{y} - \int_1^\infty \frac{e^{-y}\,\mathrm{d}y}{y}\right| &\leq \int_1^{\sqrt{\log n}} \frac{|e^{-B_n(y)} - e^{-y}|\,\mathrm{d}y}{y} \\ + \int_{\sqrt{\log n}}^{\log n} \frac{e^{-B_n(y)}\,\mathrm{d}y}{y} + \int_{\sqrt{\log n}}^\infty \frac{e^{-y}\,\mathrm{d}y}{y} &=: (I) + (II) + (III), \end{aligned}$$

where

$$
\begin{aligned}
(I) \quad &\le \quad \int_1^{\sqrt{\log n}} \frac{\text{Const}}{\sqrt{\log n}} \frac{e^{-y}\,\mathrm{d}y}{y} \le \frac{\text{Const}}{\sqrt{\log n}},\\
(II) \quad &= \quad \int_{(\log n)^{-\frac{1}{2}}}^{1} \frac{\mathrm{d}x}{x(1+x/1)\dots(1+x/n)}\\
&\le \quad \int_{(\log n)^{-\frac{1}{2}}}^{1} \frac{\mathrm{d}x}{x\left(1+x\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)\right)}\\
&= \quad \log \frac{x}{1+H_n}\Big|_{(\log n)^{-\frac{1}{2}}}^{1} = \log \frac{H_n+\sqrt{\log n}}{H_n+1} = \frac{1}{\sqrt{\log n}}(1+o(1)),\\
(III) \quad &\le \quad \frac{1}{\sqrt{\log n}} \int_1^\infty e^{-y}\,\mathrm{d}y = \frac{1}{\sqrt{\log n}}.
\end{aligned}
$$

■

Proof of Lemma 1.5. The derivation of (a) is fairly straightforward by induction; it can also be recovered from Section 4 in Branson (2006).

To establish (b), note that

$$
a(n,m) \le \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \frac{1}{i_1 i_2 \dots i_m} = \left(\sum_{i=1}^{n} \frac{1}{i}\right)^m \equiv (H_n)^m
$$

and $H_n \le 1 + \log n$ for $n \ge 1$.

Finally, to prove (c), let

$$
\tilde{a}(n,m) := \sum_{1\le i_1<i_2<\dots<i_m\le n} \frac{1}{i_1 i_2 \dots i_m}
$$

(observe that here all i_k's must be distinct). From equation (3.2) in Grünberg (2006) it follows that for a fixed m satisfy

$$
\tilde{a}(n,m) = \frac{(\log n)^m}{m!} + \frac{\gamma(\log n)^{m-1}}{(m-1)!} + \frac{(\gamma^2-\zeta(2))\log^{m-2} n}{(m-2)!\,2} + \dots
$$

On the other hand,

$$
\begin{aligned}
0 \quad &< \quad a(n,m) - \tilde{a}(n,m) = \sum_{r=1}^{m-1} \sum_{1\le i_1<i_2<\dots<i_r=i_{r+1}\le i_{r+2}\le\dots\le i_m\le n} \frac{1}{i_1 i_2 \dots i_m}\\
&\le \quad \sum_{r=1}^{m-1} \left(\sum_{k=1}^{n} \frac{1}{k^2}\right) \sum_{1\le i_1<\dots<i_{r-1}\le i_{r+2}\le\dots\le i_m\le n} \frac{1}{i_1 i_2 \dots i_m}\\
&\le \quad 2m\,a(n,m-2) \le 2m(\log n+1)^{m-2}
\end{aligned}
$$

Therefore,

$$
a(n,m) = \frac{(\log n)^m}{m!} + \frac{\gamma(\log n)^{m-1}}{(m-1)!} + O(\log^{m-2} n)
$$

similar to $\tilde{a}(n,m)$. ■

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