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Fast LCMV-based Methods for Fundamental Frequency Estimation

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***Abstract*—Recently, optimal linearly constrained minimum variance (LCMV) filtering methods have been applied to fundamental frequency estimation. Such estimators often yield preferable performance but suffer from being computationally cumbersome as the resulting cost functions are multimodal with narrow peaks, as well as requiring matrix inversions for each point in the search grid. In this paper, we therefore consider fast implementations of LCMV-based fundamental frequency estimators, exploiting the estimators' inherently low displacement rank of the used Toeplitz-like data covariance matrices, using as such either the classic time domain averaging covariance matrix estimator, or, if aiming for an increased spectral resolution, the covariance matrix resulting from the application of the recent iterative adaptive approach (IAA). The proposed, exact implementations reduce the required computational complexity with several orders of magnitude, but, as we show, further computational savings can be obtained by the adoption of an approximative IAA-based data covariance matrix estimator, reminiscent to the recently proposed Quasi-Newton IAA technique. Furthermore, it is shown how the considered pitch estimators can be efficiently updated when new observations become available. The resulting time-recursive updating can reduce the computational complexity even further. The experimental results show that the performances of the proposed methods are comparable or better than that of other competing methods in terms of spectral resolution. Finally, it is shown that the time-recursive implementations are able to track pitch fluctuations of synthetic as well as real-life signals.**

***Index Terms*—Fundamental frequency estimation, optimal filtering, data adaptive estimators, efficient algorithms.**

I. Introduction

There exists a multitude of signal processing applications in which the fundamental frequency is an essential parameter, including, for instance, parametric coding of audio and speech, automatic music transcription, musical genre classification, tuning of musical instruments, separation and enhancement of audio and speech sources, and hearing aids. Due to the importance of knowing the fundamental frequency, numerous approaches and methods have been proposed for estimating this parameter, see, e.g., [?], [?], [?], [?], [?], [?], [?] and the references therein. Typically, such estimators rely on an estimate of the sample covariance matrix, or its inverse, both commonly being formed by partitioning the available measurement into sub-vectors and forming the outer-product covariance matrix estimate. As is well-known, this approach will adversely affect the achievable spectral resolution due to the necessarily limited length of the used sub-vectors. In an effort to circumvent this limitation, recent work have explored methods allowing for a higher spectral resolution. One example of such a method is the linearly constrained minimum variance (LCMV) filtering method proposed in [?], which, like the MUSIC and expectation-maximization-based methods also proposed therein, is shown to offer high-resolution estimates of the fundamental frequency of closely spaced sources. The LCMV method holds this property at a much lower complexity compared to the other methods, though. Regrettably, the LCMV method still suffers from being computationally cumbersome due to its cost function being multimodal with narrow peaks and requiring matrix inversion for each point in the search grid.

In this work, we examine both efficient implementations of the LCMV method as well as of novel high-resolution versions based on the covariance matrix obtained via the iterative adaptive approach (IAA). Using the IAA-based covariance estimate in connection with the LCMV method was originally proposed in [?]. Here, this improvement is further explored and improved upon. The IAA was originally presented in [?] to provide sparse signal representation for passive sensing, channel estimation, and single-antenna radar applications, but have since found applications in areas as diverse as MIMO radar [?], missing data recovery [?], non-uniformly sampled spectral analysis [?], coherence and polyspectral estimation [?], spectroscopy [?], and blood velocity estimation using ultrasound [?]. Notably, the IAA estimate is a non-parametric data-dependent spectral estimate that does not require partitioning of the measurements. The estimate is instead formed iteratively by estimating the spectral amplitudes of the measurement as well as the covariance matrix formed from this amplitude spectrum. Due to the lack of partitioning, the LCMV estimate resulting from exploiting the IAA-based covariance matrix estimate offers substantially higher spectral resolution than what is normally achievable using the outer-product estimate, although the improvement comes at the price of

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a considerable increase of the computational complexity. To reduce this drawback, we here propose fast implementations of the LCMV estimators, exploiting the inherently low displacement rank of the necessary product of Toeplitz-like matrices, proposing both exact and approximative implementations. The exact implementations generalizes the results in [?], [?], and have a significantly lower computational complexity than our implementations recently introduced in [?] as the inverse covariance matrix is not needed explicitly in the proposed implementations. Moreover, we propose an even faster approximative implementation of the IAA-based LCMV method using an approximative low order extension of the inverse data covariance matrix, reminiscent to the one developed in [?]. Finally, we show how the considered pitch estimators can be updated efficiently when new observations become available. By using such time-recursive implementations, the computationally complexity can be reduced much further as compared to batch processing, especially if time hopping is allowed.

In the following section, we first briefly recall the LCMV-based fundamental frequency estimate and the IAA-based covariance matrix estimate. Then, in Sections III and IV, we introduce the proposed efficient exact and approximative implementations of the LCMV and the IAA-based LCMV methods. Section V discusses time-recursive updating of the estimates, followed, in Section VI, by extensive simulation examples on synthetic data illustrating the performance of the proposed implementations. Finally, Section VII concludes on the presented work.

II. Fundamental Frequency Estimation using Optimal Filtering

As audio and speech signals are quasi-periodic, one may well model such signals as (see, e.g., [?])

$$x(n) = \sum_{l=1}^{L} \alpha_l e^{jl\omega_0 n} + w(n), \tag{1}$$

for $n = 0, \dots, N-1$, where L is the number of harmonics, $\alpha_l = A_l e^{j\phi_l}$, with $A_l > 0$ and ϕ_l denoting the real-valued amplitude and the phase of the lth harmonic, respectively. Furthermore, $\omega_0 \in [0; 2\pi/L]$ denotes the sought fundamental frequency, and $w(n)$ is complex-valued additive noise. For simplicity, we will here assume that the model order, L, is known, noting that this may otherwise be obtained using a model order estimator [?], [?], [?], [?], or by forming the model order and fundamental frequency estimation jointly, reminiscent to the ideas presented in [?], [?], [?]. The problem of interest is thus estimating ω_0 from $x(n)$ without making any strong assumptions on the statistics of the noise process.

A. Harmonic LCMV Method

Fundamental frequency estimation may, for instance, be conducted using the optimal filtering method introduced in [?], being based on an optimal LCMV filter. Consider an $(M-1)$th order FIR filter of which the output is given by

$$y(n) = \sum_{m=0}^{M-1} h(m)x(n-m) = \mathbf{h}_M^H \mathbf{x}_M(n), \tag{2}$$

TABLE I
HARMONIC LCMV METHOD

$$\hat{\mathbf{R}}_M = \frac{1}{N-M+1} \sum_{n=M-1}^{N-1} \mathbf{x}_M(n)\mathbf{x}_M^H(n)$$
$$\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega_0) = \mathbf{Z}_{M,L}^H(\omega_0)\hat{\mathbf{R}}_M^{-1}\mathbf{Z}_{M,L}(\omega_0)$$
$$\hat{\omega}_0 = \arg\max_{\omega_0 \in \Omega_0} \mathbf{1}_L^T \left[\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega_0)\right]^{-1} \mathbf{1}_L$$

for $n = M-1, \dots, N-1$, where

$$\mathbf{h}_M = \begin{bmatrix} h(0) & \cdots & h(M-1) \end{bmatrix}^H, \tag{3}$$

$$\mathbf{x}_M(n) = \begin{bmatrix} x(n) & \cdots & x(n-M+1) \end{bmatrix}^T, \tag{4}$$

with $(\cdot)^T$ and $(\cdot)^H$ denoting the transpose and conjugate transpose, respectively. The output power of the filter is

$$\text{E}\{|y(n)|^2\} = \mathbf{h}_M^H \mathbf{R}_M \mathbf{h}_M, \tag{5}$$

where

$$\mathbf{R}_M = \text{E}\{\mathbf{x}_M(n)\mathbf{x}_M^H(n)\}, \tag{6}$$

with $\text{E}\{\cdot\}$ denoting the statistical expectation. The optimal filter response is found using the LCMV principle, such that the filter is designed to have a unit gain at the harmonic frequencies while having maximum noise suppression. This design procedure can also be formulated as

$$\min_{\mathbf{h}_M} \mathbf{h}_M^H \mathbf{R}_M \mathbf{h}_M \quad \text{subj. to } \mathbf{h}_M^H \mathbf{z}_M(l\omega_0) = 1 \ , \quad \text{for } l = 1, \dots, L, \tag{7}$$

where

$$\mathbf{z}_M(\omega_0) = \begin{bmatrix} 1 & e^{-j\omega_0} & \cdots & e^{-j(M-1)\omega_0} \end{bmatrix}^T. \tag{8}$$

The solution to the quadratic optimization problem with multiple equality constraints in (7) is [?]

$$\mathbf{h}_M^{\text{LCMV}}(\omega_0) = \mathbf{R}_M^{-1}\mathbf{Z}_{M,L}(\omega_0)\left[\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega_0)\right]^{-1}\mathbf{1}_L, \tag{9}$$

with $\mathbf{1}_L \in \mathbb{R}^L$ denoting a vector of ones,

$$\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega_0) \triangleq \mathbf{Z}_{M,L}^H(\omega_0)\mathbf{R}_M^{-1}\mathbf{Z}_{M,L}(\omega_0), \tag{10}$$

and where

$$\mathbf{Z}_{M,L}(\omega_0) = \begin{bmatrix} \mathbf{z}_M(\omega_0) & \cdots & \mathbf{z}_M(L\omega_0) \end{bmatrix}. \tag{11}$$

An estimate of the fundamental frequency may thus be found by inserting (9) into (5) and maximize the output power, yielding

$$\hat{\omega}_0 = \arg\max_{\omega_0 \in \Omega_0} \mathbf{1}_L^T \left[\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega_0)\right]^{-1} \mathbf{1}_L, \tag{12}$$

where Ω_0 is a set of candidate fundamental frequencies. Here, we term the estimator in (12) the LCMV fundamental frequency estimator[1]. The covariance matrix $\mathbf{R}_M$ is generally

[1]Note that similar estimators are also referred to as MVDR or Capon methods in the literature.

TABLE II
IAA-BASED HARMONIC LCMV METHOD

Iterate until convergence

$$\hat{\alpha}_k = \frac{\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}\mathbf{x}_N}{\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}\mathbf{z}_N(\omega_k)}$$
$$\tilde{\mathbf{R}}_N = \sum_{k=0}^{K-1} |\hat{\alpha}_k|^2 \mathbf{z}_N(\omega_k)\mathbf{z}_N^H(\omega_k)$$

End

$$\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0) = \mathbf{Z}_{N,L}^H(\omega_0)\tilde{\mathbf{R}}_N^{-1}\mathbf{Z}_{N,L}(\omega_0)$$
$$\hat{\omega}_0 = \arg\max_{\omega_0\in\Omega_0} \mathbf{1}_L^T \left[\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0)\right]^{-1}\mathbf{1}_L$$

unknown, and is commonly replaced by the sample covariance matrix

$$\hat{\mathbf{R}}_M = \frac{1}{N-M+1}\sum_{n=M-1}^{N-1} \mathbf{x}_M(n)\mathbf{x}_M^H(n). \tag{13}$$

The resulting fundamental frequency estimation algorithm is summarized in Tab. I. To ensure that $\hat{\mathbf{R}}_M$ is invertible, the length of the sub-vectors, $\mathbf{x}_M(n)$, are restricted to $M < N/2+1$, thereby limiting the resolution to being on the order of $1/M$ [?]. A direct implementation of the estimator requires roughly

$$\mathcal{C}^{\text{cov}} \approx M^3 + M^2\bar{N} + \bar{F}\left(ML^2 + LM^2 + L^3\right) \tag{14}$$

operations, where $\bar{N} \triangleq N - M + 1$ and $\bar{F} \triangleq F/L$, with $F = |\Omega_0|$ being the size of the uniformly spaced grid of frequencies where the search for the optimum ω_0 is conducted. Typically, $F \gg N$, and due to the nature of the problem, the search is limited to frequencies up to $\bar{F}$.

B. IAA-based Harmonic LCMV Method

We proceed to recall the IAA-based covariance matrix estimate, which is then used in conjunction with the LCMV method for fundamental frequency estimation. However, it should be stressed that this covariance matrix estimate could similarly be used in conjunction with other covariance based fundamental frequency estimators, thereby offering a similar improved spectral resolution. Following the usual IAA notation, let

$$\mathbf{x}_N = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1)\end{bmatrix}^T . \tag{15}$$

Then, the IAA estimate is formed by iteratively estimating the complex spectral amplitudes, $\alpha(\omega_k) \triangleq \alpha_k$, and the corresponding covariance matrix, $\tilde{\mathbf{R}}_L$, until practical convergence, as (see [?], [?] for further details)

$$\hat{\alpha}_k = [\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}\mathbf{z}_N(\omega_k)]^{-1}\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}\mathbf{x}_N, \tag{16}$$
$$\tilde{\mathbf{R}}_N = \sum_{k=0}^{K-1} |\hat{\alpha}_k|^2 \mathbf{z}_N(\omega_k)\mathbf{z}_N^H(\omega_k), \tag{17}$$

for $k = 0, 1, \ldots, K-1$, with K denoting the size of the grid of frequencies utilized in the IAA implementation, and $\tilde{\mathbf{R}}_N$ being initialized to the identity matrix, $\mathbf{I}_N$. This implies that the complex amplitudes are initialized using the FFT of the entire measurement vector. Typically, in practice, 10-15 iterations are sufficient for convergence [?]. The expression in (16) can also be interpreted as a filtering operation, i.e.,

$$\hat{\alpha}_k = \left[\mathbf{h}_N^{\text{IAA}}(\omega_k)\right]^H \mathbf{x}_N , \tag{18}$$

where the IAA filter, $\mathbf{h}_N^{\text{IAA}}(\omega_k)$, is defined as

$$\mathbf{h}_N^{\text{IAA}}(\omega_k) = [\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}\mathbf{z}_N(\omega_k)]^{-1}\tilde{\mathbf{R}}_N^{-1}\mathbf{z}_N(\omega_k), \tag{19}$$

from which it may be noted that the IAA filter resembles the optimal filter used in the traditional Capon method for spectrum estimation [?]. In [?], we instead proposed the IAA-based optimal LCMV (IAA-LCMV) filter, formed as

$$\mathbf{h}_N^{\text{IAA-LCMV}}(\omega_0) = \tilde{\mathbf{R}}_N^{-1}\mathbf{Z}_{N,L}(\omega_0)\left[\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0)\right]^{-1}\mathbf{1}_L, \tag{20}$$

where

$$\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0) \triangleq \mathbf{Z}_{N,L}^H(\omega_0)\tilde{\mathbf{R}}_N^{-1}\mathbf{Z}_{N,L}(\omega_0). \tag{21}$$

That is, we use the filter design in (9) together with the IAA covariance matrix estimate obtained after convergence has been achieved. Combining (18) and (20), one obtains an estimate of the output power of the IAA-LCMV filter as

$$\begin{aligned}\hat{P}_{\text{IAA-LCMV}}(\omega_0) =& \mathbf{1}_L^T\left[\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0)\right]^{-1}\mathbf{Z}_{N,L}^H(\omega_0)\tilde{\mathbf{R}}_N^{-1}\mathbf{X}_N \\ &\times \tilde{\mathbf{R}}_N^{-1}\mathbf{Z}_{N,L}(\omega_0)\left[\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0)\right]^{-1}\mathbf{1}_L,\end{aligned} \tag{22}$$

with $\mathbf{X}_N = \mathbf{x}_N\mathbf{x}_N^H$. By taking the expected value of the output power, we get

$$\text{E}\left\{\hat{P}_{\text{IAA-LCMV}}(\omega_0)\right\} = \mathbf{1}_L^T\left[\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega_0)\right]^{-1}\mathbf{1}_L, \tag{23}$$

from which the expectation-based fundamental frequency estimate is obtained as

$$\hat{\omega}_0 = \arg\max_{\omega_0\in\Omega_0} \text{E}\left\{\hat{P}_{\text{IAA-LCMV}}(\omega_0)\right\}. \tag{24}$$

A summary of the algorithm is found in Tab. II. A direct implementation of (24) requires roughly

$$\mathcal{C}^{\text{IAA}} \approx m(N^3 + 3N^2K) + \bar{F}\left(NL^2 + LN^2 + L^3\right) \tag{25}$$

operations, where m is the number of IAA iterations and, usually, $K \leq F$.

III. EXACT EFFICIENT IMPLEMENTATIONS

Efficient and exact implementations of (12) and (24) may alternatively be formed by exploiting the inherent displacement structure of the estimator, forming the implementation using Gohberg-Semencul (GS) factorizations of the involved inverse covariance matrices. Consider a Hermitian matrix $\mathbf{P}_M \in \mathcal{C}^{M\times M}$, and define the lower shifting matrix

$$\mathbf{D}_M = \begin{bmatrix} \mathbf{0}^T & 0 \\ \mathbf{I}_{M-1} & \mathbf{0}\end{bmatrix} . \tag{26}$$

Clearly, $(\mathbf{D}_M)^M = \mathbf{0}$. Then, the displacement of $\mathbf{P}_M$, denoted as $\nabla_{\mathbf{D}_M,\mathbf{D}_M^T}\mathbf{P}_M$, with respect to $\mathbf{D}_M$ and $\mathbf{D}_M^T$ is defined as

$$\nabla_{\mathbf{D}_M,\mathbf{D}_M^T}\mathbf{P}_M \triangleq \mathbf{P}_M - \mathbf{D}_M\mathbf{P}_M\mathbf{D}_M^T. \tag{27}$$

Suppose that there exist integers ρ and $\sigma_i \in \{-1, 1\}$, for $i = 1, 2, \ldots, \rho$, such that (see also [?], [?], [?])

$$\nabla_{\mathbf{D}_M, \mathbf{D}_M^T} \mathbf{P}_M = \sum_{i=1}^{\rho} \sigma_i \mathbf{t}_M^{(i)} \mathbf{t}_M^{(i)H} = \mathbf{T}_{M,\rho} \mathbf{\Sigma}_\rho \mathbf{T}_{M,\rho}^H, \quad (28)$$

where

$$\mathbf{T}_{M,\rho} = \begin{bmatrix} \mathbf{t}_M^{(1)} & \cdots & \mathbf{t}_M^{(\rho)} \end{bmatrix}, \quad (29)$$

$$\mathbf{\Sigma}_\rho = \text{diag}\left\{ \begin{bmatrix} \sigma_1 & \cdots & \sigma_\rho \end{bmatrix}^T \right\}, \quad (30)$$

with diag$(\mathbf{x})$ denoting the diagonal matrix formed with the vector $\mathbf{x}$ along its diagonal, and with $\mathbf{t}_M^{(i)}$ being the ith so-called generator vector. Then, the GS factorization of $\mathbf{P}_M$ may be expressed as

$$\mathbf{P}_M = \sum_{i=1}^{\rho} \sigma_i \mathcal{L}\left(\mathbf{D}_M, \mathbf{t}_M^{(i)}\right) \mathcal{L}^H\left(\mathbf{D}_M, \mathbf{t}_M^{(i)}\right), \quad (31)$$

where $\mathcal{L}(\mathbf{D}_M, \mathbf{b}_M)$ denotes a Krylov matrix of the form $\mathcal{L}(\mathbf{D}_M, \mathbf{b}_M) = [\mathbf{b}_M \ \mathbf{D}_M \mathbf{b}_M \ \mathbf{D}_M^2 \mathbf{b}_M \ \cdots \ \mathbf{D}_M^{M-1} \mathbf{b}_M]$. Given the GS factorization of $\mathbf{P}_M$, the coefficients of the trigonometric polynomial

$$\psi(\omega) \triangleq \mathbf{z}_M^H(\omega) \mathbf{P}_M \mathbf{z}_M(\omega) = \sum_{\kappa=-M+1}^{M-1} c_\kappa e^{-j\kappa\omega} \quad (32)$$

can then be formed at a cost of $\mathcal{O}(\rho M \log_2(2M))$ using the method detailed in [?]. Extending the results presented therein, we proceed on proposing a fast method for the estimation of the coefficients of the matrix valued trigonometric polynomials associated with the inverse covariance matrices for the here considered LCMV and IAA-LCMV fundamental frequency estimators, (10) and (21), respectively. We thus consider the matrix valued trigonometric polynomial associated to $\mathbf{P}_M$, defined as

$$\mathbf{\Psi}_L(\omega) \triangleq \mathbf{Z}_{M,L}^H(\omega) \mathbf{P}_M \mathbf{Z}_{M,L}(\omega), \quad (33)$$

where

$$[\mathbf{\Psi}_L(\omega)]_{l_1,l_2} \triangleq \psi_{l_1,l_2}(\omega) = \mathbf{z}_M^H(l_1\omega) \mathbf{P}_M \mathbf{z}_M(l_2\omega). \quad (34)$$

Notice that when $l_1 = l_2 = 1$, $\psi_{1,1}(\omega)$ coincides with (32) and thus can be handled using [?]. As it has been recently shown in [?], the coefficients of the remaining polynomials (34) can be computed by summing up the diagonals of matrices constructed by proper expansion of $\mathbf{P}_M$, where the matrix $\mathbf{P}_M$ itself was reconstructed from the displacement representation (28) at hand, at a cost of $(\rho + 0.5L^2)N^2$ operations. The computed polynomial coefficients were subsequently utilized for the evaluation of (34) on the unit circle at a cost of $0.25L^2 F \log_2 F$. In the following, we proceed by presenting a novel method for the evaluation of the matrix valued trigonometric polynomial in (33) directly from the displacement representation of the associated inverse covariance matrix and the displacement representation of its increased order matrix expansion, circumventing the need for reconstructing the inverse covariance matrix from its given displacement representation. This results in an even faster scheme than the one proposed in [?].

TABLE III
FAST HARMONIC LCMV METHOD

1) Estimate the displacement vectors $\mathbf{t}_M^{(i)}$, $i = 1, 2, 3, 4$ of $\hat{\mathbf{R}}_M^{-1}$ using the generalized Levinson algorithm [?] at a complexity of $4.5M^2 + 1.5N \log_2 N$.
2) Compute $\{\mathbf{t}_M^{(1)}, \mathbf{t}_M^{(2)}, \mathbf{t}_M^{(3)}, \mathbf{t}_M^{(4)}\} \rightarrow \{\mathbf{t}_{M+1}^{(1)}, \mathbf{t}_{M+1}^{(2)}, \mathbf{t}_{M+1}^{(3)}, \mathbf{t}_{M+1}^{(4)}\}$. using single step generalized Levinson recursions at a cost of $8M$
3) Compute the coefficients of $\psi_{1,1}^{\text{cov}}(\omega) = \mathbf{z}_M^H(\omega) \hat{\mathbf{R}}_M^{-1} \mathbf{z}_M(\omega)$ and evaluate $\psi_{1,1}^{\text{cov}}(\omega)$ on a set of F frequencies, equally spaced on the unit circle as [?] at a cost of $9M \log_2(2M) + 0.5F \log_2 F$.
4) Evaluate $\varphi_1^{(i)}(\omega) = \mathbf{z}_{M+1}^H(\omega) \mathbf{t}_{M+1}^{(i)}$, $i = 1, 2, 3, 4$ on a set of F frequencies, equally spaced on the unit circle at a cost of $2F \log_2 F$.
5) Compute $\psi_{l_1,l_1}^{\text{cov}}(\omega)$ and $\psi_{l_1,l_2}^{\text{cov}}(\omega)$, for $l_1 = 1, 2 \ldots L$ and $l_2 = l_1, \ldots L$ as dictated by (40)-(45) for equally spaced frequencies in the range $]0; 2\pi/L]$ at a cost of $2.5FL$.
6) Compute $\mathbf{1}_L^T \left[\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega)\right]^{-1} \mathbf{1}_L$ for equally spaced frequencies in the range $]0; 2\pi/L]$ at a cost of FL^2.

A. Fast Harmonic LCMV Method

Restricting the set of candidate fundamental frequencies, Ω_0, to the frequencies uniformly spanned on the unit circle, the maximization of (12) may be performed indirectly by exhaustive searching. This results in the evaluation of the trigonometric matrices in (10), and the computation of their inverses over the set of uniformly spaced frequencies of interest, which enables the use of the Fast Fourier Transform (FFT) for computational speed-up. First, the generalized Levinson algorithm is employed for the computation of a displacement representation,

$$\mathbf{T}_{M,4} = [\mathbf{t}_M^{(1)}, \mathbf{t}_M^{(2)}, \mathbf{t}_M^{(3)}, \mathbf{t}_M^{(4)}], \quad (35)$$

$$\Sigma_4 = \text{diag}\{1, -1, 1, -1\}, \quad (36)$$

of $\hat{\mathbf{R}}_M$ in (13), as detailed in [?]. The computation of the displacement representation requires about

$$\mathcal{C}^{\text{FCOV}}(M, N) \approx 4.5M^2 + 1.5N \log_2 N. \quad (37)$$

Using $\hat{\mathbf{R}}_M^{-1}$ in place of $\mathbf{P}_M$ in (33) results in (10), which component wise is expressed as

$$[\boldsymbol{\mathcal{G}}_L^{\text{cov}}(\omega)]_{l_1,l_2} \triangleq \psi_{l_1,l_2}^{\text{cov}}(\omega) = \mathbf{z}_M^H(l_1\omega) \hat{\mathbf{R}}_M^{-1} \mathbf{z}_M(l_2\omega). \quad (38)$$

When $l_1 = l_2 = 1$, (38) implies that

$$\psi_{1,1}^{\text{cov}}(\omega) \triangleq \mathbf{z}_M^H(\omega) \hat{\mathbf{R}}_M^{-1} \mathbf{z}_M(\omega). \quad (39)$$

The coefficients of $\psi_{1,1}^{\text{cov}}(\omega)$ can be computed directly from (35) and (36) at a cost of $9M \log_2(2M)$ using the method detailed in [?] and $\psi_{1,1}^{\text{cov}}(\omega)$ can be subsequently evaluated on the unit circle using an FFT at a cost of $0.5F \log_2 F$, where F is the number of frequencies uniformly spanning the unit circle. As the search in (12) is restricted to a set of equally spaced frequencies on the unit circle for frequencies ω up to $2\pi/L$, we are able to evaluate the remaining polynomials across the main diagonal of (38) at no extra cost, by downsampling (39) in the frequency domain as

$$\psi_{l_1,l_1}^{\text{cov}}(\omega) = \psi_{1,1}^{\text{cov}}(l_1\omega), \quad (40)$$

TABLE IV
FAST IAA-BASED HARMONIC LCMV METHOD

1) Estimate the displacement vectors $\mathbf{t}_N^{(i)}$, $i = 1, 2$ of $\mathbf{R}_N^{-1}$ and the coefficients of $\psi_{1,1}^{\text{IAA}}(\omega) = \mathbf{z}_N^H(\omega)\tilde{\mathbf{R}}_N^{-1}\mathbf{z}_N(\omega)$ using the fast IAA algorithm [**?**], [**?**] at a cost of $m[N^2 + 12N\log_2(2N) + 1.5K\log_2 K]$.
2) Compute $\{\mathbf{t}_N^{(1)}\} \to \{\mathbf{t}_{N+1}^{(1)}\}$ using single step Levinson recursion at a cost of $2N$.
3) Evaluate $\psi_{1,1}^{\text{IAA}}(\omega)$ on a set of F frequencies, equally spaced on the unit circle as at a cost of $0.5F\log_2 F$.
4) Evaluate $\varphi^{(i)}(\omega) = \mathbf{z}_{N+1}^H(\omega)\mathbf{t}_{N+1}^{(i)}$, $i = 1, 2$ on a set of F frequencies, equally spaced on the unit circle at a cost of $F\log_2 F$.
5) Compute $\psi_{l_1,l_1}^{\text{IAA}}(\omega)$ and $\psi_{l_1,l_2}^{\text{IAA}}(\omega)$, for $l_1 = 1, 2\ldots L$ and $l_2 = l_1, \ldots L$ as dictated by (40)-(45) for equally spaced frequencies in the range $]0; 2\pi/L]$ at a cost of $1.5FL$.
6) Compute $\mathbf{1}^T\left[\boldsymbol{\mathcal{G}}_L^{\text{IAA}}(\omega)\right]^{-1}\mathbf{1}$ for equally spaced frequencies in the range $]0; 2\pi/L]$ at a cost of FL^2.

for $l_1 = l_2$ and $l_1 = 2, \ldots, L$. The off diagonal polynomials in (38) can be efficiently computed from the displacement representation of an increased order matrix defined as

$$\hat{\hat{\mathbf{R}}}_{M+1} \triangleq \begin{bmatrix} \hat{\mathbf{R}}_M^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}, \tag{41}$$

which, as is shown in the Appendix, can be expressed as

$$\begin{aligned}\nabla_{\mathbf{D}_{M+1},\mathbf{D}_{M+1}^T}\hat{\hat{\mathbf{R}}}_{M+1} &\triangleq \begin{bmatrix} \hat{\mathbf{R}}_M^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \hat{\mathbf{R}}_M^{-1} \end{bmatrix} \\ &= \sum_{i=1}^{4}\sigma_i \mathbf{t}_{M+1}^{(i)}\mathbf{t}_{M+1}^{(i)H}, \end{aligned}\tag{42}$$

where the displacement vectors $\mathbf{t}_{M+1}^{(i)}$ are be computed from those already available in (35) using single step generalized Levinson recursions. Finally, by using (38) and (42) for $l_1 \neq l_2$, one may write

$$\psi_{l_1,l_2}^{\text{cov}}(\omega)\left[1 - e^{j(l_1-l_2)\omega}\right] = \sum_{i=1}^{4}\sigma_i\varphi_{l_1}^{(i)}(\omega)\varphi_{l_2}^{(i)*}(\omega), \tag{43}$$

where

$$\varphi_{l_1}^{(i)}(\omega) \triangleq \mathbf{z}_{M+1}^H(l_1\omega)\mathbf{t}_{M+1}^{(i)}. \tag{44}$$

This implies that

$$\psi_{l_1,l_2}^{\text{cov}}(\omega) = \frac{1}{1 - e^{j(l_1-l_2)\omega}}\sum_{i=1}^{4}\sigma_i\varphi_1^{(i)}(l_1\omega)\varphi_1^{(i)*}(l_2\omega), \tag{45}$$

for the set of frequencies of interest up to $2\pi/L$ with $\psi_{l_1,l_2}^{\text{cov}}(0) = \psi_{1,1}^{\text{coc}}(0)$. Note that $\varphi_1^{(i)}(l_1\omega)$ can be obtained by downsampling $\varphi_1^{(i)}(\omega)$ in the frequency domain. The algorithm constituting the fast harmonic LCMV method is summarized in Tab. III. Its computational complexity is dominated by the complexity of computing 4 FFT's of size F and the inversion of F/L, $L \times L$ linear systems of equations, yielding an overall complexity

$$\begin{aligned}\mathcal{C}^{\text{FLCMV}} \approx\ &\mathcal{C}^{\text{FCOV}}(M, N) + 9M\log_2(2M) \\ &+ 2.5F\log_2 F + 2.5FL + FL^2.\end{aligned}\tag{46}$$

Compared to [**?**], the new approach provides a faster way for the computation of the pertinent variables, circumventing the need for estimating $\hat{\mathbf{R}}_M^{-1}$ explicitly.

B. Fast IAA-based LCMV Method

Estimation of the fundamental frequency using the IAA-LMCV method is performed by maximizing (24). Restricting the search to an equally spaced set of frequencies on the unit circle, this task can be efficiently tackled by means of the FFT as in the LCMV approach discussed before. First, though, the covariance matrix $\tilde{\mathbf{R}}_N$ is estimated using the IAA as described by (16) and (17), which can efficiently implemented without the need of direct estimation of the covariance matrix and its inverse. This can be accomplished using the celebrated Levinson-Durbin (LD) algorithm and some fast techniques for the evaluation of trigonometric polynomials related to structured matrices as detailed in [**?**], [**?**]. In this way, given the available data set $\mathbf{x}_N$, the displacement representation of the Toeplitz matrix in (17), as well as the displacement representation of its inverse, $\tilde{\mathbf{R}}_N^{-1}$, are iteratively estimated at a cost of

$$\begin{aligned}\mathcal{C}^{\text{FIAA}}(N, m) \approx m[&N^2 + 12N\log_2(2N) \\ &+ 1.5K\log_2 K]\end{aligned}\tag{47}$$

operations. With the displacement representation of $\tilde{\mathbf{R}}_N^{-1}$ at hand, denoted as $\mathbf{t}_N^{(1)}$ and $\mathbf{t}_N^{(2)}$, the displacement representation of the increased order matrix

$$\hat{\tilde{\mathbf{R}}}_{N+1} \triangleq \begin{bmatrix} \tilde{\mathbf{R}}_N^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

is found as

$$\begin{bmatrix} \tilde{\mathbf{R}}_N^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \tilde{\mathbf{R}}_N^{-1} \end{bmatrix} = \sum_{i=1}^{2}\sigma_i\mathbf{t}_{N+1}^{(i)}\mathbf{t}_{N+1}^{(i)} \tag{48}$$

as shown in the Appendix, where

$$\mathbf{t}_{N+1}^{(1)} = \mathbf{A}_{N+1}, \quad \sigma_1 = 1, \tag{49}$$

$$\mathbf{t}_{N+1}^{(2)} = \mathbf{J}_{N+1}\mathbf{A}_{N+1}^*, \quad \sigma_2 = -1, \tag{50}$$

with the matrices $\mathbf{A}_{N+1}$ and $\mathbf{J}_{N+1}$ defined in the Appendix. By combining (48)–(50) with a scheme similar to (38)–(39) and (43)–(45), one obtains an efficient way of computing diagonal and off-diagonal elements of the matrix valued trigonometric polynomial in (21). This algorithm is detailed in Tab. IV, and it has a computational complexity of

$$\mathcal{C}^{\text{FIAA-LCMV}} \approx \mathcal{C}^{\text{FIAA}}(N, m) + F\log_2 F + 1.5FL + FL^2. \tag{51}$$

IV. FAST APPROXIMATIVE IAA-BASED LCMV METHOD

Substantial computational savings can be achieved by using the approximative IAA algorithm recently proposed in [**?**] for the estimation of the covariance matrix and its inverse required in (24). In [**?**], a superfast implementation of the IAA algorithm, using the preconditioned conjugates gradient method and a Quasi Newton (QN) approach, was proposed for the formulation of an appropriate preconditioning. The

QN approach was also used for the implementation of an approximative IAA spectral estimation algorithm, where the inverse of the data covariance matrix was computed in terms of a lower order counterpart. Building on these results, we here present a novel, approximative, fundamental frequency estimation method. The proposed approach is motivated by the QN algorithms formulated in [?], [?], where the inverse of Toeplitz-like matrices is approximated by extrapolating the inverse of a lower size matrix. The lower size matrix is treated as if it has been associated with an autoregressive (AR) model of lower order $q \leq N$. Thus, instead of computing $\tilde{\mathbf{R}}^{-1}$, a lower order extrapolated estimate is adopted. This results in an approximate IAA algorithm, where $\alpha(\omega_k)$ and $\tilde{\mathbf{R}}$ are estimated iteratively as

$$\hat{\alpha}_k = [\mathbf{z}_N^H(\omega_k)\mathbf{Q}_N^{-1}\mathbf{z}_N(\omega_k)]^{-1}\mathbf{z}_N^H(\omega_k)\mathbf{Q}_N^{-1}\mathbf{x}_N, \quad (52)$$

$$\tilde{\mathbf{R}}_q = \sum_{k=0}^{K-1} |\hat{\alpha}_k|^2 \mathbf{z}_q(\omega_k)\mathbf{z}_q^H(\omega_k), \quad (53)$$

for $k = 0, 1, \ldots, K-1$, until practical convergence. The qth order autocorrelation matrix $\tilde{\mathbf{R}}_q$ is initialized to the identity matrix, $\mathbf{I}_q$, and

$$\mathbf{Q}_N^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0}^T \\ \mathbf{0} & \tilde{\mathbf{R}}_q^{-1} \end{bmatrix} + \mathbf{A}_{N,N-q+1}\mathbf{A}_{N,N-q+1}^H, \quad (54)$$

with

$$\mathbf{A}_{N,N-q+1} \triangleq \begin{bmatrix} \bar{\mathbf{a}}_N & \mathbf{D}\bar{\mathbf{a}}_N & \ldots & \mathbf{D}^{N-q}\bar{\mathbf{a}}_N \end{bmatrix}, \quad (55)$$

$$\bar{\mathbf{a}}_N = [\hat{\mathbf{a}}_q \; \mathbf{0}_{N-q}^T]^T, \quad (56)$$

where $\hat{\mathbf{a}}_q$ is the displacement generator associated with the power-normalized forward predictor of $\tilde{\mathbf{R}}_q$, as detailed in [?]. Then, the resulting approximative QN-IAA-based harmonic LCMV (QN-IAA-LCMV) method is formed by considering the cost function related to the estimate of $\mathbf{Q}_N$ as

$$\mathrm{E}\{\hat{P}_{\text{QN-IAA-LCMV}}(\omega_0)\} = \mathbf{1}_L^T \left[\boldsymbol{\mathcal{G}}_L^{\text{QN-IAA}}(\omega_0)\right]^{-1} \mathbf{1}_L, \quad (57)$$

where

$$\boldsymbol{\mathcal{G}}_L^{\text{QN-IAA}}(\omega_0) \triangleq \mathbf{Z}_{N,L}^H(\omega_0)\mathbf{Q}_N^{-1}\mathbf{Z}_{N,L}(\omega_0). \quad (58)$$

The fundamental frequency is then estimated by maximizing the output power using

$$\hat{\omega}_0 = \arg\max_{\omega_0 \in \Omega_0} \mathrm{E}\left\{\hat{P}_{\text{QN-IAA-LCMV}}(\omega_0)\right\}. \quad (59)$$

Choosing $q \ll N$, a significant computation reduction can be achieved at the expense of a possible degradation in the quality of the spectrum estimate. The displacement representation of the approximate inverse covariance matrix $\mathbf{Q}_N^{-1}$ is estimated from the available data $\mathbf{x}_N$ using the QN-IAA algorithm detailed in [?], where the LD algorithm is employed for the computation of the displacement representation of $\tilde{\mathbf{R}}_q^{-1}$. This representation is subsequently utilized in (54), at a cost of

$$\mathcal{C}^{\text{QN-IAA}}(m, N, q) \approx m\left[q^2 + 12q\log_2(2q) + N\log_2(N) + 1.5K\log_2(K)\right] \quad (60)$$

operations. Evaluation of (58) on the frequency grid of interest can be performed efficiently as in the IAA-based LCMV case detailed in the previous subsection, using the displacement representation of an extended matrix as

$$\begin{bmatrix} \mathbf{Q}_N^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Q}_N^{-1} \end{bmatrix} = \sum_{i=1}^{2} \sigma_i \bar{\mathbf{t}}_{N+1}^{(i)} \bar{\mathbf{t}}_{N+1}^{(i)}, \quad (61)$$

where

$$\bar{\mathbf{t}}_{N+1}^{(1)} = \begin{bmatrix} \bar{\mathbf{a}}_N^T & 0 \end{bmatrix}^T, \quad \sigma_1 = 1, \quad (62)$$

$$\bar{\mathbf{t}}_{N+1}^{(2)} = \mathbf{J}_{N+1}\left(\bar{\mathbf{t}}_{N+1}^{(1)}\right)^*, \quad \sigma_2 = -1. \quad (63)$$

This can be exploited to obtain a fast scheme similar to that described in Table IV, with the fast QN-IAA algorithm being used instead of the fast IAA algorithm for the estimation of the displacement representation of the data covariance matrix and the associated trigonometric polynomial in Step 1, and with $\bar{\mathbf{t}}_{N+1}^{(1)}$ and $\bar{\mathbf{t}}_{N+1}^{(2)}$ being used instead of $\mathbf{t}_{N+1}^{(1)}$ and $\mathbf{t}_{N+1}^{(2)}$ in Step 2. Thus, the overall computational complexity of the proposed QN-IAA-LCMV method is given by

$$\mathcal{C}^{\text{QN-IAA-LCMV}} \approx \mathcal{C}^{\text{QN-IAA}}(N, m, q) + F\log_2(F) + FL + FL^2. \quad (64)$$

V. Time-Recursive Implementations

We proceed to examine how the discussed methods may be efficiently updated as additional measurements becomes available.

A. Fast Harmonic LCMV

To allow for such an updating, the required covariance matrices should be replaced by suitable time-recursive estimates. Ta achieve this, an exponentially forgetting window approximation may be formed in place of (6) as

$$\hat{\mathbf{R}}_M(n) = \sum_{m=0}^{n} \lambda^{n-m}\mathbf{x}_M(m)\mathbf{x}_M^H(m) \quad (65)$$

$$= \lambda\hat{\mathbf{R}}_M(n-1) + \mathbf{x}_M(n)\mathbf{x}_M^H(n), \quad (66)$$

where $\lambda \in (0,1)$ is the forgetting factor controlling the memory fading of the recursive estimator, with $\hat{\mathbf{R}}_M(-1)$ initialized by the scaled identity matrix $\sigma\mathbf{I}_M$, for $\sigma > 0$ (see also [?]). Exponentially forgetting updating is here selected in favor of a rectangular sliding window updating as the former allows for a computationally simpler algorithm as well as that the associated spectral variables are then updated in a more stable manner [?]. As shown in [?], $\hat{\mathbf{R}}_M^{-1}(n)$ obeys a particularly interesting identity of the form

$$\begin{bmatrix} \hat{\mathbf{R}}_M^{-1}(n) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \lambda\hat{\mathbf{R}}_M^{-1}(n) \end{bmatrix} + \mathbf{t}_M^{(1)}(n)\mathbf{t}_M^{(1)H}(n) - \mathbf{t}_M^{(2)}(n)\mathbf{t}_M^{(2)H}(n) + \mathbf{t}_M^{(3)}(n)\mathbf{t}_M^{(3)H}(n), \quad (67)$$

where the vectors $\mathbf{t}_M^{(1)}(n)$, $\mathbf{t}_M^{(2)}(n)$, and $\mathbf{t}_M^{(3)}(n)$ are defined in terms of the power-normalized forward and backward predictors as well as the Kalman gain vector (often denoted $\mathbf{a}_M(n)$, $\mathbf{b}_M(n)$, and $\mathbf{w}_M(n)$ in the adaptive signal processing nomenclature) associated with the sample covariance matrix $\hat{\mathbf{R}}_M(n)$ at time instant n. Using (67) in conjunction with (10) results in an efficient way for the component-wise estimation of the matrix $\mathcal{G}_L^{\text{cov}}(n,\omega)$ at time instant n, required in (12). The (l_1, l_2)th component of this matrix can also be written as

$$[\mathcal{G}_L^{\text{cov}}(n,\omega_0)]_{l_1,l_2} = \psi_{l_1,l_2}^{\text{cov}}(n,\omega_0) \tag{68}$$

where $\psi_{l_1,l_2}^{\text{cov}}(n,\omega_0) \triangleq \mathbf{z}_M^H(l_1\omega_0)\hat{\mathbf{R}}_M^{-1}(n)\mathbf{z}_M(l_2\omega_0)$, which, using (67), takes the form

$$\psi_{l_1,l_2}^{\text{cov}}(n,\omega_0) = \frac{1}{1-\lambda e^{-j(l_1-l_2)\omega_0}}\Big[\varphi_{l_1}^{(1)}(n,\omega_0)\varphi_{l_2}^{(1)*}(n,\omega_0) - \varphi_{l_1}^{(2)}(n,\omega_0)\varphi_{l_2}^{(2)*}(n,\omega_0) + \varphi_{l_1}^{(3)}(n,\omega_0)\varphi_{l_2}^{(3)*}(n,\omega_0)\Big], \tag{69}$$

with

$$\varphi_l^{(i)}(n,\omega_0) \triangleq \mathbf{z}_M^H(l\omega_0)\mathbf{t}_M^{(i)}(n). \tag{70}$$

As the resulting search is restricted on a set of equally spaced frequencies on the unit circle for frequencies ω up to $2\pi/L$, one may write

$$\varphi_l^{(i)}(n,\omega) = \varphi_1^{(i)}(n,l\omega). \tag{71}$$

The time-varying generator vectors of $\hat{\mathbf{R}}_M^{-1}(n)$, namely $\mathbf{t}_M^{(i)}(n)$, for $i = 1,2,3$, are computed using a standard recursive least squares (RLS) algorithm at a cost of approximately $2M^2$ operations. Alternatively, a stabilized fast RLS (FRLS) algorithm can be employed at a reduced cost of $7M$ (see also [?] and references therein). Summarizing, the proposed time-recursive fast harmonic LCMV method consists of the following steps:

1) computation of the time varying generator vectors $\mathbf{t}_M^{(i)}(n)$, for $i = 1,2,3$, using the standard RLS or FRLS algorithm, or any other well behaved method,
2) element-wise computation of (69) using three FFTs as implied by (69) and (70), and
3) the search for the optimal fundamental frequency using (12).

It is worth noting that step 1) only needs to be updated at each time instant in a time-recursive way. The computations involved in the remaining steps 2) and 3), albeit their time-varying formulation, are not truly time-recursive in nature, since these depend on variables at the current time instant n only. This feature enables the development of time-recursive fundamental frequency algorithms with time hopping, in cases when the estimation of the fundamental frequency is not required at each time instant n, but only at every K_{hop} time units instead. The computational complexity per processed sample of the proposed time recursive harmonic LCMV method is therefore

$$\mathcal{C}_{\text{TR}}^{\text{FLCMV}} \approx 2M^2 + \left[1.5F\log_2 F + (2L+L^2)F\right]/K_{\text{hop}} \tag{72}$$

operations, when the RLS is used at step 1), or

$$\mathcal{C}_{\text{TR-F}}^{\text{FLCMV}} \approx 7M + \left[1.5F\log_2 F + (2L+L^2)F\right]/K_{\text{hop}} \tag{73}$$

operations, when the FRLS is employed instead.

B. Fast IAA-based LCMV

Regrettably, the time frequency interleaving imposed by the iterative scheme in (16) and (17) of the IAA-based approach does not allow for a pure time-recursive estimation of the IAA-based covariance matrix and its subsequent use for a time-recursive computation of (21), required by the IAA-based cost function in (24). However, the development of a time-recursive scheme for the IAA-based fundamental frequency estimation is still feasible. As recently proposed in [?], the estimate of the covariance matrix at time instant n should be approximately equal to the estimate of the covariance matrix at time instant $n-1$ upon convergence. Thus, an approximative time-recursive update of the covariance matrix in (17) may be constructed as

$$\hat{\alpha}_k(n) = \frac{\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}(n-1)\mathbf{x}_N(n)}{\mathbf{z}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}(n-1)\mathbf{z}_N(\omega_k)}, \tag{74}$$

$$\tilde{\mathbf{R}}_N(n) = \sum_{k=0}^{K-1} |\hat{\alpha}_k(n)|^2 \mathbf{z}_N(\omega_k)\mathbf{z}_N^H(\omega_k), \tag{75}$$

where $\mathbf{x}_N(n) = [x(n-N+1)\ x(n-N+2)\ \ldots\ x(n)]^T$ is the data vector at time instant n. Although $\mathcal{G}_L^{\text{IAA}}(n,\omega_0)$, resulting from (21) and (75), is time-dependent, the required computations are not time-recursive in nature. This enables time hopping in the IAA-based fundamental frequency estimation case as well if desired. The computational complexity of the proposed time-recursive, IAA-based, harmonic LCMV scheme is therefore approximately given by

$$\mathcal{C}_{\text{TR}}^{\text{FIAA-LCMV}} \approx \mathcal{C}^{\text{FIAA}}(N,1) + F\left[\log_2 F + 1.5L + L^2\right]/K_{\text{hop}} \tag{76}$$

The time-recursive, QN-, and IAA-based, harmonic LCMV methods are organized in a similar way.

VI. Experimental Results

The experimental results obtained during the evaluation of the proposed methods are divided into three parts. First, we evaluate the computational complexities of the implementations considered herein. Then, we evaluate the statistical performance of the pitch estimators proposed for batch processing, and, finally, we evaluate the tracking performance of the proposed time-recursive pitch estimators.

A. Computational Complexities

In Fig. 1, we have depicted the computational complexities as a function of the number of samples, N, for the different implementations described in the previous sections. The computational complexities are obtained using the herein presented theoretical expressions. First, we considered the computational complexities for batch processing as shown in Figs. 1a and 1b. From these figures, we can see that the fast implementations indeed have lower complexities than the direct implementations of the LCMV and IAA-LCMV methods. Furthermore, we observe that the implementations of the IAA-LCMV method generally have higher complexities than the corresponding implementations of the LCMV method, and that the fast implementations proposed herein reduce the computational complexity significantly compared to the

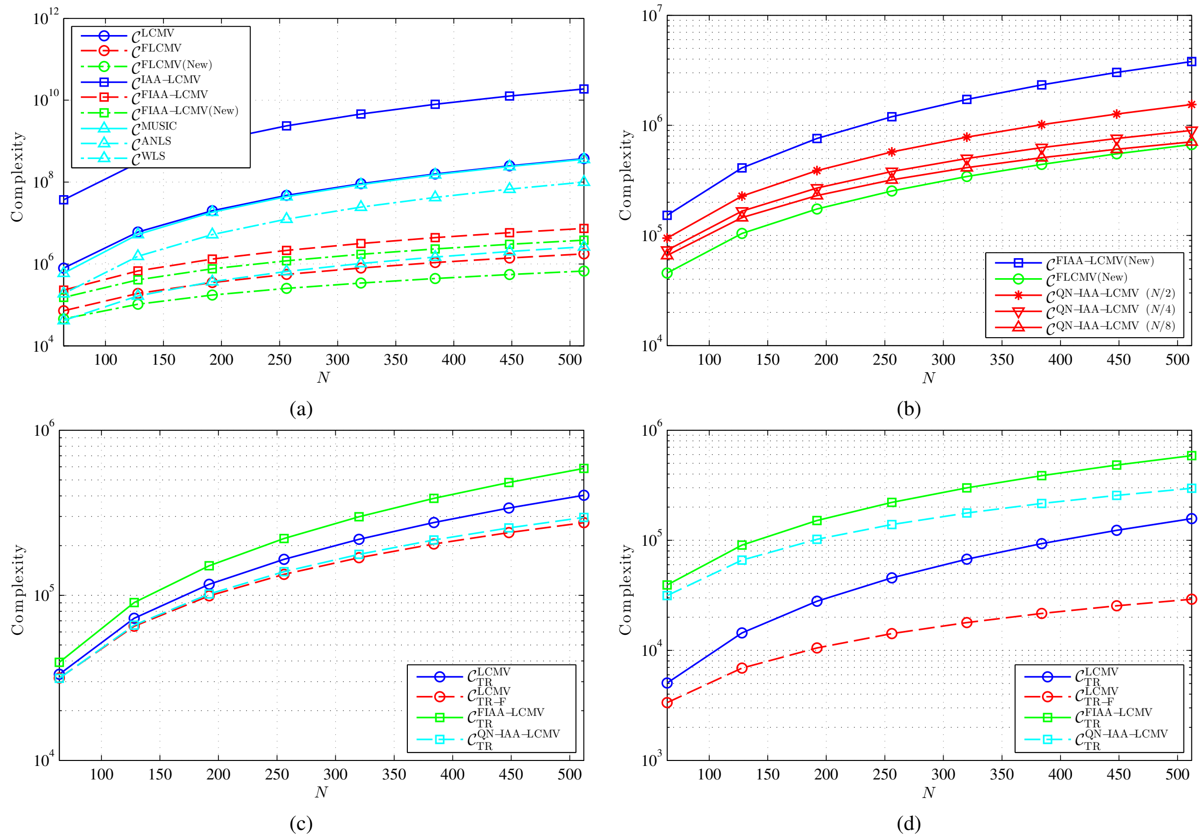

Fig. 1. Computational complexities of the different harmonic LCMV fundamental frequency estimation algorithms for (a)–(b) batch processing and (c)–(d) time-recursive processing. In all cases, $M = N/2 + 1$, $m = 10$, $K = 4N$, $F = 10N$, $L = 5$ $L_{\rm i} = 0$. Time hopping with $K_{\rm hop} = 1$ and $K_{\rm hop} = 10$ are considered in (c) and (d), respectively. The complexities for the fast exact implementations proposed herein is denoted by (New) in the superscript, while the corresponding fast implementations proposed in [**?**] has no extra superscript.

implementations proposed in [**?**]. Finally, we note that the QN-based approximative implementation of the IAA-LCMV methods can have computational complexity comparable to that of the fast implementation of the LCMV method if q is allowed to be small, although it should be noted that this may have a negative impact on the fundamental frequency estimation accuracy. The above observations imply the following relationship in the considered scenarios: $\mathcal{C}^{\text{IAA-LCMV}} > \mathcal{C}^{\text{LCMV}} > \mathcal{C}^{\text{FIAA-LCMV}} > \mathcal{C}^{\text{QN-IAA-LCMV}} > \mathcal{C}^{\text{FLCMV}}$. Then, we considered the computational complexities for the proposed time-recursive fundamental frequency estimators in Figs. 1c–1d. When time hopping with $K_{\rm hop} = 1$ is used, we observe the following relationship between the complexities: $\mathcal{C}_{\text{TR}}^{\text{FIAA-LCMV}} > \mathcal{C}_{\text{TR}}^{\text{LCMV}} > \mathcal{C}_{\text{TR}}^{\text{QN-IAA-LCMV}} > \mathcal{C}_{\text{TR-F}}^{\text{LCMV}}$. In other words, the implementations of the IAA-LCMV method have the highest computational complexity, but by using the QN-based approximative implementation, the complexity gets close to that of the fast exact implementation of the LCMV method. When using time hopping with $K_{\rm hop} = 10$, the computational complexities of the IAA-based methods are somewhat higher in relation to the sample covariance based methods, i.e., $\mathcal{C}_{\text{TR}}^{\text{FIAA-LCMV}} > \mathcal{C}_{\text{TR}}^{\text{QN-IAA-LCMV}} > \mathcal{C}_{\text{TR}}^{\text{LCMV}} > \mathcal{C}_{\text{TR-F}}^{\text{LCMV}}$.

B. Batch Processing Implementations

We proceed to evaluate the estimation accuracy of the fundamental frequency estimators considered herein, investigating the influence of K, N, the expected fundamental frequency, and the spacing between fundamental frequencies (the last in a two source scenario). For these experiments, the number of candidate fundamental frequencies was $|\Omega_0| = 2^{16}$. Initially, we consider a noisy, harmonic signal as in the previous investigation. Fig. 2a shows the measured mean squared error (MSE) of the IAA-LCMV and QN-IAA-LCMV estimators as a function of K, with the fundamental frequency being samples from $\mathcal{U}(0.3, 0.4)$. For this and all the following experiments in this subsection, all MSEs were obtained from 500 simulations with different noise realizations. The results in Fig. 2a show the performance of the estimators for two different sample lengths, i.e., $N = 40$ and $N = 80$. As is clear from the figure, one needs more frequency points when N is increased to achieve the maximum possible performance for both the IAA-LCMV and the QN-IAA-LCMV methods. Note that, from this figure, it seems that the QN-IAA-LCMV has the best performance. This is only the case for low Ks where IAA-LCMV suffers from line splitting, whereas, for high Ks, the IAA-LCMV method shows the best, general performance as expected. For $N = 40$, $K \approx 400$ seems to be sufficient, whereas at least $K \approx 1200$ frequency points are needed for $N = 80$. In this and all the following experiments, the order of the autocorrelation matrix was lowered to $q = \lfloor N/2 \rfloor$ in the QN-IAA-LCMV implementation. Then, in Fig. 2b, the MSE of the LCMV-based filtering methods is shown as function of

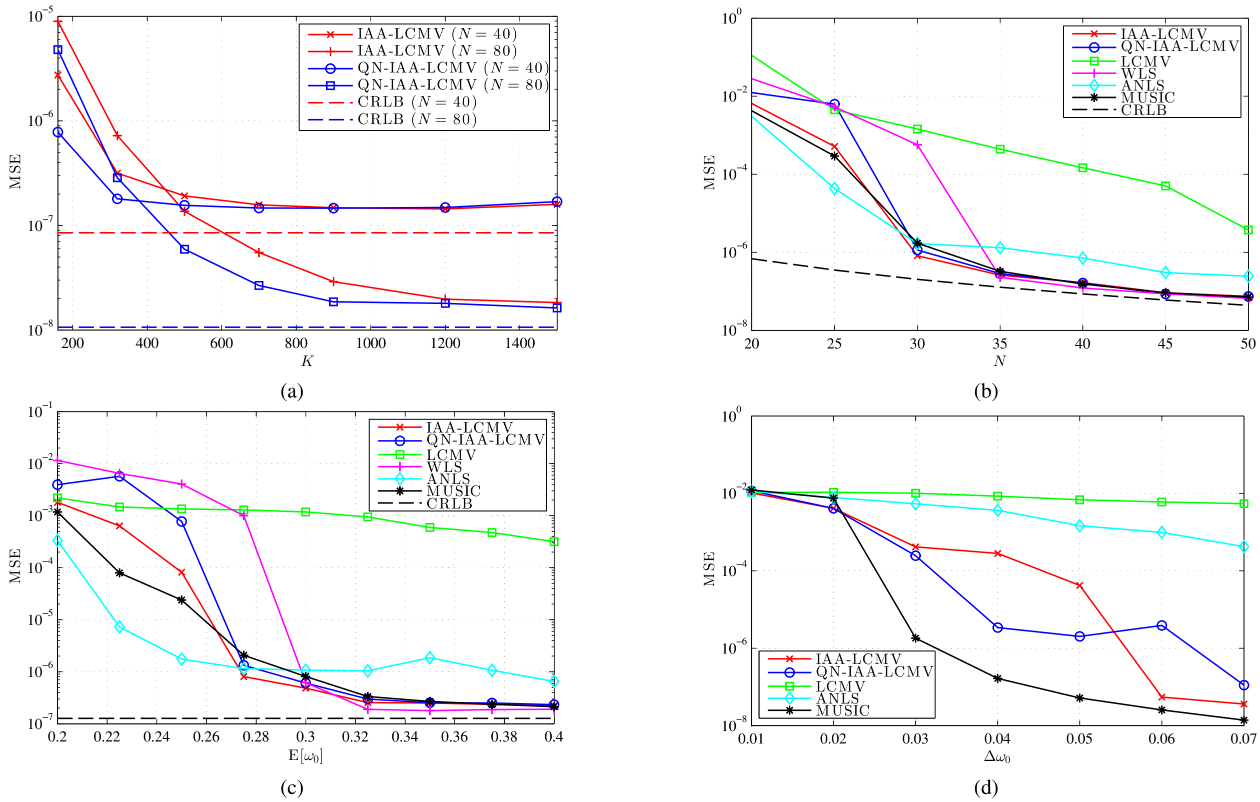

Fig. 2. Mean squared errors of different fundamental frequency estimators as a function of (a) the number of frequency grid points used for the IAA-based covariance matrix estimate, (b) the number of available samples, (c) the expected fundamental frequency, and (d) the spacing between fundamental frequencies in a two source scenario. Moreover, the Cramér-Rao lower bound (CRLB) is depicted in (b) and (c).

N, for $K = 1000$ frequency grid points, and it is compared with a WLS method [**?**], [**?**], an approximate NLS (ANLS) method [**?**], and a MUSIC-based method [**?**]. For comparison, the computational complexities of the MUSIC, ANLS, and WLS methods are

$$C^{\text{MUSIC}} \approx M^2\bar{N} + M^3 + \bar{F}(LM + L)(M - L - L_{\text{i}}), \quad (77)$$

$$C^{\text{ANLS}} \approx \bar{F}(LN + L), \quad (78)$$

$$C^{\text{WLS}} \approx M^2\bar{N} + M^3 + \bar{F}(M^2 + M) + 6L, \quad (79)$$

where L_{i} is the number of interfering sinusoids, and for the MUSIC and WLS methods, M denotes the subvector length. It is not straightforward to compare this complexity with those of the LCMV methods, though, since the MUSIC method requires not only the model order of the desired signal, but also the model order of all interfering signals to be known. In practice, it would require a search across all possible model orders, for every source to obtain all these model order estimates, which does not appear directly from the computational complexity expressions above for the known model order scenario. Also, the MUSIC method will be vulnerable against erroneous model order estimates for the interfering sources, which is not a problem for the LCMV methods. The provided complexity for the WLS method, assumes that the unconstrained frequency estimates is found using the MUSIC method. In Fig. 1a, the complexities of the MUSIC, ANLS, and WLS methods are compared to those of the implementations considered in this paper. In all simulations considered in this section, the filter length used in the LCMV method was $\lfloor N/4 \rfloor$, and the subvector length used in the MUSIC-based method was $\lfloor N/2 \rfloor$. One may note from Fig. 2b that the IAA-based estimators outperform all but the MUSIC method for data lengths in the interval, say, $30 < N < 35$. For larger data lengths, the IAA estimators outperform the ANLS and LCMV methods, while their performance is similar to that of the WLS and MUSIC method. Examining the influence of the fundamental frequency, Fig. 2c shows the MSE as a function of the expected fundamental frequencies, $\text{E}[\omega_0]$, where, in each simulation, the fundamental frequency was sampled from $\text{E}[\omega_0] + \mathcal{U}(-0.001, 0.001)$, using $N = 35$ and $K = 1000$. As is clear from the results, the IAA estimators outperforms the LCMV and ANLS methods for $\text{E}[\omega_0] > 0.28$, while their performances are comparable to those of the MUSIC and WLS methods for $\text{E}[\omega_0] > 0.3$. Finally, we compared the discussed methods in a scenario with two harmonic sources, examining two sources with $L = 3$ unit amplitude harmonics. The ratio between each of the sources and a white Gaussian noise source was 40 dB. In each simulation, the fundamental frequency ω_0^1 of first source was sampled from $\mathcal{U}(0.299, 0.301)$ and the fundamental frequency of the second source was $\omega_0^2 = \omega_0^1 + \Delta\omega_0$, where $\Delta\omega_0$ is the spacing, using $N = 60$, and $K = 1,000$. As seen in Fig. 2d, the performances of the IAA methods are generally better than those of the LCMV and ANLS methods, while the MUSIC method shows the best performance (although it should be recalled that the MUSIC estimate assumes perfect a

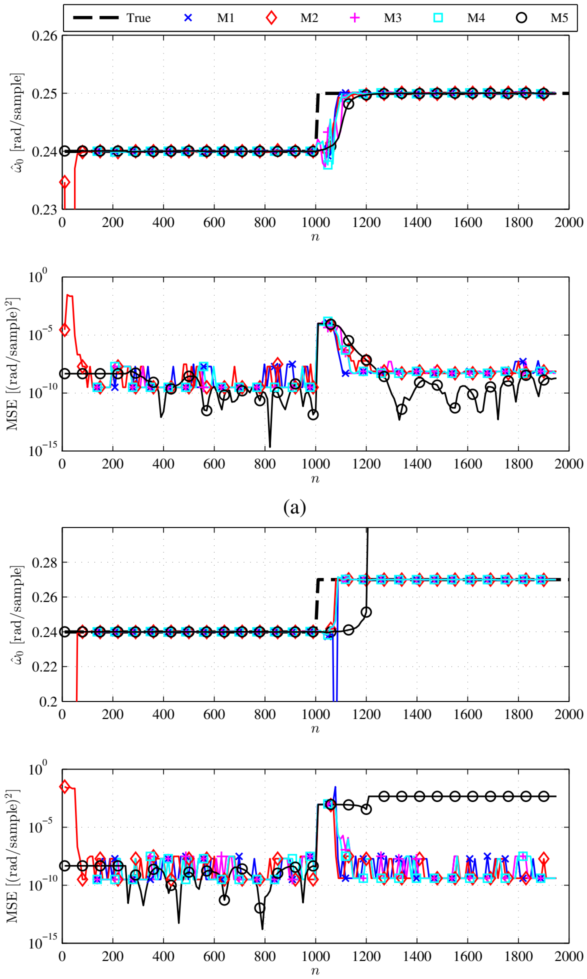

Fig. 3. Plot of (top) the true pitch and pitch estimates obtained using M1-M5, and (bottom) the MSE associated with the pitch estimates.

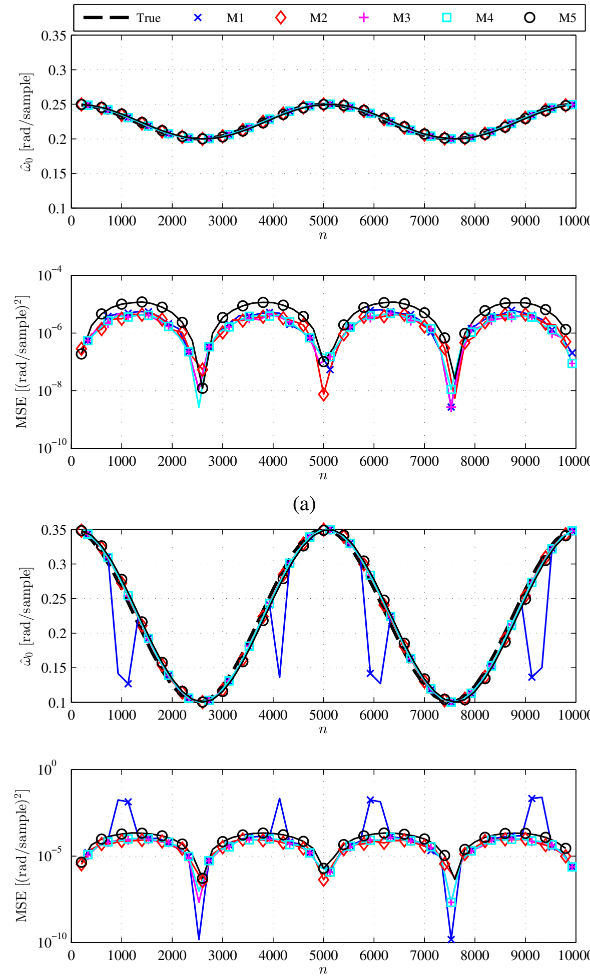

Fig. 4. Plots of (top) the true pitch and pitch estimates obtained using M1-M5, and (bottom) the MSE associated with the pitch estimates.

priori knowledge of all model orders). In summary and maybe most importantly, all the results obtained from the statistical evaluation show that the IAA can be used to improve the spectral resolution of the LCMV method that uses the sample covariance matrix estimate as we claimed in the introduction.

C. Time-Recursive Implementations

We proceed to evaluate the performance of the time-recursive implementations. These implementations are evaluated qualitatively on synthetic and real-life signals. In the evaluation, we consider

- **M1** the LCMV method implemented by applying (12) on rectangular sliding windowed data,
- **M2** the LCMV method implemented by (67)-(71),
- **M3** the IAA-LCMV method implemented by (74)-(75),
- **M4** the IAA-LCMV method implemented by (74)-(75) with an exponential forgetting factor on the IAA amplitude spectrum estimate,
- **M5** the NLS pitch tracker proposed in [**?**] without Kalman filtering.

The NLS pitch tracker (M5) is included in the evaluation for benchmarking purposes. In all methods, observed data vectors

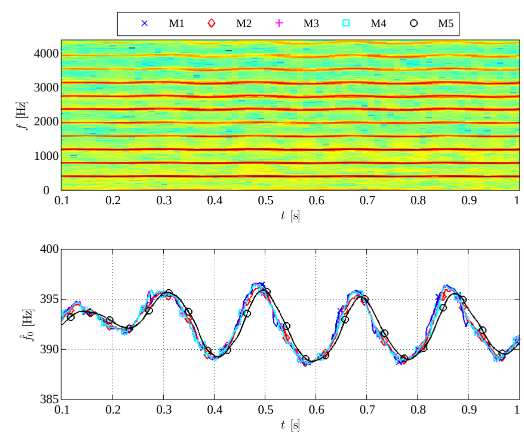

Fig. 5. Plots of (top) the spectrogram of a real-life violin signal with vibrato, and (bottom) pitch estimates obtained using M1-M5 when applied on the violin signal.

of length $N = 128$ were considered. In M1–M2, a filter length of $M = 50$ were used, and the forgetting factor λ in the RLS algorithm of M2 was set to 0.98. The IAA-based methods (M3–M4) were set up with $K = 2000$ and 10 iterations for initialization. The forgetting factor in M4 was also set to $\lambda = 0.98$. In all methods, only pitch candidates in the interval $2\pi[0.004, 0.1]$ were considered, with $|\Omega_0| = 40000$ for M1–

M4 and $|\Omega_0| = 2^{14}$ for M5. Using the above setup, we first applied the methods under evaluation on a synthetic signal constituted by a harmonic signal with $L = 5$ with an abrupt pitch change and white Gaussian noise; the SNR was 20 dB. A total of 2000 samples of the signal was observed. For the first 1000 samples, the true pitch was $\omega_0 = 0.24$, after which it changed to $\omega_0 = 0.24 + \delta$. In Fig. 3a and 3b, we show simulation results for $\delta = 0.01$ and $\delta = 0.03$, respectively. For $\delta = 0.01$, all methods show similar tracking performance. The NLS tracker (M5) obtains the lowest MSEs, which is explained by the fact that it obtains the pitch estimate using a gradient search rather than using a grid search as in the methods M1–M4. For a larger change in pitch ($\delta = 0.03$), the NLS tracker (M5) fails to track the pitch compared to the proposed methods. This can also be explained by the gradient search. We also evaluated the methods on a synthetic signal with smooth pitch changes. Again, the number of harmonics was $L = 5$, the noise was white Gaussian, and the SNR was 20 dB. Using frequency modulation, we obtained a harmonic signal with a pitch of $\omega_{0,\text{FM}}(n) = \omega_0 + \delta \cos(2\pi f_{\text{FM}} n / N_{\text{total}})$ at time instance n, where $\omega_0 = 0.225$, $f_{\text{FM}} = 2$ samples^{-1} is the modulation frequency, and $N_{\text{total}} = 10000$ is the total number observed samples. Simulation results for $\delta = 0.025$ and $\delta = 0.125$ are depicted in 4a and 4b, respectively. For $\delta = 0.025$, the performance in terms of MSE is similar for the methods M1–M4, whereas that MSE is generally larger for M5. The conclusions are the same for $\delta = 0.125$, except that the LCMV method with a sliding rectangular window (M1) has problems with pitch halving. Finally, we applied the methods on a real-life violin signal with vibrato. The spectrogram of the signal and the estimation results are shown in Fig. 5. As it appears from these results, all methods were able to track the pitch fluctuations of this real-life signal.

VII. Conclusions

In this paper, we consider fast implementations of two recently proposed pitch estimators. The estimators considered are both based on optimal filtering using the linearly constrained minimum variance (LCMV) principle, but uses different estimates of the data covariance matrix; in one of the estimators, the sample covariance matrix estimate is used, whereas an iterative adaptive approach (IAA) estimate is used in the other. We propose fast implementations for both of the pitch estimators, exploiting the low displacement rank of the necessary products of Toeplitz-like matrices. As shown, this reduces the computational complexity by several orders of magnitude. We also propose an approximative fast implementation, using covariance matrices of lower size and extrapolation. This implementation has an even lower computational complexity. Finally, we propose time-recursive implementations of both estimators. These provide yet another mean for reducing the complexity. Our quantitative evaluation show that the considered IAA-based estimator outperforms other state-of-the-art pitch estimators in terms of mean squared error in many scenarios. Moreover, the qualitative evaluations show that the proposed time-recursive implementations can be used for tracking abrupt as well as smooth pitch changes of both synthetic and real-life signals. Obvious and relevant, future extensions of the methods proposed herein are, e.g., to also incorporate model order estimation, and to generalize the IAA-based methods to support estimation of the covariance matrix of the observed signal from more than one snapshot. Both extensions are likely to increase the robustness of the presented methods.

Appendix

Displacement representation of $\hat{\bar{\mathbf{R}}}_{M+1}$

Consider the unscaled sampled covariance matrix (13) corresponding to an increased order filter as

$$\bar{\mathbf{R}}_{M+1} \triangleq \sum_{n=M}^{N-1} \mathbf{x}_{M+1}(n)\mathbf{x}_{M+1}(n), \tag{80}$$

where scaling by the factor $1/(N-M)$ is omitted for reasons of notation simplicity. (80) is partitioned in an upper and a lower form as

$$\begin{bmatrix} \bar{\mathbf{R}}_M^{\text{up}} & \mathbf{r}_M^{\text{b}} \\ \mathbf{r}_M^{\text{b}H} & r_M^{\text{bo}} \end{bmatrix} = \begin{bmatrix} r_M^{\text{fo}} & \mathbf{r}_M^{\text{f}H} \\ \mathbf{r}_M^{\text{f}} & \bar{\mathbf{R}}_M^{\text{dn}} \end{bmatrix}, \tag{81}$$

and where

$$\bar{\mathbf{R}}_M^{\text{up}} = \bar{\mathbf{R}}_M - x_M(M-1)\mathbf{x}_M^H(M-1), \tag{82}$$

$$\bar{\mathbf{R}}_M^{\text{dn}} = \bar{\mathbf{R}}_M - x_M(N)\mathbf{x}_M^H(N), \tag{83}$$

noting that $\bar{\mathbf{R}}_M$ that appears above corresponds to the unscaled version of $\hat{\mathbf{R}}_M$ defined by (13). Application of the matrix inversion lemma for partitioned matrices and for low rank modified matrices, [?] results in

$$\begin{bmatrix} \left(\bar{\mathbf{R}}_M^{\text{up}}\right)^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \mathbf{B}_{M+1}\mathbf{B}_{M+1}^H = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \left(\bar{\mathbf{R}}_M^{\text{dn}}\right)^{-1} \end{bmatrix} + \mathbf{A}_{M+1}\mathbf{A}_{M+1}^H \tag{84}$$

where

$$\mathbf{B}_{M+1} = \begin{bmatrix} -\left(\bar{\mathbf{R}}_M^{\text{up}}\right)^{-1} \mathbf{r}_M^{\text{b}} \\ 1 \end{bmatrix} / \sqrt{\alpha_M^{\text{b}}}, \tag{85}$$

$$\alpha_M^{\text{b}} = r_M^{\text{bo}} - \mathbf{r}_M^{\text{b}H} \left(\bar{\mathbf{R}}_M^{\text{up}}\right)^{-1} \mathbf{r}_M^{\text{b}}, \tag{86}$$

$$\mathbf{A}_{M+1} = \begin{bmatrix} 1 \\ -\left(\bar{\mathbf{R}}_M^{\text{dn}}\right)^{-1} \mathbf{r}_M^{\text{f}} \end{bmatrix} / \sqrt{\alpha_M^{\text{f}}}, \tag{87}$$

$$\alpha_M^{\text{f}} = r_M^{\text{fo}} - \mathbf{r}_M^{\text{f}H} \left(\bar{\mathbf{R}}_M^{\text{dn}}\right)^{-1} \mathbf{r}_M^{\text{f}}, \tag{88}$$

and $(\bar{\mathbf{R}}_M^{\text{up}})^{-1} = \bar{\mathbf{R}}_M^{-1} + \mathbf{v}_M\mathbf{v}_M^H/\alpha_M^v$, $(\bar{\mathbf{R}}_M^{\text{dn}})^{-1} = \bar{\mathbf{R}}_M^{-1} + \mathbf{w}_M\mathbf{w}_M^H/\alpha_M^w$, with

$$\mathbf{w}_M = -\bar{\mathbf{R}}_M^{-1}\mathbf{x}_M(N), \tag{89}$$

$$\alpha_M^w = 1 - \mathbf{x}_M^H(N)\bar{\mathbf{R}}_M^{-1}\mathbf{x}_M(N), \tag{90}$$

$$\mathbf{v}_M = -\bar{\mathbf{R}}_M^{-1}\mathbf{x}_M(M-1), \tag{91}$$

$$\alpha_M^v = 1 - \mathbf{x}_M^H(M-1)\bar{\mathbf{R}}_M^{-1}\mathbf{x}_M(M-1). \tag{92}$$

Combining (84)-(92), one obtains

$$\begin{bmatrix} \bar{\mathbf{R}}_M^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \bar{\mathbf{R}}_M^{-1} \end{bmatrix} = \mathbf{A}_{M+1}\mathbf{A}_{M+1}^H$$
$$-\mathbf{B}_{M+1}\mathbf{B}_{M+1}^H + \mathbf{W}_{M+1}\mathbf{W}_{M+1}^H - \mathbf{V}_{M+1}\mathbf{V}_{M+1}^H. \tag{93}$$

where $\mathbf{W}_{M+1} = \begin{bmatrix} 0 & \mathbf{w}_M \end{bmatrix}^T$ and $\mathbf{V}_{M+1} = \begin{bmatrix} \mathbf{v}_M & 0 \end{bmatrix}^T$. Finally, proper scaling yields

$$\begin{bmatrix} \hat{\mathbf{R}}_M^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \hat{\mathbf{R}}_M^{-1} \end{bmatrix} = \sum_{i=1}^{4} \sigma_i \mathbf{t}_{M+1}^{(i)} \mathbf{t}_{M+1}^{(i)}, \quad (94)$$

where

$$\mathbf{t}_{M+1}^{(1)} = \mathbf{A}_{M+1}\sqrt{L}, \quad \sigma_1 = 1, \quad (95)$$
$$\mathbf{t}_{M+1}^{(2)} = \mathbf{B}_{M+1}\sqrt{L}, \quad \sigma_2 = -1, \quad (96)$$

$$\mathbf{t}_{M+1}^{(3)} = \mathbf{W}_{M+1}\sqrt{L}, \quad \sigma_1 = 1, \quad (97)$$
$$\mathbf{t}_{M+1}^{(4)} = \mathbf{V}_{M+1}\sqrt{L}, \quad \sigma_2 = -1, \quad (98)$$

where $L = N - M + 1$, noting that variables (85)–(88) and (89)–(92) can all be computed once the displacement representation of $\mathbf{R}_M^{-1}$ is available [**?**] at a cost of $O(M)$.

DISPLACEMENT REPRESENTATION OF $\hat{\tilde{\mathbf{R}}}_{N+1}$

Consider the increased order covariance matrix associated with the IAA-based LCMV fundamental frequency estimator (17) and introduce the upper and lower partitions of the form

$$\tilde{\mathbf{R}}_{N+1} = \begin{bmatrix} r_0 & \mathbf{r}_N^{\mathrm{f}H} \\ \mathbf{r}_N^{\mathrm{f}} & \tilde{\mathbf{R}}_N \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{R}}_N & \mathbf{J}_N \mathbf{r}_N^{\mathrm{f}*} \\ \mathbf{r}_N^{\mathrm{f}T}\mathbf{J}_N & r_0 \end{bmatrix}, \quad (99)$$

where $\mathbf{J}_N$ is the exchange matrix. Application of the matrix inversion lemma for partitioned matrices (see, e.g., [**?**]) results in

$$\tilde{\mathbf{R}}_{N+1}^{-1} = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \tilde{\mathbf{R}}_N^{-1} \end{bmatrix} + \mathbf{A}_{N+1}\mathbf{A}_{N+1}^H \quad (100)$$
$$= \begin{bmatrix} \tilde{\mathbf{R}}_N^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \mathbf{J}_{N+1}\mathbf{A}_{N+1}\mathbf{A}_{N+1}^T\mathbf{J}_{N+1}, \quad (101)$$

where $\mathbf{A}_{N+1} = [1 \ \ -\tilde{\mathbf{R}}_N^{-1}\mathbf{r}_N^{\mathrm{f}}]^T/(\alpha_N^{\mathrm{f}})^{0.5}$ and $\alpha_N^{\mathrm{f}} = r_0 - \mathbf{r}_N^{\mathrm{f}H}\tilde{\mathbf{R}}_N^{-1}\mathbf{r}_N^{\mathrm{f}}$. Subtracting (100) from (101) yields

$$\begin{bmatrix} \tilde{\mathbf{R}}_N^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \tilde{\mathbf{R}}_N^{-1} \end{bmatrix} = \sum_{i=1}^{2} \sigma_i \mathbf{t}_{N+1}^{(i)} \mathbf{t}_{N+1}^{(i)}, \quad (102)$$

where

$$\mathbf{t}_{N+1}^{(1)} = \mathbf{A}_{M+1}, \quad \sigma_1 = 1, \quad (103)$$
$$\mathbf{t}_{N+1}^{(2)} = \mathbf{J}_{N+1}\mathbf{A}_{N+1}^*, \quad \sigma_2 = -1. \quad (104)$$

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