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ABSTRACT

In this paper, we introduce a non-parametric 2-D spectral estimator for smooth spectra, allowing for irregularly sampled measurements. The estimate is formed by assuming that the spectrum is smooth and will vary slowly over the frequency grids, such that the spectral density inside any given rectangle in the spectral grid may be approximated well as a plane. Using this framework, the 2-D spectrum is estimated by finding the solution to a convex covariance fitting problem, which has an analytic solution. Numerical simulations indicate the achievable performance gain as compared to the Blackman-Tukey estimator.

Index Terms— 2-D frequency estimation, smooth spectrum, irregular sampling.

1. INTRODUCTION

Limited attention has been dedicated to the problem of finding estimators of smooth spectra, especially when formed from irregularly sampled data. In contrast, a vast number of contributions have been made on the estimation on sparse spectra, including 2-D spectra, e.g., [1–4]. Recently, in [5, 6], a smooth spectral estimator was introduced, assuming a piecewise linear spectral density and formulating the spectral estimation as a maximum likelihood optimization. We expanded on this idea in [7], where we allowed for (non-uniformly sampled) time-varying signals, forming a smooth time-frequency representation of the signal. In this work, we further this development, proposing a *non-parametric* estimator of smooth 2-D spectra. To incorporate the assumption of smoothness, we expand on the piecewise linear assumption in [5, 6] to instead assume each 2-D frequency rectangle to be well represented as a plane. Forming a transformation tensor over the frequency grid, a covariance representation of the smooth spectrum is formed, which is then fitted to a covariance estimate based on the data. Different from many other forms of spectral estimators, such as, for instance, ARMA based estimators, the proposed method does not require any *a priori* model order information.

The remainder of the paper is organized as follows: In the next section, we introduce the assumed data model including our smoothness assumption. Then, in section 3, we formulate the proposed spectral estimate, followed, in section 4, by the derivation of the analytic solution. Finally, we conclude, in section 5, by illustrating the achievable performance of the proposed estimator.

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2. DATA MODEL

Consider a stationary signal $y(t,s)$, where $t = t_0, \dots, t_{N_t-1}$ and $s = s_0, \dots, s_{N_s-1}$ denote the (possibly non-uniform) sampling times, having a two-dimensional spectrum denoted $\Phi(\omega_1, \omega_2)$. Furthermore, the spectrum is assumed to be band-limited within $[-B_1, B_1]$ and $[-B_2, B_2]$, as well as being alias-free and smooth. Any frequency point, (ω_1, ω_2), in the spectrum is assumed well modeled as

$$\begin{aligned}\Phi(\omega_1,\omega_2) =& \frac{\Phi(\omega_{1,j+1},\omega_{2,k}) - \Phi(\omega_{1,j},\omega_{2,k})}{\Delta_1}(\omega_1-\omega_{1,j}) + \\ & \frac{\Phi(\omega_{1,j},\omega_{2,k+1}) - \Phi(\omega_{1,j},\omega_{2,k})}{\Delta_2}(\omega_2-\omega_{2,k}) + \\ & \Phi(\omega_{1,j},\omega_{2,k}) \\ =& \frac{\omega_1-\omega_{1,j}}{\Delta_1}\Phi(\omega_{1,j+1},\omega_{2,k}) + \\ & \frac{\omega_2-\omega_{2,k}}{\Delta_2}\Phi(\omega_{1,j},\omega_{2,k+1}) + \\ & \left(1 - \frac{\omega_1-\omega_{1,j}}{\Delta_1} - \frac{\omega_2-\omega_{2,k}}{\Delta_2}\right)\Phi(\omega_{1,j},\omega_{2,k})\end{aligned} \tag{1}$$

where $\omega_{n,j}$ and $\omega_{n,j+1}$ are the closest two points in the n:th dimension, for $n = 1$ and 2, and with Δ_1 and Δ_2 denoting the width of the first and second frequency grid surface, respectively. In other words, any point inside a rectangle in the grid structure is assumed to lie on the plane defined by the lower left three corners of the rectangle. Furthermore, the covariance function is defined as

$$\boldsymbol{R}(\tau_1,\tau_2) = \int_{-B_1}^{B_1}\int_{-B_2}^{B_2} \Phi(\omega_1,\omega_2)\, e^{i\omega_1\tau_1 + i\omega_2\tau_2} d\omega_1 d\omega_2 \tag{2}$$

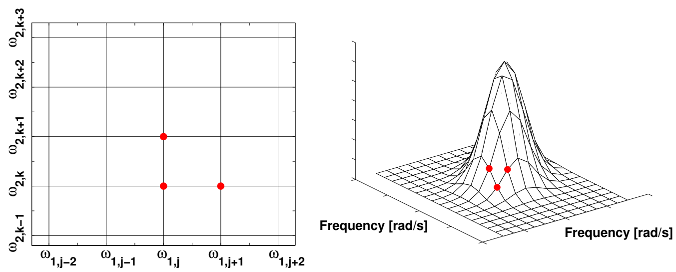

Fig. 1: Example of the proposed grid structure. The figure on the left shows the grid structure that the frequency plane is divided into. Each point inside a rectangle is assumed to lie on the plane defined by the lower left three corners. The right figure shows the corresponding structure on a 2-D power spectrum.

3. SMOOTH 2D SPECTRAL ESTIMATOR

In order to form the smooth spectral estimate, the 2-D spectral plane is divided into a grid structure with $M_1 \times M_2$ grid points. An example of such a grid structure can be seen in Figure 1. The smaller values of M_1 and M_2 that are assigned, the smoother one assumes the spectrum to be. The spectrum is estimated by minimizing the distance between the corresponding covariance function and a transform of the spectrum in the correlation domain. Combining (1) and (2) together with the fact that the rectangles in the grid structure are disjoint, allowing for the double integral to be divided into a double sum over the gird structure, yields (13), given at the top of the next page. The transform from the frequency domain to the correlation domain is thus constructed from the coefficients in (13). One may find the transform by identifying each coefficient

$$\boldsymbol{\mathcal{F}}_{j,k} = \int_{\omega_{2,k}}^{\omega_{2,k+1}} \int_{\omega_{1,j}}^{\omega_{1,j+1}} \frac{\omega_1 - \omega_{1,j}}{\Delta_1} e^{i\omega_1\tau_1 + i\omega_2\tau_2} d\omega_1 d\omega_2 \tag{3}$$

$$\boldsymbol{\mathcal{G}}_{j,k} = \int_{\omega_{2,k}}^{\omega_{2,k+1}} \int_{\omega_{1,j}}^{\omega_{1,j+1}} \frac{\omega_2 - \omega_{2,k}}{\Delta_2} e^{i\omega_1\tau_1 + i\omega_2\tau_2} d\omega_1 d\omega_2 \tag{4}$$

$$\boldsymbol{\mathcal{H}}_{j,k} = \int_{\omega_{2,k}}^{\omega_{2,k+1}} \int_{\omega_{1,j}}^{\omega_{1,j+1}} e^{i\omega_1\tau_1 + i\omega_2\tau_2} d\omega_1 d\omega_2 \tag{5}$$

where it should be noted that the integrands are separable in terms of ω_1 and ω_2. These can then be sorted into four different cases, being

$$\beta_1(n,\ell) = \int_{\omega_{n,\ell}}^{\omega_{n,\ell+1}} d\omega_n = \Delta_n \tag{6}$$

$$\beta_2(n,\ell) = \int_{\omega_{n,\ell}}^{\omega_{n,\ell+1}} \frac{\omega_n - \omega_{n,\ell}}{\Delta_n} d\omega_n = \frac{\Delta_n}{2} \tag{7}$$

$$\beta_3(n,\ell) = \int_{\omega_{n,\ell}}^{\omega_{n,\ell+1}} e^{i\omega_n\tau_n} d\omega_n = \frac{e^{i\omega_{n,\ell}\tau_n}}{i\tau_n}\left(e^{i\Delta_n\tau_n} - 1\right) \tag{8}$$

$$\begin{aligned}\beta_4(n,\ell) = &\int_{\omega_{n,\ell}}^{\omega_{n,\ell+1}} \frac{\omega_n - \omega_{n,\ell}}{\Delta_n} e^{i\omega_n\tau_n} d\omega_n \\ =&e^{i\omega_{n,\ell+1}}\left(\frac{1}{i\tau_n} + \frac{1}{\Delta_n\tau_n^2}[1 - e^{i\Delta_n\tau_n}]\right)\end{aligned} \tag{9}$$

Using these, one obtains

$$\boldsymbol{\mathcal{F}}_{j,k} = \begin{cases} \beta_2(1,j)\beta_1(2,k) & \text{if } \tau_1 = \tau_2 = 0 \\ \beta_2(1,j)\beta_3(2,k) & \text{if } \tau_1 = 0,\ \tau_2 \neq 0 \\ \beta_4(1,j)\beta_1(2,k) & \text{if } \tau_1 \neq 0,\ \tau_2 = 0 \\ \beta_4(1,j)\beta_3(2,k) & \text{if } \tau_1 \neq 0,\ \tau_2 \neq 0 \end{cases} \tag{10}$$

$$\boldsymbol{\mathcal{G}}_{j,k} = \begin{cases} \beta_1(1,j)\beta_2(2,k) & \text{if } \tau_1 = \tau_2 = 0 \\ \beta_1(1,j)\beta_4(2,k) & \text{if } \tau_1 = 0,\ \tau_2 \neq 0 \\ \beta_3(1,j)\beta_2(2,k) & \text{if } \tau_1 \neq 0,\ \tau_2 = 0 \\ \beta_3(1,j)\beta_4(2,k) & \text{if } \tau_1 \neq 0,\ \tau_2 \neq 0 \end{cases} \tag{11}$$

$$\boldsymbol{\mathcal{H}}_{j,k} = \begin{cases} \beta_1(1,j)\beta_1(2,k) & \text{if } \tau_1 = \tau_2 = 0 \\ \beta_1(1,j)\beta_3(2,k) & \text{if } \tau_1 = 0,\ \tau_2 \neq 0 \\ \beta_3(1,j)\beta_1(2,k) & \text{if } \tau_1 \neq 0,\ \tau_2 = 0 \\ \beta_3(2,j)\beta_3(2,k) & \text{if } \tau_1 \neq 0,\ \tau_2 \neq 0 \end{cases} \tag{12}$$

Collecting the terms from (3)-(12) into one transform tensor $\boldsymbol{C}_{j,k}$, the covariance structure in (13) may be expressed as

$$R(\tau_1,\tau_2) = \sum_{j=1}^{M_1}\sum_{k=1}^{M_2} \boldsymbol{C}_{j,k}\Phi(\omega_{1,j},\omega_{2,k}) \tag{14}$$

Using this smooth and band-limited version of the Fourier transform, one may form a 2-D spectral estimate by expressing it as a covariance fitting problem. By minimizing the distance between an initial covariance function estimate, $\hat{\boldsymbol{R}}$, and the covariance function evaluated with the transform, one may formulate the problem as

$$\begin{aligned} \underset{\boldsymbol{\Phi}}{\text{minimize}} \quad & \Big|\Big|\hat{\boldsymbol{R}} - \sum_j\sum_k \boldsymbol{\mathcal{C}}_{j,k}\boldsymbol{\Phi}\Big|\Big|_F^2 \\ \text{subject to} \quad & \boldsymbol{\Phi} - \boldsymbol{\Gamma}_1\boldsymbol{\Phi} = 0 \\ & \boldsymbol{\Phi} - \boldsymbol{\Phi}\boldsymbol{\Gamma}_2 = 0 \end{aligned} \tag{15}$$

where $||\cdot||_F$ denotes the Frobenius norm, and $\boldsymbol{\Gamma}_\ell$ is an exchange matrix of dimension $(M_\ell+1)\times(M_\ell+1)$, for $\ell = 1$ and 2. The two constraints in (15) ensures that the spectral estimate is mirrored in both dimensions and can be dropped if the signal is complex valued.

$$\boldsymbol{R}(\tau_1,\tau_2)=\sum_{j=1}^{M_1}\sum_{k=1}^{M_2}\int_{\omega_{2,k}}^{\omega_{2,k+1}}\int_{\omega_{1,j}}^{\omega_{1,j+1}}\Bigg(\frac{\omega_1-\omega_{1,j}}{\Delta_1}\Phi(\omega_{1,j+1},\omega_{2,k})+\frac{\omega_2-\omega_{2,k}}{\Delta_2}\Phi(\omega_{1,j},\omega_{2,k+1})+ \left(1-\frac{\omega_1-\omega_{1,j}}{\Delta_1}-\frac{\omega_2-\omega_{2,k}}{\Delta_2}\right)\Phi(\omega_{1,j},\omega_{2,k})\Bigg)e^{i\omega_1\tau_1+i\omega_2\tau_2}d\omega_1 d\omega_2 \tag{13}$$

4. SOLVING THE OPTIMIZATION PROBLEM

We proceed by introducing an approximate formulation of (15), which creates more robustness in the estimate

$$\underset{\boldsymbol{Z}}{\text{minimize}}\quad \left|\left|\hat{\boldsymbol{R}}-\sum_j\sum_k \boldsymbol{\mathcal{C}}_{j,k}\boldsymbol{Z}\right|\right|_F^2+ \lambda_1\left|\left|\boldsymbol{Z}-\boldsymbol{\Gamma}_1\boldsymbol{Z}\right|\right|_F^2+\lambda_2\left|\left|\boldsymbol{Z}-\boldsymbol{Z}\boldsymbol{\Gamma}_2\right|\right|_F^2 \tag{16}$$

where λ_1 and λ_1 are regularization parameters. Since all the terms in (16) are convex the problem is convex, thus there is a matrix $\boldsymbol{Z}^*$ that minimizes (16). Furthermore, all the terms in (16) are differentiable which indicates that a analytic solution may be available. One problem with finding the solution is that $\boldsymbol{Z}$ is multiplied from the right in the second term and from the left in the third term of (16). To eliminate this problem, we proceed by rewriting the problem on vector form. Let $\boldsymbol{z}=\text{vec}(\boldsymbol{Z})$ and $\boldsymbol{R}_v=\text{vec}(\boldsymbol{R})$, where $\text{vec}(\cdot)$ denotes the vectorization operator, stacking the columns of the matrix on top of each other, and let

$$\boldsymbol{H}^{(1)}=\boldsymbol{C}_v \tag{17}$$

$$\boldsymbol{H}^{(2)}=\left[\ \boldsymbol{I}_{M_2+1}\otimes\left[\boldsymbol{I}_{M_1+1}-\boldsymbol{\Gamma}_1\right]\ \right] \tag{18}$$

$$\boldsymbol{H}^{(3)}=\left[\ \left[\boldsymbol{I}_{M_2+1}-\boldsymbol{\Gamma}_2\right]^T\otimes\boldsymbol{I}_{M_1+1}\ \right] \tag{19}$$

where $\boldsymbol{I}_{M_\ell+1}$ is the $(M_\ell+1)\times(M_\ell+1)$ identity matrix for $\ell=1$ and 2, $(\cdot)^T$ denotes the transpose, $\boldsymbol{C}_v$ is $\boldsymbol{\mathcal{C}}$ rearranged to account for the vectorization of $\hat{\boldsymbol{R}}$ and $\boldsymbol{Z}$, and $\otimes$ denotes the Kronecker product. We may now write (16) as

$$\underset{\boldsymbol{z}}{\text{minimize}}\quad \left|\left|\hat{\boldsymbol{R}}_v-\boldsymbol{H}^{(1)}\boldsymbol{z}\right|\right|_F^2+ \lambda_1\left|\left|\boldsymbol{H}^{(2)}\boldsymbol{z}\right|\right|_F^2+\lambda_2\left|\left|\boldsymbol{H}^{(3)}\boldsymbol{z}\right|\right|_F^2 \tag{20}$$

Differentiating (20) with respect to $\boldsymbol{z}$ and putting it equal to zero yields

$$-\boldsymbol{H}^{(1)^T}\left(\boldsymbol{R}_v-\boldsymbol{H}^{(1)}\boldsymbol{z}\right)+\lambda_1\boldsymbol{H}^{(2)^T}\boldsymbol{H}^{(2)}\boldsymbol{z}+ \lambda_2\boldsymbol{H}^{(3)^T}\boldsymbol{H}^{(3)}\boldsymbol{z}=0 \tag{21}$$

Solving (21) for $\boldsymbol{z}$ yields

$$\boldsymbol{z}^*=\boldsymbol{G}^{-1}\left(\boldsymbol{H}^{(1)^T}\hat{\boldsymbol{R}}_v\right) \tag{22}$$

where

$$\boldsymbol{G}=\left(\boldsymbol{H}^{(1)^T}\boldsymbol{H}^{(1)}+\lambda_1\boldsymbol{H}^{(2)^T}\boldsymbol{H}^{(2)}+\lambda_2\boldsymbol{H}^{(3)^T}\boldsymbol{H}^{(3)}\right) \tag{23}$$

The regularization parameters λ_1 and λ_2 governs the symmetry in the two frequencies dimensions, i.e., the larger value the more symmetry. We coin the resulting estimator the Covariance-fitting Approach for smooth Two-dimensional Signals (CATS).

5. NUMERICAL RESULTS

We proceed to examine the performance of the proposed CATS method by testing it on simulated 2-D data and comparing it to the Blackman-Tukey (BT) estimate, defined as

$$\hat{\boldsymbol{\Phi}}_{BT}(\omega_1,\omega_2)=\boldsymbol{\Phi}_0(\omega_1,\omega_2)\star\boldsymbol{H}(\omega_1,\omega_2) \tag{24}$$

where

$$\boldsymbol{\Phi}_0(\omega_1,\omega_2)=\left[\frac{1}{N_1N_2}\sum_{j=0}^{N_1-1}\sum_{k=0}^{N_2-1}y(j,k)e^{-i\omega_1\frac{j}{N_1}-i\omega_2\frac{k}{N_2}}\right] \tag{25}$$

and with $\star$ denoting convolution, where $\boldsymbol{H}(\omega_1,\omega_2)$ is the spectral representation of the two-dimensional separable composition of Blackman windows, such that (the corresponding time window)

$$h(j,k)=h_1(j)h_2(k) \tag{26}$$

with

$$h_\ell(k)=0.42+0.5\cos\left(\frac{\pi k}{M_\ell-1}\right)+0.08\cos\left(\frac{\pi k}{M_\ell-1}\right)$$

The considered data is formed as a 2-D ARMA(p_1,p_2,q_1,q_2) process defined as

$$y(t,s)=-\sum_{\substack{k=0\\ k=m\neq 0\text{ simultaneously}}}^{p1}\sum_{m=0}^{p2}a_{k,m}y(t-k,s-m)\ + +\sum_{k=1}^{q1}\sum_{m=1}^{q2}b_{k,m}e(t-k,s-m) \tag{27}$$

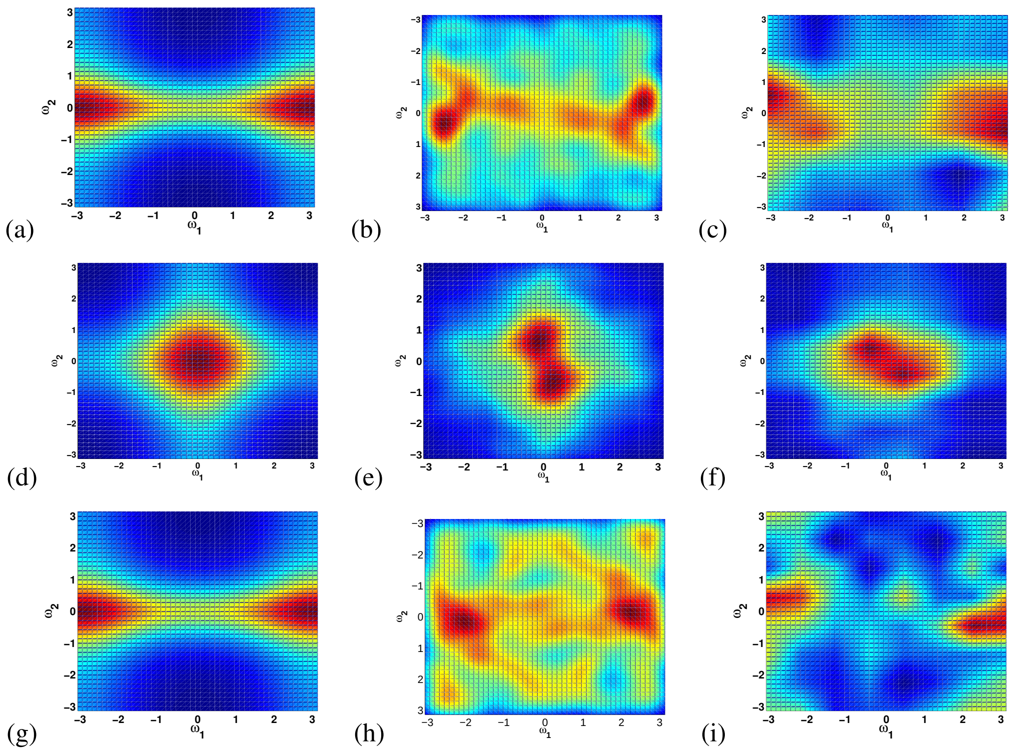

Fig. 2: The figure show the three discussed examples, one on each row, with the leftmost figure showing the true spectrum, the middle the Blackman-Tukey, and the rightmost the estimate obtained by the proposed CATS estimator.

where $a_{k,m}$ and $b_{k,m}$ are the ARMA coefficient, and $e(t,s)$ is a 2-D Gaussian white noise with variance σ^2. The 2-D spectrum of $y(t,s)$ is thus given as

$$\mathbf{\Phi}_y(\omega_1,\omega_2)=\frac{\left|\sum_{k=0}^{q_1}\sum_{m=0}^{q_2} b_{k,m}e^{-i(k\omega_1+m\omega_2)}\right|^2}{\left|\sum_{k=0}^{q_1}\sum_{m=0}^{q_2} a_{k,m}e^{-i(k\omega_1+m\omega_2)}\right|^2}\sigma^2 \tag{28}$$

In all simulations, the regularization parameters of (16) are set to $\lambda_1 = \lambda_2 = 1$. In the first example, the simulation was done using regularly sampled data. Table 1 shows the resulting performance of the proposed estimator as compared to the BT estimate, obtained using for 100 Monte-Carlo runs, where the distance to the true spectrum was measured in three different ways, namely using the mean squared error (MSE) defined as

$$\text{MSE}=\frac{1}{N_{\omega_1}N_{\omega_2}}\sum_{j=1}^{N_{\omega_1}}\sum_{k=1}^{N_{\omega_2}}\left(\Phi(\omega_{1,j},\omega_{2,k})-\hat{\Phi}(\omega_{1,j},\omega_{2,k})\right)^2$$

where N_{ω_1} and N_{ω_2} are the number of frequency points in the BT estimate. Furthermore, the Hausdorff distance is defined as

$$H=\max\left\{\sup_{\omega_{1,j},\omega_{2,k}}\inf_{\omega_{1.j},\omega_{2,k}}\left|\hat{\Phi}(\omega_{1,j},\omega_{2,k})-\Phi(\omega_{1,j},\omega_{2,k})\right|,\right.$$
$$\left.\inf_{\omega_{1,j},\omega_{2,k}}\sup_{\omega_{1.j},\omega_{2,k}}\left|\hat{\Phi}(\omega_{1,j},\omega_{2,k})-\Phi(\omega_{1,j},\omega_{2,k})\right|\right\}$$

and the log-spectral distance (log-SD) as

$$LSD=\sqrt{\frac{1}{N_{\omega_1}N_{\omega_2}}\sum_{j=1}^{N_{\omega_1}}\sum_{k=1}^{N_{\omega_2}}100\log_{10}\left(\frac{\hat{\Phi}(\omega_{1,j},\omega_{2,k})}{\Phi(\omega_{1,j},\omega_{2,k})}\right)^2}$$

The proposed methods is evaluated using three different data sets. In all three cases the observed data is of dimension (50×45), from which the covariance function estimate is formed. The proposed method estimates the spectrum over a grid of size (6×6), and BT uses a window of size 15 (this was found to yield the best achievable performance of all examined windows). The first row of Figure 2 shows a typical realization of the two estimators, where the leftmost is the true spectral density, the middle one depicts the BT estimate, whereas the rightmost shows the result of the proposed method. The corresponding error measures can be seen in

	CATS	Blackman-Tukey
MSE	$0.0216(\pm 0.0064)$	$0.0904(\pm 0.0105)$
Hausdorff	$1.5098(\pm 0.2945)$	$3.3773(\pm 0.1992)$
Log-SD	$0.6275(\pm 0.0919)$	$1.6365(\pm 0.1062)$
MSE	$0.0876(\pm 0.0367)$	$0.0634(\pm 0.0098)$
Hausdorff	$2.4324(\pm 0.6795)$	$3.0231(\pm 0.1732)$
Log-SD	$1.4402(\pm 0.3371)$	$1.8810(\pm 0.1257)$
MSE	$0.0597(\pm 0.0195)$	$0.0997(\pm 0.0117)$
Hausdorff	$2.7433(\pm 0.5706)$	$3.5013(\pm 0.2417)$
Log-SD	$1.1567(\pm 0.2140)$	$1.6601(\pm 0.1090)$

Table 1: The mean estimation error of the spectral estimators in three different measures for the three examples. The standard deviation is given in parenthesis. The top two groups are the results for two different spectral densities estimated from regularly sampled data. The lower group are the results for the third example where the data is observed irregularly.

the top group in Table 1, where the proposed method clearly outperforms the BT estimates. We proceed to examine an alternative spectrum containing a low-frequency tone, as shown in the second row of Figure 2. As seen in the middle part of Table 1 and Figure 2, the proposed method outperforms the BT estimator in the Hausdorff and Log-SD measure, but not in MSE, although the results are similar. Finally, we examine the performance on irregularly sampled data, with the resulting spectral densities depicted in the third row of Figure 2. The data was simulated by randomly removing half of the columns and rows of the data matrix. The lower part of Table 1 and the third row of Figure 2 show the results, where one can see that the proposed method has lowered its performance but is yielding clearly preferable performance as compared to the BT estimator.

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